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On a Reduced Load Equivalence Under Heavy Tail Assumptions *

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Abstract

We propose a general framework for obtaining asymptotic distributional bounds on the stationary backlog $W^{A_1+A_2,c}$ in a buffer fed by a combined fluid process A_1+A_2 and drained at a constant rate c. The fluid process A_1 is an (independent) on-off source with average and peak rates ρ_1 and r_1, respectively, and with distribution G for the activity periods. The fluid process A_2 of average rate ρ_2 is arbitrary but independent of A_1. These bounds are used to identify subexponential distributions G and fairly general fluid processes A_2 such that the asymptotic equivalence $\mathbf{P}\left[W^{A_1+A_2,c} > x\right] \sim \mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]$ $(x \to \infty)$ holds under the stability condition $\rho_1+\rho_2 < c$ and under the non-triviality condition $c-\rho_2 < r_1$. The stationary backlog $W^{A_1,c-\rho_2}$ in these asymptotics results from feeding source A_1 into a buffer drained at *reduced* rate $c-\rho_2$. This reduced load asymptotic equivalence extends to a larger class of distributions G a result obtained by Jelenkovic and Lazar [18] in the case when G belongs to the class of regular intermediate varying distributions.

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1 Introduction

On-off sources provide a natural and versatile tool for modeling incoming traffic at a node (router) or gateway of a network. An on-off source is characterized as a stochastic process which alternates between periods of silence (off) and activity (on). During on periods such a source generates data continuously at a constant rate, and becomes silent during off periods; a more acurate definition will be given in Section 2.

As traffic flows generated by multiple on-off sources are typically multiplexed onto a single link, it is of great practical importance to address the corresponding buffering issues in an effort to make efficient use of network resources. This is often carried out in the context of the following simple model for a multiplexer: The superposition of these on-off sources is offered to a single infinite capacity buffer which is drained at a constant rate. If W denotes the resulting stationary backlog (assumed to exist), then it is expected that its probability distribution crucially depends on the statistics of the on periods. For instance, it is well known that for a single *exponential* on-off source, i.e., a source with on (and off) periods which are exponentially distributed, the tail distribution $\mathbf{P}[W > x]$ decays exponentially fast as x tends to infinity [1]. This exponential decay property is preserved under multiplexing in the sense that when several independent exponential on-off sources are combined, the tail distribution $\mathbf{P}[W > x]$ still decays exponentially fast [12]. Both situations discussed so far are instances of a class of Markov modulated fluid models which has been extensively studied [1, 12, 23, 27] since the seminal work of Kosten [20]. A fairly comprehensive theory has been developed for such sources, and algorithms are now available for the numerical evaluation of the tail distribution $\mathbf{P}[W > x]$ for all values of x [2, 27].

On the other hand, the situation is quite different when at least one of the on-off sources has heavy-tailed on periods. The need for considering such models with heavy-tailed components can be traced back to recent measurements of network traffic [21] which exhibit long-range dependence and burstiness over an extremely wide range of time scales. Along these lines, for a single on-off source with *subexponential* distribution G for the on periods, Jelenkovic and Lazar [18, Thm 9] have shown that

$$\mathbf{P}[W > x] \sim K_0 \int_{x/\alpha_0}^{\infty} (1 - G(u))\, du \quad (x \to \infty) \tag{1.1}$$

for appropriate constants $K_0, \alpha_0 > 0$ determined by the source statistics (such as average and peak rates) and the buffer release rate (Remark 4.1); typical examples of subexponential distributions include the Weibull, log-normal and generalized Pareto distributions [14]. In fact, the asymptotic (1.1) extended a result obtained earlier by Boxma [6, Thm 5.1] for generalized Pareto (or regularly varying) distributions G of the form

$$1 - G(x) = x^{-\beta} L(x), \quad x > 0$$

with $\beta > 0$ and some regularly varying $L : \mathbb{R}_+ \to \mathbb{R}_+$ [5]. Recently, the case $1 < \beta < 2$ has been viewed with particular interest since it corresponds to the input process being long-range dependent.

Extensions of (1.1) to multiple on-off sources have been considered in the literature, with a survey of related recent results available in [7]. In [6, Thm 6.1] Boxma already showed that if a single on-off source (source 1) with regularly varying on periods shares the buffer with an exponential on-off source (source 2), then the former dominates the behavior of the buffer. This result was extended by Jelenkovic and Lazar [18, Thm 10] to two on-off sources with more general statistics. It was shown that source 1 dominates the behavior of the buffer if its on periods have an *intermediate regular varying* distribution [10] as long as the tail of the on periods of source 1 is heavier (in some precise technical sense) than that of source 2. A noteworthy byproduct of the results in [6, 18] is that the source 2 contributes to the *asymptotic* behavior of $\mathbf{P}[W > x]$ only through its *average* fluid generation rate ρ_2. In fact, the following rephrasing of these results was first pointed out by Jelenkovic and Lazar [18, Thm 10]: Let A_1 and A_2 denote the on-off sources 1 and 2 with average rates ρ_1 and ρ_2, and peak rates r_1 and r_2, respectively. The combined (or multiplexed) fluid process $A_1 + A_2$ is offered to a buffer which is drained at rate c (fluid units/sec) under the stability condition $\rho_1 + \rho_2 < c$. Under the non-triviality condition $c - \rho_2 < r_1$, the corresponding stationary backlog $W^{A_1+A_2,c}$ has the property

$$\mathbf{P}\left[W^{A_1+A_2,c} > x\right] \sim \mathbf{P}\left[W^{A_1,c-\rho_2} > x\right] \quad (x \to \infty) \tag{1.2}$$

where the stationary backlog $W^{A_1,c-\rho_2}$ results from feeding source A_1 into a buffer drained at the *reduced* rate $c - \rho_2$. This asymptotic equivalence (1.2) reflects the following intuitive notion: The tail of Pareto distributed activity periods of source 1 is considerably heavier than the exponential tails governing the exponential on-off source 2. As a result, a single on period for source 1 is likely to correspond to a large number of successive on and off periods in source 2. Such a disparity in time scales is enough for the Law of Large Numbers to kick in for source 2, effectively averaging out random fluctuations about the mean ρ_2 and replacing them by the average behavior of the source 2. It is now a small step to believe in the plausibility of (1.2).

The *reduced load (asymptotic) equivalence* (1.2) suggests a natural way of approximating the distribution of $W^{A_1+A_2,c}$ with that of $W^{A_1,c-\rho_2}$. As this latter quantity is associated with the single source A_1, a reduction in computational efforts may result, at least in principle. For instance, when applicable, (1.1) and (1.2) together imply

$$\mathbf{P}\left[W^{A_1+A_2,c} > x\right] \sim K_1 \int_{x/\alpha_1}^{\infty} (1 - G(u))\, du \quad (x \to \infty)$$

for appropriate constants $K_1, \alpha_1 > 0$ determined by the statistics of A_1 and by the release rate $c - \rho_2$. Given its asymptotic basis, this approximation will become increasingly accurate with x large, a property that might make it well suited in various contexts for evaluating very small cell loss probabilities via buffer overflow probabilities. However, the accuracy of the resulting estimates remains an open question, with some indications that it might be poor.

Leaving aside these computational issues, we note that the class of intermediate regular varying distributions in [18] includes regularly varying distributions, but does *not* contain the log-normal and Weibull distributions. Hence, the validity of (1.2) is already in question when the activity period of source 1 is characterized by these standard subexponential distributions. However, the plausibility argument made for (1.2) in the Pareto case, if indeed correct, holds out the possibility that the range of validity for (1.2) extends beyond the class of intermediate regular varying distributions. In particular, in the same way that Boxma's result for single fluid with regularly varying activity periods in [6] was generalized as (1.1) by Jelenkovic and Lazar [18], it is natural to speculate whether (1.2) holds more generally for the class of subexponential distributed activity periods. This question forms the motivation behind the developments presented here as we revisit the model in [18] when source 1 has a *subexponential* activity period.

To help the reader navigate the many technical sections of the paper, we summarize below the approach taken to establishing (1.2) and some of the paper's main contributions: The arguments articulate around lower and upper bounds, which in the best of cases take the form

$$1 \leq \liminf_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]} \quad \text{and} \quad \limsup_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]} \leq 1. \tag{1.3}$$

Our point of departure for establishing such bounds is the representation

$$W^{A_1+A_2,c} =_{st} \sup_{t\geq 0} \left(A_1(t) + A_2(t) - ct\right) \tag{1.4}$$

which holds under the usual assumptions (Section 2). As in [18] we introduce perturbations $h : \mathbb{R}_+ \to \mathbb{R}$ in order to write (1.4) as

$$\begin{aligned} W^{A_1+A_2,c} &=_{st} \sup_{t\geq 0} \left(A_1(t) - (c-\rho_2)t + A_2(t) - \rho_2 t\right) \\ &= \sup_{t\geq 0} \left(A_1(t) - (c-\rho_2)t + h(t) + A_2(t) - \rho_2 t - h(t)\right). \end{aligned} \tag{1.5}$$

While only *linear* perurbations sufficed in [18], we shall need *general* perturbations for handling the broader class of subexponential distributions. This decomposition (1.5) is then invoked in Section 2 in order to derive generic bounds. These bounding arguments hold in a fairly general framework, and are given in terms of the "perturbed" backlog associated with source 1 after load reduction, namely

$$W^{A_1,c-\rho_2,h} := \sup_{t\geq 0} \left(A_1(t) - (c-\rho_2)t + h(t)\right).$$

Asymptotic bounds on the tail distribution of the random variable (rv) $W^{A_1,c-\rho_2,h}$ are established in Section 4 when the source A_1 is a standard independent on-off source with subexponential activity periods; the needed facts on subexponential distributions are collected in Section 3. These asymptotic bounds on $W^{A_1,c-\rho_2,h}$ can now be used in conjunction with the bounds developed for $W^{A_1+A_2,c}$ in Section 2. This is done in Sections 5, 6 and 7 when the source A_1 is a standard independent on-off source with subexponential activity periods and source A_2 is a fairly general fluid source satisfying at minimum a Central Limit Theorem. The bounds are optimized by considering perturbations of the form εh with ε small and letting ε go to zero in the resulting bounds.

Still we need to identify the best perturbation "direction" h, best in the sense of making the lower (resp. upper) bound largest (resp. smallest). In the determination of a suitable h for either the lower or upper bound, a crucial and central role is played by the mapping $m_G : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$m_G(x) := \frac{\int_x^\infty (1 - G(t))\, dt}{1 - G(x)}, \quad x \geq 0$$

with G denoting the distribution of the activity period. In fact, when G is subexponential but not regularly varying, the requirements imposed on h for optimizing the bounds coincide with well-known properties enjoyed by distributions in the maximum domain of attraction of the Gumbel distribution [14], most notably properties characterizing the so-called *auxiliary* function associated with such distributions. The Weibull, log-normal and Bentkander distributions belong to this maximum domain of attraction, thereby allowing us to take $h(x) \sim m_G(x)$ $(x \to \infty)$ in these cases and to conclude to the validity of (1.2) under certain conditions; these examples are discussed in Section 8. The importance of the function m_G is reinforced by a negative result of Dumas and Simonian [11] to the effect that the asymptotic equivalence cannot hold if $\lim_{x\to\infty} m_G(x)/\sqrt{x} = 0$. In particular, while (1.2) holds for all log-normal distributions, it holds only for the Weibull distributions that have heavy enough tails as in (8.17).

Generalized Pareto rvs belong to the maximum domain of attraction of the Féchet distribution, and *not* of the Gumbel distribution [14]. However, the approach presented here applies to that case as well; in fact, $m_G(x) \sim x$ $(x \to \infty)$ and points of contact with Extreme Value Theory also emerge since the arguments of Jelenkovic and Lazar [18] were based on using linear perturbations. The somewhat singular nature of the regularly varying case is made increasingly apparent in Section 9 where we provide various extensions for it: Following [18] we show that the validity of (1.2) also holds when G belongs to the larger class of intermediate regular varying rvs [9]. The asymptotic "scale invariance" property of these rvs is shown to imply the validity of the upper bound in (1.3) *without* independence of the sources.

In Section 10.1 we consider the case when A_1 is a superposition of independent on-off sources. Exact asymptotics for this case are notoriously hard to obtain in general; we contend ourselves with a lower bound that extends an earlier result of Choudhury and Whitt [8]. Finally, in Section 10.2 we discuss the situation often encountered in practice where A_2 itself is an aggregation of several sources, typically with a simpler probabilistic structure (e.g., independent on-off sources); we show how the necessary technical conditions on the component sources transfer to the aggregate source. Section 11 closes the paper with a list of open problems associated with the reduced load equivalence (1.2).

To facilitate the reading of this long paper, we have relegated proofs of major technical results to several appendices. A word on the notation in use: Throughout $\Longrightarrow_t$ denotes the convergence in law with t going to infinity. Equivalence in law or in distribution between rvs and stochastic processes is denoted by $=_{st}$, and we use $\leq_{st}$ for the strong stochastic ordering between rvs. Also for mappings $f, g : \mathbb{R}_+ \to \mathbb{R}$, the relation $f(x) \sim g(x)$ is understood as $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, the qualifier $(x \to \infty)$ being omitted for the sake of notational simplicity. For any scalar x, we write $x^+ = \max(x, 0)$ and with any mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}$, we associate the mapping $\varphi^+ : \mathbb{R}_+ \to \mathbb{R}_+$ defined through

$\varphi^+(x) := (\varphi(x))^+$ for all x in $\mathbb{R}$.

2 Generic bounds

We adopt the following framework in order to develop several generic bounds: A fluid process (or source) is defined as any $\mathbb{R}_+$-valued stochastic process $A = \{A(t),\, t \geq 0\}$ with non-decreasing right-continuous sample paths such that

$$\lim_{t\to\infty} \frac{A(t)}{t} = \rho \quad \text{a.s.};$$

as usual the non-negative constant ρ represents the average rate of the source.

We interpret $A(t)$ as the amount of fluid generated by the source in the interval $[0, t)$. If the fluid process $A := \{A(t),\, t \geq 0\}$ is offered to an infinite capacity buffer from which it is drained at the constant rate of c (fluid units/sec), then it is well known that the corresponding backlog at time $t \geq 0$ is given by

$$W^{A,c}(t) = \sup_{0\leq s\leq t} (A(t) - A(s) - c(t-s))$$

provided the buffer is empty at time $t = 0$.

The following facts are well known [22]: If the arrival process A has *stationary* increments, then $W^{A,c}(t) \Longrightarrow_t W^{A,c}$ where the *stationary backlog* rv $W^{A,c}$ has the representation

$$W^{A,c} = \sup_{t\geq 0} (A_R(t) - ct) \tag{2.1}$$

with $A_R := \{A_R(t),\, t \geq 0\}$ denoting the *time-reversed* process associated with A. Moreover, the rv $W^{A,c}$ is a.s. finite under the stability condition $\rho < c$.

The fluid models encountered in practice have stationary increments. In such cases, in dealing with (2.1), with a slight abuse of notation we denote the process A_R by A instead, or equivalently, we interpret the arrival process A backwards in time. We note that in many important instances (Section 4), the distributional equivalence $\{A_R(t),\, t \geq 0\} =_{st} \{A(t),\, t \geq 0\}$ holds.

Except in a few isolated cases (e.g., [1, 23, 27]), characterizing the distribution of the stationary backlog rv $W^{A,c}$ is a difficult task, not to say an impossible one. As a result, we resort to studying the tail behavior of $W^{A,c}$, and we do so by deriving lower and upper bounds on the quantity of interest in structured situations. In particular, we have in mind situations where several fluid sources are multiplexed onto a single link. As the bounds are obtained by a perturbation technique, we find it useful to generalize the definition (2.1) by associating with any mapping $h : \mathbb{R}_+ \to \mathbb{R}$ the rv $W^{A,c,h}$ given by

$$W^{A,c,h} := \sup_{t\geq 0} (A(t) - ct + h(t))\,.$$

In due course, several assumptions will be imposed on such a *perturbation* mapping h.

In Sections 2.1 and 2.2, we consider the case when the fluid process A is obtained as the superposition of two fluid processes A_1 and A_2 as understood earlier in this section, with no additional assumptions. For each $i = 1, 2$, let ρ_i denote the average rate of source A_i and let $c > 0$ denote the release rate of the fluid.

2.1 Generic lower bounds

We begin with a generic lower bound on the tail distribution of the stationary backlog.

Proposition 2.1 *Assume the arrival process A to be the superposition $A = A_1 + A_2$ of two independent fluid processes such that*

$$A_1(t) \le r_1 t, \quad t \ge 0 \tag{2.2}$$

Then, for any Borel mapping $h : \mathbb{R}_+ \to \mathbb{R}$, it holds for each $x \ge 0$ that

$$\mathbf{P}\left[W^{A_1+A_2,c} > x\right] \ge \left(\inf_{\{t:(r_1+\rho_2-c)t+h(t)>x\}} \mathbf{P}\left[A_2(t) \ge \rho_2 t + h(t)\right]\right) \mathbf{P}\left[W^{A_1,c-\rho_2,h} > x\right]. \tag{2.3}$$

Proof: Fix $d > 1$ and pick $0 < \delta < 1 - d^{-1}$. Fix $x \ge 0$. There exists an a.s. finite and nonnegative rv τ_δ such that

$$\min\left(dx, W^{A_1,c-\rho_2,h}\right) \le \frac{1}{1-\delta}\left(A_1(\tau_\delta) - (c-\rho_2)\tau_\delta + h(\tau_\delta)\right).$$

This can be seen by considering separately the cases $W^{A_1,c-\rho_2,h} = \infty$ and $W^{A_1,c-\rho_2,h} < \infty$; the rv τ_δ may depend on x.

Let $a(t) = \alpha t + h(t)$ $(t \ge 0)$ with $\alpha := r_1 + \rho_2 - c$. The rv τ_δ being independent of A_2, it holds that

$$\begin{aligned}
&\mathbf{P}\left[W^{A,c} > x\right] \\
&\ge \mathbf{P}\left[A_1(\tau_\delta) - (c-\rho_2)\tau_\delta + h(\tau_\delta) + A_2(\tau_\delta) - \rho_2\tau_\delta - h(\tau_\delta) > x\right] \\
&\ge \mathbf{P}\left[A_1(\tau_\delta) - (c-\rho_2)\tau_\delta + h(\tau_\delta) > x, A_2(\tau_\delta) - \rho_2\tau_\delta - h(\tau_\delta) \ge 0\right] \\
&= \mathbf{P}\left[A_1(\tau_\delta) - (c-\rho_2)\tau_\delta + h(\tau_\delta) > x, a(\tau_\delta) > x, A_2(\tau_\delta) - \rho_2\tau_\delta - h(\tau_\delta) \ge 0\right] \\
&= \int_{\{t\,:\,a(t)>x\}} \mathbf{P}\left[A_2(t) \ge \rho_2 t + h(t)\right] \mathbf{P}\left[A_1(t) - (c-\rho_2)t + h(t) > x \mid \tau_\delta = t\right] \mathbf{P}\left[\tau_\delta \in dt\right] \\
&\ge \left(\inf_{\{t:a(t)>x\}} \mathbf{P}\left[A_2(t) \ge \rho_2 t + h(t)\right]\right) \mathbf{P}\left[A_1(\tau_\delta) - (c-\rho_2)\tau_\delta + h(\tau_\delta) > x\right] \\
&\ge \left(\inf_{\{t:a(t)>x\}} \mathbf{P}\left[A_2(t) \ge \rho_2 t + h(t)\right]\right) \mathbf{P}\left[\min\left(dx, W^{A_1,c-\rho_2,h}\right) > \frac{x}{1-\delta}\right] \\
&= \left(\inf_{\{t:a(t)>x\}} \mathbf{P}\left[A_2(t) \ge \rho_2 t + h(t)\right]\right) \mathbf{P}\left[W^{A_1,c-\rho_2,h} > \frac{x}{1-\delta}\right]
\end{aligned}$$

where the first equality made use of the constraint (2.2) on source A_1, and the last inequality follows from the definition of τ_δ. Letting $\delta \downarrow 0$ in this last inequality yields the desired conclusion. ∎

The interest in this lower bound resides in the fact that the two sources have now been *decoupled*, with source A_1 (resp. A_2) entering only the second (resp. first) factor. In Section 4, we focus on the evaluation of the first factor in the more restricted context of stationary independent on-off sources.

2.2 Generic upper bounds

The upper bounds derived in this paper all flow from the following observation:

Lemma 2.1 *Assume the arrival process A to be the superposition $A = A_1 + A_2$ of two fluid processes. For any mapping $h : \mathbb{R}_+ \to \mathbb{R}$, it holds that*

$$W^{A,c} \leq W^{A_1,c-\rho_2,h} + W^{A_2,\rho_2,-h}. \tag{2.4}$$

Proof: We note that

$$\begin{aligned} W^{A,c} &= \sup_{t\geq 0} \left(A_1(t) - (c-\rho_2)t + h(t) + A_2(t) - \rho_2 t - h(t)\right) \\ &\leq \sup_{t\geq 0} \left(A_1(t) - (c-\rho_2)t + h(t)\right) + \sup_{t\geq 0} \left(A_2(t) - \rho_2 t - h(t)\right) \end{aligned} \tag{2.5}$$

and the conclusion (2.4) follows. ∎

This upper bound is interesting only when the rvs $W^{A_1,c-\rho_2,h}$ and $W^{A_2,\rho_2,-h}$ are a.s. finite; a necessary condition on h for this to happen is given by

$$0 \leq \liminf_{t\to\infty} \frac{h(t)}{t} \leq \limsup_{t\to\infty} \frac{h(t)}{t} \leq c - (\rho_1 + \rho_2)$$

with the condition becoming sufficient if both outmost inequalities hold as strict inequalities. When some of these outmost inequalities hold only as equalities, the conclusion depends on the growth behavior of h at infinity, such situations being discussed in Section 6.3.

3 Preliminaries

For easy reference, we collect below some definitions and technical facts that are used throughout the paper. The proofs of the various lemmas are provided in Appendix A.

We first recall the definitions of various classes of probability distributions on $\mathbb{R}_+$ which are of interest here: An $\mathbb{R}_+$-valued rv X is said to have a *long tail*, denoted $X \in \mathcal{L}$, if

$$\lim_{x\to\infty} \frac{\mathbf{P}[X > x-y]}{\mathbf{P}[X > x]} = 1, \quad y \in \mathbb{R}; \tag{3.1}$$

a *subexponential tail*, denoted $X \in \mathcal{S}$, if

$$\lim_{x\to\infty} \frac{\mathbf{P}[X + X' > x]}{\mathbf{P}[X > x]} = 2$$

where X' is an independent copy of X.

Additional material on these classes of distributions can be found in the monograph [14]. It is well known that the class $\mathcal{S}$ is a subclass of $\mathcal{L}$ [13]. Also, it is worth noting that (3.1) holds for all $y \ge 0$ if and only if it holds for all $y \le 0$.

We follow up with some standard (and some less standard) results on long-tailed distributions.

Lemma 3.1 *Let X, Y, Z and T denote four mutually independent $\mathbb{R}$-valued rvs such that X, Y and Z are nonnegative.*

(1) If $X \in \mathcal{S}$ (resp. $\mathcal{L}$) and $\mathbf{P}[Y > x] \sim c\,\mathbf{P}[X > x]$ for some positive constant c, then $Y \in \mathcal{S}$ (resp. $\mathcal{L}$);

(2) If $X \in \mathcal{L}$, then $\mathbf{P}[X - Y + d > x] \sim \mathbf{P}[X > x]$ for any scalar d;

(3) If $Z \in \mathcal{S}$, $\mathbf{P}[X > x] \sim c_1\,\mathbf{P}[Z > x]$ and $\mathbf{P}[T > x] \sim c_2\,\mathbf{P}[Z > x]$ for constants $c_1 \ge 0$ and $c_2 \ge 0$, then $\mathbf{P}[X + T > x] \sim (c_1 + c_2)\,\mathbf{P}[Z > x]$. In particular, $(X+T)^+ \in \mathcal{S}$ if $c_1 + c_2 > 0$.

Next, with any $\mathbb{R}_+$-valued rv X with $0 < \mathbf{E}[X] < \infty$, we associate the $\mathbb{R}_+$-valued rv $X^\star$ whose distribution is the *integrated tail* distribution of X, namely

$$\mathbf{P}[X^\star \le x] = \frac{1}{\mathbf{E}[X]} \int_0^x \mathbf{P}[X > u]\,du, \quad x \ge 0. \tag{3.2}$$

Some useful facts concerning $X^\star$ are contained in the following lemma.

Lemma 3.2 *Let X and Y be independent $\mathbb{R}_+$-valued rvs. If $X \in \mathcal{L}$ with $0 < \mathbf{E}[X] < \infty$, then we have*

$$\mathbf{P}[X > x] = o(\mathbf{P}[X^\star > x]) \tag{3.3}$$

and

$$\int_x^\infty \mathbf{P}[X - Y > u]\,du \sim \mathbf{E}[X]\,\mathbf{P}[X^\star > x]. \tag{3.4}$$

The next two lemmas address the transfer of tail properties of the rv X to that of a transformed rv $\varphi(X)$ for some mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. References to absolute continuity are given in [19, p. 336].

Lemma 3.3 *Consider a Borel mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}$ such that*

(i) φ is strictly increasing in the limit, i.e., there exists $x_0 \geq 0$ such that the restriction $\varphi : [x_0, \infty) \to \mathbb{R}$ is strictly increasing, with $\lim_{x\to\infty} \varphi(x) = \infty$ and

$$\Phi_0 := \sup_{0 \leq x \leq x_0} \varphi(x) < \infty; \tag{3.5}$$

(ii) φ is absolutely continuous on (x_0, ∞) and the limit

$$\lim_{x\to\infty} \varphi'(x) =: \Phi \tag{3.6}$$

exists and is finite.

Then, for any $\mathbb{R}_+$-valued rv X with $0 < \mathbf{E}[X]$, $\mathbf{E}[\varphi^+(X)] < \infty$, we have

$$\frac{\mathbf{P}[(\varphi^+(X))^\star > x]}{\mathbf{P}[\varphi^+(X^\star) > x]} \sim \Phi \frac{\mathbf{E}[X]}{\mathbf{E}[\varphi^+(X)]}. \tag{3.7}$$

Lemma 3.4 *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a mapping strictly increasing and convex in the limit, i.e., there exists $x_0 \geq 0$ such that the restriction $\varphi : [x_0, \infty) \to \mathbb{R}$ is strictly increasing and convex. Under the finiteness condition (3.5), it holds that $\varphi(X) \in \mathcal{L}$ (resp. $\mathcal{S}$) if $X \in \mathcal{L}$ (resp. $\mathcal{S}$).*

We close this section with facts that will help us identify the appropriate perturbation mappings needed to apply the generic bounds of Sections 2.1 and 2.2. For any $\mathbb{R}_+$-valued rv X, with

$$\mu(X) := \sup\{x \geq 0 : \mathbf{P}[X \leq x] < 1\},$$

we introduce the function $m_X : [0, \mu(X)) \to \mathbb{R}_+$ given by

$$m_X(x) := \mathbf{E}[X] \frac{\mathbf{P}[X^\star > x]}{\mathbf{P}[X > x]}, \quad x \in [0, \mu(X)). \tag{3.8}$$

If $\mu(X) = \infty$, then $\mu(X^\star) = \infty$ and m_X is defined on the entirety of $\mathbb{R}_+$. This situation is not restrictive for our purpose as we have $\mu(X) = \infty$ whenever $X \in \mathcal{L}$, in which case $\lim_{x\to\infty} m_X(x) = \infty$ by Lemma 3.2.

Lemma 3.5 *Consider an $\mathbb{R}_+$-valued rv X with $\mu(X) = \infty$ and $0 < \mathbf{E}[X] < \infty$. Assume that*

$$\lim_{x\to\infty} \frac{\mathbf{P}[X > x + y\varphi(x)]}{\mathbf{P}[X > x]} = \gamma(y), \quad y \in \mathbb{R} \tag{3.9}$$

for mappings $\gamma : \mathbb{R} \to \mathbb{R}_+$ *and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\lim_{x\to\infty} \varphi(x) = \infty$.

Then, for y *in* $\mathbb{R}$, *it also holds that*

$$\lim_{x\to\infty} \frac{\mathbf{P}\left[X^\star > x + y\varphi(x)\right]}{\mathbf{P}\left[X^\star > x\right]} = \gamma(y) \tag{3.10}$$

provided

$$\lim_{x\to\infty} \frac{m_X(x + y\varphi(x))}{m_X(x)} = 1. \tag{3.11}$$

Such limits are invariant under asymptotic equivalence, i.e., if φ and ψ are mappings $\mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(x) \sim \psi(x)$, then φ satisfies (3.9) with $\lim_{x\to\infty} \varphi(x) = \infty$ if and only ψ does with $\lim_{x\to\infty} \psi(x) = \infty$, and the limiting function γ is the same.

Limits of the type (3.9) are well known in Extreme Value Theory [14, 25] where they occur in the characterization of maximum domains of attraction; we refer the reader to [14, Chap. 3] [25, Chap. 1] for additional information on this topic. Of particular interest are several technical facts which are summarized below for easy reference. Recall the definition of the Gumbel distribution Λ as the distribution on $\mathbb{R}$ given by

$$\Lambda(x) = e^{-e^{-x}}, \quad x \in \mathbb{R}.$$

The needed results are culled from [14, Thm. 3.3.27, p. 143], [16, Thm. 2.5.1], [25, Lem. 1.3, p. 41] and [25, Cor. 1.7, p. 46], and are specialized below to $\mathbb{R}_+$-valued rvs with infinite support.

Lemma 3.6 *The* $\mathbb{R}_+$*-valued rv* X *with* $\mu(X) = \infty$ *and* $) < \mathbf{E}[X] < \infty$ *belongs to the maximum domain of attraction of* Λ*, denoted* $X \in MDA(\Lambda)$*, if and only if there exists a mapping* $\varphi : \mathbb{R}_+ \to (0, \infty)$ *such that (3.9) holds with*

$$\lim_{x\to\infty} \frac{\mathbf{P}\left[X > x + y\varphi(x)\right]}{\mathbf{P}\left[X > x\right]} = e^{-y}, \quad y \in \mathbb{R}.$$

A possible choice is $\varphi = m_X$ *given by (3.8), in which case (3.11) holds.*

The log-normal, Weibull and Benktander distributions belong to MDA(Λ) [14, pp. 149-150], among others.

In view of Lemmas 3.5 and 3.6, we conclude that for any $\mathbb{R}_+$-valued rv X with $\mu(X) = \infty$ and $0 < \mathbf{E}[X] < \infty$, membership in $MDA(\Lambda)$ implies

$$\lim_{x\to\infty} \frac{\mathbf{P}\left[X^\star > x + y m_X(x)\right]}{\mathbf{P}\left[X^\star > x\right]} = e^{-y}, \quad y \in \mathbb{R}. \tag{3.12}$$

As we shall see in Section 8, the conclusions of Lemma 3.6 hold *mutatis mutandis* for the class of generalized Pareto distributions; interestingly enough these are exactly the rvs which are in the maximum domain of attraction of Fréchet distributions Φ_β ($\beta > 0$) [14, p. 121].

4 On-off sources

4.1 Preliminaries

An *independent on-off* source with peak rate r is characterized by a succession of cycles, each such cycle comprising an off-period followed by an on-period. During the on-periods the source is active and produces fluid at constant rate r (unit fluid/unit time); the source is silent during the off-periods. The on-period durations $\{B_n,\ n = 0, 1, \ldots\}$ and the off-period durations $\{I_n,\ n = 0, 1, \ldots\}$ are mutually independent sequences, each composed of i.i.d. rvs such that

$$0 < \mathbf{E}[B_n], \mathbf{E}[I_n] < \infty, \quad n = 0, 1, \ldots \tag{4.1}$$

It is convenient to introduce the sequence of epochs $\{T_n,\ n = 0, 1, \ldots\}$ marking the beginning of successive cycles, namely $T_0 := 0$ and $T_{n+1} := \sum_{k=0}^{n} I_k + B_k$ for each $n = 0, 1, \ldots$. Thus, at time T_n begins the $(n+1)^{st}$ cycle with of-period duration I_n and on-period duration B_n. The activity of the source is characterized by the $\{0, 1\}$-valued process $\xi := \{\xi(t),\ t \geq 0\}$ given by

$$\xi(t) := \sum_{n=0}^{\infty} \mathbf{1}[T_n + I_n \leq t < T_{n+1}], \quad t \geq 0, \tag{4.2}$$

with the source active (resp. silent) at time t if $\xi(t) = 1$ (resp. $\xi(t) = 0$). The total amount of fluid generated in $[0, t)$ by the on-off source is now simply given by

$$A(t) = r \int_0^t \xi(s)ds, \quad t \geq 0. \tag{4.3}$$

The integrability condition (4.1) ensures these auxiliary quantities to be well defined and finite. Both processes $\{A(t),\ t \geq 0\}$ and $\{\xi(t),\ t \geq 0\}$ have right-continuous sample paths.

Under these assumptions, the following facts are well known: The process ξ admits a (time) stationary version which we still denote by $\{\xi(t),\ t \in \mathbb{R}\}$ (with a slight abuse of notation). Moreover, its time-reversed version $\{\xi(-t),\ t \in \mathbb{R}\}$ is statistically indistinguishable from $\{\xi(t),\ t \in \mathbb{R}\}$ itself. Consequently, the distributional equivalence $\{A_R(t), t \geq 0\} =_{st} \{A(t), t \geq 0\}$ does hold here and (2.1) takes the simpler form

$$W^{A,c} =_{st} \sup_{t \geq 0} (A(t) - ct) \tag{4.4}$$

where $A = \{A(t), t \geq 0\}$ computed through (4.3) with the stationary version of (4.2). From now on, with a slight abuse of terminology we refer to A so defined as a stationary (independent) on-off source.

We shall find it handy in the sequel to use the following construction of this stationary (independent) on-off source: We postulate rvs $\{I_n, B_n,\ n = 0, 1, \ldots\}$ describing the alternating sequence of off- and on-period durations starting with an off-period of duration I_0; if $I_0 = 0$ the source is construed as starting in an on-period. In the stationary regime considered here, standard renewal-theoretic considerations require that (i) (I_0, B_0), $\{I_n,\ n = 1,\ldots\}$ and $\{B_n,\ n = 1,\ldots\}$ be mutually independent families of rvs; (ii) the rvs $\{I_n,\ n = 1,\ldots\}$ (resp. $\{B_n,\ n = 1,\ldots\}$) be i.i.d. rvs with I_1 (resp. B_1) distributed as the generic off-period (resp. on-period), and (iii) the relations

$$[(I_0, B_0) \mid I_0 > 0] =_{st} (I_1^\star, B_1) \quad \text{and} \quad [(I_0, B_0) \mid I_0 = 0] =_{st} (0, B_1^\star) \tag{4.5}$$

hold with $I_1^\star$ independent of B_1. Under such assumptions, we check that

$$[I_0 + B_0 \mid I_0 > 0] =_{st} I_1^\star + B_1 \quad \text{and} \quad [I_0 + B_0 \mid I_0 = 0] =_{st} B_1^\star \tag{4.6}$$

and

$$\mathbf{P}\left[I_0 = 0\right] = \frac{\mathbf{E}\left[B_1\right]}{\mathbf{E}\left[B_1\right] + \mathbf{E}\left[I_1\right]} =: p.$$

The average rate ρ of the source is

$$\rho := \lim_{t\to\infty} \frac{A(t)}{t} = pr$$

and an independent on-off source is a particular instance of a fluid process as defined in Section 2.

The construction developed here for a stationary on-off source differs from the usual one (e.g., [17]) in two respects: The first cycle always starts at time $t = 0$ with an off-period (albeit of possible duration $I_0 = 0$) so that *every* cycle contains an activity period. Moreover, the model is prescribed through the requirement (4.5), instead of the more usual requirement (4.6) (implied by it). These features will simplify the presentation and discussion of several results by permitting direct sample path arguments, notably in Proposition 4.1.

4.2 Bounds for on-off sources

As is already apparent from the generic bounds discussed in Sections 2.1 and 2.2, it will be useful to investigate the tail behavior of $W^{A,\gamma,h}$ under a wide class of perturbations $h : \mathbb{R}_+ \to \mathbb{R}$. This will be done by first establishing bounds on $W^{A,\gamma,h}$, and then by identifying the tail asymptotics of the bounds.

With any mapping $h : \mathbb{R}_+ \to \mathbb{R}$, we associate the auxiliary mappings $a, b : \mathbb{R}_+ \to \mathbb{R}$ given by

$$a(t) := (r - \gamma)t + h(t) \quad \text{and} \quad b(t) := -\gamma t + h(t), \quad t \geq 0. \tag{4.7}$$

The $\mathbb{R}$-valued rvs $\{X_k,\ k = 0, 1, \ldots\}$ can now be defined by

$$X_k := a(B_k) + b(I_k), \quad k = 0, 1, \ldots \tag{4.8}$$

and set

$$V^{A,\gamma,h} := \max\left(h(0),\ \sup_{n=0,1,\ldots} \sum_{k=0}^{n} X_k\right). \tag{4.9}$$

Under the enforced assumptions, the rv X_0 is independent of the i.i.d. rvs $\{X_k,\ k = 1, 2, \ldots\}$.

A mapping $h : \mathbb{R}_+ \to \mathbb{R}$ is said to satisfy condition (Hr), $r = 1, 2$, if

(H1) h is absolutely continuous on $\mathbb{R}_+$ with derivative h' satisfying $\gamma - r \leq h'(t)$ a.e. on $\mathbb{R}_+$;

(H2) h is absolutely continuous on $\mathbb{R}_+$ with derivative h' satisfying $h'(t) \leq \gamma$ a.e. on $\mathbb{R}_+$.

Proposition 4.1 *Consider a Borel mapping $h : \mathbb{R}_+ \to \mathbb{R}$.*

1. *If h is superadditive, then $V^{A,\gamma,h} \leq W^{A,\gamma,h}$;*
2. *If both (H1) and (H2) hold, and h is subadditive, then $W^{A,\gamma,h} \leq V^{A,\gamma,h}$.*

A proof of Proposition 4.1 is available in Appendix B.

4.3 Tail asymptotics of the rv $V^{A,\gamma,h}$

With the help of Proposition 4.1, we can bound the tail of the rv $W^{A,\gamma,h}$ with that of $V^{A,\gamma,h}$. Hence, of particular interest are the following asymptotics of the tail distribution of the rv $V^{A,\gamma,h}$.

Proposition 4.2 *Consider a mapping $h : \mathbb{R}_+ \to \mathbb{R}$ which satisfies (H2) together with the conditions*

(H3) The limit $\lim_{x\to\infty} h'(x) =: h'(\infty)$ exists and is finite;

(H4) The mapping a given by (4.7) is strictly increasing in the limit with $\lim_{x\to\infty} a(x) = \infty$.

Further, assume the following conditions:

(H5) The rv X_1 is integrable with $\mathbf{E}[X_1] < 0$;

(H6) $a^+(B_1) \in \mathcal{L}$;

(H7) $a^+(B_1^\star) \in \mathcal{S}$.

Then, it holds that $V^{A,\gamma,h} \in \mathcal{S}$ with

$$\mathbf{P}\left[V^{A,\gamma,h} > x\right] \sim (p + K(h))\mathbf{P}\left[a^+(B_1^\star) > x\right] \tag{4.10}$$

where

$$K(h) := ((r - \gamma) + h'(\infty))\frac{\mathbf{E}[B_1]}{-\mathbf{E}[X_1]}. \tag{4.11}$$

A proof of this result is given in Appendix C. Two useful consequences emerge by combining the bounds of Section 4.2 with the asymptotics of Proposition 4.2. Indeed, Propositions 4.1(1) and 4.2 together imply the following lower bound asymptotics on the tail of $W^{A,h,\gamma}$.

Corollary 4.1 *For any superadditive mapping $h : \mathbb{R}_+ \to \mathbb{R}$ which satisfies (H2)-(H7), it holds that*

$$\liminf_{x\to\infty} \frac{\mathbf{P}\left[W^{A,\gamma,h} > x\right]}{\mathbf{P}\left[a^+(B_1^\star) > x\right]} \geq p + K(h).$$

Upper bound asymptotics on the tail of $W^{A,h,\gamma}$ can also be obtained once we combine Propositions 4.1(2) and 4.2. We omit a formal statement of this result as we do not use it in that form in the sequel.

4.4 Comments

We close this section with remarks that will be useful in the sequel:

Remark 4.1 We note that $h = 0$ satisfies the assumptions of *both* parts of Proposition 4.1, whence $W^{A,\gamma} = V^{A,\gamma,0}$. Moreover, the assumptions of Proposition 4.2 are automatically satisfied *provided* $B_1 \in \mathcal{L}$ and $B_1^\star \in \mathcal{S}$, in which case specializing (4.10) yields the exact asymptotics

$$\mathbf{P}\left[W^{A,\gamma} > x\right] \sim \frac{(1-p)\rho}{\gamma - \rho}\mathbf{P}\left[(r-\gamma)B_1^\star > x\right]. \tag{4.12}$$

This result was first obtained by Jelenkovic and Lazar [18, Theorem 9] by resorting to the Palm theory of stationary processes. This is to be contrasted with the direct approach taken here.

Remark 4.2 In the assumptions of Proposition 4.2, if we add in (H4) the requirement that the mapping a is convex in the limit, then (H6) and (H7) are implied by the conditions (H6bis) and (H7bis), respectively, with

(H6bis) $B_1 \in \mathcal{L}$;

(H7bis) $B_1^\star \in \mathcal{S}$.

This is a simple consequence of Lemma 3.4 applied to $\varphi = a^+$.

Remark 4.3 A convex (resp. concave) mapping $h : \mathbb{R}_+ \to \mathbb{R}$ with $h(0) = 0$ is absolutely continuous, and necessarily superadditive (resp. subadditive). Consequently, a convex (resp. concave) perturbation function h is a natural choice when considering the lower (resp. upper) bound. In fact, many of the needed conditions appearing in Propositions 4.1 and 4.2 are then easily verified with the help of Jensen's inequality together with the observation that the limits

$$\lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} h'(t) = h'(\infty)$$

take place monotonically (thus exist) and are finite. Such a discussion is provided in Section 8.

5 Towards bounds for two independent sources

We now deal with the situation where the arrival process A is the superposition of two *independent* fluid processes A_1 and A_2, i.e., $A = A_1 + A_2$, where A_1 is a stationary "independent on-off" as understood in Section 4.1 and A_2 is arbitrary. Source A_1 has peak rate $r_1 > c - \rho_2$ and its generic activity period B_1 has the property that $B_1 \in \mathcal{L}$ and $B_1^\star \in \mathcal{S}$. Its generic inactivity period I_1 has an arbitrary distribution with the only requirement that $0 < \mathbf{E}[I_1] < \infty$.

Before presenting the lower and upper bounds in Sections 6 and 7, we pause to introduce some notation that simplifies the presentation of the results: In the context of the two sources described above, we write

$$\alpha := r_1 + \rho_2 - c \tag{5.1}$$

and with any mapping $h : \mathbb{R}_+ \to \mathbb{R}$, we recast the earlier definitions (4.7), (4.8) and (4.11). The mapping $a : \mathbb{R}_+ \to \mathbb{R}$ is given by

$$a(x) := \alpha x + h(x), \quad x \geq 0, \tag{5.2}$$

and we set

$$K(h) := (\alpha + h'(\infty)) \frac{\mathbf{E}[B_1]}{-\mathbf{E}[X_1]} \tag{5.3}$$

with

$$X_1 := \alpha B_1 - (c - \rho_2) I_1 + h(B_1) + h(I_1). \tag{5.4}$$

We also write

$$L(h) := \frac{c - (\rho_1 + \rho_2)}{(1-p)\rho_1} (p + K(h)) \tag{5.5}$$

with $p := \mathbf{E}[B_1] (\mathbf{E}[B_1] + \mathbf{E}[I_1])^{-1}$ denoting the stationary probability that source 1 is active. Note that $L(0) = 1$ if $h = 0$.

Let

$$R(x; h) := \frac{\mathbf{P}[a^+(B_1^\star) > x]}{\mathbf{P}[\alpha B_1^\star > x]}, \quad x \geq 0.$$

Under (H4), we see that

$$R_-(h) := \liminf_{x\to\infty} R(x; h) \quad = \quad \liminf_{x\to\infty} \frac{\mathbf{P}[a^+(B_1^\star) > a^+(x)]}{\mathbf{P}[\alpha B_1^\star > a^+(x)]}$$

$$= \liminf_{x\to\infty} \frac{\mathbf{P}\left[B_1^\star > x\right]}{\mathbf{P}\left[\alpha B_1^\star > a(x)\right]}, \tag{5.6}$$

and similarly,

$$R_+(h) := \limsup_{x\to\infty} R(x;h) = \limsup_{x\to\infty} \frac{\mathbf{P}\left[B_1^\star > x\right]}{\mathbf{P}\left[\alpha B_1^\star > a(x)\right]}. \tag{5.7}$$

We draw the reader's attention to the similarity of the limits (5.6) and (5.7) with those guaranteed by Lemmas 3.5 and 3.6 (with particular reference to (3.12)). This connection will be exploited to identify the appropriate perturbation functions h in the bounds developed thus far.

While the quantities $L(h)$, $R_-(h)$ and $R_+(h)$ will help quantify the impact of the first source A_1, the generic lower bound (2.3) suggests that the contribution of the source A_2 will be expressed through the quantity

$$\Delta(h) := \liminf_{t\to\infty} \mathbf{P}\left[A_2(t) \geq \rho_2 t + h(t)\right]. \tag{5.8}$$

Its alternate expression

$$\Delta(h) = \liminf_{t\to\infty} \mathbf{P}\left[\frac{A_2(t)}{t} - \rho_2 \geq \frac{h(t)}{t}\right]$$

indicates already the possibility that its value will be determined by refinements to the assumed "Law of Large Numbers"

$$\lim_{t\to\infty} \frac{A_2(t)}{t} = \rho_2 \quad a.s. \tag{5.9}$$

defining the average rate ρ_2 for source A_2. Such refinements include the Central Limit Theorem (CLT) in the form

$$Z_2(t) := \sqrt{t}\left(\frac{A_2(t)}{t} - \rho_2\right) \Longrightarrow_t \sigma U \tag{5.10}$$

with $\sigma > 0$ and U denoting a Gaussian rv with zero mean and unit variance. This condition is not prohibitive for it holds in great generality for a variety of on-off sources (as implied by similar results on renewal processes [15]) and for superpositions thereof. Moreover, the relation

$$\mathbf{P}\left[A_2(t) - \rho_2 t > h(t)\right] = \mathbf{P}\left[Z_2(t) > \frac{h(t)}{\sqrt{t}}\right], \quad t \geq 0 \tag{5.11}$$

points to the need to impose constraints on the behavior of $\frac{h(t)}{\sqrt{t}}$ for large t in order to get non-trivial limits in (5.8).

6 Upper bounds for two independent sources

6.1 A basic upper bound for two sources

We begin with an intermediate result.

Proposition 6.1 *Consider a subadditive mapping $h : \mathbb{R}_+ \to \mathbb{R}$ which satisfies (H1)–(H7) [with $\gamma \leftarrow c - \rho_2$ and $\rho \leftarrow \rho_1$]. If*

$$\mathbf{P}\left[W^{A_2,\rho_2,-h} > x\right] = o\left(\mathbf{P}\left[a^+(B_1^\star) > x\right]\right), \tag{6.1}$$

then it holds that

$$\limsup_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]} \leq L(h)R_+(h) \tag{6.2}$$

with $L(h)$ and $R_+(h)$ given by (5.5) and (5.7), respectively.

In (6.1), the finiteness of the rv $W^{A_2,\rho_2,-h}$ is implicitly *assumed* for it is not necessarily guaranteed under the conditions (H1)–(H7) imposed on h. This finiteness issue and the tail behavior of the rv $W^{A_2,\rho_2,-h}$ are quite delicate when

$$\lim_{t\to\infty} \frac{h(t)}{t} = 0 \tag{6.3}$$

as occurs in many interesting instances. This point is explored in some detail in Section 6.3.

Proof: Combining Lemma 2.1 and the upper bound of Proposition 4.1(2) we find

$$W^{A_1+A_2,c} \leq V^{A_1,c-\rho_2,h} + W^{A_2,\rho_2,-h} \tag{6.4}$$

where the rvs $V^{A_1,c-\rho_2,h}$ and $W^{A_2,\rho_2,-h}$ are taken to be independent. From Proposition 4.2 [with $\gamma \leftarrow c - \rho_2$ and $\rho \leftarrow \rho_1$], it holds that $V^{A,\gamma,h} \in \mathcal{S}$ with

$$\mathbf{P}\left[V^{A_1,c-\rho_2,h} > x\right] \sim (p + K(h))\mathbf{P}\left[a^+(B_1^\star) > x\right]$$

where $K(h)$ is given by (5.3).

Parts (1) and (3) of Lemma 3.1 readily ensure under (6.1) that

$$\mathbf{P}\left[V^{A_1,c-\rho_2,h} + W^{A_2,\rho_2,-h} > x\right] \sim (p + K(h))\mathbf{P}\left[a^+(B_1^\star) > x\right],$$

and it immediately follows from (6.4) that

$$\limsup_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[a^+(B_1^\star) > x\right]} \leq p + K(h). \tag{6.5}$$

Remark 4.1 applied to source A_1 [with $\gamma \leftarrow c - \rho_2$ and $\rho \leftarrow \rho_1$] yields

$$\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right] \sim \frac{(1-p)\rho_1}{c - (\rho_1 + \rho_2)}\mathbf{P}\left[\alpha B_1^\star > x\right] \tag{6.6}$$

and the desired conclusion (6.2) follows upon combining (6.5) and (6.6). ∎

6.2 An improved upper bound for two sources

Proposition 6.1 begs the question of how to choose the perturbation mapping h. In view of the form of the upper bound (6.2), an obvious criterion for selecting h is that $L(h)R_+(h)$ be made as small as possible in order to yield the best upper bound. To gain some insights on how this could be achieved, we note that if h assumed non-negative values, i.e., $h : \mathbb{R}_+ \to \mathbb{R}_+$, then amongst its scaled versions $\{\varepsilon h,\ \varepsilon \geq 0\}$, the smallest value of $K(\varepsilon h)$ is achieved for $\varepsilon = 0$ and that $L(0) = \lim_{\varepsilon \downarrow 0} L(\varepsilon h) = 1$. This remark leads to the following improvement to Proposition 6.1; its proof is straightforward and therefore omitted for the sake of brevity.

Proposition 6.2 *Consider a subadditive mapping $h : \mathbb{R}_+ \to \mathbb{R}$ such that whenever $0 < \varepsilon < \varepsilon^\star$ for some $\varepsilon^\star > 0$, the scaled mapping εh satisfies (H1)-(H7) [with $\gamma \leftarrow c - \rho_2$ and $\rho \leftarrow \rho_1$]. If*

$$\mathbf{P}\left[W^{A_2,\rho_2,-\varepsilon h} > x\right] = o\left(\mathbf{P}\left[a_\varepsilon^+(B_1^\star) > x\right]\right), \quad 0 < \varepsilon < \varepsilon^\star \tag{6.7}$$

with the mapping $a_\varepsilon : \mathbb{R}_+ \to \mathbb{R}$ given by $a_\varepsilon(x) := \alpha x + \varepsilon h(x)$ $(x \geq 0)$, then it holds that

$$\limsup_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]} \leq \lim_{\varepsilon \downarrow 0} R_+(\varepsilon h). \tag{6.8}$$

In order to improve the upper bound (6.8) we need only select the perturbation direction h that makes $\lim_{\varepsilon\downarrow 0} R_+(\varepsilon h)$ as small as possible. For $h \geq 0$, we see that $R_+(\varepsilon h) \geq 1$, whence $\lim_{\varepsilon\downarrow 0} R_+(\varepsilon h) \geq 1$, and it is therefore tempting to seek $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{\varepsilon\downarrow 0} R_+(\varepsilon h) = 1$, in which case (6.8) becomes

$$\limsup_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]} \leq 1. \tag{6.9}$$

But, by virtue of Lemmas 3.5 and 3.6 (and (3.12)), whenever $B_1 \in MDA(\Lambda)$, it holds that

$$R_+(\varepsilon m_{B_1}) = \lim_{x\to\infty} \frac{\mathbf{P}\left[B_1^\star > x\right]}{\mathbf{P}\left[B_1^\star > x + \alpha^{-1}\varepsilon m_{B_1}(x)\right]} = e^{\alpha^{-1}\varepsilon}, \quad \varepsilon > 0$$

and taking $h = m_{B_1}$, we indeed get $\lim_{\varepsilon\downarrow 0} R_+(\varepsilon h) = 1$, whence (6.9), provided the appropriate assumptions are satisfied. Thus, in most cases of interest, m_{B_1} (or an asymptotic equivalent) is expected to be the perturbation function of choice for getting the best possible upper bound (6.9) in Proposition 6.2.

6.3 On the condition (6.1)

In order to assess the range of applicability of Propositions 6.1 and 6.2, we need to focus on condition (6.1) which quantifies the situation where the second source has a “lighter tail" than source A_1. To that end, consider a mapping $h : \mathbb{R}_+ \to \mathbb{R}$ such that the limit

$$\lim_{t\to\infty} \frac{h(t)}{t} =: L \tag{6.10}$$

exists and is finite. The corresponding rv $W^{A_2,\rho_2,-h}$ will be a.s. finite (resp. infinite) if $L > 0$ (resp. $L < 0$, a case of no interest here). However, if instead, $L = 0$ (or equivalently, condition (6.3) holds), then there is no *a priori* guarantee that the rv $W^{A_2,\rho_2,-h}$ will be a.s. finite as already indicated by the following result.

Lemma 6.1 *Assume the CLT refinement (5.10) to hold for source A_2. Then, for any mapping $h : \mathbb{R}_+ \to \mathbb{R}$ such that the limit*

$$\liminf_{t\to\infty} \frac{h(t)}{\sqrt{t}} =: H \tag{6.11}$$

is finite, we have $W^{A_2,\rho_2,-h} = \infty$ a.s.

If h does satisfy (6.10) and (6.11), then $L = 0$ necessarily.

Proof: Fix $t > 0$ and x in $\mathbb{R}$. We note that

$$\sqrt{t}\left(Z_2(t) - \frac{h(t)}{\sqrt{t}}\right) \le W^{A_2,\rho_2,-h},$$

whence

$$\mathbf{P}\left[Z_2(t) - \frac{h(t)}{\sqrt{t}} \ge x\right] \le \mathbf{P}\left[W^{A_2,\rho_2,-h} \ge x\sqrt{t}\right].$$

Letting t go to infinity along a sequence for which the the liminf in (6.11) is attained in this last inequality and invoking (5.10), we find

$$\mathbf{P}\left[\sigma U - H \ge x\right] \le \lim_{t\to\infty} \mathbf{P}\left[W^{A_2,\rho_2,-h} \ge x\sqrt{t}\right] = \mathbf{P}\left[W^{A_2,\rho_2,-h} = \infty\right]. \tag{6.12}$$

Finally, we get the desired conclusion upon letting x go to $-\infty$ in (6.12). ∎

Consequently, for the rv $W^{A_2,\rho_2,-h}$ to be a.s. finite under (5.10), it is necessary that $H = \infty$. We now turn to finding sufficient conditions on h under which the rv $W^{A_2,\rho_2,-h}$ is a.s. finite. In the process, we identify its tail behavior, thereby providing the means to check (6.1).

The discussion will be carried out in the following *regenerative* framework which contains most fluid models discussed in the literature, including the independent on-off sources of Section 4: Source A_2 is a fluid process $\{A_2(t),\ t \ge 0\}$ process characterized by a succession of cycles, where for each $n = 1, 2, \ldots$, the $(n+1)^{st}$ cycle has duration C_n (with $C_n > 0$ a.s.) and is associated with source A_2 producing fluid in amount Y_n (with $Y_n > 0$ a.s.). Alternatively, we have $Y_n := A_2(T_n) - A_2(T_{n-1})$ where T_n denotes the beginning of the n^{th} cycle (with the convention $T_0 = 0$), i.e., $T_n = C_1 + \ldots + C_n$.

No additional details on the operation of source A_2 will be needed. We assume that the rvs $\{(Y_n, C_n),\ n = 1, 2, \ldots\}$ are integrable and mutually independent with $\{(Y_n, C_n),\ n = 2, 3, \ldots\}$ being *identically distributed* rvs. By the Renewal Reward Theorem, the mean rate ρ_2 given by (5.9) can be evaluated as $\rho_2 = \mathbf{E}[Y_2]/\mathbf{E}[C_2]$. A fluid process/source satisfying the above requirements will be called a *regenerative fluid process/source.*

The other key probabilistic assumption is the existence of finite *exponential* moments: There exists a constant $\theta_0 > 0$ such that

$$M_n(\theta) := \mathbf{E}\left[e^{\theta C_n}\right] < \infty \quad \text{and} \quad N_n(\theta) := \mathbf{E}\left[e^{\theta Y_n}\right] < \infty, \quad 0 \le \theta \le \theta_0,\ n = 1, 2. \tag{6.13}$$

A regenerative fluid process satisfying (6.13) also admits the CLT refinement (5.10).

Proposition 6.3 *Let $\{A_2(t),\, t \ge 0\}$ be a regenerative fluid process satisfying (6.13). Consider a nondecreasing mapping $h : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies both (6.10) with $L = 0$ and (6.11) with $H = \infty$. Define the mapping $g : \mathbb{R}_+ \to \mathbb{R}_+$ by*

$$g(t) = \sqrt{t} \inf_{s \ge t} \frac{h(s)}{\sqrt{s}}, \quad t \ge 0.$$

(1) Then, there exist finite constants $\gamma_1, \gamma_2 > 0$ and $\gamma_3 \ge 0$ such that

$$\limsup_{x \to \infty} \frac{\mathbf{P}\left[W^{A_2,\rho_2,-h} > x\right]}{\int_{\gamma_1 x}^{\infty} e^{-\gamma_2 g^2(t)/t}\, dt} \le \gamma_3. \tag{6.14}$$

(2) Furthermore, if there exists ν_0 in $(0, 1/2)$ such that

$$\liminf_{t \to \infty} \frac{h(t)}{t^{1/2+\nu_0}} = \infty,$$

then there exist finite constants $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\limsup_{x \to \infty} \frac{\mathbf{P}\left[W^{A_2,\rho_2,-h} > x\right]}{x^{1-\nu}\, e^{-\delta_1 x^{\nu}}} \le \delta_2, \quad \nu \in (0, 2\nu_0). \tag{6.15}$$

A proof of this result is available in Appendix D. Proposition 6.3 provides a natural vehicle for checking condition (6.1) as is done in Section 8 on a variety of examples. Note that the denominator in (6.15) is vanishingly small with x large, so that this result does yield a non-trivial bound on the tail of $W^{A_2,\rho_2,-h}$.

The situation where $L > 0$ in (6.10) can be handled in a variety of ways: Indeed, under such an asymptotic linearity assumption on h, there exist constants α_h and $\beta_h > 0$ such that

$$h(t) \geq \alpha_h + \beta_h t, \quad t \geq 0$$

and the comparison

$$W^{A_2,\rho_2,-h} \leq_{st} W^{A_2,\rho_2+\beta_h} - \alpha_h$$

follows. Whenever the source A_2 belongs to the class of Markov modulated fluid sources [12] or more generally, is an exponential source, as understood in [18], then the tail behavior of $W^{A_2,\rho_2,-h}$ is at most exponential since that of $W^{A_2,\rho_2+\beta_h}$ has exponential decay. On the other hand, if A_2 is an on-off source with subexponential activity periods, then the asymptotics (4.12) [with $\gamma \leftarrow \rho_2+\beta_h, r \leftarrow \rho_2$]. can be invoked. In either case, these remarks serve as the basis for checking (6.1).

7 Lower bounds for two independent sources

7.1 A basic lower bound for two sources

Proposition 7.1 *Consider a superadditive mapping $h : \mathbb{R}_+ \to \mathbb{R}$ which satisfies (H2)-(H7) [with $\gamma \leftarrow c - \rho_2$ and $\rho \leftarrow \rho_1$]. Then, it holds that*

$$L(h)\Delta(h)R_-(h) \leq \liminf_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]} \tag{7.1}$$

with $L(h)$, $\Delta(h)$ and $R_-(h)$ given by (5.5), (5.8) and (5.6), respectively.

Proof. Fix $x \geq 0$. We conclude from Proposition 2.1 applied to h that

$$\left(\inf_{\{t:a(t)>x\}} \mathbf{P}\left[A_2(t) \geq \rho_2 t + h(t)\right]\right) R(x;h)\frac{\mathbf{P}\left[W^{A_1,c-\rho_2,h} > x\right]}{\mathbf{P}\left[a^+(B_1^\star) > x\right]} \leq \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[\alpha B_1^\star > x\right]}. \tag{7.2}$$

As we have in mind to let x go to infinity in (7.2), we note the following: By Corollary 4.1 applied to h with the source A_1 [and $\gamma = c - \rho_2$], it holds that

$$p + K(h) \leq \liminf_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1,c-\rho_2,h} > x\right]}{\mathbf{P}\left[a^+(B_1^\star) > x\right]} \tag{7.3}$$

with $K(h)$ given by (5.3). Moreover, under (H4), the mapping a is eventually strictly increasing (thus invertible) with $\lim_{x\to\infty} a(x) = \infty$, and we readily check that

$$\liminf_{x\to\infty} \inf_{\{t:a(t)>x\}} \mathbf{P}\left[A_2(t) \geq \rho_2 t + h(t)\right] = \liminf_{t\to\infty} \mathbf{P}\left[A_2(t) \geq \rho_2 t + h(t)\right] = \Delta(h). \tag{7.4}$$

Thus, letting x go to infinity in (7.2) and making use of (5.6), (7.3) and (7.4), we find

$$(p + K(h))\,\Delta(h) R_{-}(h) \le \liminf_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[\alpha B_1^\star > x\right]} \tag{7.5}$$

and the conclusion (7.1) is an immediate consequence of (7.5) and of the equivalence (6.6) noted earlier. ■

7.2 An improved lower bound for two sources

This time, in the same way that the upper bound in Proposition 6.1 leads to Proposition 6.2, we have the following result from Proposition 7.1.

Proposition 7.2 *Consider a superadditive mapping $h : \mathbb{R}_+ \to \mathbb{R}$ such that whenever $0 < \varepsilon < \varepsilon^\star$ for some $\varepsilon^\star > 0$, the scaled mapping εh satisfies (H2)–(H7) [with $\gamma \leftarrow c - \rho_2$ and $\rho \leftarrow \rho_1$]. Then, it holds that*

$$\left(\lim_{\varepsilon\downarrow 0} \Delta(\varepsilon h)\right)\left(\lim_{\varepsilon\downarrow 0} R_{-}(\varepsilon h)\right) \le \liminf_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]}. \tag{7.6}$$

The lower bound (7.6) can be improved by seeking a value for the product of the quantities $\lim_{\varepsilon\downarrow 0} \Delta(\varepsilon h)$ and $\lim_{\varepsilon\downarrow 0} R_{-}(\varepsilon h)$ that is as large as possible (if not the largest) among admissible perturbations h. While $\Delta(\varepsilon h) \le 1$ is always true, it is often possible to argue that

$$\lim_{\varepsilon\downarrow 0} \Delta(\varepsilon h) = 1 \tag{7.7}$$

for the selected h, in which case (7.6) reads

$$\lim_{\varepsilon\downarrow 0} R_{-}(\varepsilon h) \le \liminf_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]}. \tag{7.8}$$

For instance, under the condition (5.9) on source A_2, we find $\Delta(\varepsilon h) = 1$ for each $\varepsilon > 0$ whenever (6.10) holds with $L < 0$. If, instead, the condition (6.3) holds ($L = 0$), then the CLT refinement (5.10) to (5.9) needs to be brought into the picture. If h is a convex mapping with $h(0) = 0$, then (6.3) is equivalent to $h'(\infty) = 0$, and the requirements (H2)-(H4) naturally lead to taking $h = -u$ for some concave increasing mapping $u : \mathbb{R}_+ \to \mathbb{R}_+$ with $u'(\infty) = 0$. In that case (5.11) yields

$$\mathbf{P}\left[A_2(t) - \rho_2 t > \varepsilon h(t)\right] = \mathbf{P}\left[\frac{\sqrt{t}}{u(t)} Z_2(t) > -\varepsilon\right], \quad t \ge 0$$

and the conclusion $\Delta(\varepsilon h) = 1$ follows from (5.10) if $-h$ satisfies (6.11) with $H = \infty$.

We now focus on finding a perturbation function h such that $\lim_{\varepsilon \downarrow 0} R_-(\varepsilon h)$ is as large as possible. With $h = -u$ as above,

$$R_-(\varepsilon h) = \liminf_{x\to\infty} \frac{\mathbf{P}\left[B_1^\star > x\right]}{\mathbf{P}\left[B_1^\star > x - \alpha^{-1}\varepsilon u(x)\right]} \leq 1, \quad \varepsilon \geq 0,$$

and the requirement that $\lim_{\varepsilon \downarrow 0} R_-(\varepsilon h)$ is as large as possible will be met if $\lim_{\varepsilon \downarrow 0} R_-(\varepsilon h) = 1$, in which case the bound (7.8) becomes

$$1 \leq \liminf_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]}. \tag{7.9}$$

Here too, whenever $B_1 \in MDA(\Lambda)$, Lemma 3.6 yields

$$R_-(-\varepsilon m_{B_1}) = \lim_{x\to\infty} \frac{\mathbf{P}\left[B_1^\star > x\right]}{\mathbf{P}\left[B_1^\star > x - \alpha^{-1}\varepsilon m_{B_1}(x)\right]} = e^{-\alpha^{-1}\varepsilon}, \quad \varepsilon > 0$$

and taking $h = -m_{B_1}$, we get $\lim_{\varepsilon \downarrow 0} R_-(\varepsilon h) = 1$. Thus, in most cases of interest, $-m_{B_1}$ is expected to be the perturbation function of choice for getting (7.9).

8 Applications

In the examples which we now discuss, we assume that the fluid A is the superposition of two independent fluid processes A_1 and A_2, with A_1 a stationary "independent on-off". The conditions

$$\rho_1 + \rho_2 < c < r_1 + \rho_2 \tag{8.1}$$

are enforced throughout. Propositions 6.2 and 7.2 will be invoked in that context.

8.1 In the maximum domain of attraction of the Gumbel distribution

As indicated already in some of the comments following Propositions 6.2 and 7.2, a fairly comprehensive discussion should be expected when $B_1 \in MDA(\Lambda)$. Before indicating in Proposition 8.1 the extent to which this is indeed the case, we present the following technical fact.

Lemma 8.1 *Consider a mapping $u : \mathbb{R}_+ \to \mathbb{R}_+$ which is strictly increasing and concave in the limit, i.e., there exists $x_0 \geq 0$ such that the restriction $u : [x_0, \infty) \to \mathbb{R}_+$ is strictly increasing and concave. There exist an increasing concave mapping $U : \mathbb{R}_+ \to \mathbb{R}_+$, and a constant $x^\star \geq x_0$ such that $U(0) = 0$, $U'(0+) := \lim_{x\downarrow 0} U'(x)$ is finite and $u(x) = U(x)$ for all $x \geq x^\star$.*

Proof. Define the mapping $\hat{u} : \mathbb{R}_+ \to \mathbb{R}_+$ by $\hat{u}(x) = u(x_0)$ if $0 \leq x \leq x_0$ and by $\hat{u}(x) = u(x)$ if $x \geq x_0$. This mapping, while non-decreasing, is not necessarily concave. Let $\hat{u}_c$ denote the concave hull of $\hat{u}$, i.e., the smallest concave mapping $\hat{u}_c : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\hat{u} \leq \hat{u}_c$. In fact, $\hat{u}_c = -\mathrm{conv}(-\hat{u})$ where conv denotes the convex hull operation [26, p. 36]. It is easy to check that $\hat{u}_c$ is increasing and concave with $\hat{u}_c(x) = \hat{u}(x)$ from some x onward. The desired mapping U is now obtained by taking $U(x) = \min(x, \hat{u}_c(x))$ for all $x \geq 0$. ∎

The function U associated with u through Lemma 8.1 is clearly *not* unique. In specific examples the one constructed in the proof can be safely replaced by a more natural one which derives naturally from the form of u.

Proposition 8.1 *Assume B_1 and I_1 to be $\mathbb{R}_+$-valued rvs with $0 < \mathbf{E}[B_1], \mathbf{E}[I_1] < \infty$, such that $B_1 \in \mathcal{L}$ and $B_1^\star \in \mathcal{S}$. Suppose that the function m_{B_1} given by (3.8) is asymptotically equivalent to some mapping $u : \mathbb{R}_+ \to \mathbb{R}_+$ which is strictly increasing and concave in the limit, and let $U : \mathbb{R}_+ \to \mathbb{R}_+$ denote any increasing concave mapping associated with u as in Lemma 8.1.*

If $B_1 \in MDA(\Lambda)$, then the following holds:

(1) There exists a subadditive mapping $h : \mathbb{R}_+ \to \mathbb{R}$ such that whenever $0 < \varepsilon < \varepsilon^\star$ for some $\varepsilon^\star > 0$, the scaled mapping εh satisfies (H1)-(H7) [with $\gamma \leftarrow c - \rho_2$ and $\rho \leftarrow \rho_1$]. A possible choice is $h = U$, in which case

$$\lim_{\varepsilon \downarrow 0} R_+(\varepsilon h) = \lim_{\varepsilon \downarrow 0} e^{\alpha^{-1}\varepsilon} = 1; \tag{8.2}$$

(2) There exists a superadditive mapping $h : \mathbb{R}_+ \to \mathbb{R}$ such that whenever $0 < \varepsilon < \varepsilon^\star$ for some $\varepsilon^\star > 0$, the scaled mapping εh satisfies (H2)–(H7) [with $\gamma \leftarrow c - \rho_2$ and $\rho \leftarrow \rho_1$]. A possible choice is $h = -U$, in which case

$$\lim_{\varepsilon \downarrow 0} R_-(\varepsilon h) = \lim_{\varepsilon \downarrow 0} e^{-\alpha^{-1}\varepsilon} = 1. \tag{8.3}$$

A proof of this result is available in Appendix E.

We next indicate how Proposition 8.1 applies in some specific cases. It is worth pointing out that in the context of Proposition 8.1, the task of checking (6.7) over some *entire* interval $(0, \varepsilon^\star)$ can be greatly reduced: Indeed, with $h = U$, owing to the tail equivalence of $a_\varepsilon^+(B_1^\star)$ and $\alpha B_1^\star$ $(0 < \varepsilon < \varepsilon^\star)$ noted in the proof of Proposition 8.1, we get that (6.7) is equivalent to

$$\mathbf{P}\left[W^{A_2,\rho_2,-\varepsilon h} > x\right] = o\left(\mathbf{P}\left[\alpha B_1^\star > x\right]\right), \quad 0 < \varepsilon < \varepsilon^\star. \tag{8.4}$$

8.2 Log-normal

By definition, we have $B_1 =_{st} e^Z$ where Z is normally distributed with mean μ and variance δ^2. Recall that $B_1 \in \mathcal{S}$ (and therefore $B_1 \in \mathcal{L}$) and $B_1^\star \in \mathcal{S}$ [14, Example 1.4.7, p. 55]. It is also well known that $B_1 \in MDA(\Lambda)$ [14, p. 150]. Then, straightforward calculations yield the asymptotic equivalences

$$\mathbf{P}\left[B_1 > x\right] \sim \frac{\delta\, e^{-(\log x-\mu)^2/(2\delta^2)}}{\sqrt{2\pi}\,(\log x - \mu)}, \tag{8.5}$$

and

$$\mathbf{P}\left[B_1^\star > x\right] \sim \frac{\delta^3\, x e^{-(\log x-\mu)^2/(2\delta^2)}}{\mathbf{E}\left[B_1\right]\,\sqrt{2\pi}\,(\log x - \mu)^2} \quad \text{with} \quad \mathbf{E}\left[B_1\right] = \exp(\frac{1}{2}\delta^2 + \mu). \tag{8.6}$$

Consequently,

$$m_{LN}(x) \sim \delta^2 \frac{x}{\log x - \mu} \sim \delta^2 \frac{x}{\log x},$$

with m_{LN} the mapping associated through (3.8) with the rv $X = B_1$. Note that m_{LN} is asymptotically equivalent to the mapping $u_{LN} : \mathbb{R}_+ \to \mathbb{R}_+$ given by $u_{LN}(x) = \delta^2(x/\log x \vee e)$ $(x \geq 0)$. The mapping u_{LN} is concave and strictly increasing in the limit, whence Proposition 8.1 can be invoked with U_{LN} denoting the mapping associated with u_{LN} through Lemma 8.1.

Lower bound: Applying the lower bound (7.6) with $h = -U_{LN}$ and using (8.3), we find

$$\lim_{\varepsilon \downarrow 0} \Delta(\varepsilon h) \leq \liminf_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]}. \tag{8.7}$$

with

$$\Delta(\varepsilon h) = \liminf_{t\to\infty} \mathbf{P}\left[A_2(t) \geq \rho_2 t - \varepsilon\delta^2 \frac{t}{\log t}\right], \quad \varepsilon > 0.$$

Assuming the CLT refinement (5.10) to hold for source A_2, we conclude that

$$\Delta(\varepsilon h) = \liminf_{t\to\infty} \mathbf{P}\left[Z_2(t) > -\varepsilon\delta^2 \frac{\sqrt{t}}{\log t}\right] = \lim_{t\to\infty} \mathbf{P}\left[\frac{\log t}{\sqrt{t}} Z_2(t) > -\varepsilon\delta^2\right] = 1,$$

and we achieve the best possible lower bound

$$1 \leq \liminf_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]}. \tag{8.8}$$

Upper bound: Applying the upper bound (6.8) with $h = U_{LN}$ and making use of (8.2), we obtain

$$\limsup_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]} \leq 1 \tag{8.9}$$

provided the second source A_2 satisfies the condition (6.7), or equivalently (8.4). We now discuss this condition when A_2 is a regenerative source as defined in Section 6.3 under the moment conditions (6.13). With the choice $h = U_{LN}$, for each $\varepsilon > 0$ we have $\lim_{t\to\infty} \varepsilon h(t)/t = 0$ and

$\lim_{t\to\infty} \varepsilon h(t)/t^{1/2+\nu_0} = \infty$ for each ν_0 in $(0, 1/2)$. Therefore, Proposition 6.3 applies with $L = 0$ and $H = \infty$, we are lead to the conclusion that for each $\varepsilon > 0$, there exist finite constants $0 < \delta_{1,\varepsilon}$ and $\delta_{2,\varepsilon} \geq 0$ such that

$$\limsup_{x\to\infty} \frac{\mathbf{P}\left[W^{A_2,\rho_2,-\varepsilon h} > x\right]}{x^{1-\nu}\, e^{-\delta_{1,\varepsilon} x^{\nu}}} \leq \delta_{2,\varepsilon}, \quad \nu \in (0,1). \tag{8.10}$$

It is now plain from (8.6) and (8.10) that (8.4) indeed holds. We summarize the discussion as follows:

Proposition 8.2 *Let $A = A_1 + A_2$ be the superposition of two independent fluid processes A_1 and A_2. Assume A_1 to be a stationary "independent on-off" source such that $\rho_1 + \rho_2 < c < r_1 + \rho_2$ with activity period B_1 distributed according to (8.5), and let A_2 be a regenerative source (in the sense discussed in Section 6.3) under the moment conditions (6.13). Then, it holds that*

$$\lim_{x\to\infty} \frac{\mathbf{P}\left[W^{A,c} > x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]} = 1. \tag{8.11}$$

The lower bound (8.8) only requires that source 2 satisfies (5.10). Proposition 8.2 also holds when B_1 has a Benktander-type-I distribution [14, p. 149] since in that case $B_1 \in MDA(\Lambda) \cap \mathcal{S}$ with the corresponding function (3.8) being asymptotically equivalent to m_{LN}.

8.3 Weibull

With $a > 0$ and $0 < \beta < 1$,

$$\mathbf{P}\left[B_1 > x\right] = \exp\left(-a x^{\beta}\right), \quad x \geq 0. \tag{8.12}$$

Again we have that $B_1 \in \mathcal{S} \cap MDA(\Lambda)$ (and therefore $B_1 \in \mathcal{L}$ since $\mathcal{S} \subset \mathcal{L}$) [14, Example 1.4.7, p. 55] and $B_1^\star \in \mathcal{S}$ [14, p. 150]. All the moments of this distribution are finite, and in particular, we have

$$\mathbf{E}\left[B_1\right] = \frac{\Gamma(\beta^{-1})}{\beta a^{\frac{1}{\beta}}}$$

where the Gamma function $\Gamma : (0,\infty) \to \mathbb{R}_+$ is defined by

$$\Gamma(s) := \int_0^\infty x^{s-1} e^{-x} dx, \quad s > 0.$$

By appealing to properties of the incomplete Gamma function, we readily see that

$$\mathbf{P}\left[B_1^\star > x\right] \sim \frac{a^{\frac{1-\beta}{\beta}}}{\Gamma(\beta^{-1})} x^{1-\beta} \exp\left(-a x^{\beta}\right), \tag{8.13}$$

and we find

$$m_W(x) \sim \frac{1}{a\beta} x^{1-\beta}$$

where m_W denotes the mapping associated through (3.8) with the Weibull rv $X = B_1$

Note that m_W is asymptotically equivalent to $u_W : \mathbb{R}_+ \to \mathbb{R}_+$ given by $u_W(x) := (a\beta)^{-1} x^{1-\beta}$ $(x \geq 0)$. The mapping u_W is concave and strictly increasing on $\mathbb{R}_+$ with $u_W(0) = 0$, and Proposition 8.1 can be applied with U_W denoting the mapping associated with u_W through Lemma 8.1.

Lower bound: Applying the lower bound (7.6) with $h = -U_W$ and using (8.3), we also find (8.7) with

$$\Delta(\varepsilon h) = \liminf_{t\to\infty} \mathbf{P}\left[A_2(t) \geq \rho_2 t - \frac{\varepsilon}{a\beta} t^{1-\beta}\right], \quad \varepsilon > 0.$$

Under the CLT refinement (5.10) for source A_2, we see that

$$\Delta(\varepsilon h) = \liminf_{t\to\infty} \mathbf{P}\left[Z_2(t) > -\frac{\varepsilon}{a\beta} t^{\frac{1}{2}-\beta}\right] = \lim_{t\to\infty} \mathbf{P}\left[t^{\beta-\frac{1}{2}} Z_2(t) > -\frac{\varepsilon}{a\beta}\right] = 1$$

provided the condition

$$0 < \beta < \frac{1}{2} \tag{8.14}$$

holds, in which case the best possible lower bound (8.8) is achieved.

Upper bound: Applying the upper bound (6.8) with $h = U_W$ and making use of (8.2), we obtain the upper bound (8.9) provided the second source A_2 satisfies the condition (6.7). This condition is discussed now when A_2 is a regenerative source under the moment conditions (6.13). For the choice $h = U_W$, for each $\varepsilon > 0$ we have $\lim_{t\to\infty} \varepsilon h(t)/t = 0$ and $\lim_{t\to\infty} \varepsilon h(t)/t^{1/2+\nu_0} = \infty$ for each ν_0 in $(0, 1/2)$ such that

$$\beta + \nu_0 < \frac{1}{2}. \tag{8.15}$$

Hence, whenever we pick ν_0 in $(0, 1/2)$ such that (8.15) (thus (8.14)) holds, we can invoke Proposition 6.3 while still guaranteeing (8.8). Hence, with each $\varepsilon > 0$, there exist finite constants $0 < \delta_{1,\varepsilon}$ and $\delta_{2,\varepsilon} \geq 0$ such that (8.10) still holds but only for ν in the interval $(0, 1-2\beta)$. It is now plain from (8.13) that (8.4) indeed holds if there exists such an admissible value of ν with the property that

$$\limsup_{x\to\infty} \frac{x^{1-\nu} \exp\left(-\delta_{1,\varepsilon} x^{\nu}\right)}{x^{1-\beta} \exp\left(-a\alpha^{-\beta} x^{\beta}\right)} = 0, \tag{8.16}$$

a requirement equivalent to $\beta < \nu$. Therefore, (8.4) will hold if there exists ν in $(0, 1)$ such that $\beta < \nu < 1 - 2\beta$, a non-vacuous condition only if

$$\beta < \frac{1}{3}. \tag{8.17}$$

We summarize the findings as follows:

Proposition 8.3 *Let $A = A_1 + A_2$ be the superposition of two fluid processes A_1 and A_2. Assume A_1 to be a stationary "independent on-off" source such that $\rho_1 + \rho_2 < c < r_1 + \rho_2$ with activity period B_1 distributed according to (8.12) with (8.17), and let A_2 be a regenerative source under the moment conditions (6.13). Then, (8.11) holds.*

Here, the lower bound (8.8) requires only that source 2 satisfies (5.10) but under the additional condition (8.14) which defines the so-called moderately heavy-tail case. On the other hand, the upper bound (8.9) is shown to hold only under the more stringent condition (8.17); this constraint amounts to period of activity B_1 of source 1 to being heavy-tailed enough! At this point, the reader may wonder whether Proposition 8.3 still holds when the parameter β in (8.12) lies in the interval $(1/3, 1)$. Of course, such a conclusion, if correct, would have to be reached by arguments different from the ones used here. The following fact due to Dumas and Simonian [11] implies a partial negative answer to the question:

Proposition 8.4 *Let $A = A_1 + A_2$ be the superposition of two independent fluid processes A_1 and A_2. Assume A_1 to be a stationary "independent on-off" source such that $\rho_1 + \rho_2 < c < r_1 + \rho_2$ with activity period B_1, and let A_2 be a fluid source which satisfies (5.10). Whenever*

$$\lim_{x \to \infty} \frac{m_{B_1}(x)}{\sqrt{x}} = 0 \tag{8.18}$$

with m_{B_1} given in (3.8), then it holds that

$$\lim_{x \to \infty} \frac{\mathbf{P}\left[W^{A,c} > x\right]}{\mathbf{P}\left[W^{A_1, c-\rho_2} > x\right]} = \infty. \tag{8.19}$$

Dumas and Simonian establish this negative fact (8.19) through a very simple argument akin to the one used in deriving the generic lower bound in Proposition 2.1. When applied to the setup of Proposition 8.3 with $\frac{1}{2} < \beta < 1$, Proposition 8.4 implies the failure of the equivalence (8.11) since we now have

$$\lim_{x \to \infty} \frac{m_{B_1}(x)}{\sqrt{x}} = \lim_{x \to \infty} \frac{x^{1-\beta}}{\sqrt{x}} = 0.$$

A condition similar to (8.18) was also encountered in recent work by Asmusssen, Klüppelberg and Sigman [3, Thm. 4.1] on distributional properties of the sample of a process at subexponential times; there as well connections with Extreme Value Theory naturally emerge. When specialized to the family of Weibull distributions, their results do hold for β in the *entire* range $(0, 1/2)$. This analogy holds up the possibility that Proposition 8.3 might indeed be valid for β in the interval $(1/3, 1/2)$. After all, the argument behind the upper bound in Proposition 8.3 relies in an essential manner on the decay rates given in Proposition 6.3; there is no reason *a priori* to believe that they are best and cannot be improved! As a case in point, we remark that the reduced load approximation (8.11) does hold when the rvs $\{(Y_n, C_n),\ n = 2, \ldots\}$ characterizing source 2 are *deterministic* (as would be the case for an on-off source A_2 with deterministic on and off periods). Indeed, we then get (6.15) with $\nu_0 = 1/2$ as we note that $\mathbf{E}\left[e^{\theta(Y_2 - \rho_2 C_2)}\right] = 1$ in (D.13) so that $g(t)^2/t$ may be replaced by $g(t)$ in (D.18), and (8.16) holds with $\nu > \beta$.

When B_1 has a Benktander-type-II distribution [14, p. 149], it is also the case that $B_1 \in MDA(\Lambda) \cap \mathcal{S}$, and under the condition (8.14), Proposition 8.3 also holds since the corresponding function (3.8) is asymptotically equivalent to m_W.

8.4 Generalized Pareto

With $1 < \beta < 2$, this corresponds to

$$\mathbf{P}\left[B_1 > x\right] = x^{-\beta} L(x), \quad x \geq 0 \tag{8.20}$$

for some slowly varying function $L : \mathbb{R}_+ \to \mathbb{R}_+$. The rv B_1 is integrable but with infinite variance, and its integrated tail distribution is given by

$$\mathbf{P}\left[B_1^\star > x\right] \sim \mathbf{E}\left[B_1\right]^{-1} x^{-\beta+1} L(x).$$

We denote by m_P the mapping associated through (3.8) with the rv $X = B_1$.

Generalized Pareto rvs (8.20) do not belong to $MDA(\Lambda)$ but to the maximum domain of attraction of the Fréchet distribution Φ_β [14, p. 121]. The reader will find this matter discussed in [14, Thm. 3.3.7, p. 131]. Consequently, Proposition 8.1 cannot be invoked, and while this may be viewed as an unfortunate development, we shall see shortly that the case of generalized Pareto rvs is in fact easier than the cases treated thus far.

Indeed, we have

$$m_P(x) \sim x,$$

and in the spirit of Proposition 8.1, m_P is asymptotically equivalent to the (strictly increasing) linear mapping $u_P : x \to x$ (so that here we can take U_P to coincide with u_P in Lemma 8.1). This suggests taking perturbation mappings which are *linear*. For each η in $\mathbb{R}$, the mapping ηU_P does satisfy conditions (H1)-(H4) provided

$$c - (r_1 + \rho_2) < \eta < c - (\rho_1 + \rho_2),$$

a non-vacuous constraint under (8.1). With the notation (E.2) and (E.3), the integrability of the rvs B_1 and I_1 implies that of the rvs X_1^η and $a_\eta(B_1)$ (thus of $a_\eta^+(B_1)$). Under the stability condition we then find $\mathbf{E}\left[X_1^\eta\right] < 0$ for η in a small enough neighborhood of the origin, whence (H5) holds for ηU_P. Finally, $B_1 \in \mathcal{L}$ and $B_1^* \in \mathcal{S}$ [14], so that $a_\eta^+(B_1) \in \mathcal{L}$ and $a_\eta^+(B_1^\star) \in \mathcal{S}$ by linearity, hence (H6) and (H7) are satisfied. Consequently, for η in a small enough neighborhood of the origin, the mapping ηU_P does satisfy conditions (H1)-(H7)!

Lower bound: We apply Proposition 6.2 with $h = -U_P$. We readily conclude that (8.8) holds here as well upon noting

$$\Delta(\varepsilon h) = \mathbf{P}\left[\frac{A_2(t)}{t} - \rho_2 > -\varepsilon\right] = 1, \quad \varepsilon > 0$$

under the condition (5.9), and that

$$R_-(\varepsilon h) = \liminf_{x\to\infty} \frac{\mathbf{P}\left[(\alpha-\varepsilon)B_1^\star > x\right]}{\mathbf{P}\left[\alpha B_1^\star > x\right]} = \left(\frac{\alpha-\varepsilon}{\alpha}\right)^{\beta-1}, \quad \varepsilon > 0.$$

Upper bound: This time Proposition 6.2 will be applied with $h = U_P$. We have

$$R_+(\varepsilon h) = \limsup_{x\to\infty} \frac{\mathbf{P}\left[(\alpha+\varepsilon)B_1^\star > x\right]}{\mathbf{P}\left[\alpha B_1^\star > x\right]} = \left(\frac{\alpha+\varepsilon}{\alpha}\right)^{\beta-1}, \quad \varepsilon > 0 \tag{8.21}$$

so that (8.9) is also obtained provided condition (6.7) holds. Here as well, this condition takes the simplified form

$$\mathbf{P}\left[W^{A_2,\rho_2+\varepsilon} > x\right] = o\left(\mathbf{P}\left[\alpha B_1^\star > x\right]\right), \quad 0 < \varepsilon < \varepsilon^\star \tag{8.22}$$

owing to the fact that the rvs $\{(\alpha+\varepsilon)B_1^\star,\ \varepsilon > 0\}$ are all tail equivalent to $\alpha B_1^\star$ by virtue of (8.21). If the second source A_2 is a regenerative source under the moment conditions (6.13), then (8.22) always holds. In the Pareto case, the validity of (6.7), or equivalently of (8.22), holds more widely, even when source A_2 fails to have finite exponential moments. For instance, if source A_2 is an independent on-off source with generic activity period B_2 such that $B_2 \in \mathcal{L}$ and $B_2^\star \in \mathcal{S}$, then by Remark 4.1 we have

$$\mathbf{P}\left[W^{A_2,\rho_2+\varepsilon} > x\right] \sim K_\varepsilon \mathbf{P}\left[(r_2 - (\rho_2+\varepsilon))B_2^\star > x\right]$$

for some appropriate constant $K_\varepsilon > 0$ determined by the source statistics. Hence, (8.22) holds provided the condition

$$\mathbf{P}\left[(r_2 - (\rho_2+\varepsilon))B_2^\star > x\right] = o\left(\mathbf{P}\left[\alpha B_1^\star > x\right]\right), \quad 0 < \varepsilon < \varepsilon^\star$$

is met. Collecting all these remarks leads to the following result first obtained by Jelenkovic and Lazar [18].

Proposition 8.5 *Let $A = A_1 + A_2$ be the superposition of two independent fluid processes A_1 and A_2. Assume A_1 to be a stationary "independent on-off" source such that $\rho_1 + \rho_2 < c < r_1 + \rho_2$ with activity period B_1 distributed according to (8.20) with (5.9), and let A_2 be a fluid source satisfying (8.22) (e.g., source 2 is a regenerative source under the moment conditions (6.13)). Then, (8.11) holds.*

We complete the discussion by noting that if X is a generalized Pareto rv, then for any mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ which is asymptotically equivalent to m_P, we have

$$\lim_{x\to\infty} \frac{\mathbf{P}\left[X > x + y\varphi(x)\right]}{\mathbf{P}\left[X > x\right]} = (1+y)^{-\beta}, \quad y > -1$$

The reader will note the analogy with a similar limit in Lemma 3.6 for rvs in $MDA(\Lambda)$.

9 Additional results for intermediate regular varying rvs

In this section we strengthen Proposition 8.5 along several directions, thereby confirming the singular position occupied by generalized Pareto rvs and their extensions.

9.1 Intermediate Regular Variation

Jelenkovic and Lazar [18, Thm. 10] show that Proposition 8.5 remains true if B_1 belongs to the larger class of intermediate regular varying rvs: Following [10, Definition (1.2)], we say that an $\mathbb{R}_+$-valued rv X is an *intermediate regular varying* rv, denoted $X \in \mathcal{IR}$, if

$$\lim_{\lambda \downarrow 1} \liminf_{x \to \infty} \frac{\mathbf{P}[X > \lambda x]}{\mathbf{P}[X > x]} = \lim_{\lambda \downarrow 1} \limsup_{x \to \infty} \frac{\mathbf{P}[X > \lambda x]}{\mathbf{P}[X > x]} = 1. \tag{9.1}$$

The defining relation (9.1) is easily seen to be equivalent to

$$\lim_{\lambda \uparrow 1} \liminf_{x \to \infty} \frac{\mathbf{P}[X > \lambda x]}{\mathbf{P}[X > x]} = \lim_{\lambda \uparrow 1} \limsup_{x \to \infty} \frac{\mathbf{P}[X > \lambda x]}{\mathbf{P}[X > x]} = 1. \tag{9.2}$$

It is known that $\mathcal{R} \subset \mathcal{IR} \subset \mathcal{S}$, where $\mathcal{R}$ is the class of regular varying distributions (which coincides with the class of generalized Pareto rvs) [5, p. 18].

Our approach can also be used to extend the validity of Proposition 8.5 to the case when $B_1 \in \mathcal{IR}$. The proof is analogous to that of Proposition 8.5 upon selecting the mapping $h(x) = -x$ in the lower bound and $h(x) = x$ in the upper bound. The only difference is that now the limits $\lim_{\epsilon \downarrow 0} R_-(\varepsilon h) = 1$ (with $h(x) = -x$) and $\lim_{\epsilon \downarrow 0} R_+(\varepsilon h) = 1$ (with $h(x) = x$) follow directly from (9.1)-(9.2) together with the (easily checked) property that $B_1^\star \in \mathcal{IR}$ if $B_1 \in \mathcal{IR}$ with $0 < \mathbf{E}[B_1] < \infty$.

9.2 Upper bound without independence

This last result obtained by Jelenkovic and Lazar [18, Thm 10] can be improved along yet another direction.

Proposition 9.1 *Let $A = A_1 + A_2$ be the superposition of two fluid processes A_1 and A_2. Assume A_1 to be a stationary "independent on-off" source such that $\rho_1 + \rho_2 < c < r_1 + \rho_2$ with activity period $B_1 \in \mathcal{IR}$ and let A_2 be a fluid source satisfying (8.22) for some $\varepsilon^\star > 0$. Then, it holds that*

$$\limsup_{x \to \infty} \frac{\mathbf{P}\left[W^{A,c} > x\right]}{\mathbf{P}[W^{A_1, c-\rho_2} > x]} \le 1. \tag{9.3}$$

In contrast with Theorem 10 in [18] we do *not* require sources A_1 and A_2 to be independent. This is made possible by the asymptotic scale invariance implied by (9.1) and (9.2).

Proof. Pick ε in $(0, \min(c-(\rho_1+\rho_2), \varepsilon^\star))$ as in (8.22). Lemma 2.1 with $h(t) = \varepsilon t$ $(t \geq 0)$ and a standard union bound argument yield

$$\mathbf{P}\left[W^{A,c} > x\right] \leq \mathbf{P}\left[W^{A_1,c-\rho_2-\varepsilon} > \eta x\right] + \mathbf{P}\left[W^{A_2,\rho_2+\varepsilon} > (1-\eta)x\right], \quad x \geq 0 \tag{9.4}$$

for any η in $(0,1)$. Using (4.12) [with $\gamma \leftarrow c - \rho_2 - \varepsilon$, $\rho \leftarrow \rho_1$] for $B_1 \in \mathcal{IR}$ (thus $B_1^\star \in \mathcal{IR}$), we get

$$\mathbf{P}\left[W^{A_1,c-\rho_2-\varepsilon} > \eta x\right] \sim \frac{(1-p)\rho_1}{c-(\rho_1+\rho_2)-\varepsilon}\mathbf{P}\left[(\alpha+\varepsilon)B_1^\star > \eta x\right], \quad \eta \in [0,1], \ \varepsilon \in [0,\varepsilon^\star) \tag{9.5}$$

with α given by (5.1) as usual.

By writing

$$\frac{\mathbf{P}\left[W^{A_2,\rho_2+\varepsilon} > (1-\eta)x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]} = \frac{\mathbf{P}\left[W^{A_2,\rho_2+\varepsilon} > (1-\eta)x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > (1-\eta)x\right]}\frac{\mathbf{P}\left[W^{A_1,c-\rho_2} > (1-\eta)x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]},$$

we conclude from (8.22), (9.2) and (9.5) that

$$\lim_{\eta\uparrow 1}\limsup_{x\to\infty}\frac{\mathbf{P}\left[W^{A_2,\rho_2+\varepsilon} > (1-\eta)x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]} = 0. \tag{9.6}$$

On the other hand, application of (9.5) on each of the factors in

$$\frac{\mathbf{P}\left[W^{A_1,c-\rho_2-\varepsilon} > \eta x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]} = \frac{\mathbf{P}\left[W^{A_1,c-\rho_2-\varepsilon} > \eta x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > \eta x\right]}\frac{\mathbf{P}\left[W^{A_1,c-\rho_2} > \eta x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]}$$

gives

$$\begin{aligned}&\limsup_{x\to\infty}\frac{\mathbf{P}\left[W^{A_1,c-\rho_2-\varepsilon} > \eta x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]} \\ &= \frac{c-(\rho_1+\rho_2)}{c-(\rho_1+\rho_2)-\varepsilon}\limsup_{x\to\infty}\frac{\mathbf{P}\left[B_1^\star > \alpha(\alpha+\varepsilon)^{-1}x\right]}{\mathbf{P}\left[B_1^\star > x\right]}\limsup_{x\to\infty}\frac{\mathbf{P}\left[B_1^\star > \eta x\right]}{\mathbf{P}\left[B_1^\star > x\right]}.\end{aligned} \tag{9.7}$$

Hence, by appealing twice to (9.2) and using (9.7) we obtain

$$\lim_{\varepsilon\downarrow 0}\lim_{\eta\uparrow 1}\limsup_{x\to\infty}\frac{\mathbf{P}\left[W^{A_1,c-\rho_2-\varepsilon} > \eta x\right]}{\mathbf{P}\left[W^{A_1,c-\rho_2} > x\right]} \leq 1, \tag{9.8}$$

and combining now (9.4), (9.6) and (9.8) readily leads to (9.3). ∎

10 Superposition of independent fluid sources

Sections 5 onward have dealt with the multiplexing of two independent fluid sources A_1 and A_2, where A_1 was assumed to be a stationary "independent on-off" source and A_2 was arbitrary. In this section we consider the case when A_1 (respectively, A_2) is in turn obtained by the superposition of independent on-off (respectively, fluid) sources.

10.1 Superposition of independent on-off sources in A_1

We first consider the situation where the fluid process $A_1 = \{A_1(t),\, t \geq 0\}$ results from the superposition of N mutually independent fluid processes $A_{1,i} := \{A_{1,i}(t),\, t \geq 0\}$, $i = 1, 2, \ldots, N$, namely

$$A_1(t) := \sum_{i=1}^{N} A_{1,i}(t), \quad t \geq 0. \tag{10.1}$$

For each $i = 1, 2, \ldots, N$, the source $A_{1,i}$ is assumed to be a "stationary independent on-off" source with peak rate $r_{1,i}$, and we set $r_1 := \sum_{i=1}^{N} r_{1,i}$.

It is natural to seek an extension of the reduced load approximation result that follows from Propositions 6.2 and 7.2 when A_1 is a superposition of on-off sources as defined above and $r_1 > c - \rho_2$. Unfortunately, such an extension turns out to be extremely difficult, and we contend ourselves with only a lower bound. Proposition 10.2 generalizes a result due to Choudhury and Whitt [8, Thm 3] by allowing the presence of a "background" source A_2 that essentially reduces the service capacity c by its mean rate ρ_2. We present an intermediate result first.

For each $i = 1, \ldots, N$, we denote by $\{(I_{n,i}, B_{n,i}),\ n = 0, 1, \ldots\}$ the alternating sequence of off and on periods for the stationary version of the independent on-off source $A_{1,i}$ as described in Section 4.1.

Proposition 10.1 *Let $A_1 = \sum_{i=1}^{N} A_{1,i}$ be as in (10.1), and assume $r_1 > c - \rho_2$. Then, for any mapping $h : \mathbb{R}_+ \to \mathbb{R}_+$, with $a(t) := (r_1 - (c - \rho_2))t - h(t)$ $(t \geq 0)$, it holds that*

$$\mathbf{P}\left[W^{A_1, c-\rho_2, -h} > x\right] \geq \prod_{i=1}^{N} \left(\frac{\mathbf{E}\left[B_{1,i}\right]}{\mathbf{E}\left[B_{1,i}\right] + \mathbf{E}\left[I_{1,i}\right]} \right) \mathbf{P}\left[a(B_{1,i}^{\star}) > x\right], \quad x \geq 0. \tag{10.2}$$

Proof. Define $B_0' := \min_{i=1,\ldots,N} B_{0,i}'$ where $B_{0,i}' := B_{0,i} \mathbf{1}\left[I_{0,i} = 0\right]$, $i = 1, \ldots, N$. Then, starting from the obvious bound

$$A(B_0') - (c - \rho_2) B_0' - h(B_0') \leq W^{A_1, c-\rho_2, -h},$$

and noting that $A(B_0') = r_1 B_0'$, we see that

$$\begin{aligned}
\mathbf{P}\left[W^{A_1, c-\rho_2, -h} > x\right] &\geq \mathbf{P}\left[r_1 B_0' - (c - \rho_2) B_0' - h(B_0') > x\right] \\
&= \mathbf{P}\left[a(B_0') > x\right] \\
&\geq \mathbf{P}\left[\min_{i=1,\ldots,N} a(B_{0,i}') > x\right] \\
&= \prod_{i=1}^{N} \mathbf{P}\left[a(B_{0,i}') > x\right] \\
&\geq \prod_{i=1}^{N} \mathbf{P}\left[I_{0,i} = 0\right] \mathbf{P}\left[a(B_{0,i}) > x | I_{0,i} = 0\right]
\end{aligned}$$

and (10.2) follows from the fact that $[a(B_{0,i})|I_{0,i}=0] =_{st} a(B^\star_{1,i})$, $i=1,\ldots,N$. ∎

The next result follows from Proposition 10.1 and Proposition 2.1 in a manner similar to the proof of Proposition 7.1.

Proposition 10.2 *Let $A_1 = \sum_{i=1}^N A_{1,i}$ be as in (10.1), and assume $r_1 > c - \rho_2$. Then, for any mapping $h : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies (H4) [with $r \leftarrow r_1$, $\gamma \leftarrow c-\rho_2$, $h \leftarrow -h$], it holds that*

$$\liminf_{x\to\infty} \frac{\mathbf{P}\left[W^{A_1+A_2,c} > x\right]}{\prod_{i=1}^N \mathbf{P}\left[(r_1 - c + \rho_2)B^\star_{1,i} > x)\right]} \geq \Delta(h) \prod_{i=1}^N R_{-,i}(h) \prod_{i=1}^N \left(\frac{\mathbf{E}\left[B_{1,i}\right]}{\mathbf{E}\left[B_{1,i}\right] + \mathbf{E}\left[I_{1,i}\right]}\right)$$

with $\Delta(h)$ given by (5.8) and $R_{-,i}(h)$ by (5.6) with $B^\star_1$ replaced by $B^\star_{1,i}$.

When the on periods $B_{1,i}$ of $A_{1,i}$ have generalized Pareto, log-normal, or Weibull (with $0 < \beta < 1/2$) distributions and A_2 satisfies a CLT, we can obtain

$$\Delta(h) \prod_{i=1}^N R_{-,i}(h) = 1$$

as done in Section 8 by chosing an appropriate sequence of h. In that case we obtain the same constant as [8, Theorem 3]. More generally, by choosing $h \equiv 0$, and by assuming that A_2 satisfies a CLT we can obtain

$$\Delta(h) \prod_{i=1}^N R_{-,i}(h) = 1/2$$

with no further assumptions on the on periods $B_{1,i}$ of $A_{1,i}$.

10.2 Superposition of independent fluid sources in A_2

Propositions 6.2 and 7.2 require fairly mild conditions on source A_2. In practice, it is often the case that source A_2 is the superposition of independent fluid processes, say regenerative fluid sources, or even more specifically, independent on-off sources. The question thus naturally arises as whether the requisite conditions on the aggregate source A_2 which appear in Propositions 6.2 and 7.2 are implied by these conditions on the component sources.

We investigate these issues in the following context: Let $A_{2,i} := \{A_{2,i}(t),\, t \geq 0\}$, $i = 1,2,\ldots,N$, be N mutually independent fluid processes with average rates $\rho_{2,1},\ldots,\rho_{2,N}$, respectively. The fluid process resulting from the superposition of $A_{2,1},\ldots,A_{2,N}$ is the fluid process $A_2 := \{A_2(t),\, t\geq 0\}$ defined by

$$A_2(t) := \sum_{i=1}^N A_{2,i}(t), \quad t \geq 0; \tag{10.3}$$

its average rate is given by $\rho_2 := \sum_{i=1}^N \rho_{2,i}$. As we refer to the examples treated in Sections 8 and 9, we note that the choice of appropriate perturbation directions h in the lower and upper bounds are governed by the distribution of the activity period of source 1, with the selection ensuring the largest and smallest possible values for $\lim_{\varepsilon\downarrow 0} R_-(\varepsilon h)$ and $\lim_{\varepsilon\downarrow 0} R_+(\varepsilon h)$, respectively, among admissible perturbations. It remains therefore to explore how the impact of the individual sources on these bounds affects the impact of the aggregate source A_2.

Lower bound: In Proposition 7.2, the contribution of source A_2 to the lower bound (7.6) arises only through the constant $\lim_{\varepsilon\downarrow 0} \Delta(\varepsilon h)$ for some appropriate perturbation function $h : \mathbb{R}_+ \to \mathbb{R}$, and it is desirable to have (7.7), i.e., $\lim_{\varepsilon\downarrow 0} \Delta(\varepsilon h) = 1$. The mutual independence of the component sources $A_{2,1}, \ldots, A_{2,N}$ implies

$$\Delta_S(\varepsilon h) := \prod_{i=1}^{N} \liminf_{t\to\infty} \mathbf{P}\left[A_{2,i}(t) - \rho_{2,i}t \geq \frac{\varepsilon}{N} h(t)\right] \leq \Delta(\varepsilon h), \quad \varepsilon > 0$$

so that we may substitute the constant $\lim_{\varepsilon\downarrow 0} \Delta_S(\varepsilon h)$ in the left-hand side of (7.6) for $\lim_{\varepsilon\downarrow 0} \Delta(\varepsilon h)$. Hence, if for some $\varepsilon^\star > 0$, we have

$$\Delta_i(\varepsilon h) := \liminf_{t\uparrow\infty} \mathbf{P}\left[A_{2,i}(t) - \rho_{2,i}t \geq \varepsilon h(t)\right] = 1, \quad 0 < \varepsilon < \varepsilon^\star \tag{10.4}$$

for all $i = 1, 2, \ldots, N$, or even simply,

$$\Delta_i := \lim_{\varepsilon\downarrow 0} \Delta_i(\varepsilon h) = 1, \quad i = 1, 2, \ldots, N, \tag{10.5}$$

then $\lim_{\varepsilon\downarrow 0} \Delta_S(\varepsilon h) = 1$, thus $\lim_{\varepsilon\downarrow 0} \Delta(\varepsilon h) = 1$. In other words, the desired requirement (7.7) on A_2 is implied by the similar requirement (10.5) on each of the sources $A_{2,1}, \ldots, A_{2,N}$. As pointed out earlier, and as further discussed in Section 8, (10.4) or (10.5) will hold in many cases of interest when, depending on h, either the Law of Large Numbers (5.9) or the Central Limit Theorem (5.10) holds for each of the processes $A_{2,1}, \ldots, A_{2,N}$.

Upper bound: We now turn to the upper bound (6.8) in Proposition 6.2 when A_2 is given by the superposition (10.3). In that case, the required condition (6.7) with respect to some mapping $h : \mathbb{R}_+ \to \mathbb{R}$ will read as

$$\mathbf{P}\left[\sup_{t\geq 0}\left(\sum_{i=1}^{N} (A_{2,i}(t) - \rho_{2,i}t) - \varepsilon h(t)\right) > x\right] = o\left(\mathbf{P}\left[a_\varepsilon^+(B_1^\star) > x\right]\right), \quad 0 < \varepsilon < \varepsilon^\star. \tag{10.6}$$

Therefore, if the conditions

$$\mathbf{P}\left[W^{A_i^2, \rho_{2,i}, -\varepsilon h} > x/N\right] = o\left(\mathbf{P}\left[a_\varepsilon^+(B_1^\star) > x\right]\right), \quad 0 < \varepsilon < \varepsilon^\star,$$

for all $i = 1, 2, \ldots, N$, are simultaneously satisfied, then a standard union bound argument implies that

$$\begin{aligned}\mathbf{P}\left[\sup_{t\geq 0}\left(\sum_{i=1}^{N} \left(A_i^2(t) - \rho_{2,i}t\right) - \varepsilon h(t)\right) > x\right] &\leq \sum_{i=1}^{N} \mathbf{P}\left[W^{A_i^2, \rho_{2,i}, -\varepsilon h} > x/N\right] \\ &= o\left(\mathbf{P}\left[a_\varepsilon^+(B_1^\star) > x\right]\right)\end{aligned}$$

for $0 < \varepsilon < \varepsilon^\star$, and (10.6) holds. This argument does not require that the sources be independent.

11 Conclusions and open problems

Although we have succeeded in providing some conditions under which the reduced load equivalence (1.2) holds, the picture is far from complete, with many questions still left unanswered. We review some of them below:

In view of the negative result of Dumas and Simonian [11] described in Proposition 8.4, the equivalence (1.2) cannot hold under (8.18), and it is natural to speculate as to the form of the asymptotics for the tail probabilities $\mathbf{P}\left[W^{A_1+A_2,c} > x\right]$ even in the simple case when A_2 is an exponential on-off source.

Underlying the discussion presented here is the non-triviality condition $r_1 + \rho_2 > c$ to ensure that source 1 is not immediately flushed out when fluid is released at the reduced rate $c - \rho_2$. The equivalence (1.2) is therefore meaningless if $r_1 + \rho_2 \leq c$, and a completely different approach is needed for obtaining the correct asymptotics of $\mathbf{P}\left[W^{A_1+A_2,c} > x\right]$ even in the simple case when A_2 is an exponential on-off source.

The generic condition (6.1) is a natural one for establishing the upper bounds in the context of subexponential distributions, e.g., Lemma 3.1(3) and the line of argument flowing from the bound (6.4). In Section 6.3, by the intermediary of Proposition 6.3, we are now in possession of conditions to check the validity of (6.1) in terms of the rate of growth for the perturbation function h and the statistics of the source A_2. Proposition 6.3 (with linear perturbation functions) implies that the class of regenerative on-off sources constitutes a subclass of the class of exponential sources introduced by Jelenkovic and Lazar [18] to ensure their version of (6.1). However, what was needed here is an estimate on the rate of decay of tail probabilities associated with $W^{A_2,\rho_2,-h}$ for *non*-linear perturbation functions! In establishing this rate of decay, finite exponential moments were essential for allowing the repeated use of Chernoff bounds. Therefore, several questions suggest themselves very naturaly: As the discussion following Proposition 8.3 clearly indicates, a better decay rate is needed if one is to handle successfully the moderately ligh tailed case in its entirety. Also, it is of interest to find out what happens when source 2, while still regenerative, does not have finite exponential moments; a completely new approach would be required to get the appropriate version of Proposition 6.3 in that case.

Finally, in the introduction we mentionned the possibility of using the reduced load equivalence (1.2) for computational purposes. It seems intuitive that the heavier the tail of B_1 the better should the approximation be, but further work is required to confirm this fact.

12 Acknowledgments

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A Proofs of Lemmas 3.1–3.5

A proof of Lemma 3.1.

The proof of Claim 1 is straightforward when $X \in \mathcal{L}$ and available in [24, Lemma 2] when $X \in \mathcal{S}$. The proof of Claim 2 is an easy exercise based on bounded convergence and is therefore omitted.

When the rv T is non-negative, the proof of Claim 3 can be found in [4] or in [9, Thm. 1, p. 533]. When T is an $\mathbb{R}$-valued rv, the proof of Claim 3 proceeds as follows: Fix $x \geq 0$ and note that

$$\mathbf{P}\left[X+T>x\right] = \mathbf{P}\left[X+T>x\,|\,T>0\right]\,q + \mathbf{P}\left[X-|T|>x\,|\,T\leq 0\right]\,(1-q) \tag{A.1}$$

with $q := \mathbf{P}\left[T>0\right]$. Conditionally on $[T>0]$ the rvs X and T are independent with

$$\mathbf{P}\left[T>x\,|\,T>0\right] = \frac{1}{q}\mathbf{P}\left[T>x\right] \sim \frac{c_2}{q}\mathbf{P}\left[Z>x\right]$$

and

$$\mathbf{P}\left[X>x\,|\,T>0\right] \sim \mathbf{P}\left[X>x\right] \sim c_1\mathbf{P}\left[Z>x\right].$$

Thus, applying Lemma 3.1(3) to the conditional rvs $[X|\,T>0](=_{st} X)$ and $[T|\,T>0]$ which are non-negative, we find

$$\mathbf{P}\left[X+T>x\,|\,T>0\right] \sim \left(c_1+\frac{c_2}{q}\right)\mathbf{P}\left[Z>x\right].$$

In a similar way, conditionally on $[T\leq 0]$ the rvs X and T are independent rvs with

$$\mathbf{P}\left[X>x\,|\,T\leq 0\right] \sim \mathbf{P}\left[X>x\right] \sim c_1\mathbf{P}\left[Z>x\right].$$

Consequently, by Lemma 3.1(1), the rv $[X\,|\,T\leq 0]$ belongs to $\mathcal{S}$, thus to $\mathcal{L}$, while $[|T|\,|\,T\leq 0]$ has support in $\mathbb{R}_+$. A straighforward application of Lemma 3.1(2) to these conditional rvs yields

$$\mathbf{P}\left[X-|T|>x\,|\,T\leq 0\right] \sim \mathbf{P}\left[X>x\right] \sim c_1\mathbf{P}\left[Z>x\right]. \tag{A.2}$$

The proof is completed upon collecting (A.1)-(A.2). ∎

A proof of Lemma 3.2.

It is plain from (3.2) that

$$\frac{\mathbf{P}\left[X>x\right]}{\mathbf{P}\left[X^\star>x\right]} = \mathbf{E}\left[X\right]\left(\int_0^\infty \frac{\mathbf{P}\left[X>x+t\right]}{\mathbf{P}\left[X>x\right]}dt\right)^{-1}, \quad x>0. \tag{A.3}$$

Because $X \in \mathcal{L}$, we also have $\lim_{x\to\infty} \frac{\mathbf{P}[X>x+t]}{\mathbf{P}[X>x]} = 1$ for each $t \geq 0$, whence

$$\lim_{x\to\infty} \int_0^\infty \frac{\mathbf{P}[X > x+t]}{\mathbf{P}[X > x]} dt = \infty \tag{A.4}$$

by Fatou's lemma, and the conclusion (3.3) is now a straightforward consequence of (A.3)-(A.4).

Next, for each $x \geq 0$, we note that

$$\begin{aligned} \mathbf{E}[X]\,\mathbf{P}[X^\star > x] - \int_x^\infty \mathbf{P}[X - Y > u]\, du &= \int_x^\infty (\mathbf{P}[X > u] - \mathbf{P}[X > u + Y])du \\ &= \int_x^\infty \mathbf{P}[X > u] \left(1 - \frac{\mathbf{P}[X > u+Y]}{\mathbf{P}[X > u]}\right) du. \end{aligned}$$

By Lemma 3.1(2), X and $X - Y$ have the same right tail, i.e., for every $\varepsilon > 0$, there exists $u^\star = u^\star(\varepsilon) > 0$ such that

$$0 \leq 1 - \frac{\mathbf{P}[X > u+Y]}{\mathbf{P}[X > u]} \leq \varepsilon, \quad u \geq u^\star.$$

Consequently,

$$0 \leq 1 - \frac{1}{\mathbf{E}[X]} \frac{\int_x^\infty \mathbf{P}[X - Y > u]\, du}{\mathbf{P}[X^\star > x]} \leq \varepsilon, \quad x \geq u^\star$$

and the conclusion (3.4) immediately follows. ∎

A proof of Lemma 3.3.

Under the assumptions on φ, there exists $x^\star > 0$ such that on the interval $[x^\star, \infty)$, φ is strictly increasing with the sets $\varphi([x^\star,\infty)) = [\varphi(x^\star), \infty)$ and $\varphi([0, x^\star))$ being non-intersecting. Moreover, $x^\star$ can always be selected large enough so that $x^\star > x_0$ and $\varphi(x^\star) > 0$. Consequently, the restriction of φ to $[x^\star, \infty)$ is a.e. differentiable and invertible with $\lim_{y\to\infty} \varphi^{-1}(y) = \infty$ and we have $\{y \in \mathbb{R}_+ : \varphi(y) > u\} = (\varphi^{-1}(u), \infty)$ as soon as $\varphi(x^\star) \leq u$.

For $x \geq \varphi(x^\star)$, we note that

$$\begin{aligned} \mathbf{E}[\varphi^+(X)]\,\mathbf{P}[(\varphi^+(X))^\star > x] &= \int_x^\infty \mathbf{P}[\varphi(X) > u]\, du \\ &= \int_{\varphi^{-1}(x)}^\infty \mathbf{P}[\varphi(X) > \varphi(v)]\, \varphi'(v) dv \\ &= \int_{\varphi^{-1}(x)}^\infty \mathbf{P}[X > v]\, \varphi'(v) dv. \end{aligned} \tag{A.5}$$

The lower bound

$$\frac{\mathbf{P}[(\varphi^+(X))^\star > x]}{\mathbf{P}[X^\star > \varphi^{-1}(x)]} \geq \inf\{\varphi'(v) :\ v \geq \varphi^{-1}(x)\} \frac{\mathbf{E}[X]}{\mathbf{E}[\varphi^+(X)]}$$

is immediate and letting x go to infinity in it, we conclude that

$$\liminf_{x\to\infty} \frac{\mathbf{P}\left[(\varphi^+(X))^\star > x\right]}{\mathbf{P}\left[X^\star > \varphi^{-1}(x)\right]} \geq \lim_{x\to\infty} \inf\{\varphi'(v) : \ v \geq \varphi^{-1}(x)\} \frac{\mathbf{E}\left[X\right]}{\mathbf{E}\left[\varphi^+(X)\right]}. \tag{A.6}$$

Similarly, we have the upper bound

$$\frac{\mathbf{P}\left[(\varphi^+(X))^\star > x\right]}{\mathbf{P}\left[X^\star > \varphi^{-1}(x)\right]} \leq \sup\{\varphi'(v) : \ v \geq \varphi^{-1}(x)\} \frac{\mathbf{E}\left[X\right]}{\mathbf{E}\left[\varphi^+(X)\right]},$$

so that

$$\limsup_{x\to\infty} \frac{\mathbf{P}\left[(\varphi^+(X))^\star > x\right]}{\mathbf{P}\left[X^\star > \varphi^{-1}(x)\right]} \leq \lim_{x\to\infty} \sup\{\varphi'(v) : \ v \geq \varphi^{-1}(x)\} \frac{\mathbf{E}\left[X\right]}{\mathbf{E}\left[\varphi^+(X)\right]}. \tag{A.7}$$

The result (3.7) readily follows from (A.6) and (A.7) under the existence of the limit (3.6). ∎

A proof of Lemma 3.4.

We first establish the result in the special case when the mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and convex on $\mathbb{R}_+$ with $\varphi(0) = 0$. Under these assumptions, φ is continuous and has a uniquely defined inverse φ^{-1} on $\mathbb{R}_+$ with $\varphi^{-1}(0) = 0$. The convexity of φ implies the concavity of φ^{-1}, and we have $\lim_{x\to\infty} \varphi(x) = \lim_{x\to\infty} \varphi^{-1}(x) = \infty$.

(Claim 1): It is always the case that

$$\limsup_{x\to\infty} \frac{\mathbf{P}\left[\varphi(X) > x + y\right]}{\mathbf{P}\left[\varphi(X) > x\right]} \leq 1, \quad y \geq 0.$$

On the other hand, the concavity of φ^{-1} coupled with $\varphi^{-1}(0) = 0$ implies the subadditivity of φ^{-1}, i.e., $\varphi^{-1}(x + y) \leq \varphi^{-1}(x) + \varphi^{-1}(y)$ for all $x, y \geq 0$. Hence, fixing $y \geq 0$, we find

$$\begin{aligned} \mathbf{P}\left[\varphi(X) > x + y\right] &= \mathbf{P}\left[X > \varphi^{-1}(x + y)\right] \\ &\geq \mathbf{P}\left[X > \varphi^{-1}(x) + \varphi^{-1}(y)\right], \quad x \geq 0. \end{aligned}$$

Consequently,

$$\frac{\mathbf{P}\left[\varphi(X) > x + y\right]}{\mathbf{P}\left[\varphi(X) > x\right]} \geq \frac{\mathbf{P}\left[X > \varphi^{-1}(x) + \varphi^{-1}(y)\right]}{\mathbf{P}\left[X > \varphi^{-1}(x)\right]}, \quad x \geq 0$$

and using the fact that $X \in \mathcal{L}$, we see that

$$\liminf_{x\to\infty} \frac{\mathbf{P}\left[\varphi(X) > x + y\right]}{\mathbf{P}\left[\varphi(X) > x\right]} \geq 1, \quad y \geq 0.$$

The desired conclusion $\varphi(X) \in \mathcal{L}$ follows.

(Claim 2): This time, the convexity of φ and the condition $\varphi(0) = 0$ imply φ superadditive, i.e., $\varphi(x+y) \geq \varphi(x) + \varphi(y)$ for all $x, y \geq 0$. Consequently, with a rv X' distributed like X but independent of it, we have

$$\mathbf{P}\left[\varphi(X) + \varphi(X') > x\right] \leq \mathbf{P}\left[\varphi(X + X') > x\right], \quad x \geq 0$$

so that

$$\frac{\mathbf{P}\left[\varphi(X) + \varphi(X') > x\right]}{\mathbf{P}\left[\varphi(X) > x\right]} \leq \frac{\mathbf{P}\left[X + X' > \varphi^{-1}(x)\right]}{\mathbf{P}\left[X > \varphi^{-1}(x)\right]}, \quad x \geq 0.$$

The condition $X \in \mathcal{S}$ yields

$$\limsup_{x\to\infty} \frac{\mathbf{P}\left[\varphi(X) + \varphi(X') > x\right]}{\mathbf{P}\left[\varphi(X) > x\right]} \leq 2,$$

and the conclusion $\varphi(X) \in \mathcal{S}$ is now immediate once we note that it is always the case that

$$\liminf_{x\to\infty} \frac{\mathbf{P}\left[\varphi(X) + \varphi(X') > x\right]}{\mathbf{P}\left[\varphi(X) > x\right]} \geq 2.$$

We now turn to the general case by considering a mapping φ which satisfies the weaker assumptions of the lemma. By convexity, $\lim_{x\to\infty} \varphi(x) = \infty$, and by the finiteness of Φ_0, there exists $x^\star \geq x_0$ such that on the interval $[x^\star, \infty)$, φ is strictly increasing and convex with $\Phi_0 \leq \varphi(x^\star)$ and $0 < \varphi(x^\star)$ (hence $0 < \varphi(x)$ for $x > x^\star$). Now, consider the interpolated mapping $\varphi^\star : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\varphi^\star(x) = \frac{\varphi(x^\star)}{x^\star} x$ $(0 \leq x \leq x^\star)$ and $\varphi^\star(x) = \varphi(x)$ $(x^\star \leq x)$. By construction, the mapping $\varphi^\star : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and convex on $\mathbb{R}_+$ with $\varphi^\star(0) = 0$. Therefore, by the first part of the proof, whenever X belongs to $\mathcal{L}$ (resp. to $\mathcal{S}$), it follows that $\varphi^\star(X)$ is an element of $\mathcal{L}$ (resp. of $\mathcal{S}$). The desired conclusion on $\varphi(X)$ now follows from Lemma 3.1(1) once we observe that the rvs $\varphi(X)$ and $\varphi^\star(X)$ have equivalent right tails, i.e.,

$$\lim_{x\to\infty} \frac{\mathbf{P}\left[\varphi(X) > x\right]}{\mathbf{P}\left[\varphi^\star(X) > x\right]} = 1,$$

a fact readily verified from the construction of $h^\star$ under the assumptions on h. ∎

A proof of Lemma 3.5. Fix $x \geq 0$ and y in $\mathbb{R}$. The desired conclusion (3.10) is an immediate consequence of the obvious relation

$$\frac{\mathbf{P}\left[X^\star > x + y\varphi(x)\right]}{\mathbf{P}\left[X^\star > x\right]} = \frac{m_X(x + y\varphi(x))}{m_X(x)} \frac{\mathbf{P}\left[X > x + y\varphi(x)\right]}{\mathbf{P}\left[X > x\right]}$$

and of the assumed limit (3.11). ∎

B A proof of Proposition 4.1

We drop the superscripts A, h and γ from the notation as these quantities are fixed throughout the discussion. As in Section 4.1 we write $T_0 = 0$ and $T_{n+1} := \sum_{k=0}^{n}(I_k + B_k)$ for $n = 0, 1, \ldots$. We also note that $A(T_{n+1}) = r\sum_{k=0}^{n} B_k$ for $n = 0, 1, \ldots$.

(Claim 1): Restricting attention to the epochs $\{T_n,\ n = 0, 1, \ldots\}$ in the supremum entering the definition of W and noting that

$$h(T_n) \geq \sum_{k=0}^{n-1} h(I_k) + h(B_k), \quad n = 1, 2, \ldots \tag{B.1}$$

by the superadditivity of h, we obtain the bound

$$\begin{aligned} W &\geq \sup_{n=0,1,\ldots} \left(A(T_n) - \gamma T_n + h(T_n)\right) \\ &\geq \max\left(h(0), \sup_{n=1,2,\ldots} \left(\sum_{k=0}^{n-1} (rB_k - \gamma(I_k + B_k) + h(I_k) + h(B_k))\right)\right) \\ &= \max\left(h(0), \sup_{n=0,1,\ldots} \left(\sum_{k=0}^{n} X_k\right)\right) = V. \end{aligned}$$

(Claim 2): Condition (H1) (resp. (H2)) ensures that the mapping a (resp. b) is monotone increasing (resp. decreasing), and it is therefore easy to check that

$$W = \sup_{n=0,1,\ldots} \left(A(T_n) - \gamma T_n + h(T_n)\right), \tag{B.2}$$

where this time around, the subadditivity of h yields

$$h(T_n) \leq \sum_{k=0}^{n-1} h(I_k) + h(B_k), \quad n = 1, 2, \ldots. \tag{B.3}$$

Consequently, by an argument similar to that given in the proof of Claim 1, we get

$$\begin{aligned} W &\leq \max\left(h(0), \sup_{n=1,2,\ldots} \left(\sum_{k=0}^{n-1} (rB_k - \gamma(I_k + B_k) + h(I_k) + h(B_k))\right)\right) \\ &= \max\left(h(0), \sup_{n=0,1,\ldots} \left(\sum_{k=0}^{n} X_k\right)\right) = V. \end{aligned}$$

■

C A proof of Proposition 4.2

Here as well, we drop the superscripts A, h and γ from the notation. Note from (4.9) the relation

$$\mathbf{P}[V > x] = \mathbf{P}[X_0 + M > x], \quad x \geq h(0) \tag{C.1}$$

with X_0 independent of the rv M which is given by

$$M := \left(\sup_{n=1,2,\ldots} \sum_{k=1}^{n} X_k \right)^+ . \tag{C.2}$$

As we have in mind to invoke Lemma 3.1(3), we consider in turn the asymptotic behavior of each the rvs X_0 and M.

(Step 1) The discussion will make use of several technical facts which we now develop: Under (H2) the inequality

$$h(t) - h(0) = \int_0^t h'(s)ds \leq \gamma t, \quad t \geq 0$$

holds, so that

$$b(t) \leq b(0), \quad t \geq 0 \tag{C.3}$$

and

$$a^+(t) \leq ((r-\gamma)t + h(0) + \gamma t)^+ \leq h(0)^+ + rt, \quad t \geq 0. \tag{C.4}$$

As a result of (C.4), the integrability of B_1 implies that of $a^+(B_1)$; Tchebychev's inequality now yields

$$x\mathbf{P}\left[a^+(B_1) > x\right] \leq \mathbf{E}\left[a^+(B_1)\right], \quad x > 0 \tag{C.5}$$

and the rv $a^+(B_1)$ being long-tailed by (H6), we can select x large enough to conclude

$$0 < \mathbf{E}\left[a^+(B_1)\right] < \infty. \tag{C.6}$$

Next, under (H3) and (H4) it follows from (C.4) that a^+ satisfies the conditions (i) and (ii) of Lemma 3.3. Thus, applying Lemma 3.3 [with $\varphi = a^+$ and $X = B_1$], we get

$$\mathbf{P}\left[(a^+(B_1))^\star > x\right] \sim \left((r-\gamma) + h'(\infty)\right) \frac{\mathbf{E}[B_1]}{\mathbf{E}[a^+(B_1)]} \mathbf{P}\left[a^+(B_1^\star) > x\right] \tag{C.7}$$

given (4.1) and (C.6).

(Step 2) The tail of asymptotics of M will be identified through a well-known result of Veraverbeke [28, Theorem 2(B), p. 35]. To prepare for it, with the definition (4.8) of X_1, we remark the inequality $X_1 \leq a(B_1) + b(0)$ via (C.3), whence $X_1^+ \leq a^+(B_1) + b(0)^+$, and the integrability of X_1^+ is implied by that of $a^+(B_1)$. Also, appealing to (C.3) again, we get

$$\begin{aligned} \mathbf{P}[X_1 > x] &= \mathbf{P}\left[a^+(B_1) - \mid b(I_1) - b(0) \mid + b(0) > x\right] \\ &\sim \mathbf{P}\left[a^+(B_1) > x\right] \end{aligned} \tag{C.8}$$

where the equivalence is validated by Lemma 3.1(2) [with $X = a^+(B_1)$, $Y = \mid b(0) - b(I_1^\star) \mid$ and $d = b(0)$] under the independence of the rvs B_1 and I_1. In conclusion, we have

$$\mathbf{P}\left[X_1^+ > x\right] \sim \mathbf{P}\left[a^+(B_1) > x\right] \tag{C.9}$$

so that $X_1^+ \in \mathcal{L}$ under (H6). Now, the same arguments based on Tchebychev's inequality which gave (C.6), when applied to the rv X_1^+, yield

$$0 < \mathbf{E}\left[X_1^+\right] < \infty. \tag{C.10}$$

Consequently, the integrated tail rv $(X_1^+)^\star$ associated with X_1^+ is well defined, and the equivalence (C.9) gives

$$\int_x^\infty \mathbf{P}\left[X_1^+ > u\right] du \sim \int_x^\infty \mathbf{P}\left[a^+(B_1) > u\right] du. \tag{C.11}$$

Hence, under (C.6), by the definition of the integrated tail of the rv $a^+(B_1)$, we conclude that

$$\begin{aligned} \int_x^\infty \mathbf{P}\left[X_1 > u\right] du &\sim \int_x^\infty \mathbf{P}\left[a^+(B_1) > u\right] du \\ &\sim \mathbf{E}\left[a^+(B_1)\right] \mathbf{P}\left[(a^+(B_1))^\star > x\right] \qquad \text{(C.12)} \\ &\sim ((r-\gamma) + h'(\infty))\, \mathbf{E}\left[B_1\right] \mathbf{P}\left[a^+(B_1^\star) > x\right] \qquad \text{(C.13)} \end{aligned}$$

where the asymptotic equivalence (C.13) follows from (C.7).

Using (H7) we conclude from (C.13) and Lemma 3.1 that the rv X_1^+ has an integrated tail in $\mathcal{S}$. Since $\mathbf{E}\left[X_1\right] < 0$ under (H5), Theorem 2(B) of [28, p. 35] yields

$$-\mathbf{E}\left[X_1\right] \mathbf{P}\left[M > x\right] \sim \int_x^\infty \mathbf{P}\left[X_1 > u\right] du \tag{C.14}$$

and upon substituting (C.13) into (C.14), we readily obtain the asymptotics

$$\mathbf{P}\left[M > x\right] \sim K(h)\mathbf{P}\left[a^+(B_1^\star) > x\right] \tag{C.15}$$

where $K(h)$ is given by (4.11).

(Step 3) To discover the tail asymptotics of the rv X_0, we observe from (4.5) that

$$\mathbf{P}\left[X_0 > x\right] = (1-p)\mathbf{P}\left[a(B_1) + b(I_1^\star) > x\right] + p\mathbf{P}\left[a(B_1^\star) + b(0) > x\right], \quad x \geq 0 \tag{C.16}$$

with $I_1^\star$, B_1 and $B_1^\star$ independent rvs.

Under (H6), the rv $a^+(B_1)$ belongs to $\mathcal{L}$. Thus, recalling (C.3) and applying Lemma 3.1(2) [with $X = a^+(B_1)$, $Y = \mid b(0) - b(I_1^\star) \mid$ and $d = b(0)$], we get

$$\begin{aligned} \mathbf{P}\left[a(B_1) + b(I_1^\star) > x\right] &\sim \mathbf{P}\left[a^+(B_1) - \mid b(0) - b(I_1^\star) \mid + b(0) > x\right] \\ &= \mathbf{P}\left[a^+(B_1) > x\right]. \qquad \text{(C.17)} \end{aligned}$$

Lemma 3.2 [with $X = a^+(B_1)$] and (C.7) successively yield

$$\begin{aligned} \mathbf{P}\left[a^+(B_1) > x\right] &= o(\mathbf{P}\left[(a^+(B_1))^\star > x\right]) \\ &= o(\mathbf{P}\left[a^+(B_1^\star) > x\right]); \end{aligned} \tag{C.18}$$

the required moment conditions $0 < \mathbf{E}[B_1]$, $\mathbf{E}[a^+(B_1)] < \infty$ in Lemma 3.2 hold owing to (4.1) and (C.6). It is now plain from (C.17) and (C.18) that

$$\mathbf{P}[a(B_1) + b(I_1^\star) > x] = o(\mathbf{P}\left[a^+(B_1^\star) > x\right]). \tag{C.19}$$

On the other hand, $a^+(B_1^\star)$ belongs to $\mathcal{S}$ under (H7) (thus to $\mathcal{L}$), whence asymptotically equivalent to $a^+(B_1^\star) + b(0)$, i.e.,

$$\mathbf{P}[a(B_1^\star) + b(0) > x] \sim \mathbf{P}\left[a^+(B_1^\star) > x\right]. \tag{C.20}$$

Combining (C.16), (C.19) and (C.20) we find

$$\mathbf{P}[X_0 > x] \sim p\mathbf{P}\left[a^+(B_1^\star) > x\right]. \tag{C.21}$$

(Step 4) Collecting (C.15) and (C.21), we readily conclude to (4.10) and (4.11) by an application of Lemma 3.1(3) [with $X = M$, $T = X_0$, $Z = a^+(B_1^\star)$, $c_1 = K(h)$, $c_2 = 1$]. By Lemma 3.1(1), membership of V in $\mathcal{S}$ follows from that of $a^+(B_1^\star)$ in $\mathcal{S}$. ∎

D A proof of Proposition 6.3

We need the following fact later in the proof:

Lemma D.1 *Consider an $\mathbb{R}$-valued rv X such that $\mathbf{E}[X] = 0$ and*

$$\mathbf{E}\left[e^{\theta X}\right] < \infty, \quad |\theta| \le \theta^\star \tag{D.1}$$

for some $\theta^\star > 0$. Then, there exists $\theta^{\star\star}$ in the interval $(0, \theta^\star)$ and $\xi > 0$ such that

$$\mathbf{E}\left[e^{\theta X}\right] < e^{\frac{\xi}{2}\theta^2}, \quad |\theta| \le \theta^{\star\star}. \tag{D.2}$$

Proof: Fix θ in $\mathbb{R}$. It is a simple matter to check the identity

$$e^{\theta x} = 1 + \theta x + \theta^2 \int_0^x \left(\int_0^t e^{\theta s} ds\right) dt, \quad x \in \mathbb{R}.$$

The zero-mean condition implies

$$\mathbf{E}\left[e^{\theta X}\right] = 1 + \theta^2 K(\theta) \quad \text{with} \quad K(\theta) := \mathbf{E}\left[\int_0^X \left(\int_0^t e^{\theta s} ds\right) dt\right],$$

and we conclude

$$\mathbf{E}\left[e^{\theta X}\right] \leq 1 + \theta^2 |K(\theta)| \leq e^{\theta^2 |K(\theta)|}.$$

The proof of (D.2) is now completed by noting that condition (D.1) ensures the existence of some $\theta^{\star\star}$ in the interval $(0, \theta^\star)$ such that

$$\xi := \frac{1}{2} \sup_{|\theta| \leq \theta^{\star\star}} |K(\theta)| < \infty.$$

■

To proceed with the proof of Proposition 6.3, we set $\overline{C} := \mathbf{E}[C_2] > 0$, and pick scalars

$$x \geq 0, \quad \alpha, \varepsilon > 0, \quad \eta, \mu \in (0,1) \quad \text{and} \quad \beta \in (0, \overline{C}).$$

We also write

$$z(x) := \frac{(1-\eta)}{\varepsilon} x \quad \text{and} \quad n(x) := \lfloor \frac{(1-\mu) z(x)}{\overline{C} + \alpha} \rfloor$$

and for easy reference, we set

$$M_i := \sup_{0 \leq \theta \leq \theta_0} M_i(\theta) = M_i(\theta_0) \quad \text{and} \quad N_i := \sup_{0 \leq \theta \leq \theta_0} N_i(\theta) = N_i(\theta_0), \quad i = 1, 2.$$

The definition of the rv $W^{A_2, \rho_2, -h}$ and the monotone character of h immediately yield

$$W^{A_2, \rho_2, -h} \leq \sup_{n=0,1,\ldots} \left(A_2(T_{n+1}) - \rho_2 T_n - h(T_n)\right),$$

so that information on the tail of the rv $W^{A_2, \rho_2, -h}$ can in principle be obtained by considering the tail of the maximum associated with the "perturbed random walk" $\{A_2(T_{n+1}) - \rho_2 T_n - h(T_n),\ n = 0, 1, \ldots\}$. However, the matter is not straightforward, especially when $L = 0$ in (6.10), as we note then that

$$\lim_{n \to \infty} \frac{1}{n} \left(A_2(T_{n+1}) - \rho_2 T_n - h(T_n)\right) = 0 \quad a.s.$$

Although a Large Deviations Principle will hold for $\{A_2(T_{n+1}) - \rho_2 T_n,\ n = 0, 1, \ldots\}$ (and even for the perturbed random walk under some additional conditions on h), this is not enough to guarantee exponential decay for the tail of the rv $W^{A_2, \rho_2, -h}$. The basic idea of the proof will be to first "extract" the exponential tails associated with the various underlying Large Deviations Principles. What remains will provide us with a way to capture the effect of the perturbation h.

By the union bound, it is then plain that

$$\begin{aligned}\mathbf{P}\left[W^{A_2,\rho_2,-h} > x\right] &\leq \sum_{n=0}^{\infty} \mathbf{P}\left[A_2(T_{n+1}) - \rho_2 T_n - h(T_n) > x\right] \\ &\leq A(x) + B(x) \end{aligned} \tag{D.3}$$

with

$$A(x) := \sum_{n=0}^{\infty} \mathbf{P}\left[A_2(T_{n+1}) - (\rho_2 + \varepsilon)T_n > \eta x\right]$$

and

$$B(x) := \sum_{n=0}^{\infty} \mathbf{P}\left[A_2(T_{n+1}) - \rho_2 T_n - h(T_n) > x, A_2(T_{n+1}) - \rho_2 T_n \leq \varepsilon T_n + \eta x\right].$$

The remainder of the proof consists in bounding each of the terms $A(x)$ and $B(x)$.

(Step 1) Using a Chernoff bound argument on each term of $A(x)$ and the independence assumptions on the rvs $\{(C_n, Y_n),\ n = 1, 2, \ldots\}$, we conclude that

$$\begin{aligned} A(x) &= \mathbf{P}\left[Y_1 > \eta x\right] + \sum_{n=1}^{\infty} \mathbf{P}\left[Y_{n+1} + \sum_{i=1}^{n}(Y_i - (\rho_2+\varepsilon)C_i) > \eta x\right] \\ &\leq e^{-\theta\eta x}\, N_1 \left(1 + N_2 \sum_{n=0}^{\infty} \left(\mathbf{E}\left[e^{\theta(Y_2 - (\rho_2+\epsilon)C_2)}\right]\right)^n\right), \quad 0 < \theta < \theta_0. \end{aligned}$$

For θ in $[0, \theta_0]$, $r_1(\theta) := \mathbf{E}\left[e^{\theta(Y_2-(\rho_2+\epsilon)C_2)}\right] \leq N_2(\theta) < \infty$ under (6.13), and we get $r_1'(0) = -\varepsilon \mathbf{E}\left[C_2\right] < 0$ by the definition of ρ_2. Hence, $r_1(\theta) < 1$ for all θ in $(0, \theta_1)$ with $0 < \theta_1 \leq \theta_0$, so that

$$A(x) \leq D_1(\theta)\, e^{-\theta\eta x} \quad \text{with} \quad D_1(\theta) := N_1\left(1 + \frac{N_2}{1 - r_1(\theta)}\right) < \infty. \tag{D.4}$$

(Step 2) To handle $B(x)$ we note the following: Whenever $A_2(T_{n+1}) - \rho_2 T_n - h(T_n) > x$ *and* $A_2(T_{n+1}) - \rho_2 T_n \leq \varepsilon T_n + \eta x$, then necessarily $A_2(T_{n+1}) - \rho_2 T_n > h(T_n)$ and $h(T_n) + x < \varepsilon T_n + \eta x$, the latter inequality implying $z(x) < T_{n+1}$. Hence,

$$\begin{aligned} B(x) &\leq \sum_{n=0}^{\infty} \mathbf{P}\left[A_2(T_{n+1}) - \rho_2 T_n > h(T_n), z(x) < T_{n+1}\right] \\ &\leq C(x) + D(x) \end{aligned} \tag{D.5}$$

with

$$C(x) := \sum_{n=0}^{\infty} \mathbf{P}\left[(n+1)(\overline{C} + \alpha) + \mu z(x) < T_{n+1}\right]$$

and

$$D(x) := \sum_{n=0}^{\infty} \mathbf{P}\left[A_2(T_{n+1}) - \rho_2 T_n > h(T_n), z(x) < T_{n+1} \leq (n+1)(\overline{C} + \alpha) + \mu z(x)\right].$$

A Chernoff bound argument on each term of $C(x)$ yields

$$\begin{aligned} C(x) &\leq \sum_{n=0}^{\infty} e^{-\theta\mu z(x)} \mathbf{E}\left[e^{\theta(T_{n+1}-(n+1)(\overline{C}+\alpha))}\right] \\ &\leq M_1 e^{-\theta(\mu z(x)+\overline{C}+\alpha)} \sum_{n=0}^{\infty} r_2(\theta)^n \end{aligned}$$

for θ in $[0,\theta_0]$ where we have set $r_2(\theta) := \mathbf{E}\left[\exp(\theta(C_2-\overline{C}-\alpha))\right]$. Since $r_2(0)=0$ and $r_2'(0)=-\alpha<0$, there exists $\theta_2>0$ such that $r_2(\theta)<1$ on $(0,\theta_2)$. Hence, on the range $0<\theta<\theta_3:=\min\{\theta_1,\theta_2\}$, we have

$$C(x) \leq D_2(\theta)\, e^{-\theta\mu z(x)} \quad \text{with} \quad D_2(\theta) := \frac{M_1 e^{-\theta(\overline{C}+\alpha)}}{(1-r_2(\theta))} < \infty. \tag{D.6}$$

(Step 3) Next, we note that the condition $z(x) < T_{n+1} \leq (n+1)(\overline{C}+\alpha)+\mu z(x)$ is vacuous unless $(n+1)(\overline{C}+\alpha)+\mu z(x) > z(x)$, or equivalently, unless $n(x)\leq n$, whence the first $n(x)$ terms in the sum $D(x)$ equal zero. With $\beta_1 := \overline{C}-\beta>0$, we get

$$D(x) \leq E(x) + F(x) \tag{D.7}$$

with

$$E(x) := \sum_{n=n(x)}^{\infty} \mathbf{P}\left[T_n < n\beta_1\right]$$

and

$$F(x) := \sum_{n=n(x)}^{\infty} \mathbf{P}\left[A_2(T_{n+1}) - \rho_2 T_n > h(T_n), n\beta_1 \leq T_n\right].$$

The usual Chernoff bound argument now yields

$$E(x) \leq e^{\theta\beta_1} \sum_{n=n(x)}^{\infty} r_3(\theta)^{n-1} \tag{D.8}$$

for all $\theta>0$, where $r_3(\theta) := \mathbf{E}\left[e^{\theta(\beta_1-C_2)}\right]$. Since $r_3(0)=1$ and $r_3'(0)=-\beta<0$, there exists θ_4 in $(0,\theta_0)$ such that $r_3(\theta)<1$ for all θ in $(0,\theta_4)$. Hence, on the range $0<\theta<\theta_5:=\min\{\theta_3,\theta_4\}$, we deduce from (D.8) that

$$E(x) \leq D_3(\theta)\, r_3(\theta)^{n(x)} \quad \text{with} \quad D_3(\theta) := \frac{e^{\theta\beta_1}}{r_3(\theta)\,(1-r_3(\theta))} < \infty. \tag{D.9}$$

(Step 4) Collecting the bounds (D.3), (D.5) and (D.7), we find

$$\mathbf{P}\left[W^{A_2,\rho_2,-h} > x\right] \leq R(x) + F(x) \tag{D.10}$$

with

$$R(x) := A(x) + C(x) + E(x).$$

Making use of (D.4), (D.6) and (D.9), it follows that

$$R(x) \leq D_1(\theta)\, e^{-\theta\eta x} + D_2(\theta)\, e^{-\theta\mu z(x)} + D_3(\theta)\, r_3(\theta)^{n(x)}$$

whenever $0 < \theta < \theta_5$. Let γ_0 be a constant such that $n(x) \geq \gamma_0 x$ for x large enough, e.g., select γ_0 as $\gamma_0 := (1-\mu)(1-\eta)/(2\varepsilon(\overline{C}+\alpha)$. With this notation we get

$$R(x) \leq D(\theta)\, e^{-dx} \tag{D.11}$$

for x large enough, where $0 < d := \min(\theta\eta, -\gamma_0 \log(r_3(\theta)), \theta\mu(1-\eta)/\varepsilon)$ and $D(\theta) := \sum_{i=1}^{3} D_i(\theta)$.

(Step 5) For the last term $F(x)$, observe from the monotone character of h that $h(T_n) \geq h(n\beta_1)$ if $n\beta_1 \leq T_n$, whence

$$F(x) \leq \sum_{n=n(x)}^{\infty} \mathbf{P}\left[A_2(T_{n+1}) - \rho_2 T_n - h(n\beta_1) > 0\right]. \tag{D.12}$$

Pick θ in the interval $(0, \theta_0)$. For each $n = 1, 2, \ldots$, a Chernoff bound argument gives

$$\begin{aligned}
&\mathbf{P}\left[A_2(T_{n+1}) - \rho_2 T_n - h(n\beta_1) > 0\right] \\
&\leq \quad e^{-\theta h(n\beta_1)} \mathbf{E}\left[e^{\theta(Y_1 - \rho_2 C_1)}\right] \mathbf{E}\left[e^{\theta Y_{n+1}}\right] \mathbf{E}\left[e^{\theta(Y_2 - \rho_2 C_2)}\right]^{n-1} \\
&\leq \quad N_1 N_2 e^{-\theta h(n\beta_1)} \mathbf{E}\left[e^{\theta(Y_2 - \rho_2 C_2)}\right]^{n-1}.
\end{aligned} \tag{D.13}$$

By Lemma D.1, under the moment relation $\mathbf{E}[Y_2] = \rho_2 \mathbf{E}[C_2]$, there exist $\xi > 0$ and θ_6 in $(0, \theta_0)$ such that

$$\mathbf{E}\left[e^{\theta(Y_2 - \rho_2 C_2)}\right] \leq e^{\frac{\xi}{2}\theta^2}, \quad \theta \in (0, \theta_6). \tag{D.14}$$

Combining (D.13) and (D.14), we find

$$\mathbf{P}\left[A_2(T_{n+1}) - \rho_2 T_n - h(n\beta_1) > 0\right] \leq N_1 N_2 e^{-\left(h(n\beta_1)\theta - \frac{n\xi}{2}\theta^2\right)}. \tag{D.15}$$

This last upper bound is best, i.e., smallest, for $\theta = h(\beta_1 n)/n\xi$, a quantity that can be made arbitrary small since (6.10) holds with $L = 0$, hence smaller than θ_6, for n large enough, in which case (D.15) can be tightened to

$$\mathbf{P}\left[A_2(T_{n+1}) - \rho_2 T_n - h(n\beta_1) > 0\right] \leq N_1 N_2 e^{-\frac{h(n\beta_1)^2}{2n\xi}}. \tag{D.16}$$

In particular, for x large enough, we conclude from (D.12) and (D.16) that

$$F(x) \leq N_1 N_2 \sum_{n=n(x)}^{\infty} e^{-\frac{h(n\beta_1)^2}{2n\xi}}. \tag{D.17}$$

Consequently, for x large enough, we have $n(x) \geq \gamma_0 x$, and with $\gamma_1 := \beta_1\gamma_0$ and $\gamma_2 := \beta_1/2\xi$, a standard bounding argument shows that

$$\begin{aligned}
\sum_{n=n(x)}^{\infty} e^{-\frac{h(n\beta_1)^2}{2n\xi}} &\leq \sum_{n=n(x)}^{\infty} e^{-\frac{\gamma_2 g(n\beta_1)^2}{n\beta_1}} \\
&\leq \int_{\gamma_0 x}^{\infty} e^{-\gamma_2 g(\beta_1 t)^2/\beta_1 t}\, dt \\
&= \frac{1}{\beta_1}\int_{\gamma_1 x}^{\infty} e^{-\gamma_2 g(t)^2/t}\, dt =: \frac{1}{\beta_1} R_g(x) \qquad \text{(D.18)}
\end{aligned}$$

as we note that $t \to g(t)/\sqrt{t} = \inf_{s\geq t} h(s)/\sqrt{s}$ is nondecreasing. Reporting (D.18) into (D.17) yields $F(x) \leq MR_h(x)$ with $M := N_1N_2/\beta_1$, so that (D.10) and (D.11) together now imply

$$\mathbf{P}\left[W^{A_2,\rho_2,-h} > x\right] \leq D(\theta)\, e^{-dx} + MR_g(x) \qquad \text{(D.19)}$$

for x large enough.

(Step 6) We are now in position to prove (6.14) and (6.15). In view of (D.19), (6.14) will follow with $\gamma_3 = M$ if we show that $\lim_{x\to\infty} e^{-ax}/R_g(x) = 0$ for any $a > 0$, or equivalently that $\lim_{x\to\infty} R_g(x)e^{ax} = \infty$ for any $a > 0$. To that end, with

$$G(t) := \frac{a}{\gamma_1}t - \gamma_2\frac{g(t)^2}{t}, \quad t > 0$$

we get

$$\begin{aligned}
R_g(x)e^{ax} &= e^{ax}\int_{\gamma_1 x}^{\infty} e^{G(t)}\, e^{-at/\gamma_1}\, dt \\
&\geq \left(\inf_{t\geq\gamma_1 x} e^{G(t)}\right)\cdot\int_{\gamma_1 x}^{\infty} e^{ax-at/\gamma_1}\, dt = \frac{\gamma_1}{a}\inf_{t\geq\gamma_1 x} e^{G(t)}.
\end{aligned}$$

The desired conclusion is obtained if we show that $\lim_{t\to\infty} G(t) = \infty$, a fact which follows from $L = 0$ in (6.10), the fact $g(t)/t \leq h(t)/t$ $(t > 0)$ and the identity

$$G(t) = t\left(\frac{a}{\gamma_1} - \gamma_2\left(\frac{g(t)}{t}\right)^2\right), \quad t > 0.$$

We now turn to the proof of (6.15). For any given constant $C > 0$, assumption (2) implies that $h(t)^2/t \geq Ct^{2\nu_0} \geq Ct^{\nu}$ for t large enough whenever ν lies in $(0, 2\nu_0)$. Therefore, for x large enough, a simple bounding argument and the well-known asymptotics of the incomplete Gamma function lead to

$$R_g(x) \leq \int_{\gamma_1 x}^{\infty} e^{-\delta t^{\nu}}\, dt \sim \delta_0\,(\gamma_1 x)^{1-\nu}\, e^{-\delta(\gamma_1 x)^{\nu}} \qquad \text{(D.20)}$$

with $\delta := C\,\gamma_2$ and $\delta_0 := \delta^{(1-\nu)/\nu}\Gamma(\nu)$. Combining (6.14) and (D.20) gives (6.15) with $\delta_1 = \delta\gamma_1^{\nu}, \delta_2 = \gamma_3\delta_0\gamma_1^{(1-\nu)}$.

∎

E A proof of Proposition 8.1

We show that the choices $h = U$ and $h = -U$ meet the requirements of Claims 1 and 2, respectively. To avoid unnecessary repetitions, we start with some comments that are common to both Claims.

(Fact 1): Because $\lim_{x\to\infty} m_{B_1}(x) = \infty$ by Lemma 3.2, the same property holds for u by the asymptotic equivalence $m_{B_1} \sim u$, hence for U. The mapping U being concave and increasing on $\mathbb{R}_+$, the convergence

$$\lim_{t\to\infty} \frac{U(t)}{t} = \lim_{t\to\infty} \frac{u(t)}{t} = \lim_{t\to\infty} u'(t) =: u'(\infty) \tag{E.1}$$

takes place in a monotonically decreasing manner, whence the limit exists and is finite.

(Fact 2): With (5.2) and (5.4) in mind for the situations at hand, for every η in $\mathbb{R}$ we write

$$a_\eta(x) := \alpha x + \eta U(x), \quad x \geq 0, \tag{E.2}$$

and

$$X_1^\eta := \alpha B_1 - (c - \rho_2) I_1 + \eta(U(B_1) + U(I_1)). \tag{E.3}$$

By Jensen's inequality we have $0 \leq \mathbf{E}[U(B_1)] \leq U(\mathbf{E}[B_1]) < \infty$, with similar inequalities for I_1, whence the rvs $U(B_1)$ and $U(I_1)$ are both integrable. Thus, the rvs $a_\eta(B_1)$ and X_1^η are integrable; we have $\mathbf{E}[X_1^\eta] < 0$ for small enough η in view of the fact that $\lim_{\eta\downarrow 0} \mathbf{E}[X_1^\eta] < 0$ by the stability condition.

(Fact 3): For a given η in $\mathbb{R}$, whenever the mapping a_η is *strictly increasing in the limit*, we have

$$\begin{aligned}
\lim_{x\to\infty} \frac{\mathbf{P}\left[a_\eta^+(B_1) > x\right]}{\mathbf{P}[\alpha B_1 > x]} &= \lim_{x\to\infty} \frac{\mathbf{P}\left[a_\eta^+(B_1) > a_\eta^+(x)\right]}{\mathbf{P}\left[\alpha B_1 > a_\eta^+(x)\right]} \\
&= \lim_{x\to\infty} \frac{\mathbf{P}[B_1 > x]}{\mathbf{P}\left[\alpha B_1 > a_\eta^+(x)\right]} = e^{\alpha^{-1}\eta}
\end{aligned} \tag{E.4}$$

by virtue of Lemma 3.6 since $B_1 \in MDA(\Lambda)$ (and $m_{B_1} \sim U$). A similar argument, making use this time of the consequence (3.12) of Lemma 3.6, also yields

$$\lim_{x\to\infty} \frac{\mathbf{P}\left[a_\eta^+(B_1^\star) > x\right]}{\mathbf{P}[\alpha B_1^\star > x]} = \lim_{x\to\infty} \frac{\mathbf{P}[B_1^\star > x]}{\mathbf{P}\left[\alpha B_1^\star > a_\eta^+(x)\right]} = e^{\alpha^{-1}\eta}. \tag{E.5}$$

In other words, whenever the mapping a_η is strictly increasing in the limit, we conclude from (E.4) that the rvs αB_1 and $a_\eta^+(B_1)$ are both tail equivalent, while (E.5) implies that the rvs $\alpha B_1^\star$ and $a_\eta^+(B_1^\star)$ are tail equivalent. By Lemma 3.1 (1) the rvs $a_\eta^+(B_1)$ and $a_\eta^+(B_1^\star)$ are elements of $\mathcal{L}$ and $\mathcal{S}$, respectively.

We are now ready to discuss Claims 1 and 2.

(Claim 1): Define $h := U$ and fix $\varepsilon > 0$. The mapping h is subadditive and absolutely continuous by concavity (with $h(0) = 0$). The scaled mapping εh automatically satisfies (H3) and (H4) (under the second half of (8.1)). From this latter condition we conclude via Fact 3 that $a^+_\varepsilon(B_1)$ and $a^+_\varepsilon(B^\star_1)$ are elements of $\mathcal{L}$ and $\mathcal{S}$, respectively, i.e., conditions (H6) and (H7) both hold for εh. Conditions (H1), (H2) and (H3) on εh are equivalent to

$$c - (r_1 + \rho_2) \leq \varepsilon U'(x) \leq c - \rho_2 \quad a.e. \text{ on } \mathbb{R}_+ \tag{E.6}$$

with $U'(\infty)$ finite. Because U is a non-decreasing and concave function, it follows that $0 \leq U'(\infty) \leq U'(x) \leq U'(0+)$ a.e. on $\mathbb{R}_+$ with $U'(0+)$ finite by Lemma 8.1. The constraints (E.6) are therefore implied by requiring

$$c - (r_1 + \rho_2) \leq \varepsilon U'(\infty) \leq \varepsilon U'(0+) \leq c - \rho_2 \quad a.e. \text{ on } \mathbb{R}_+ \tag{E.7}$$

and under (8.1) this is obviously satisfied if ε is chosen sufficiently small. If ε is taken small enough, we see from the discussion above that (H5) holds as well.

(Claim 2): This time, define $h := -U$ and fix $\varepsilon > 0$. The absolutely continuous mapping h is now superadditive. Since U is non-decreasing, the scaled mapping εh automatically satisfies (H2), while condition (H3) holds by the remarks leading to (E.1). Moreover, by Fact 2 we see that εh satisfies (H5).

The mapping $a_{-\varepsilon}$ will be strictly increasing if $a'_{-\varepsilon}(x) > 0$ a.e. on $\mathbb{R}_+$, or equivalently,

$$a'_{-\varepsilon}(x) = r_1 + \rho_2 - c - \varepsilon U'(x) > 0 \quad a.e. \text{ on } \mathbb{R}_+. \tag{E.8}$$

By remarks made in the proof of Claim 1, this last requirement will hold if

$$a'_{-\varepsilon}(x) \geq r_1 + \rho_2 - c - \varepsilon U'(0+) > 0 \quad a.e. \text{ on } \mathbb{R}_+ \tag{E.9}$$

by virtue of the concavity of U. This is always possible owing to (8.1) by selecting ε sufficiently small, in which case $a_{-\varepsilon}$ is strictly increasing on $\mathbb{R}_+$ with $\lim_{x\to\infty} a_{-\varepsilon}(x) = \infty$.

Consequently, for ε sufficiently small, it is the case that εh satisfies (H4), hence (H6) and (H7) by appealing to Fact 3. ∎

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