TECHNICAL RESEARCH REPORT

Closed-Loop Monitoring Systems for Detecting Incipient Instability

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T.R. 98-49

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Manuscript: 9 September 1998

Abstract

Monitoring systems are proposed for the detection of incipient instability in uncertain nonlinear systems. The work employs generic features associated with the response to noise inputs of systems bordering on instability. These features, called "noisy precursors" in the work of Wiesenfeld, also yield information on the type of bifurcation that would be associated with the predicted instability. The closed-loop monitoring systems proposed in the paper have several advantages over simple open-loop monitoring. The advantages include the ability to influence the frequencies at which the noisy precursors are observed, and the ability to simultaneously monitor and control the system.

1 Introduction

In this paper, we propose monitoring systems for detection of incipient instability in uncertain nonlinear systems. Our aim is to develop automatic monitoring systems that provide a warning that a system is operating dangerously close to an instability. Such a warning mechanism can be of great value especially when no accurate system model is available and the system is being operated in a stressed condition. If an exact model were available, then the stability boundary could be calculated off-line, and there would be no need for an on-line monitoring system for detecting incipient instability.

Generically, loss of stability results in bifurcation of new steady states from the nominal one [6], [17],[19]. The type of bifurcation that occurs depends on the manner in which the system loses stability. Since bifurcations involve changes neglected in linearized models, it is not surprising that linear control design methods have often been found to be inadequate for control law design for stressed systems. Bifurcation control methods have arisen to address stabilization of systems in such situations [4].

In this paper, we contribute to this problem area by considering the design of monitoring systems for detection of incipient instability. We also make observations on the design of combined monitoring and control systems for physical systems susceptible to bifurcation and loss of stability.

To develop an on-line approach to the detection of incipient instability in the absence of an accurate system model, we harness the effects of external continuously acting disturbances. The presence of these disturbances facilitates determination of measured signal features associated with nearness to instability. The presence of disturbance inputs, which can occur naturally or be injected, is crucial. Without continuous disturbances, a system at equilibrium would remain at equilibrium until an instability occurs, with no possibility of an on-line warning signal. We take the continuously acting disturbances to be white noise inputs. This allows us to make use of previous work of Wiesenfeld and co-workers [20, 11, 21, 22, 24, 14, 23]. Wiesenfeld was interested in features that can be observed in the power spectrum of a measured output of a system that is operating close to an incipient bifurcation. He focused on nonlinear systems operating along a steady state limit cycle. He referred to the distinguishing features in the power spectrum as "noisy precursors" of the bifurcations. Noisy precursors are aspects of the power spectral density of a measured output that arise in the vicinity of an instability.

The monitoring systems we propose do not employ or require a system model. Rather, use of the noisy precursors notion allows a nonparametric approach based on general features of noise-driven systems operating close to instability. Moreover, the monitoring systems we develop are closed-loop, in the sense that they involve both sensing and actuation. This has important advantages over the direct open-loop approach of simply monitoring a system output and deciding if it exhibits features of a noisy precursor. Closed-loop monitoring systems can enhance our confidence in deciding that system operation is indeed near an instability as well as in determining the nature of the instability.

For many engineering system models, the normal operating condition is an equilibrium point rather than a periodic solution. Thus, we extend the theory of noisy precursors to systems operating at an equilibrium point. This forms the basis for our design of monitoring systems for detection of incipient instability.

Our results apply to situations that share the following general characteristics. A physical system is operating at or near a nominal stable steady state. The system depends on a set of parameters, some of which change slowly with time. Outside a certain range of parameter values (the "design range"), model uncertainty impedes reliable determination of system stability. However, there are circumstances in which the parameters will move outside of the design range. Moreover, in these circumstances *it is crucial that system operability be maintained as far as possible outside the design range.* The parameter changes may occur due to action of the system operator, or may be exogenous. We refer to operation outside the design range as "stressed operation."

There are many important examples in which systems need to be operated in off-design, stressed conditions. The driving factors depend on the application, but in general they entail a desire to achieve increased performance without re-design or expansion of the system. Often, stressed operation leads to a reduced margin of stability. Thus, stressed operation can be unsafe, in that small uncertainties or disturbances can lead to loss of stability, i.e., to system failure. Examples include electric power system voltage collapse [18], chemical reactor runaway [12], jet engine stall [13], aircraft stall at high angle-of-attack [5], and laser system instability [8]. For each of these examples, precise models are difficult to obtain, especially outside the design range.

The paper is organized as follows. In Section 2, we study noisy precursors for instability for systems operating at an equilibrium point. In Section 3, we introduce a basic monitoring system that facilitates use of precursors to detect incipient instability. In Section 4, we redesign the monitoring system to ensure that the bifurcation occurring in the overall system is supercritical. The monitoring systems of Section 3 and Section 4 require full state feedback. In Section 5, we alleviate the full state feedback requirement for plants that can be viewed as singularly perturbed (two time-scale) systems. In Section 6, we relax another assumption, namely that the system equilibrium point is known. In Section 7, we give a simple example. In Section 8, we collect our conclusions.

2 Noisy Precursors for Nonlinear System Instabilities

In this section, we extend the noisy precursor analysis of Wiesenfeld [20] to systems operating at an equilibrium point. Wiesenfeld considered systems driven by white noise and operating near a periodic steady state. He showed that the power spectrum of a measured output for such a system exhibits sharply growing peaks near certain frequencies as the system nears a bifurcation. The particulars depend on the type of bifurcation that the system was approaching. He used the results to show that bifurcating systems could be used as selective-frequency amplifiers [11], [22], [23].

Consider a nonlinear dynamic system ("the plant")

$$\dot{\tilde{x}} = f(\tilde{x}, \mu) + N(t) \tag{1}$$

where $\tilde{x} \in R^n$, μ is a bifurcation parameter, and $N(t) \in R^n$ is a zero-mean vector white Gaussian noise process. Let the system possess an equilibrium point $\tilde{x}_0$. For small perturbations and noise, the dynamical behavior of the system can be described by the linearized system in the vicinity of the equilibrium point $\tilde{x}_0$. The linearized system corresponding to (1) with a small noise forcing $N(t)$ is given by

$$\dot{x} = Df(\tilde{x}_0, \mu)x + N(t) \tag{2}$$

where $x := \tilde{x} - \tilde{x}_0$ and $N(t) \in R^n$ is a vector white Gaussian noise having zero mean. For the results of the linearized analysis to have any bearing on the original nonlinear model, we must assume that the noise is of small amplitude. This assumption of small noise will be explicated below, in terms of smallness of correlation and cross-correlation coefficients. The distinct notation for the system state $\tilde{x}$ and the linearized system state x was used here for clarity. In the sequel, we will simply use the notation x and the meaning will be clear from the context.

The noise $N(t)$ can occur naturally or can be injected using available controls. To facilitate consideration of cases in which the noise is intentionally injected, we write $N(t)$ in the general form

$$N(t) = Bn(t) \tag{3}$$

where $B \in R^{n \times m}$ and $n(t) \in R^m$ is a vector white Gaussian noise. This notation allows us to easily consider cases in which noise is injected in different equations through available actuation means. The noise vector $N(t)$ is a white Gaussian with zero mean as long as $n(t)$ is a white Gaussian with zero mean.

We view the system (2) as being in steady state and driven only by the noise process. Thus, we solve for the evolution of the state assuming a zero initial condition. The solution of equation (2) at time t with a zero initial condition is

$$x(t) = e^{At} \int_0^t e^{-As} N(s) ds \tag{4}$$

where $A := DF(x_0)$. For our analysis, we assume that x_0 is an asymptotically stable equilibrium point, i.e., all the eigenvalues of A have negative real part. We can express (4) in terms of the eigenvectors and eigenvalues of A (normalized, and assumed distinct):

$$\begin{aligned} x(t) &= \sum_{j=1}^{n} l^j \int_0^t \sum_{k=1}^{n} l^k N(s) e^{\lambda_k s} r^k ds e^{\lambda_j t} r^j \\ l^i r^j &= \delta_{ij} \end{aligned}$$

where r^i and l^i are right and left eigenvectors, respectively, of A corresponding to eigenvalue λ_i, and where δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{5}$$

Thus, the i-th component of $x(t)$ is given by

$$x_i(t) = \sum_{j=1}^{n} e^{\lambda_j t} r_i^j \int_0^t l^j N(s) e^{-\lambda_j s} ds \tag{6}$$

Since the power spectrum is the Fourier transform of the autocovariance function, we calculate the autocovariance for $x_i(t)$:

$$\begin{aligned} \langle x_i(t) x_i(t+\tau) \rangle &= \sum_{j=1}^{n} \sum_{k=1}^{n} e^{\lambda_j t} e^{\lambda_k (t+\tau)} r_i^j r_i^k \int_0^{t+\tau} \int_0^t e^{-\lambda_j s_1} e^{-\lambda_k s_2} \\ &\cdot \sum_{o=1}^{n} \sum_{p=1}^{n} l_o^j l_p^k \langle N_o(s_1) N_p(s_2) \rangle ds_1 ds_2 \end{aligned}$$

Let the noise have autocorrelation function

$$\langle N_i(t) N_j(t+\tau) \rangle = \nu_{ij} \delta(\tau) \tag{7}$$

where $\delta(\cdot)$ is the Dirac delta function and the ν_{ij} are constants for all i, j. Moreover, ν_{ij} should be small enough such that linearized analysis is valid. Again, (7) is satisfied for the linearized system as long as $n(t)$ satisfies

$$\langle n_i(t) n_j(t+\tau) \rangle = \psi_{ij} \delta(\tau) \tag{8}$$

where the ψ_{ij} are constants for all i, j. Then

$$\langle x_i(t) x_i(t+\tau) \rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} e^{\lambda_j t} e^{\lambda_k (t+\tau)} r_i^j r_i^k \int_0^{t+\tau} \int_0^t e^{-\lambda_j s_1} e^{-\lambda_k s_2}$$

$$
\begin{aligned}
&\cdot \quad \sum_{o=1}^{n}\sum_{p=1}^{n} l_o^j l_p^k \nu_{op}\delta(s_1 - s_2)ds_1 ds_2 \\
&= \quad \sum_{j=1}^{n}\sum_{k=1}^{n} e^{\lambda_j t} e^{\lambda_k (t+\tau)} r_i^j r_i^k \int_0^t e^{-\lambda_j s} e^{-\lambda_k s} \\
&\cdot \quad \sum_{o=1}^{n}\sum_{p=1}^{n} l_o^j l_p^k \nu_{op} ds
\end{aligned} \tag{9}
$$

Note that the upper limit of integration changes from $t+\tau$ to t because the impulse $\delta(s_1-s_2)$ occurs for $s_1 = s_2$, and $s_1 < t$ in the inner integral.

For a dynamic system that depends on a single parameter, there are two basic types of bifurcation from an equilibrium point. One is stationary bifurcation in which a new equilibrium emerges or the original equilibrium point suddenly disappears at the bifurcation. The other is Hopf bifurcation, where a periodic orbit emerges from the equilibrium point at bifurcation. In stationary bifurcation, a real eigenvalue of the linearized system becomes zero as the parameter varies. In Hopf bifurcation, a complex conjugate pair of eigenvalues crosses the imaginary axis.

Consider first the Hopf bifurcation. Assume that a complex conjugate pair of eigenvalues (denote them as $\lambda \equiv \lambda_1, \bar{\lambda} \equiv \lambda_2$) close to the imaginary axis has relatively smaller negative real part in absolute value compared to other system eigenvalues:

$$
|Re(\lambda_1)|, |Re(\lambda_2)| \ll |Re(\lambda_i)| \tag{10}
$$

for $i = 3, \ldots, n$. Since the integrand in (9) is the product of decaying exponentials (due to the asymptotic stability assumption) and bounded, terms involving λ_1 and λ_2 dominate (9) for large t:

$$
\begin{aligned}
\langle x_i(t)x_i(t+\tau)\rangle \quad &\approx \quad e^{\lambda_1(2t+\tau)}(r_i^1)^2 \int_0^t e^{-2\lambda_1 s} \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^1 l_k^1 \nu_{ij} ds \\
&+ \quad e^{\lambda_2(2t+\tau)}(r_i^2)^2 \int_0^t e^{-2\lambda_2 s} \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^2 l_k^2 \nu_{ij} ds \\
&+ \quad e^{\lambda_1(t+\tau)+\lambda_2 t} r_i^1 r_i^2 \int_0^t e^{-(\lambda_1+\lambda_2)s} \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^1 l_k^2 \nu_{ij} ds \\
&+ \quad e^{\lambda_2(t+\tau)+\lambda_1 t} r_i^1 r_i^2 \int_0^t e^{-(\lambda_1+\lambda_2)s} \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^1 l_k^2 \nu_{ij} ds
\end{aligned}
$$

The power spectrum is measured with the use of a spectrum analyzer, and most practical spectrum analyzers perform both an ensemble average and a time average. Thus, the final autocovariance function is

$$C_{ii}(\tau) := Re[\overline{\langle x_i(t)x_i(t+\tau)\rangle}^t] \quad (11)$$

$$\approx \Xi \cdot \frac{2e^{-\epsilon\tau}}{\epsilon}\cos(\omega\tau) + \Upsilon \cdot [\frac{e^{-\epsilon\tau}(\epsilon\cos(\omega\tau) - \omega\sin(\omega\tau))}{2(\epsilon^2+\omega^2)}] \quad (12)$$

where $\overline{\cdot}^t$ indicates averaging over time t, and we have written C_{ii} in terms of $\epsilon, \omega > 0$ instead of in terms of $\lambda_1 = -\epsilon + j\omega$ and $\lambda_2 = -\epsilon - j\omega$. Also, Ξ and Υ are

$$\Xi := \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^1 l_k^2 \nu_{jk} r_i^1 r_i^2$$

$$\Upsilon := \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^1 l_k^1 \nu_{jk} (r_i^1)^2 + \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^2 l_k^2 \nu_{jk} (r_i^2)^2$$

Finally, taking the Fourier transform of equation (12) yields the desired power spectrum:

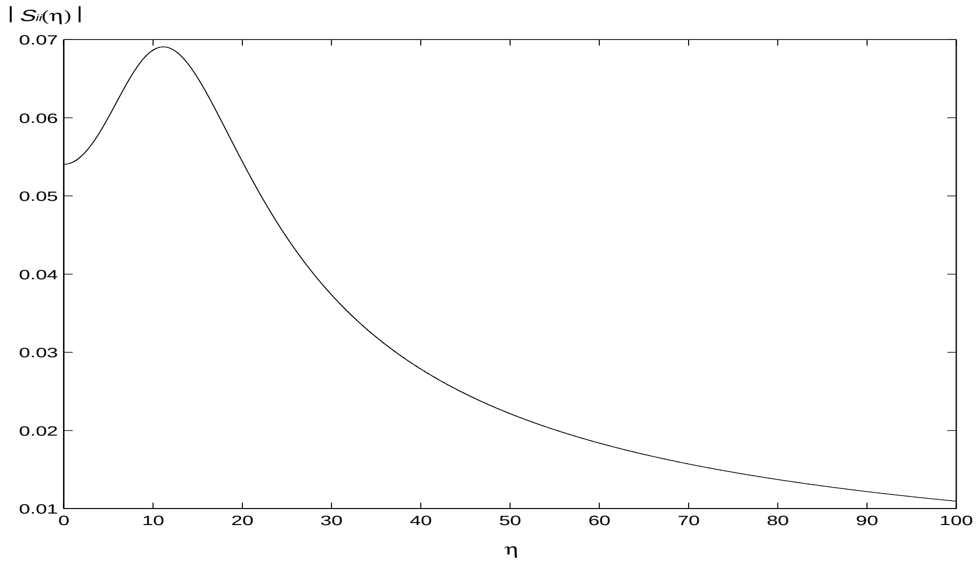
$$S_{ii}(\eta) = \Xi\frac{(j\eta+\epsilon)}{(j\eta+\epsilon)^2+\omega^2} + \Upsilon[\frac{\epsilon(j\eta+\epsilon)}{(\epsilon^2+\omega^2)((j\eta+\epsilon)^2+\omega^2))} - \frac{1}{(\epsilon^2+\omega^2)((j\eta+\epsilon)^2+\omega^2))}] \quad (13)$$

The magnitude of $S_{ii}(\eta)$ is maximum at $\eta = \omega$ and the maximum grows without bound as $\epsilon \to 0$. Moreover, as the noise power (as measured by the ν_{ij}) increases, the magnitude of $S_{ii}(\eta)$ also increases. However, since Ξ and Υ affect $S_{ii}(\eta)$ linearly and uniformly over frequency η, the shape of the magnitude $S_{ii}(\eta)$ doesn't change with increasing noise power. Of course, we have assumed that the noise is of small amplitude, so we cannot actually allow the ν_{ij} to increase without bound.

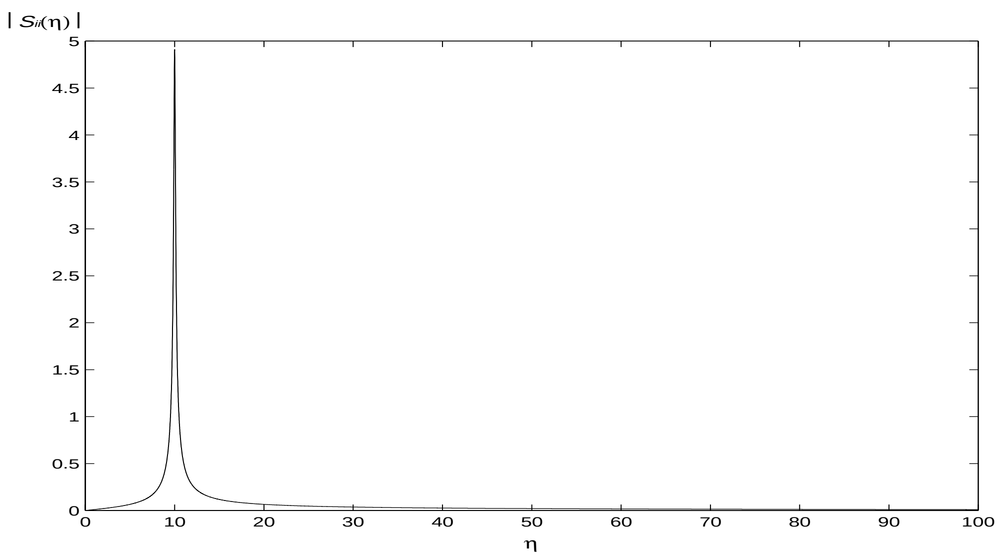
Fig. 1 shows the magnitude of $S_{ii}(\eta)$ for $\omega = 10$, for two values of ϵ. (For definiteness, Ξ and Υ have been set to 1 in constructing Fig. 1.) Note the sharp peak around $\omega = 10$ that appears as $\epsilon \to 0$. From this observation, we can conclude that the power spectrum peak near the bifurcation is located at ω, and the magnitude of this peak grows as ϵ approaches to zero. This property will be used as a precursor signaling the closeness to Hopf bifurcation.

To study the impact of noise near a stationary bifurcation, assume that a real eigenvalue close to zero (denote it as $\lambda \equiv \lambda_1$) and that it has relatively smaller negative real part in absolute value compared to the other system eigenvalues:

$$|\lambda_1| \ll |Re(\lambda_i)| \quad (14)$$



a. $\epsilon = 10$



b. $\epsilon = 0.1$

Figure 1: Power spectrum magnitude for Hopf bifurcation when $\omega = 10$ for two values of ϵ

for $i = 2, \ldots, n$. Due to (14), terms with $j = 1$ and $k = 1$ dominate the expression (9) for large t, so that

$$\langle x_i(t)x_i(t+\tau)\rangle \approx e^{\lambda_1(2t+\tau)}(r_i^1)^2 \int_0^t e^{-2\lambda_1 s} \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^1 l_k^1 \nu_{ij} ds$$

Taking the time average, we get the autocovariance function

$$C_{ii}(\tau) := \overline{\langle x_i(t)x_i(t+\tau)\rangle}^t \quad (15)$$

$$= [\sum_{j=1}^{n}\sum_{k=1}^{n} l_j^1 l_k^1 \nu_{jk}](r_i^1)^2 \frac{e^{-\epsilon\tau}}{2\epsilon} \quad (16)$$

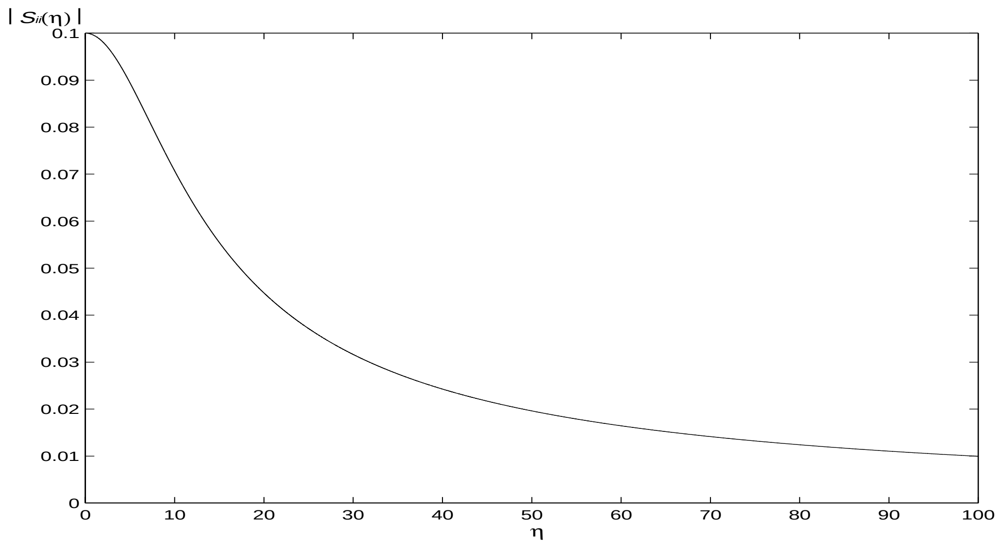
Fourier transformation of (16) gives the desired power spectrum:

$$S_{ii}(\eta) = [\sum_{j=1}^{n}\sum_{k=1}^{n} l_j^1 l_k^1 \nu_{jk}](r_i^1)^2 \frac{1}{2\epsilon(\epsilon + j\eta)} \quad (17)$$

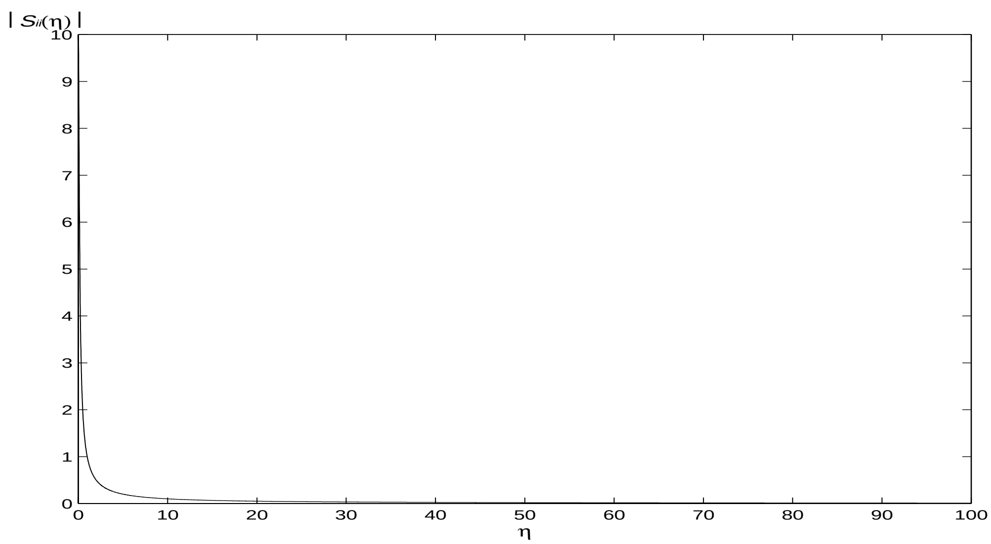
This equation shows that the magnitude of the power spectrum peak grows as ϵ approaches to zero and the location of this peak is $\eta = 0$. Fig. 2 shows the magnitude of $S_{ii}(\eta)$ (17). (For definiteness, the coefficient in square brackets in Eq. (17) has been set to 1 in constructing Fig. 2.) Note the sharp growing peak around $\omega = 0$ as $\epsilon \to 0$.

3 Monitoring System for Detecting Incipient Stationary and Hopf Bifurcation

As shown in the foregoing section, we can expect to observe a growing peak in the power spectrum of a measured output of a nonlinear system with white Gaussian noise input as the system approaches a bifurcation. In the case of Hopf bifurcation, the location of the power spectrum peak coincides with the imaginary axis crossing frequency of the critical eigenvalues. In the case of stationary bifurcation the power spectrum peak occurs at zero frequency. In this section, we use these observations to develop a monitoring system for proximity to bifurcation. Since noisy precursors associated with stationary bifurcation involve a growing peak in the power spectrum at zero frequency, these are difficult to resolve. Hence, we first propose a closed-loop monitoring system that addresses this problem by transforming a stationary bifurcation into a Hopf bifurcation. That is, the original plant augmented with the monitoring system undergoes a Hopf bifurcation. The critical frequency of the Hopf bifurcation is set by the monitoring system itself. After introducing the monitoring system and studying its use in monitoring for stationary bifurcation, we study its use in monitoring a system for proximity to a Hopf bifurcation.



a. $\epsilon = 10$



b. $\epsilon = 0.1$

Figure 2: Power spectrum magnitude for stationary bifurcation for two values of ϵ

3.1 Generating a Hopf Bifurcation from a Stationary Bifurcation

Suppose the plant of interest is susceptible to loss of stability through a stationary bifurcation. Since Hopf bifurcation is easier to detect than stationary bifurcation through noisy precursors, we introduce a monitoring system that replaces the stationary bifurcation with a Hopf bifurcation of tunable frequency.

In the absence of noise, let the plant obey the dynamics

$$\dot{x} = f(x, \mu) \tag{18}$$

Results we obtain for this model will have immediate implications for precursor-based monitoring of the system with noise effects included. Suppose the following assumptions hold:

(S1) The origin is an equilibrium point of system (18) for all values of μ.

(S2) System (18) undergoes stationary bifurcation from the origin at $\mu = \mu_c$ (i.e., there is a simple eigenvalue $\lambda(\mu)$ of $Df(0, \mu)$ such that for some value $\mu = \mu_c$, $\lambda(\mu_c) = 0$ and $\frac{d\lambda(\mu_c)}{d\mu} \neq 0$)

(S3) All other eigenvalues of $Df(0, \mu_c)$ are in the open left half complex plane.

We introduce the following *augmented system* (plant plus monitoring system) corresponding to (18):

$$\begin{aligned} \dot{x}_i &= f(x, \mu) - cy_i \\ \dot{y}_i &= cx_i \end{aligned} \tag{19}$$

Here, $y \in R^n$, $c \in R$ and $i = 1, 2, \ldots, n$. Eq. (19) will later be viewed as a basic monitoring system whose use facilitates detection of either stationary or Hopf bifurcation. Note that the state vector consists of the original physical system states x augmented with the states y of the monitoring system.

Proposition 1 *Under assumptions (S1)-(S3), the augmented system (19) undergoes a Hopf bifurcation from the origin at $\mu = \mu_c$. Moreover, if for any value of μ the origin of the original system (18) is asymptotically stable (resp. unstable), then the origin is asymptotically stable (resp. unstable) for the augmented system (19).*

Proof: Denote by A the Jacobian matrix of system (18) at the origin. Clearly, the origin $(0,0)$ in R^{2n} is an equilibrium point of the augmented system (19). The Jacobian matrix of the augmented system (19) at the origin is

$$J = \begin{bmatrix} A & -cI \\ cI & 0 \end{bmatrix} \tag{20}$$

Let α be any eigenvalue of A and r the corresponding right eigenvector. Also, denote by λ any eigenvalue of J and the associated right eigenvector by $v = [v_1 \quad v_2]^T$. Then,

$$\lambda v_1 = Av_1 - cv_2 \tag{21}$$

$$\lambda v_2 = cv_1 \tag{22}$$

We seek a solution for which $v_1 = r$. From (22), we have

$$v_2 = \frac{c}{\lambda} r \tag{23}$$

Substituting (23) into (21) and using $r \neq 0$, we get

$$\lambda^2 - \alpha\lambda + c^2 = 0 \tag{24}$$

Thus, any eigenvalue α of A has corresponding to it two eigenvalues of J, which are the solutions of the quadratic equation above:

$$\lambda = \frac{\alpha \pm \sqrt{\alpha^2 - 4c^2}}{2} \tag{25}$$

Thus, the eigenvalues of the Jacobian matrix of the augmented system (19) are

$$\lambda_{2i-1,2i} = \frac{\alpha_i \pm \sqrt{\alpha_i^2 - 4c^2}}{2} \qquad i = 1, 2, \ldots, n \tag{26}$$

where $\alpha_i,\ i = 1, \ldots, n$ are the eigenvalues of A. Let the eigenvalue of A that becomes 0 at criticality be α_1. At $\mu = \mu_c$, the eigenvalues of the augmented system associated with α_1 are (using (24)) are a pair of pure imaginary eigenvalues at μ_c

$$\lambda_1, \lambda_2 = \pm cj \tag{27}$$

Note that the pair of pure imaginary eigenvalues (27) depends on c.

For a Hopf bifurcation to occur, the transversality condition should be satisfied. That is, the eigenvalues crossing the imaginary axis should do so with nonzero speed. From the quadratic equation (24) and using the fact that $\alpha_1 = 0$ at $\mu = \mu_c$, we have

$$\frac{dRe(\lambda_1)}{d\mu} = \frac{1}{2}\frac{d\alpha_1}{d\mu} \tag{28}$$

Since $\alpha_1 = 0$ and $\frac{d\alpha_1}{d\mu} \neq 0$ at $\mu = \mu_c$ from assumption **(S2)**, (28) implies $\frac{dRe(\lambda_1)}{d\mu} = \frac{1}{2}\frac{d\alpha_1}{d\mu} \neq 0$ (i.e., the transversality condition holds for system (19)). Therefore, the augmented system (19) undergoes a Hopf bifurcation from the origin at $\mu = \mu_c$.

The last step in the proof consists in showing that all other eigenvalues of the matrix J are in the open left half complex plane. Any pair of eigenvalues of J can be obtained from (26). For a real eigenvalue of A, it is clear from (26) that the corresponding pair of eigenvalues of J have negative real part if $\alpha_i < 0$ since $\alpha_i < Re\{\sqrt{\alpha_i^2 - 4c^2}\}$. For a complex conjugate pair of eigenvalues of A (denoted $\gamma, \bar{\gamma}$), we have the following two equations:

$$\lambda^2 - \gamma\lambda + c^2 = 0 \tag{29}$$
$$\lambda^2 - \bar{\gamma}\lambda + c^2 = 0 \tag{30}$$

Multiply (29) and (30) to get the following fourth order equation:

$$\lambda^4 - (\gamma + \bar{\gamma})\lambda^3 + (2c^2 + \gamma\bar{\gamma})\lambda^2 - c^2(\gamma + \bar{\gamma})\lambda + c^4 = 0 \tag{31}$$

Denoting $\gamma = a + bj$ and $\bar{\gamma} = a - bj$, equation (31) simplifies to

$$\lambda^4 - 2a\lambda^3 + (2c^2 + a^2 + b^2)\lambda^2 - 2ac^2\lambda + c^4 = 0 \tag{32}$$

Applying the Routh-Hurwitz criterion [7] to (32), we obtain the Routh array

$$\begin{array}{cccc}
s^4 & 1 & 2c^2 + a^2 + b^2 & c^4 \\
s^3 & -2a & -2ac^2 & 0 \\
s^2 & 3c^2 + a^2 + b^2 & c^4 & 0 \\
s^1 & \frac{-2ac^2(2c^2+a62+b^2)}{3c^2+a^2+b^2} & 0 & 0 \\
s^0 & c^4 & 0 & 0
\end{array}$$

From assumption (S3), $a < 0$. This guarantees that all the entries in the first column of the Routh array are positive. Therefore, all eigenvalues of the Jacobian matrix of the augmented system have negative real part.

From the foregoing discussion, it is also clear that if any eigenvalue of A has positive real part, then the corresponding eigenvalues of J also have positive real part. This proves that if the origin is unstable for the plant, then it is also unstable for the augmented system. ■

Note that since the value c in equation (19) is adjustable, we can control the crossing frequency of the complex conjugate pair of eigenvalues of the augmented system. Thus, for detecting stationary bifurcation, we only need to monitor a frequency band around the chosen value of c. It is also possible to slowly vary c in a controlled fashion, giving added confidence in our assessment that an instability is imminent.

There are some other advantages of our monitoring system. The augmented system (19) has the same critical parameter value (μ_c) as the original system. This is actually not a luxury but a necessity for the system to be practically useful. In addition, the final part of the proof shows that augmenting the states y_i and applying the feedbacks cy_i to the original system does not change the local stability of the system. Moreover, to apply the monitoring system, we do not need knowledge of the original system. However, there are some restrictions and further considerations in applying the suggested monitoring system to general physical systems. We will discuss these in subsequent sections.

We assumed above that the stationary bifurcation was such that the transversality condition is satisfied. This means that the eigenvalue that vanishes at criticality must cross into the right half of the complex plane with nonzero speed as the parameter is varied. In a saddle node bifurcation, however, the nominal equilibrium disappears at criticality, so that the transversality condition does not hold. For a saddle-node bifurcation, the augmented system of this section results in a degenerate Hopf bifurcation. The possible bifurcation diagrams for degenerate Hopf bifurcation are more complex than for Hopf bifurcation [16], [10]. However, for the purpose of detecting incipient instability, the details of the ensuing bifurcation are not important. These details become important when we consider the system's post-bifurcation behavior.

3.2 Detecting Hopf Bifurcation using Monitoring System

In this section, we consider the effect of the monitoring system of the proceeding section on a system that undergoes Hopf bifurcation instead of stationary bifurcation. Consider again the system (18), repeated here for convenience:

$$\dot{x} = f(x, \mu) \tag{33}$$

(H1) The origin is an equilibrium point of (33) for all values of μ.

(H2) System (33) undergoes a Hopf bifurcation from the origin at $\mu = \mu_c$.

(H3) All other eigenvalues of $Df(0, \mu_c)$ are in the open left half complex plane.

As in the foregoing section, let the augmented system (plant plus monitoring system) be

$$\begin{aligned} \dot{x}_i &= f_i(x, \mu) - cy_i \\ \dot{y}_i &= cx_i \end{aligned} \tag{34}$$

where $x \in R^n$, $y \in R^n$, $c \in R$, and $i = 1, 2, \ldots, n$.

Proposition 2 *Under the assumptions (H1)-(H3), the augmented system (34) undergoes a codimension two bifurcation at $\mu = \mu_c$, in which two complex conjugate pairs of eigenvalues cross the imaginary axis. Moreover, for any value of μ if the origin of the original system is asymptotically stable (resp. unstable), then the origin is asymptotically stable (resp. unstable) for the augmented system.*

Proof: First, we show that the augmented system has two pairs of pure imaginary eigenvalues at the origin for $\mu = \mu_c$, and that these eigenvalues satisfy the transversality condition.

From assumption **(H2)**, the Jacobian matrix of the original system at the origin has a pair of pure imaginary conjugate eigenvalues (denote them by $j\omega, -j\omega$) for $\mu = \mu_c$. From the proof of Proposition 1, it is clear that each of these eigenvalues results in a pair of eigenvalues for the augmented system which are the solutions of the following equations:

$$\lambda^2 - j\omega\lambda + c^2 = 0 \tag{35}$$

$$\lambda^2 + j\omega\lambda + c^2 = 0 \tag{36}$$

By multiplying the equations above, we get a fourth order equation the solutions of which are eigenvalues of augmented system:

$$\lambda^4 + (2c^2 + \omega^2)\lambda^2 + c^4 = 0 \tag{37}$$

The four solutions of the equation above are given by

$$\lambda = \pm \frac{\sqrt{-2c^2 - \omega^2 \pm \sqrt{4c^2\omega^2 + \omega^4}}}{\sqrt{2}} \tag{38}$$

Note that $2c^2 + \omega^2 > \sqrt{4c^2\omega^2 + \omega^4}$ for all $c, \omega \in R$. Therefore, the Jacobian matrix of the augmented system has two pairs of pure imaginary conjugate eigenvalues at the critical parameter value.

To check the transversality condition, consider the eigenvalues for μ near μ_c. Near $\mu = \mu_c$, we have the following fourth order equation the solutions of which result from the pair of complex conjugate eigenvalues ($\alpha \pm \omega j$) of the original system (see (31)):

$$\lambda^4 - 2\alpha\lambda^3 + (2c^2 + \alpha^2 + \omega^2)\lambda^2 - 2c^2\alpha + c^4 = 0 \tag{39}$$

Since μ is close to μ_c, by continuity it follows that this equation has two pairs of complex conjugate eigenvalues as its solutions for μ near μ_c. Denote these as $e \pm fj, g \pm hj$. The next relationship is now easily demonstrated:

$$-2\alpha = \sum_{i=1}^{4} \lambda_i = e + g \tag{40}$$

$$-2c^2\alpha = \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{4} \lambda_i\lambda_j\lambda_k = 2g(e^2 + f^2) + 2e(g^2 + h^2) \tag{41}$$

where $\lambda_i, \lambda_j, \lambda_k$ are roots of (39). Taking the derivative of both sides of the equation above with respect to μ and evaluating at $\mu = \mu_c$ ($e = g = 0$, $\beta = 0$,$\gamma^2 = c - a$) gives

$$\begin{aligned} \frac{de}{d\mu} + \frac{dg}{d\mu} &= -2\frac{d\alpha}{d\mu} \\ h^2\frac{de}{d\mu} + f^2\frac{dg}{d\mu} &= -c^2\frac{d\alpha}{d\mu} \end{aligned} \tag{42}$$

Next we solve Eq. (42) for $\frac{dg}{d\mu}, \frac{de}{d\mu}$. Also, h and f are not 0 at the critical point from (38) and $f \neq h$ at the critical point if $c \neq 0$. These conditions guarantee that if $\frac{d\alpha}{d\mu} \neq 0$, then $\frac{dg}{d\mu} \neq 0, \frac{de}{d\mu} \neq 0$.

The last step in the proof consists in showing that all other eigenvalues of the Jacobian matrix J of (34) lie in the open left half complex plane. Note that we have the same form of matrix J as in the proof of Proposition 1:

$$J = \begin{bmatrix} A & -cI \\ cI & 0 \end{bmatrix} \tag{43}$$

where all noncritical eigenvalues of A have negative real part. We can use the same procedure as in Proposition 1 to prove that if all noncritical eigenvalues of A have negative real part, then all corresponding eigenvalues of J have negative real part.

It is also clear that if any eigenvalue of A has positive real part, then the corresponding eigenvalues of J also have positive real part. This implies that if the original system is unstable, then the augmented system is also unstable. ■

Since two pairs of eigenvalues of the augmented system cross the imaginary axis at the critical parameter value, we can expect to see *two* peaks in the power spectrum as the system nears the bifurcation point. From (38), we see that the values of the pairs of imaginary eigenvalues at criticality depend on c. Hence, we can change the location of the power spectrum peaks by changing c. Moreover, we can predict the exact locations of the peaks if the pair of eigenvalues crossing the imaginary axis in the original system is known.

Because two pairs of eigenvalues cross the imaginary axis for the augmented system, the augmented system undergoes a codimension two bifurcation. The nature of the bifurcation behavior depends strongly on $f(x, \mu)$. The possible bifurcation diagrams for the associated degenerate Hopf bifurcation can be found in [16], [10]. However, as was the case for detecting incipient stationary bifurcation, the details of the degenerate Hopf bifurcation are not important. They become important when we consider the system's post-bifurcation behavior.

4 Stabilization of Bifurcated Limit Cycle in the Augmented System

In this section, we suppose that the plant is subject to loss of stability through a stationary bifurcation. In these circumstances, the monitoring system proposed above results in a Hopf bifurcation in the augmented system. Besides being able to predict that a bifurcation is about to take place, it would be useful if the monitoring system could also ensure stability of the bifurcated solution. That is, a system that can perform both monitoring and control functions is desirable. The purpose of this section is to illustrate how the monitoring system we have proposed can be modified to serve in both capacities. Liberal use is made of the bifurcation formulas and associated results summarized in Appendix B.

4.1 Stability of the Bifurcated Limit Cycle of the Augmented System

First, we consider the relationship between the stability of bifurcated equilibrium points of the original system and stability of the bifurcated limit cycle of the augmented system. If stability of the bifurcated equilibria of the original system implies stability of the bifurcated periodic solution of the augmented system, then bifurcation control design need only be performed for the original system. We proceed to show, however, that the stability properties of the bifurcation persist in the case of scalar systems, but not generally for systems of dimension two or higher.

Let the plant be given by

$$\dot{x} = f(x, \mu) \tag{44}$$

where $x \in R^n$ is the state vector and $\mu \in R$ is the bifurcation parameter, and the noise input is neglected for the purposes of this section. Suppose that at the critical parameter value $\mu = \mu_c$, the Jacobian matrix of (44) evaluated at the equilibrium point $x_0 = 0$ has one zero eigenvalue.

For simplicity, we first consider the one-dimensional case ($n = 1$), i.e., suppose the state vector of (44) is a single variable. Also, suppose system (44) undergoes a pitchfork bifurcation. It is easy to see that the left (l) and right (r) eigenvectors corresponding to the simple zero eigenvalue at criticality can be taken as any nonzero constants. Set $r = 1$ and $l = 1$, so that r and l satisfy the normalization $lr = 1$. Lemma 1 (see Appendix B) then applies directly, allowing calculation of the associated bifurcation stability coefficients β_1 and β_2. As discussed in Appendix B, the pitchfork bifurcation is supercritical (giving stable bifurcated equilibria) if $\beta_1 = 0$ and $\beta_2 < 0$. Since system (44) is assumed to undergo a pitchfork bifurcation, $\beta_1 = 0$:

$$\beta_1 = lQ(r, r) = \frac{\partial^2 f}{\partial x^2}(0) = 0 \tag{45}$$

Thus, $Q(r, r)$ vanishes. Since $Q(r, r) = 0$, we have $x_2 = 0$ (using the notation of Appendix B). Thus, β_2 becomes

$$\beta_2 = 2lC_0(r, r, r) = \frac{2}{3!}\frac{\partial^3 f}{\partial x^3}(0) \tag{46}$$

The augmented system corresponding to the plant (44) is

$$\begin{aligned} \dot{x} &= f(x, \mu) - cy \\ \dot{y} &= cx \end{aligned} \tag{47}$$

where $x, y, \mu \in R$. We have shown that the augmented system undergoes a Hopf bifurcation if the original system undergoes a pitchfork bifurcation. To check the stability of the bifurcated periodic solution of (47), we only have to check the sign of the Hopf bifurcation stability coefficient β_2 (102). At criticality, the Jacobian matrix of (47) is

$$L_0 := \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} \tag{48}$$

The matrix L_0 has an eigenvalue cj with corresponding right eigenvector $r = \begin{bmatrix} 1 & -j \end{bmatrix}^T$ and left eigenvector $l = \frac{1}{2} \begin{bmatrix} 1 & j \end{bmatrix}$. Eigenvalue $-cj$ has right eigenvector $\bar{r}$ and left eigenvector $\bar{l}$. Note that higher order terms only come from $f(x, \mu)$ and they are not a function of y. From this observation, we have

$$\begin{aligned} Q((x,y),(x,y)) &= \begin{pmatrix} \frac{1}{2!} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x2}(0) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(0) x^2 \\ 0 \end{pmatrix} \end{aligned} \tag{49}$$

From equation (45), equation (49) implies that $Q(r, \bar{r})$ and $Q(r, r)$ both vanish. Therefore, the solutions of (100) and (101) are $a = 0$ and $b = 0$. Now, β_2 of (47) becomes

$$\beta_2 = 2Re\{\frac{3}{4} l C(r, r, \bar{r})\} = 2 \frac{3}{4} \frac{1}{3!} \frac{\partial^2 f}{\partial x^3}(0) \tag{50}$$

since higher order terms only come from $f(x, \mu)$ and none depend on y.

Note that the sign of (46) agrees with the sign of (50). The next proposition therefore follows.

Proposition 3 *Suppose the system (44) is of first order, i.e., n=1. If the plant (44) undergoes a supercritical pitchfork bifurcation (respectively a subcritical pitchfork bifurcation), then the transformed system (47) undergoes a supercritical Hopf bifurcation (respectively a subcritical Hopf bifurcation).*

Next, we consider the case $n \geq 2$, that is, the case in which the dimension of the plant is at least 2. We show using an example that the monitoring system proposed above does not necessarily preserve the stability character of the bifurcation in the plant. That is, a supercritical pitchfork bifurcation (resp. a subcritical pitchfork bifurcation) in the plant need not result in a supercritical Hopf bifurcation (resp. a subcritical Hopf bifurcation) in the augmented system.

Consider the example

$$\begin{aligned} \dot{x_1} &= -\mu x_1 - x_1^3 + x_1 x_2 \\ \dot{x_2} &= -x_2 + k x_1^2 \end{aligned} \tag{51}$$

where $\mu \in R$ is a bifurcation parameter and $k \in R$ is a constant. It is easy to see that the origin is an equilibrium point for all parameter values μ and that a pitchfork bifurcation occurs for $\mu = 0$. Moreover, a simple calculation shows that β_2 for this pitchfork bifurcation is -1. This implies that system (51) undergoes a supercritical pitchfork bifurcation at $\mu = 0$.

The augmented system corresponding to (51) is

$$\begin{aligned} \dot{x_1} &= -\mu x_1 - x_1^3 + x_1 x_2 - c y_1 \\ \dot{y_1} &= c x_1 \\ \dot{x_2} &= -x_2 + k x_1^2 - c y_2 \\ \dot{y_2} &= c x_2 \end{aligned} \tag{52}$$

As discussed in Appendix B, typically a Hopf bifurcation's stability is determined by a single bifurcation stability coefficient β_2 (this differs from the β_2 coefficient in the study of pitchfork bifurcations). The Hopf bifurcation is supercritical if the coefficient β_2 is negative, and it is subcritical if the coefficient is positive. We now calculate β_2 for the Hopf bifurcation that occurs in the augmented system (52). To facilitate application of the formulas in Appendix B, denote the state vector of (52) as $z = (z_1, z_2, z_3, z_4)^T$ where $z_1 := x_1$, $z_2 := y_1$, $z_3 := x_2$, and $z_4 := y_2$. The Jacobian matrix of (52) evaluated at the origin at criticality is

$$\begin{bmatrix} 0 & -c & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & -1 & -c \\ 0 & 0 & c & 0 \end{bmatrix} \tag{53}$$

One eigenvalue of this matrix is cj, and it has corresponding right eigenvector $r = \begin{bmatrix} 1 & -j & 0 & 0 \end{bmatrix}^T$ and left eigenvector $l = \frac{1}{2}\begin{bmatrix} 1 & j & 0 & 0 \end{bmatrix}$. The conjugate eigenvalue $-cj$ has right eigenvector $\bar{r}$ and left eigenvector $\bar{l}$. The Taylor series expansion of the right side of (52) has the following quadratic and cubic terms:

$$Q(z,z) = \begin{bmatrix} z_1 z_3 \\ 0 \\ k z_1^2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ 0 \\ k x_1^2 \\ 0 \end{bmatrix}$$

$$C(z,z,z) = \begin{bmatrix} -z_1^3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -x_1^3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{54}$$

Therefore, we have

$$\begin{aligned} Q(r,\bar{r}) &= Q(r,r) = \begin{bmatrix} 0 \\ 0 \\ k \\ 0 \end{bmatrix} \\ C(r,r,\bar{r}) &= \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{55}$$

Solving Eqs. (100) and (101) of Appendix B, we obtain

$$\begin{aligned} a &= \begin{bmatrix} 0 & 0 & 0 & -\frac{k}{c} \end{bmatrix}^T \\ b &= \begin{bmatrix} 0 & 0 & \frac{kj}{2j-3c} & \frac{k}{2j-3c} \end{bmatrix}^T \end{aligned} \tag{56}$$

Substituting these values in (102) of Appendix B, we find that β_2 for system (52) is

$$\beta_2 = -\frac{3}{4} + \frac{k}{4+9c^2} \tag{57}$$

Note that for sufficiently large k, β_2 is positive. For such values of k, the augmented system (52) therefore undergoes a subcritical Hopf bifurcation even though the plant undergoes a supercritical pitchfork bifurcation. Thus, for $n \geq 2$, the monitoring system as presented above does not necessarily preserve the stability of bifurcated solutions. We now proceed to modify the design to address this deficiency.

4.2 Redesign for Combined Monitoring and Bifurcation Stabilization

In this section, we modify our monitoring system such that the bifurcated limit cycle occurring in the augmented system is guaranteed stable regardless of the stability of the pitchfork bifurcation occurring in the plant. That is, we ensure that the Hopf bifurcation in the augmented system is supercritical, and that this holds regardless of whether the pitchfork bifurcation in the original system is supercritical or subcritical. The modification that we introduce in the monitoring system involves the addition of a nonlinear term with a gain parameter that can be tuned to ensure the desired result.

Let the plant obey the dynamics

$$\dot{x} = f(x, \mu), \tag{58}$$

which we assume undergoes a pitchfork bifurcation from the origin for $\mu = \mu_c$. Here, $x \in R^n$ and $\mu \in R$ is the bifurcation parameter. Denote by r_s and l_s the right and left eigenvectors, respectively, corresponding to the simple zero eigenvalue of the system linearization at $\mu = \mu_c$. Take the first component of r_s to be 1 and impose the normalization $l_s r_s = 1$ (following the procedure in Appendix B).

Now consider the following redesign of the augmented system:

$$\begin{aligned} \dot{x}_i &= f_i(x, \mu) - c y_i \\ \dot{y}_i &= c x_i - m x_1^2 y_i \end{aligned} \tag{59}$$

Here, c and m are real constants. At criticality, the system (59) has Jacobian matrix

$$J = \begin{bmatrix} A & -cI \\ cI & 0 \end{bmatrix} \tag{60}$$

at the origin, written in terms of A, the Jacobian matrix of (58). Employing Proposition 1, it is easy to show that the augmented system (59) undergoes Hopf bifurcation and that the matrix J has eigenvalues $\pm ci$. Moreover, the right and left eigenvectors of J corresponding to the eigenvalue cj are given by

$$\begin{aligned} r &= \begin{bmatrix} r_s & -j r_s \end{bmatrix}^T \\ l &= \frac{1}{2} \begin{bmatrix} l_s & j l_s \end{bmatrix} \end{aligned} \tag{61}$$

Also, the right and left eigenvectors corresponding to the eigenvalue $-cj$ are given by $\bar{r}$ and $\bar{l}$, respectively.

The stability of the bifurcated periodic solution of the augmented system is determined by the sign of the bifurcation stability coefficient β_2 (Eq. (102), Appendix B):

$$\beta_2 = 2Re\{2lQ_0(r, a) + lQ_0(\bar{r}, b) + \frac{3}{4} lC(r, r, \bar{r})\} \tag{62}$$

Since there are no quadratic terms in y_i in the augmented system (59), β_2 simplifies to

$$\beta_2 = 2Re\{2l_s Q_f(r,a) + l_s Q_f(\bar{r},b) + \frac{3}{4} lC(r,r,\bar{r})\} \tag{63}$$

where $Q_f(\cdot,\cdot)$ denotes the quadratic terms in the Taylor expansion of $f(x,\mu_c)$. Moreover, we can simplify

$$\begin{aligned} lC(r,r,\bar{r}) &= l_s C_f(r,r,\bar{r}) + \frac{j}{2} l_s C_y(r,r,\bar{r}) \\ &= l_s C_f(r,r,\bar{r}) + \frac{m}{6} \sum_{i=1}^{n} l_s^i r_s^i (r_s^1)^2 \end{aligned} \tag{64}$$

where $C_f(x,x,x)$ denotes the cubic terms in the Taylor expansion of $f(x,\mu_c)$, and r_s^i and l_s^i denote the i-th component of r_s and l_s, respectively. Since the first component of r_s^1 is 1 and $l_s r_s = 1$, (64) reduces to

$$lC(r,r,\bar{r}) = l_s C_f(r,r,\bar{r}) - \frac{m}{6} \tag{65}$$

Hence, β_2 becomes

$$\beta_2 = 2Re\{2l_s Q_f(r,a) + l_s Q_f(\bar{r},b) + \frac{3}{4} l_s C_f(r,r,\bar{r})\} - \frac{3m}{12} \tag{66}$$

By choosing m positive and sufficiently large, we can ensure that β_2 will be negative. This will imply that the Hopf bifurcation occurring in the augmented system (59) is supercritical.

Here, we have suggested only one of many possible designs that render the Hopf bifurcated supercritical. The method is robust, since the efficacy of the design does not depend on the details of the plant model. For any given plant, a sufficiently large feedback gain m will result in supercriticality of the Hopf bifurcation. Note that we added a nonlinear term only to the dynamics of the augmented states y_i not to those of the physical system states x_i. We therefore have considerable freedom in choosing the nonlinear feedback gain m.

5 Reduced Order Monitoring System

The closed-loop monitoring systems introduced in the preceding two sections entail the use of full state feedback. In this section, we alleviate this requirement for plant models that can be viewed as singularly perturbed (or two time-scale) systems. We design a monitoring system in which only the slow states are fed back to the controls.

Consider a plant given by a singularly perturbed system of the form

$$\dot{x} = f(x, z, \mu, \epsilon)$$
$$\epsilon\dot{z} = g(x, z, \mu, \epsilon) \tag{67}$$

where $x \in R^n$, $z \in R^m$, $\mu, \epsilon \in R$ and ϵ is small but positive. The reduced system is obtained by formally setting $\epsilon = 0$ in (67), giving

$$\dot{x} = f(x, z, \mu, 0)$$
$$0 = g(x, z, \mu, 0) \tag{68}$$

Let $m_0 = (0, z_0)$ be an equilibrium point of the reduced system. Also, assume

(SP1) $m_0 = (0, z_0)$ is an equilibrium point of (68) for all values of μ.

(SP2) f, g, are $C^r (r \geq 5)$ in x, z, μ, ϵ in a neighborhood of $(m_0, 0, 0)$.

(SP3) No eigenvalue of $D_2 g(0, z_0, 0, 0)$ has zero real part.

(SP4) The reduced system undergoes a stationary bifurcation at m_0 for the critical parameter value $\mu = \mu_c$.

Let the augmented system (plant plus monitoring system) corresponding to (67) be

$$\begin{aligned} \dot{x} &= f(x, z, \mu, \epsilon) - cy \\ \dot{y} &= cx \\ \epsilon\dot{z} &= g(x, z, \mu, \epsilon) \end{aligned} \tag{69}$$

Proposition 4 *Let* **(SP1)**-**(SP4)** *above hold. Then there is an* $\epsilon_0 > 0$ *and for each* $\epsilon \in [0, \epsilon_0]$ *the augmented system (69) undergoes a Hopf bifurcation at an equilibrium* $m_0^{\mu_c^\epsilon, \epsilon}$ *near* m_0 *for a critical parameter value* μ_c^ϵ *near* μ_c.

Proof: By virtue of Theorem in [1] on persistence of Hopf bifurcation under singular perturbation, we need to verify two conditions. The first is that the reduced system corresponding to (69) undergoes a Hopf bifurcation at $(0, 0, z_0)$ at the critical parameter value $\mu = \mu_c$. The second condition is that the Jacobian matrix of g with respect to the fast variables z does not possess any eigenvalues with zero real part. The reduced system

$$\begin{aligned} \dot{x} &= f(x, z, \mu, 0) - cy \\ \dot{y} &= cx \\ 0 &= g(x, z, \mu, 0) \end{aligned} \tag{70}$$

Since the original reduced system (68) undergoes a stationary bifurcation, we can apply Proposition 1 to (70) to show that the reduced augmented system (70) undergoes Hopf bifurcation at the critical parameter value $\mu = \mu_c$. The result now follows from [1].■

Proposition 4 is useful because it implies that we only have to augment and feed back slow states in a two-time scale system to transform stationary bifurcation into Hopf bifurcation.

6 Monitoring System for Nonzero Equilibrium Point

Although the results of the preceding sections do not depend on availability of an accurate model of the plant, they do require knowledge of the nominal equilibrium. In this section, we alleviate this requirement through a re-design of the monitoring system. Not surprisingly, the increased generality comes with some cost, mainly in the simplicity of the observed instability precursor.

The requirement of a known equilibrium is embodied in assumption (S1), which states that the nominal equilibrium point of the plant is fixed at the origin for all parameter values. Through a simple parameter-dependent coordinate change, it is clear that the results of the preceding sections still apply under the milder assumption that the equilibrium is a known function of the parameter.

Assumption (S1) was invoked so that the equilibrium point of the plant is not changed upon applying state feedback. A standard control technique for exactly preserving an equilibrium despite model uncertainty involves the use of washout filters [15]. Our revised designs in this section entail adjoining a washout filter to the previous monitoring system designs.

Denote by $x_0(\mu)$ the nominal equilibrium point of system (18). We now allow the equilibrium to depend in some unknown fashion on the parameter μ.

The following re-designed augmented system involves two sets of additional variables: the vector y, which appears in the original design (19); and the vector z, the washout filter states:

$$\begin{aligned} \dot{x}_i &= f_i(x,\mu) - cy_i \\ \dot{y}_i &= cx_i + az_i \\ \dot{z}_i &= y_i \end{aligned} \tag{71}$$

for $i = 1, 2, \ldots, n$, where $a, c \in R$.

Proposition 5 *Assume the original system (18) satisfies* **(S2)** *and* **(S3)** *at an equilibrium point $x_0(\mu)$ not necessarily at the origin. Then the augmented system (71) undergoes a* *codimension 2 bifurcation at $\mu = \mu_c$. At criticality, the linearization of (71) possesses one simple zero eigenvalue and a pair of pure imaginary eigenvalues.*

Proof: The equilibrium point of the augmented system (71) is $(x_0, 0, z_0)$, where z_0 is solution of $cx_i + az_i = 0$. Note that new augmented system keeps x_0 as a component of this equilibrium point. The Jacobian matrix of (71) evaluated at this equilibrium point is

$$J = \begin{bmatrix} A & -cI & 0 \\ cI & 0 & aI \\ 0 & I & 0 \end{bmatrix} \tag{72}$$

where A is the Jacobian matrix of the original system evaluated at x_0. Let α be any eigenvalue of A and r corresponding eigenvector. Also, assume λ is an eigenvalue of J with eigenvector $v = [v_1^T \quad v_2^T \quad v_3^T]^T$. Then

$$\lambda v_1 = A v_1 - c v_2 \tag{73}$$

$$\lambda v_2 = c v_1 + a v_3 \tag{74}$$

$$\lambda v_3 = v_2 \tag{75}$$

Attempt a solution v for which $v_1 = r$. Solve (74) and (75) for v_2 and v_3 in terms of r, we get

$$v_2 = \frac{c\lambda}{\lambda^2 - a} r$$

$$v_3 = \frac{c}{\lambda^2 - a} r$$

Substituting the equation for v_2 into (73) and using $r \neq 0$, gives

$$\lambda^3 - \alpha\lambda^2 + (c^2 - a)\lambda + a\alpha = 0 \tag{76}$$

Since one eigenvalue of A becomes 0 at $\mu = \mu_c$, we can set $\alpha = 0$ to get following equation for the expected pair of eigenvalues system at criticality:

$$\lambda^3 + (c^2 - a)\lambda = 0 \tag{77}$$

If we choose $a < 0$, then J has eigenvalues $0, \pm\sqrt{c^2 - a}\, j$ which correspond to the zero eigenvalue of the original system criticality.

Next, we check the transversality condition. Equation (76) which corresponds to crossing simple real eigenvalue of original system has one real and a pair of complex conjugate as its solution near the critical point. Denote δ as the real eigenvalue and $\beta \pm \gamma j$ as the pair of complex conjugate eigenvalue. Solving this notation in (76) and separating real and imaginary parts, we obtain

$$\begin{aligned} \delta + 2 * \beta &= -\alpha \\ \delta(\beta^2 + \gamma^2) &= \alpha a \end{aligned} \tag{78}$$

Differentiating these equations with respect to μ, gives

$$\begin{aligned} \frac{d\delta}{d\mu} + 2\frac{d\beta}{d\mu} &= -\frac{d\alpha}{d\mu} \\ \frac{d\delta}{d\mu}(\beta^2 + \gamma^2) + 2\frac{d\beta}{d\mu}\beta\delta + 2\frac{d\gamma}{d\mu}\gamma\delta &= \frac{d\alpha}{d\mu}a \end{aligned} \tag{79}$$

At the critical parameter value $\mu = \mu_c$, $\delta = 0$, $\beta = 0$, and $\gamma^2 = c^2 - a$. Thus, at $\mu = \mu_c$,

$$\frac{d\delta}{d\mu} + 2\frac{d\beta}{d\mu} = -\frac{d\alpha}{d\mu} \tag{80}$$

$$\frac{d\delta}{d\mu}(c^2 - a) = \frac{d\alpha}{d\mu}a \tag{81}$$

From equation (81), $\frac{d\alpha}{d\mu} \neq 0$. Solving these equations for $\frac{d\beta}{d\mu}$, gives

$$\frac{d\beta}{d\mu} = -\frac{1}{2}\frac{c}{c^2 - a}\frac{d\alpha}{d\mu} \tag{82}$$

which is nonzero if $\frac{d\alpha}{d\mu} \neq 0$ and $a < 0$.

As was the case with Proposition 1, the final step in the proof consists of showing that all other eigenvalues of the matrix J are in the open left half complex plane (C_-). There are three eigenvalues of J which correspond to one negative real value eigenvalue of A and these eigenvalues are solutions of the (76). By using the Routh-Hurwitz criterion, we can show that solutions of equation (76) in C_- if the corresponding real eigenvalue of A is in C_-. For the complex conjugate pair of eigenvalues of A $(\gamma, \bar{\gamma})$, we have following two equations

$$\lambda^3 - \gamma\lambda^2 + (c^2 - a)\lambda + a\gamma = 0 \tag{83}$$

$$\lambda^3 - \bar{\gamma}\lambda^2 + (c^2 - a)\lambda + a\bar{\gamma} = 0 \tag{84}$$

Multiply (83) and (84) to get the sixth order equation

$$\begin{aligned}\lambda^6 &- (\gamma+\bar{\gamma})\lambda^5 + (2(c^2-a)+\gamma\bar{\gamma})\lambda^4 + (\gamma+\bar{\gamma})(2a-c^2)\lambda^3 \\ &+ ((c^2-a)^2 - 2a\gamma\bar{\gamma})\lambda^2 + (c^2-a)a(\gamma+\bar{\gamma})\lambda + a^2\gamma\bar{\gamma} = 0 \end{aligned} \tag{85}$$

By applying the Routh-Hurwitz criterion to (85), we can show that all solutions of (85) are in the open left half complex plane if $Re(\gamma)$ is negative (details are in Appendix A). ■

We have proved that the new augmented system (71) with nominal equilibrium not necessarily at the origin replaces a stationary bifurcation with a codimension two bifurcation. Note that the design gives the same critical parameter value for the plant and the augmented system. In addition, the crossing eigenvalues at critical point are located 0 and $\pm\sqrt{c^2-a}\,j$. Also, note that an original simple zero eigenvalue persists under the augmentation. Thus, the monitoring system's effectiveness has to do with its introduction of a purely imaginary pair of eigenvalues at criticality in addition to the zero eigenvalue. Near bifurcation, we expect power spectrum peaks to be located at 0 and $\sqrt{c^2-a}$. By varying c and a (both tunable parameters), we can tune the location of the peak at $\sqrt{c^2-a}$ as desired. This flexibility increases our assurance that the power spectrum peak is caused by closeness to instability rather than by other factors (such as noise). However, the new augmented system (71) also comes with some disadvantages compared to the system in Proposition 1. In Proposition 1, we transform a stationary bifurcation into a Hopf bifurcation. In other words, the system has a limit cycle as its solution instead of a new equilibrium point near of bifurcation. In comparison to the previous augmented system design (19), the new augmented system (71) shows more complicated bifurcation behavior [9]. The system is no longer guaranteed to have a periodic orbit as a solution near bifurcation. Either a periodic orbit or a new equilibrium point could result at bifurcation. The bifurcation diagram depends strongly on the vector field $f(x,\mu)$. However, it may be possible that augmented system has desired bifurcation diagram by introducing some nonlinear terms into augmented states. Of course, to do that we have detail knowledge on $f(x,\mu)$. Details on codimension two bifurcations can be found in [9]. However, for the purpose of monitoring, it is enough to have a discernible power spectrum peak when the system approaches instability.

The next proposition asserts that the new augmented system also works for singularly perturbed systems using fewer states for feedback. The only difference from the previous results on singularly perturbed systems is that we no longer require **(SP1)** of Section 5. Using the same notation as in Section 5, we have the following proposition.

Proposition 6 *Let (SP2)-(SP4) of Section 5 hold for the system (18). Then there is an* $\epsilon_0 > 0$ *and for each* $\epsilon_0 \in [0, \epsilon_0]$ *the following extended system undergoes a codimension two* *(one real and a pair of complex eigenvalues crossing) bifurcation at an equilibrium* $m_0^{\mu_c^\epsilon,\epsilon}$ *near* m_0 *for a critical parameter value* μ_c^ϵ *near* μ_c*:*

$$\begin{aligned} \dot{x}_i &= f_i(x, z, \mu, \epsilon) - cy_i \\ \dot{y}_i &= cx_i + aw_i \\ \dot{w}_i &= y_i \\ \epsilon \dot{z} &= g(x, z, \mu, \epsilon) \end{aligned} \tag{86}$$

where $i = 1, 2, \ldots, n$.

Proof: Follows directly from Proposition 5 and Theorem 7 of [1]. ■

7 An Example

Consider again the simple system (51) which undergoes a pitchfork bifurcation. For convenience, we rewrite the equations for the plant (including a noise term) augmented with a monitoring system:

$$\begin{aligned} \dot{x}_1 &= -\mu x_1 - x_1^3 + x_1 x_2 - cy_1 + N(t) \\ \dot{y}_1 &= cx_1 \\ \dot{x}_2 &= -x_2 + 5x_1^2 - cy_2 \\ \dot{y}_2 &= cx_2 \end{aligned} \tag{87}$$

Here, $N(t)$ is a white Gaussian noise. The system (87) undergoes a Hopf bifurcation at $\mu = 0$, which is the parameter value where a pitchfork bifurcation occurs for the original system (51). The origin loses stability as μ is decreased through $\mu = 0$.

The simulation results in this section were obtained by the MATLAB Simulink package. Figure 3 shows the location of the power spectrum peak in frequency as the parameter c is (quasistatically) changed. The simulations were done for a parameter value of $\mu = 0.1$, which is before the origin loses stability. Note from the figure that the location of the power spectrum peak obtained from simulation (shown as an asterisk in Figure 3) agrees well with the predicted location (straight line in Figure 3).

8 Conclusions

We have proposed closed-loop monitoring systems for detection of incipient instability in uncertain nonlinear plants. These systems make use of characteristics of the power spectrum

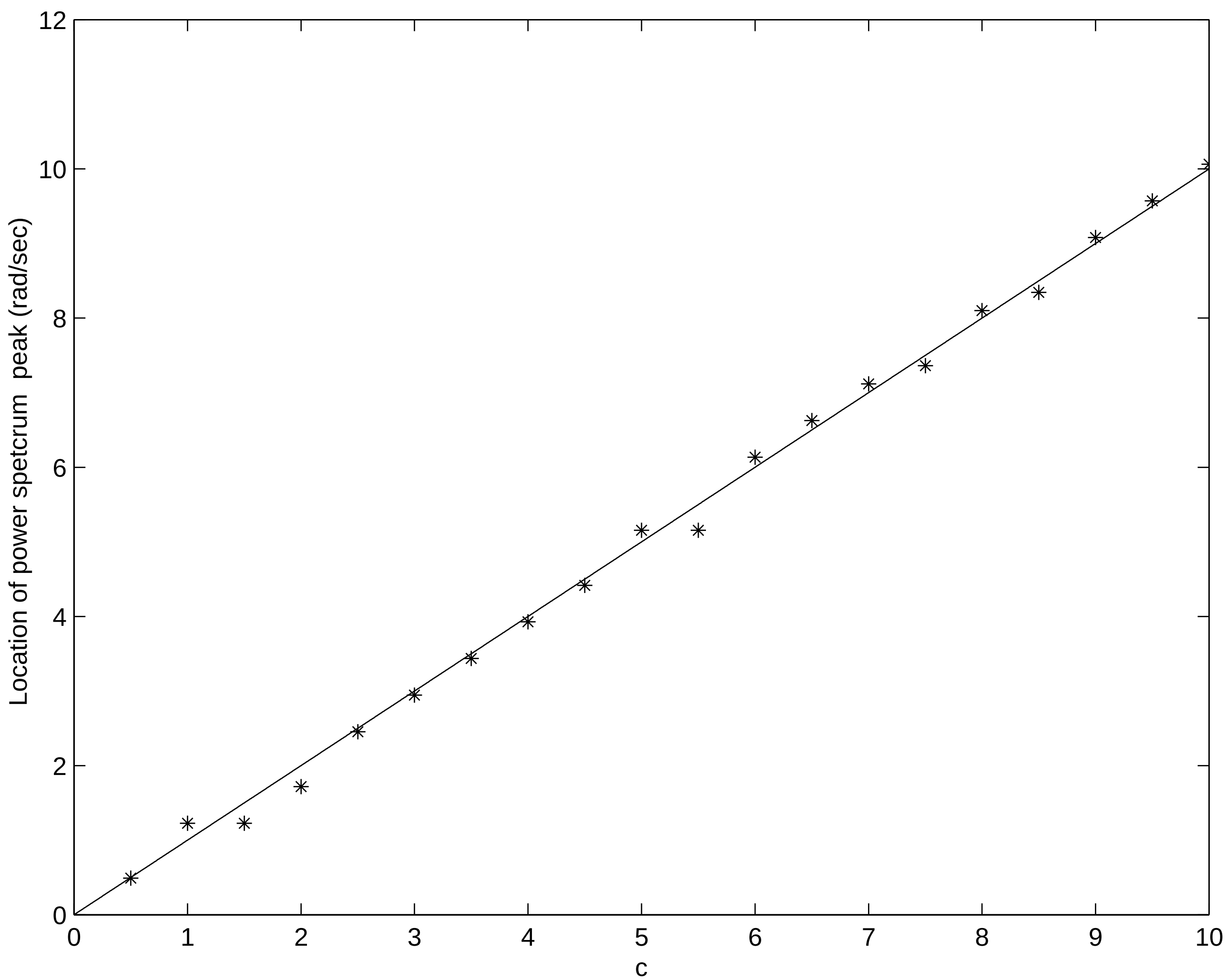


Figure 3: Variation of location of power spectrum peak with c $(\mu = 0.1)$

of a measured output in the vicinity of an instability. By employing closed-loop designs, we are able to more reliably monitor for incipient instability through on-line tuning of control parameters. Two time-scale models were used to reduce the number of measurements fed back in the closed-loop monitoring system. We have studied the impact of the monitoring systems on stability of bifurcations that occur when stability is lost, and proposed design modifications to ensure stability of bifurcated solutions in a robust fashion.

A Routh-Hurwitz Calculation for Proposition 5

Using the same notation as in Proposition 5, and letting $\gamma = \sigma + j\omega$ and $\bar{\gamma} = \sigma - j\omega$, Eq. (85) becomes

$$\begin{aligned} \lambda^6 - 2\sigma\lambda^5 + (2(c^2 - a) + \sigma^2 + \omega^2)\lambda^4 + 2\sigma(2a - c^2)\lambda^3 &\quad + \\ ((c^2 - a)^2 - 2a(\sigma^2 + \omega^2))\lambda^2 + 2(c^2 - a)a\sigma\lambda + a^2(\sigma^2 + \omega^2) &\quad = \quad 0 \end{aligned}$$

Applying the Routh-Hurwitz criterion to the equation above, we obtain the Routh array

$$\begin{array}{lccccc} s^6 & 1 & 2(c-a)+\sigma & (c-a)^2 - 2a(\sigma^2+\omega^2) & a^2(\sigma^2+\omega^2) \\ s^5 & -2\sigma & 2\sigma(2a-c) & 2\sigma a(c-a) & 0 \\ s^4 & c+\sigma^2+\omega^2 & c^2 - ac - 2a(\sigma^2+\omega^2) & a^2(\sigma^2+\omega^2) & 0 \\ s^3 & \frac{2\sigma c(a-(\sigma^2+\omega^2))}{c+\sigma^2+\omega^2} & \frac{2\sigma ac(c-a+\sigma^2+\omega^2)}{c+\sigma^2+\omega^2} & 0 & 0 \\ s^2 & \Lambda & a^2(\sigma^2+\omega^2) & 0 & 0 \\ s^1 & \frac{-2ac^3\sigma(c+\sigma^2+\omega^2)}{a(\sigma^2+\omega^2+c)-(a+c)^2} & 0 & 0 & 0 \\ s^0 & a^2(\sigma^2+\omega^2) & 0 & 0 & 0 \end{array}$$

where $\Lambda = \frac{a(\sigma^2+\omega^2)(c+\sigma^2+\omega^2)-(a+c)^2(\sigma^2+\omega^2)}{a-(\sigma^2+\omega^2)}$. If $a < 0$ and $\sigma < 0$, then all entries in the first column of this array are positive. Therefore, under this condition all solutions of (85) have negative real part.

B Stability of Bifurcated Solutions

In this appendix, we recall formulas for determining the local stability of bifurcated solutions. Details can be found in Abed and Fu [2],[3].

Consider a one-parameter family of nonlinear autonomous systems

$$\dot{x} = f(x, \mu) \tag{88}$$

where $x \in R^n$ is the vector state and μ is a real-valued parameter. Let $f(x, \mu)$ be sufficiently smooth in x and μ and let $x_{0,\mu}$ be the nominal equilibrium point of the system as a function of the parameter μ.

First, we consider the case of stationary bifurcation. For simplicity, we take the critical parameter value to be $\mu_c = 0$ in the statement of the next hypothesis.

(S) The Jacobian matrix of system (88) at the equilibrium $x_{0,\mu}$ has a simple zero eigenvalue $\lambda_1(\mu)$ with $\lambda_1'(0) \neq 0$, and the remaining eigenvalues lie in the open left half of the complex plane for $\mu = 0$.

The Stationary Bifurcation Theorem asserts that hypothesis (S) implies a stationary bifurcation from $x_{0,\mu}$ at $\mu = 0$ for (88). A new equilibrium branch bifurcates from $x_{0,\mu}$ at $\mu = 0$. The theorem states that near the point $(x_{0,0}, 0)$ of the $(n+1)$-dimensional (x, μ)-space, there exists a locally unique curve of critical points $(x(\epsilon), \mu(\epsilon))$, distinct from $x_{0,\mu}$ and passing through $(x_{0,0}, 0)$, such that for all sufficiently small $|\epsilon|$, $x(\epsilon)$ is an equilibrium point of (88) when $\mu = \mu(\epsilon)$. (Here, ϵ is an auxiliary small parameter.)

The series expansions of $x(\epsilon), \mu(\epsilon)$ can be written as

$$\mu(\epsilon) = \mu_1 \epsilon + \mu_2 \epsilon^2 + \cdots \tag{89}$$

$$x(\epsilon) = x_{0,\mu} + x_1 \epsilon + x_2 \epsilon^2 + \cdots \tag{90}$$

If $\mu_1 \neq 0$, the system undergoes a transcritical bifurcation from $x_{0,\mu}$ at $\mu = 0$. That is, there is a second equilibrium point besides $x_{0,\mu}$ for both positive and negative values of μ with $|\mu|$ small. If $\mu_1 = 0$ and $\mu_2 \neq 0$, the system undergoes a pitchfork bifurcation for $|\mu|$ sufficiently small. That is, there are two new equilibrium points existing simultaneously, either for positive or for negative values of μ with $|\mu|$ small. The new equilibrium points have an eigenvalue $\beta(\epsilon)$ which vanishes at $\mu = 0$. The series expansion $\beta(\epsilon)$ is given by

$$\beta(\epsilon) = \beta_1 \epsilon + \beta_2 \epsilon^2 + \cdots \tag{91}$$

We have the exchange of stability formula:

$$\beta_1 = -\mu_1 \lambda'(0) \tag{92}$$

Moreover, in case $\beta_1 = 0$, β_2 is given by

$$\beta_2 = -2\mu_2 \lambda'(0) \tag{93}$$

Eqs. (92),(93) are not explicit formulas for β_1 and β_2. Explicit formulas are given Lemma 1.

The bifurcation stability coefficients β_1 and β_2 can be obtained using eigenvector computations and series expansion of the vector field. System (88) can be written in the series form

$$\begin{aligned} \dot{\tilde{x}} &= L_\mu \tilde{x} + Q_\mu(\tilde{x},\tilde{x}) + C_\mu(\tilde{x},\tilde{x},\tilde{x}) + \cdots \\ &= L_0\tilde{x} + \mu L_1 \tilde{x} + \mu^2 L_2 \tilde{x} + \cdots \\ &\quad + Q_0(\tilde{x},\tilde{x}) + \mu Q_1(\tilde{x},\tilde{x}) + \cdots \\ &\quad + C_0(\tilde{x},\tilde{x},\tilde{x}) + \cdots \end{aligned} \tag{94}$$

where $\tilde{x} = x - x_{0,0}, L_\mu, L_1, L_2$ are $n \times n$ matrices, $Q_\mu(x,x), Q_0(x,x), Q_1(x,x)$ are vector-valued quadratic forms generated by symmetric bilinear forms, and $C_\mu(x,x,x), C_0(x,x,x)$ are vector-valued cubic forms generated by symmetric trilinear forms.

By assumption, the Jacobian matrix L_0 has a simple zero eigenvalue with the remaining eigenvalues stable. Denote by l and r the left (row) and right (column) eigenvectors, respectively, of the matrix L_0 associated with the simple zero eigenvalue, where the first component of r is set to 1 and the left eigenvector l is chosen such that $lr = 1$. (Setting the first component of r to 1 sometimes requires a re-ordering of the state variables.) The following well known fact is used in the statement of the next lemma:

$$\lambda'(0) = lL_1 r \tag{95}$$

The two lemmas that follow give stability criteria for the bifurcated equilibria of system (94). The first addresses to pitchfork bifurcation, while the second addresses transcritical bifurcation.

Lemma 1 *Let hypothesis (S) hold. Then*

$$\beta_1 = lQ_0(r,r) \tag{96}$$

Also, if $\beta_1 = 0$, then

$$\beta_2 = 2l\{2Q_0(r,x_2) + C_0(r,r,r)\} \tag{97}$$

where x_2 solves

$$L_0 x_2 = -Q_0(r,r) \tag{98}$$

The bifurcated equilibrium points of (94) near x_μ^0 for μ near 0 are asymptotically stable if $\beta_1 = 0$ and $\beta_2 < 0$, but they are unstable if $\beta_1 = 0$ and $\beta_2 > 0$.

Lemma 2 *Let hypothesis (S) hold, and suppose that* $\beta_1 \neq 0$ *(this can be checked using (96)). Then the bifurcated solution is asymptotically stable on one side of* $\mu = 0$ *and is unstable on the other. For any given value of* μ *near 0, the stability of the bifurcated solution is opposite that of the nominal equilibrium.*

Now consider system (88) under for the following hypothesis, which implies occurrence of Hopf bifurcation. Again, for simplicity the critical value of the parameter is taken as $\mu_c = 0$.

(H) The Jacobian matrix of system (88) at the equilibrium $x_{0,\mu=0}$ has a pair of pure imaginary eigenvalues $\lambda_1(0) = j\omega_c$ and $\bar{\lambda_1}(0) = -j\omega_c$ with $\omega_c \neq 0$, the transversality condition $\frac{\partial Re[\lambda(0)]}{\partial \mu} \neq 0$ is satisfied, and all the remaining eigenvalues lie in the open left half complex plane.

Under these conditions, the Hopf Bifurcation Theorem asserts the existence of a one-parameter family $p_\epsilon, 0 < \epsilon \leq \epsilon_0$ of nonconstant periodic solutions of system (88) emerging from $x = x_{0,\mu}$ at the parameter value 0 for sufficiently small $|\mu|$. Exactly one of the characteristic exponents of p_ϵ governs the asymptotic stability and is given by a real, smooth and even function

$$\beta(\epsilon) = \beta_2\epsilon^2 + \beta_4\epsilon^4 + \cdots \tag{99}$$

Specifically, p_ϵ is orbitally stable if $\beta(\epsilon) < 0$ but is unstable if $\beta(\epsilon) > 0$. Generically the local stability of the bifurcated periodic solution p_ϵ is decided by the sign of the coefficient β_2. It happens that the sign of β_2 also determines the stability of the critical equilibrium point $x_{0,\mu}$. An algorithm for computing the stability coefficient β_2 follows.

Step1 Express (88) in the Taylor series form (94). Let r be the right eigenvector of L_0 corresponding to eigenvalue $j\omega_c$ with the first component of r set to 1. Let l be the left eigenvector of L_0 corresponding to the eigenvalue $j\omega_c$, normalized such that $lr = 1$.

Step2 Solve the equations

$$L_0 a = -\frac{1}{2}Q_0(r, \bar{r}) \tag{100}$$

$$(2j\omega_c I - L_0)b = \frac{1}{2}Q_0(r, r) \tag{101}$$

for a and b.

Step 3 The stability coefficient β_2 is given by

$$\beta_2 = 2Re\{2lQ_0(r, a) + lQ_0(\bar{r}, b) + \frac{3}{4}lC(r, r, \bar{r})\} \tag{102}$$

Acknowledgments

This research has been supported in part by the Air Force Office of Scientific Research under Grant F49620-96-1-0161 and by the Office of Naval Research under Multidisciplinary University Research Initiative (MURI) Grant N00014-96-1-1123.

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