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Discrete-Time Risk-Sensitive Filters with Non-Gaussian Initial Conditions and their Ergodic Properties

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Discrete-time risk-sensitive filters with non-Gaussian initial conditions and their ergodic properties

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Abstract

In this paper, we study asymptotic stability properties of risk-sensitive filters with respect to their initial conditions. In particular, we consider a linear time-invariant systems with initial conditions that are not necessarily Gaussian. We show that in the case of Gaussian initial conditions, the optimal risk-sensitive filter asymptotically converges to any suboptimal filter initialized with an incorrect covariance matrix for the initial state vector in the mean square sense provided the incorrect initializing value for the covariance matrix results in a risk-sensitive filter that is asymptotically stable, that is, results in a solution for a Riccati equation that is asymptotically stabilizing. For non-Gaussian initial conditions, we derive the expression for the risk-sensitive filter in terms of finite number of parameters. Under a boundedness assumption satisfied by the fourth order absolute moment of the initial state variable and a slow growth condition satisfied by a certain Radon-Nikodym derivative, we show that a suboptimal risk-sensitive filter initialized with Gaussian initial conditions asymptotically approaches the optimal risk-sensitive filter for non-Gaussian initial conditions in the mean square sense.

1 Introduction

Risk-sensitive filtering optimizes an exponential of quadratic (or more general convex) cost criterion. As opposed to L_2 filtering, risk-sensitive filtering penalizes the higher order moments of the estimation error energy, thus making the filters useful in uncertain plant and noise environments. It also allows a trade-off between optimal filtering for the nominal model case and the average noise situation, and robustness to worst case noise and model uncertainty by weighting the index of the exponential by a risk-sensitive parameter. For example, it has been shown in [1] that discrete-time risk-sensitive filters for hidden Markov models (HMM) with finite-discrete states perform better than standard HMM filters in situations involving uncertainties in the noise statistics. A more recent work [2] shows that such risk-sensitive filters

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enjoy an error bound which is the sum of two terms, the first of which coincides with an upper bound on the error one would obtain if one knew exactly the underlying probability model, while the second term is a measure of the distance between the true and design probability models. Although risk-sensitive filtering was introduced for discrete-time linear systems in [3], the term "risk-sensitive filtering" was introduced in [4] and more general discrete-time nonlinear systems were treated, using similar techniques of [5] in the context of risk-sensitive control. Apart from the potential usefulness of risk-sensitive filters in uncertain environments, risk-sensitive problems, in the small noise-limit, have been shown to be closely related to estimation/control problems in a deterministic worst-case noise scenario given from a differential game (H_∞ estimation/control problems for linear systems) [6] [7] [5] [8] [9].

It is well known that the mean of the conditional density of the state given the observations for a stochastic state space signal model achieves the minimum variance filter. For a linear Gaussian system with known or Gaussian distributed initial conditions, the conditional density is Gaussian and given by its mean and covariance (which can be calculated off-line from a Riccati differential or difference equation). This is also popularly known as a Kalman filter. On the other hand, the optimal estimation problem becomes an essentially nonlinear problem if the initial condition is not Gaussian distributed. However, for linear Gaussian systems, it has been shown [10] [11] that the optimal filter (or its density) can be given by a finite number of statistics, which constitute the optimal (in the minimum variance sense) filter for an augmented linear system. The initial condition is often not known and it is often unrealistic to assume that the initial condition has a Gaussian density. However, it has been shown in [11](continuous-time) [12] (discrete-time) that the conditional density filter forgets the initial condition asymptotically in an exponential rate. In other words, one can assume a Gaussian density for the initial condition and use a suboptimal Kalman filter which asymptotically becomes optimal, provided the actual density of the initial condition has finite first and second order moments. Exponential stability results for discrete-time filters have been shown in [13] and for Benes filters [14] in [15]. Also, stability results for filters based on Lyapunov exponents have been explored in [16] [17] [18].

It is also well known that the optimal risk-sensitive filter for a discrete-time linear Gaussian system with a Gaussian initial condition is an H_∞ filter [3] [4]. Analogous results for continuous-time systems can be found in [19] [20] [21]. In the case of a non-Gaussian initial condition, the risk-sensitive estimation problem, as can be expected, becomes a nonlinear problem in general.

In this paper, we consider the problem of risk-sensitive estimation for discrete-time linear Gaussian time-invariant systems with non-Gaussian initial conditions. Our objective is to study the effects of initial conditions on the risk-sensitive estimates and asymptotic stability or forgetting properties of such estimates with respect to initial conditions. We first consider the case of arbitrary Gaussian initial conditions, i.e., arbitrary initial covariance matrices (the mean of the Gaussian distribution is taken to be zero without loss of generality), a suboptimal risk-sensitive estimate (initialized with an incorrect covariance matrix) asymptotically approaches the optimal risk-sensitive estimate (initialized with the true covariance matrix) provided the incorrect initial covariance matrix results in a stabilizing solution of the H_∞-like Riccati equation. The case with non-Gaussian initial conditions is slightly more complicated.

We first derive an expression for the risk-sensitive estimate that is finite-dimensional, and a sum of two quantities, the first of which asymptotically approaches the risk-sensitive estimate for the Gaussian initial condition (with arbitrary but stabilizing initial covariance matrix) and the second term approaches zero asymptotically under a boundedness condition satisfied by the fourth order absolute moment of the initial state variable and a slow growth condition satisfied by a the fourth order moment of a certain Radon-Nikodym derivative. These convergence results are derived in the mean square sense.

In Section 2, we introduce the signal model, the risk-sensitive estimation problem and reformulate it under a new probability measure. In Section 3, we briefly present the optimal risk-sensitive filter for linear Gaussian systems with Gaussian initial conditions and show the asymptotic stability of these filters with respect to arbitrary Gaussian initial conditions in the mean square sense. Section 4 deals with non-Gaussian initial conditions where we first derive the optimal risk-sensitive filter using the information state approach and then we show the asymptotic mean square convergence properties of such filters with respect to their initial conditions. Section 5 presents some concluding remarks.

2 Signal model

Consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ on which we define the following stochastic linear time-invariant discrete-time state space model:

$$\begin{aligned} x_{k+1} &= F x_k + G w_{k+1}, x_0 \sim \Pi_0(x_0) \\ y_k &= H x_k + v_k \end{aligned} \tag{2.1}$$

Here $x_k \in \mathbb{R}^n, y_k \in \mathbb{R}^p, k \in \mathbb{N}$. The process noise $w_k \in \mathbb{R}^n$ and the measurement noise $v_k \in \mathbb{R}^p$ are *i.i.d.* Gaussian distributed with zero mean and covariance I_n and I_p respectively. Also, $GG^* = \Sigma_w > 0$. Π_0 is not necessarily Gaussian.

We assume that x_0, w_k, v_k are mutually independent and that (F, G) are stabilizable and (F, H) are detectable.

Denote the complete filtration generated by the observation σ-algebra, namely, $\sigma\{y_0, y_1, \ldots, y_k\}$ as $\{\mathcal{Y}_k\}$, the complete filtration generated by $\sigma\{x_0\} \vee \sigma\{w_1, \ldots, w_{k-1}\}$ as $\{\mathcal{F}_k\}$ and the complete filtration generated by $\sigma\{y_0, \ldots, y_k\} \vee \sigma\{x_0\} \vee \sigma\{w_0, \ldots, w_{k-1}\}$ as $\{\mathcal{G}_k\}$.

Risk-sensitive estimation

We define the risk-sensitive estimation problem for the discrete-time system (2.1) as to obtain a $\mathcal{Y}_k$-measurable process $\hat{x}_k \in \mathbb{R}^n$ (assumed to be well-defined) such that

$$\hat{x}_k \in \underset{\zeta}{\operatorname{argmin}} \, E[\exp \mu\{\sum_{l=0}^{k-1} l(x_l, \hat{x}_l) + l(x_k, \zeta)\} \mid \mathcal{Y}_t] \tag{2.2}$$

Here, $E[.]$ denotes expectation under P, $\mu > 0$ and $l : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is measurable in $(x, \hat{x})$ and continuous in $\hat{x}$ and is of the following form

$$l(x, \hat{x}) = \frac{1}{2}(x - \hat{x})^*(x - \hat{x}) \tag{2.3}$$

Remark 2.1 Please note above that while estimating $\hat{x}_k$, we do not obtain new estimates of $x_l,\ l < k$. Hence this is a strict filtering problem. However, we consider a sum of estimation error costs in the index of the exponential. Note that considering just the cost at one time point (instead of the sum) will give rise to a different optimization problem. In the linear Gaussian case, this results in a Kalman filter whereas a cost like (2.2) results in an H_∞ filter. For more discussion on this, see [8] [4].

Next, we work under a probability measure $\bar{P}$ such that under $\bar{P}$, $\{y_k\}$ is a sequence of i.i.d. Gaussian distributed random variables with zero mean and covariance I_p and independent of x_k (and hence x_0). Using a change of measure argument and a discrete-time version of Girsanov's theorem, the risk-sensitive estimation can be re-formulated as

$$\hat{x}_k \in \operatorname*{argmin}_{\zeta} \bar{E}[\Lambda_k \exp \mu\{\sum_{l=0}^{k-1} l(x_l, \hat{x}_l) + l(x_k, \zeta)\} \mid \mathcal{Y}_k] \tag{2.4}$$

where $\Lambda_k = \prod_{l=0}^{k} \exp((Hx_k)' y_k - \frac{1}{2}(Hx_k)'(HX_k))$. For details on this particular application of change of probability measure technique, see [22](discrete-time) and [23] [20] (continuous-time).

3 Discrete-time risk-sensitive estimation with Gaussian initial condition

In this section, we present the risk-sensitive estimation results for discrete-time linear Gaussian systems with Gaussian initial conditions and study the asymptotic forgetting property of the estimates with respect to initial conditions. Without loss of generality (see [10]), we take the mean of the initial density to be zero. It is with respect to the covariance matrix of the initial state that we study the asymptotic convergence properties.

The following theorem summarizes the risk-sensitive estimation results for the linear Gaussian systems with Gaussian initial condition (for similar proofs, see [4].

Theorem 3.1 *Consider the signal model (2.1) and the risk-sensitive cost given by (2.2), (2.3). Suppose* $x_0 \sim N(0, \Sigma)$. *The optimal risk-sensitive estimate* $\hat{x}_k^G$ *is then given by the following stochastic difference equation*

$$\begin{aligned} \hat{x}_k^G &= F\hat{x}_{k-1}^G + \Sigma_k H'(y_k - HF\hat{x}_{k-1}^G) \\ \hat{x}_0^G &= (H'H + \Sigma^{-1})^{-1} H' y_0 \end{aligned} \tag{3.1}$$

where Σ_k *satisfies the following discrete-time Riccati equation:*

$$\Sigma_k^{-1} = H'H + [\Sigma_w + F(\Sigma_{k-1}^{-1} - \mu I)^{-1} F']^{-1},\ \Sigma_0 = (H'H + \Sigma^{-1})^{-1} \tag{3.2}$$

Proof A similar proof can be found in [4] and is not repeated here. □

Remark 3.1 It is implicitly assumed above that μ is small enough such that $\Sigma_w + F(\Sigma_{k-1}^{-1} - \mu I)^{-1}F' > 0,\ \forall k > 0$.

3.1 Asymptotic optimality of discrete-time risk-sensitive filters with Gaussian initial conditions

In this subsection, we present the results for the asymptotic optimality of the discrete-time risk-sensitive filters with respect to arbitrary Gaussian initial conditions. Without loss of generality ([10]), we take the mean of the Gaussian density to be 0. It is well known (from H_∞ filtering theory) that the solutions to the Riccati equation (3.2) are not necessarily stabilizing under the previous stabilizability and detectability assumptions (unlike Kalman filtering Riccati equations). If the initial value of Σ_0 is chosen such that $\lim_{k\to\infty} \Sigma_k$ (assuming that the limit exists) which is the steady state solution of the algebraic Riccati equation associated with (3.2), is stabilizing, then we consider that value of Σ_0 to be a candidate for an arbitrary initial value for solving (3.2). We denote the set of such admissible initial choices for Σ_0 as $\mathcal{D}$.

The steady state solution Σ_∞ is the solution to the following algebraic Riccati equation:

$$\bar{P}^{-1} = H'H + [\Sigma_w + F(\bar{P}^{-1} - \mu I)^{-1}F']^{-1} \tag{3.3}$$

In what follows, we will always consider initializing values for Σ_0 that result in a stabilizing solution Σ_∞.

Consider the following suboptimal risk-sensitive estimate β_k^G which satisfies the following stochastic difference equation:

$$\begin{aligned} \beta_k^G &= F\beta_{k-1}^G + Q_k H'(y_k - HF\beta_{k-1}^G) \\ \beta_0^G &= (H'H + Q^{-1})^{-1}H'y_0 \end{aligned} \tag{3.4}$$

where Q_k satisfies the following Riccati difference equation:

$$Q_k^{-1} = H'H + [\Sigma_w + F(Q_{k-1}^{-1} - \mu I)^{-1}F']^{-1},\ Q_0 = (H'H + Q^{-1})^{-1},\ Q \in \mathcal{D} \tag{3.5}$$

In other words, (3.4), (3.2) describe a suboptimal risk-sensitive estimate with an arbitrary initial covariance matrix $Q \in \mathcal{D}$. We will show that β_k^G converges to $\hat{x}_k^G$ in the mean square sense, that is as $k \to \infty$, $E|\hat{x}_k^G - \beta_k^G|^2 \to 0$.

We make the following assumptions:

Assumption 3.1 *We assume that $\hat{x}_k^G$ exists for all $k \in \mathbb{N}$ and μ is small enough such that*

$$\lim_{T\to\infty} \frac{1}{T} \log E[\exp \mu\{\sum_{l=0}^{T} l(x_l, \hat{x}_l^G)\} \mid \mathcal{Y}_T] \le c_1 < \infty,\ for\ some\ c_1 > 0 \tag{3.6}$$

Assumption 3.2 *There exists a bounded symmetric positive definite solution Σ_k to (3.2) ($\Sigma_0 \in \mathcal{D}$) for all $k > 0$, such that $lim_{k\to\infty}\Sigma_k = \Sigma_\infty$ exists and $\rho(F - \Sigma_\infty H'HF) < 1$.*

Remark 3.2 The above assumption implies that the following discrete-time unforced time-varying linear system

$$\Psi_{k+1} = (F - \Sigma_k H'HF)\Psi_k,\ \Psi_0 = I \tag{3.7}$$

is exponentially stable. This follows from the fact that (see [12]) $\lim_{k\to\infty} \frac{1}{k} \ln \lambda_{\max}(\Psi_k'\Psi_k) \le 2\ln\rho(F - \Sigma_\infty H'HF)$.

Hence we have the following proposition:

Proposition 3.1 *There exists a* $|\ln\rho(F - \Sigma_\infty H'HF)| > \sigma > 0,\ M_\sigma^1 > 0$ *such that*

$$\|\Psi_k \Psi_j^{-1}\| \le M_\sigma^1 \exp(-\sigma(k-j)),\ \forall k > j \ge 0 \tag{3.8}$$

We also state the following result without proof, that can be derived from Assumption 3.2. For similar proofs in continuous-time literature, see [11] and the references therein. Related results in monotonicity and stability properties of discrete-time Riccati equations can be found in [24].

Proposition 3.2 *There exists a* $\sigma > 0,\ M_\sigma^2 > 0,\ k_0 \ge 0$ *such that*

$$\|\Sigma_k - Q_k\| \le M_\sigma^2 \exp(-\sigma k),\ \forall k \ge k_0 \tag{3.9}$$

In other words, it follows that both $\lim_{k\to\infty} Q_k \to \Sigma_\infty$, $\lim_{k\to\infty} \Sigma_k \to \Sigma_\infty$ exponentially fast and also that the unforced linear system

$$\tilde{\Psi}_{k+1} = (F - Q_k H'HF)\tilde{\Psi}_k \tag{3.10}$$

is exponentially stable.

Define the following quantity:

Definition 3.1

$$\varepsilon_k = y_k - HF\beta_{k-1}^G \tag{3.11}$$

Using Propositions 3.2 and the fact that Σ_k is a stable matrix for all $k \in \mathbb{N}$, one can prove the following Lemma:

Lemma 3.1

$$\sup_k E[|\varepsilon_k|^2] \le D_1^2 < \infty \tag{3.12}$$

Proof Using (3.1), (3.4), (2.1), one can write

$$\varepsilon_k = HF\nu_{k-1} + v_k + HGw_k \tag{3.13}$$

where

$$\nu_k = (F - Q_k H'HF)\nu_{k-1} + (G - Q_k H'HG)w_k - Q_k H' v_k \tag{3.14}$$

Since, x_0, v_k and w_k are mutually independent and $\sup_k \|Q_k\|$ is bounded, one can show from stability of time-varying discrete-time systems that $\sup_k E|\nu_k|^2$ is uniformly bounded. Note that the exponential stability of (3.10) plays an important role.

Once we obtain the above, using similar arguments, one can show that (3.12) holds. □

Now, we present the main theorem of this section:

Theorem 3.2 *Consider the risk-sensitive optimization problem given by (2.2), (2.3) with $x_0 \sim N(0, \Sigma)$. Consider also the evolution equations for the optimal and a suboptimal estimates given by (3.1), (3.4) and the associated Riccati difference equations (3.2), (3.5). Suppose Assumptions 3.1, 3.2 holds. Then,*

$$\lim_{k\to\infty} E|\hat{x}_k^G - \beta_k^G|^2 \to 0 \tag{3.15}$$

Proof Using (3.1), (3.4), one can write

$$e_k^G = (F - \Sigma_k H'HF)e_{k-1}^G + (\Sigma_k - Q_k)H'\varepsilon_k, \; e_0^G = \hat{x}_0^G - \beta_0^G \tag{3.16}$$

where $e_k^G \triangleq \hat{x}_k^G - \beta_k^G$.

Now, the solution to (3.16) can be written as

$$e_k^G = \Psi_k e_0^G + \sum_{j=1}^{k} \Psi_k \Psi_j^{-1} \eta_j \tag{3.17}$$

where $\eta_k \triangleq (\Sigma_k - Q_k)H'\varepsilon_k$.

Using Minkowski's inequality, one can write

$$E|e_k^G|^2 \leq \left(\sqrt{\|\Psi_k\|^2 E|e_0^G|^2} + \sqrt{E|\sum_{j=1}^{k} \Psi_k \Psi_j^{-1} \eta_j|^2} \right)^2 \tag{3.18}$$

One can easily show using the facts that x_0, v_0 are Gaussian distributed and mutually independent, that there exists a $0 \leq M_3 < \infty$ such that $E|e_0^G|^2 \leq M_3$. Using Proposition 3.1, one can then write

$$E|e_k^G|^2 \leq \left(M_\sigma^4 \exp(-\sigma k) + \sqrt{E|z_k|^2} \right)^2 \tag{3.19}$$

where $z_k = \sum_{j=1}^{k} \Psi_k \Psi_j^{-1} \eta_j$.

Now, one can use Propositions 3.1, 3.2 again and the C_r-inequality [25] for $r = 2$ to obtain

$$\begin{aligned} E|z_k|^2 &\leq M_\sigma^5 E[\sum_{j=1}^{k} \exp(-\sigma(k-j)) \exp(-\sigma j)|\varepsilon_j|]^2 \\ &= M_\sigma^5 \exp(-2\sigma k) E[\sum_{j=1}^{k} |\varepsilon_j|]^2 \\ &\leq M_\sigma^5 \exp(-2\sigma k) k E[\sum_{j=1}^{k} |\varepsilon_j|^2] \\ &\leq M_\sigma^5 D_1^2 k^2 \exp(-2\sigma k) \end{aligned} \tag{3.20}$$

Here M_σ^4, M_σ^5 are constants independent of k. Also, we have assumed without loss of generality that $k_0 = 0$. This is so because if $k_0 > 0$, one can use the fact that $E|e_{k_0}|^2 < \infty,\ E|\varepsilon_{k_0}|^2 < \infty$ and carry on the analysis from there as if $k = k_0$ is the initial time point.

It is obvious from the above that $E|z_k|^2 \to 0$ as $k \to \infty$. Using this together with (3.19), we have (3.15). □

One can obtain the following corollary to the previous theorem:

Corollary 3.1 *Consider the risk-sensitive optimization problem given by (2.2), (2.3) with* $x_0 \sim N(0, \Sigma)$. *Consider also the evolution equations for the optimal risk-sensitive estimate given by (3.1), (3.2). Consider the following suboptimal risk-sensitive estimate given by*

$$\begin{aligned} \tilde{\beta}_k^G &= F\tilde{\beta}_{k-1}^G + \tilde{\Sigma}_k H'(y_k - HF\tilde{\beta}_{k-1}^G) \\ \tilde{\beta}_0^G &= 0 \end{aligned} \tag{3.21}$$

where $\tilde{\Sigma}_k$ *satisfies the following Riccati difference equation:*

$$\tilde{\Sigma}_k^{-1} = H'H + [\Sigma_w + F(\tilde{\Sigma}_{k-1}^{-1} - \mu I)^{-1}F']^{-1},\ \tilde{\Sigma}_0 = 0 \tag{3.22}$$

Suppose Assumptions 3.2, 3.1 hold. Then,

$$\lim_{k\to\infty} E|\hat{x}_k^G - \tilde{\beta}_k^G|^2 \to 0 \tag{3.23}$$

Similarly,

$$\lim_{k\to\infty} E|\beta_k^G - \tilde{\beta}_k^G|^2 \to 0 \tag{3.24}$$

Remark 3.3 Note that $\tilde{\Sigma}_0 = 0$ implies $\tilde{\Sigma}_1 = [H'H + \Sigma_w^{-1}]^{-1}$ and we assume $\tilde{\Sigma}_k > 0,\ \forall k > 0$.

4 Discrete-time risk-sensitive estimation with non-Gaussian initial conditions

In this section, we first derive the optimal risk-sensitive estimate for discrete-time linear time-invariant systems with non-Gaussian initial conditions. We derive a recursive update formula for a modified information state and express the optimal risk-sensitive as a function of the parameters of the information state and the non-Gaussian distribution of the initial condition. Throughout this section, we assume that $x_0 \sim \Pi_0(x_0)$, where $\Pi_0(x_0)$ is not Gaussian but has zero mean and satisfies certain properties. We will make the formal assumptions later on. Also, the superscript NG will stand for estimates with non-Gaussian initial condition.

Now, we define the risk-sensitive information state conditioned on the initial state. Note that this is a slightly modified definition than the one used in [4].

Definition 4.1 *Define the unnormalized conditional measure $q_k(x,\xi)$ where*

$$\begin{aligned} q_k(x,\xi)dx &= \bar{E}[\Lambda_k \exp(\mu \sum_{l=0}^{k-1} l(x_l,\hat{x}_l)) I(x_k \in dx) \mid \mathcal{Y}_k, x_0 = \xi] \\ q_0(x,\xi) &= \exp[-\frac{1}{2}\xi' H' H \xi + \xi' H' y_0]\delta(x-\xi) \end{aligned} \tag{4.1}$$

Remark 4.1 Note that the risk-sensitive information state defined in [4] can be written as $q_k(x)$ (which is only conditioned on $\mathcal{Y}_k$) where

$$q_k(x) = \int_{\mathbf{R}^n} q_k(x,\xi)\Pi_0(\xi)d\xi \tag{4.2}$$

Using the Definition 4.1, one can easily prove the following Lemma:

Lemma 4.1 *The information state $q_k(x,\xi)$ obeys the following recursive equation:*

$$\begin{aligned} q_k(x,\xi) &= Z_w \exp[(Hx)'y_k - \frac{1}{2}(Hx)'(Hx)] \int_{\mathbf{R}^n} \exp[-\frac{1}{2}(x-Fz)'\Sigma_w^{-1}(x-Fz)] \\ &\qquad \exp[\frac{\mu}{2}(z-\hat{x}_{k-1})'(z-\hat{x}_{k-1})]q_{k-1}(z,\xi)dz \end{aligned} \tag{4.3}$$

where $Z_w = \frac{1}{(2\pi|\Sigma_w|)^{\frac{n}{2}}}$.

Proof The proof simply follows from Definition 4.1. A similar proof can be found in [4] and is not repeated here. □

It also follows from the Definition 4.1 that the optimal risk-sensitive estimate is given by

$$\hat{x}_k \in \operatorname*{argmin}_{\zeta} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} q_k(x,\xi)\exp(\mu l(x,\zeta))\Pi_0(\xi)dxd\xi \tag{4.4}$$

It is obvious from the above Definition 4.1 that the information state achieves an expression similar to that for the information state with a known initial state vector. It is also well known that for known initial state vectors, the information state achieves an unnormalized Gaussian expression for linear Gaussian systems. The proof follows by induction. We can use similar proof techniques to prove the following Theorem:

Theorem 4.1 *The risk-sensitive information state defined as the unnormalized conditional measure in Definition 4.1 for the linear time invariant system (2.1) is given by*

$$q_k(x,\xi) = s_k(\xi)\exp(-\frac{1}{2}(x-m_k(\xi))'\tilde{\Sigma}_k^{-1}(x-m_k(\xi))) \tag{4.5}$$

where $\tilde{\Sigma}_k$ satisfies the Riccati difference equation (3.22). such that $\tilde{\Sigma}_k > 0$, $\forall k > 0$ and $V_{k-1}^{-1} = F'\Sigma_w^{-1}F + \tilde{\Sigma}_{k-1}^{-1} - \mu I > 0$, $\forall k > 1$. $m_k(\xi)$ is given by the following equations:

$$\begin{aligned} m_k(\xi) &= \beta_k^{NG} + \Phi_k\xi,\ m_0(\xi) = \xi \\ \tilde{\Sigma}_k^{-1}\beta_k^{NG} &= \Sigma_w^{-1}FV_{k-1}(\tilde{\Sigma}_{k-1}^{-1}\beta_{k-1}^{NG} - \mu\hat{x}_{k-1}^{NG}) + H'y_k,\ \beta_0^{NG} = 0 \end{aligned} \tag{4.6}$$

where Φ_k is given by

$$\Phi_k = [F - \tilde{\Sigma}_k H'HF + \mu\tilde{\Sigma}_k\Sigma_w^{-1}FV_{k-1}]\Phi_{k-1},\ \Phi_0 = I \tag{4.7}$$

Also $s_k(\xi)$ is given by

$$\begin{aligned}
s_k(\xi) &= \gamma_k \exp\left[-\frac{1}{2}\xi' L_k \xi + \xi' \rho_k^{NG}\right] \\
\gamma_k &= \gamma_{k-1} Z_k S_k(\hat{x}_{k-1}, \beta_{k-1}, y_k),\ \gamma_0 = 1 \\
L_k &= L_{k-1} + \Phi'_{k-1}(\tilde{\Sigma}_{k-1}^{-1} - \tilde{\Sigma}_{k-1}^{-1} V_{k-1} \tilde{\Sigma}_{k-1}^{-1})\Phi_{k-1} - \Phi'_k \tilde{\Sigma}_k^{-1}\Phi_k,\ L_0 = H'H \\
\rho_k^{NG} &= \rho_{k-1}^{NG}\Phi'_k \tilde{\Sigma}_k^{-1}\beta_k^{NG} - \Phi'_{k-1}(\tilde{\Sigma}_{k-1}^{-1} - \tilde{\Sigma}_{k-1}^{-1} V_{k-1}\tilde{\Sigma}_{k-1}^{-1})\beta_{k-1}^{NG} - \mu\Phi'_{k-1}\tilde{\Sigma}_{k-1}^{-1} V_{k-1}\hat{x}_{k-1}^{NG} \\
\rho_0^{NG} &= H' y_0
\end{aligned} \tag{4.8}$$

where Z_k is a deterministic constant and $S_k(\hat{x}_{k-1}, \beta_{k-1}, y_k)$ is a function involving exponential of quadratic expressions of $\hat{x}_{k-1}, \beta_{k-1}, y_k$.

Proof First of all, one can obtain an unnormalized Gaussian expression like (4.5) for $q_1(x,\xi)$ using the expression for $q_0(x,\xi)$ given in (4.1). $q_0(x,\xi)$ also gives us the expressions for $\tilde{\Sigma}_0, \beta_0^{NG}, \Phi_0, \gamma_0, L_0, \rho_0^{NG}$. One can then apply the method of induction to obtain the expression for $q_k(x,\xi)$ for any k using Lemma 4.1.

In view of the fact we are considering a linear Gaussian system with exponential of quadratic cost, the mean of the information state (as a function of ξ) naturally assumes an affine structure like that given in (4.6) (this approach is similar to that in [23]). $\beta_k^{NG}, \rho_k^{NG}, \hat{x}_k^{NG}$ bear the superscript NG to denote that we are dealing with non-Gaussian initial conditions.

The recursive expressions for $\beta_k^{NG}, \tilde{\Sigma}_k, \gamma_k, L_k$ and ρ_k^{NG} are obtained equating two sides of (4.3) and expressing the right hand side of (4.3) in the form of the left hand side. □

Remark 4.2 Note that the above theorem expresses the information state in terms of finite number of parameters $\beta_k^{NG}, \tilde{\Sigma}_k, \gamma_k, L_k$ and ρ_k^{NG}. Also, $\Phi_k, \tilde{\Sigma}_k$ and L_k can be calculated off-line.

Also note that $\tilde{\Sigma}_0 = 0$ merely implies that the initial condition is known.

One can now apply the above theorem to obtain the expression for the optimal risk-sensitive estimate using (4.4), which we state in the following theorem.

Theorem 4.2 *Consider the linear time-invariant system given by (2.1). Consider also the cost objective given by (2.2) , (2.3). Suppose $x_0 \sim \Pi_0(x_0)$. Then the optimal risk-sensitive estimate denoted by $\hat{x}_k^{NG}$ is given by*

$$\hat{x}_k^{NG} = \beta_k^{NG} + \Phi_k D_k(N_k^{NG}) \tag{4.9}$$

where $D_k(N_k^{NG})$ is given by

$$D_k(N_k^{NG}) = \frac{\int_{\mathbf{R}^n} \xi \exp[-\frac{1}{2}\xi' M_k \xi + \xi' N_k^{NG}]\Pi_0(\xi) d\xi}{\int_{\mathbf{R}^n} \exp[-\frac{1}{2}\xi' M_k \xi + \xi' N_k^{NG}]\Pi_0(\xi) d\xi} \tag{4.10}$$

and M_k, N_k^{NG} are given by

$$\begin{aligned}
M_k &= L_k + \Phi'_k \tilde{\Sigma}_k^{-1}\Phi_k - \Phi'_k \tilde{\Sigma}_k^{-1}(\tilde{\Sigma}_k^{-1} - \mu I)^{-1}\tilde{\Sigma}_k^{-1}\Phi_k,\ M_0 = L_0 - \mu I \\
N_k^{NG} &= \rho_k^{NG} - \Phi'_k \tilde{\Sigma}_k^{-1}\beta_k^{NG} + \Phi'_k \tilde{\Sigma}_k^{-1}(\tilde{\Sigma}_k^{-1} - \mu I)^{-1}\tilde{\Sigma}_k^{-1}\beta_k^{NG} - \mu\Phi'_k \tilde{\Sigma}_k^{-1}(\tilde{\Sigma}_k^{-1} - \mu I)^{-1}\hat{x}_k^{NG} \\
N_0^{NG} &= \rho_0^{NG} - \mu\hat{x}_0^{NG}
\end{aligned} \tag{4.11}$$

Proof The proof follows easily by using (4.4) along with the expression for $q_k(x,\xi)$ given by (4.5), (4.6). Differentiating with respect to $\hat{x}_k^{NG}$ and equating the derivative equal to zero, some algebraic manipulations result in (4.10), (4.11). The fact that the cost function is convex and approaches ∞ as $|\hat{x}_k^{NG}| \to \infty$, implies that the solution is a minimum and the desired solution. It also guarantees the existence of $N_k^{NG}, \ \forall k$ from above. $\square$

Remark 4.3 Note that the difficulty in obtaining a closed form expression for $\hat{x}_k^{NG}$ is that it is given by an implicit equation. This makes the analysis for asymptotic optimality of such estimates difficult and to simplify the analysis, we make certain assumptions in the next subsection. Although these assumptions are sufficient to guarantee the asymptotic optimality of risk-sensitive filters with respect to non-Gaussian initial conditions, it is essentially hard to verify the some of the assumptions in practice. However, for $\mu = 0$, one can solve the implicit equations explicitly to obtain solutions for risk-neutral estimation and similar results as in [10] can be obtained.

Note above that one can express $D_k(N_k^{NG})$ as the conditional mean of x_0 under a different probability measure $\hat{P}$ such that $\frac{d\hat{P}}{dP} = \bar{\Lambda}_k$ where $\bar{\Lambda}_k$ is a $\{\mathcal{Y}_k \vee \sigma\{x_0\}\}$-adapted process given by

$$\bar{\Lambda}_k = \frac{\exp[-\frac{1}{2}x_0' M_k x_0 + x_0' N_k^{NG}]}{\int_{\mathbf{R}^n} \exp[-\frac{1}{2}\xi' M_k \xi + \xi' N_k^{NG}]\Pi_0(\xi)d\xi} \tag{4.12}$$

Hence, $D_k(N_k^{NG}) = E_{x_0}[x_0\bar{\Lambda}_k|\mathcal{Y}_k]$ and also, $E_{x_0}[\bar{\Lambda}_k] = 1$.

4.1 Asymptotic optimality of risk-sensitive filters for non-Gaussian initial conditions

In this section, we present the results on the mean square asymptotic convergence of the optimal risk-sensitive estimate to a suboptimal risk-sensitive estimate with a Gaussian initial condition assumption with zero mean and arbitrary covariance matrix $Q \in \mathcal{D}$ (defined as β_k^G in the previous Section).

Before presenting the main theorem on the convergence result, we make the following assumptions:

Assumption 4.1 *μ is chosen small enough and $\Pi_0(.)$ has such regularity properties that $\int_{\mathbf{R}^n} \exp[-\frac{1}{2}\xi' M_k \xi + \xi' N_k^{NG}]\Pi_0(\xi)d\xi$ is well-defined for all k.*

Denote $F - \tilde{\Sigma}_k H'HF + \mu\tilde{\Sigma}_k\Sigma_w^{-1}FV_{k-1} \triangleq A_k$. Then the existence of $\lim_{k\to\infty} A_k = A_\infty$ follows from the fact that $\lim_{k\to\infty} \tilde{\Sigma}_k = \tilde{\Sigma}_\infty$ exists.

Assumption 4.2 $\rho(A_\infty) < 1$.

Remark 4.4 Obviously, the above assumption guarantees that the following linear time-varying unforced linear system (see (4.7)):

$$\Phi_k = [F - \tilde{\Sigma}_k H'HF + \mu\tilde{\Sigma}_k\Sigma_w^{-1}FV_{k-1}]\Phi_{k-1}, \ \Phi_0 = I \tag{4.13}$$

is exponentially stable, i.e., there exists a $|\ln\rho(A_\infty)| > \sigma_1 > 0$, $M_{\sigma_1} > 0$ such that

$$\|\Phi_k\| \le M_{\sigma_1}\exp(-\sigma_1 k) \tag{4.14}$$

Assumption 4.3 *There exists a* $0 < M_x < \infty$ *such that* $E[|x_0|^4] < M_x$.

Assumption 4.4 *There exists a* $M_d > 0$ *and* $0 < \sigma_d < \sigma_1$ *for some* $0 < \sigma_1 < |\ln\rho(A_\infty)|$ *such that* $\overline{\Lambda}_k$ *is a* $\{\mathcal{Y}_k \bigvee \sigma\{x_0\}\}$*- adapted process where* $\sup_k E[\overline{\Lambda}_k^{\,4}] \le M_d \exp(4\sigma_d t)$.

Remark 4.5 Note that Assumption 4.3 and Assumption 4.4 together imply that $E|D_k(N_k^{NG})|^2 \le M_z\exp(2\sigma_d t),\ \forall k \in \mathbb{N}$ where $M_z > 0$ is a constant. To see this, note that $|D_k(N_k^{NG})|^2 \le E_{x_0}[|x_0|^2\overline{\Lambda}_k^2 \mid \mathcal{Y}_k]$ from Jensen's inequality. Hence $E|D_k(N_k^{NG})|^2 \le E[|x_0|^2\overline{\Lambda}_k^2] \le \sqrt{E[|x_0|^4]}\sqrt{E[\overline{\Lambda}_k^4]}$ where the last step follows from Schwartz's inequality. Now, using Assumption 4.3 and Assumption 4.4, it follows that $E|D_k(N_k^{NG})|^2 \le M_z\exp(2\sigma_d t),\ \forall k \in \mathbb{N}$. One can possibly look for a sufficient condition by imposing regularity properties on $\Pi_0(.)$ and boundedness properties on the process N_k^{NG} such that Assumption 4.4 is satisfied. But due to the complicated nature of the process N_k^{NG} we postpone such investigation for the time being. However, it is clearly seen that Assumption 4.4 is not that restrictive since it allows an exponential growth (slow enough).

With the above assumptions and Assumption 3.2 holding, one can summarize the main result of this section in the following theorem:

Theorem 4.3 *Consider the signal model (2.1) where* $x_0 \sim \Pi_0(x_0)$, Π_0 *being non-Gaussian. Consider also the risk-sensitive estimation problem given by (2.2), (2.3). Suppose Assumptions 3.2, 4.3, 4.4, 4.2 hold. Then the optimal risk-sensitive estimate given by (4.9), (4.10), (4.11) asymptotically approaches a suboptimal risk-sensitive estimate given by (3.21), (3.22) in the mean square sense, i.e.,*

$$\lim_{k\to\infty} E|\hat{x}_k^{NG} - \tilde{\beta}_k^G|^2 \to 0 \tag{4.15}$$

Proof Note that one can write

$$\hat{x}_k^{NG} - \tilde{\beta}_k^G = (\hat{x}_k^{NG} - \beta_k^{NG}) + (\beta_k^{NG} - \tilde{\beta}_k^G)$$

which implies

$$|\hat{x}_k^{NG} - \tilde{\beta}_k^G|^2 \le [|\hat{x}_k^{NG} - \beta_k^{NG}| + |\beta_k^{NG} - \tilde{\beta}_k^G|]^2$$

Now, one can apply Minkowski's inequality to obtain

$$E|\hat{x}_k^{NG} - \tilde{\beta}_k^G|^2 \le \left[\sqrt{E|\hat{x}_k^{NG} - \beta_k^{NG}|^2} + \sqrt{E|\beta_k^{NG} - \tilde{\beta}_k^G|^2}\right]^2 \tag{4.16}$$

Also, from Remark 4.5 and (4.14), we have

$$E|\hat{x}_k^{NG} - \beta_k^{NG}|^2 \le M_{\sigma_1}M_z\exp(-2\sigma_\gamma k) \tag{4.17}$$

where $0 < \sigma_\gamma = \sigma_1 - \sigma_d$.

Now, consider the process $\tilde{e}_k \triangleq \beta_k^{NG} - \tilde{\beta}_k^G$. Using (4.6), (3.21), one can write

$$\tilde{e}_k = (F - \tilde{\Sigma}_k H'HF)\tilde{e}_{k-1} - \mu\tilde{\Sigma}_k\Sigma_w^{-1}FV_{k-1}\Phi_{k-1}D_{k-1}(N_{k-1}^{NG}),\ \tilde{e}_0 = 0 \tag{4.18}$$

Denoting $\mu\tilde{\Sigma}_k\Sigma_w^{-1}FV_{k-1}$ as R_{k-1}, one can then obtain

$$E|\tilde{e}_k|^2 = E|\sum_{j=0}^{k-1}\tilde{\Psi}_k\tilde{\Psi}_{j+1}^{-1}R_j\Phi_jD_j|^2 \tag{4.19}$$

where $\tilde{\Psi}_k$ is the transition matrix associated with $F - \tilde{\Sigma}_k H'HF$. Using the assumption that $\tilde{\Sigma}_k$ is stable for all $k > 0$, one can have $\sup_k \|R_k\| < \infty$ which implies

$$\begin{aligned}
E|\tilde{e}_k|^2 &\leq K_aE[\sum_{j=0}^{k-1}\|\tilde{\Psi}_k\tilde{\Psi}_{j+1}^{-1}\|\|\Phi_j\||D_j|]^2 \\
&\leq M_{\sigma,\sigma_1}E[\sum_{j=0}^{k-1}\exp[-\sigma(k-j-1)]\exp(-\sigma_1 j)|D_j|]^2 \\
&\leq M_{\sigma,\sigma_1}^a kE[\sum_{j=0}^{k-1}\exp[-2\sigma(k-j)]\exp(-2\sigma_1 j)|D_j|^2] \\
&\leq M_{\sigma,\sigma_1}^a M_z k\exp(-2\sigma k)\left[\sum_{j=0}^{k-1}\exp[2(\sigma-\sigma_\gamma)j]\right] \\
&\leq M_{\sigma,\sigma_1,\sigma_\gamma}M_z k\exp(-2\sigma_\gamma k)[1-\exp[-2(\sigma-\sigma_\gamma)k]] \\
&\leq M_{\sigma,\sigma_1,\sigma_\gamma}M_z k\exp(-2\sigma_\gamma k)
\end{aligned} \tag{4.20}$$

where we have assumed $\sigma > \sigma_\gamma$. If $\sigma < \sigma_\gamma$, we have the following expression for the above bound

$$E|\tilde{e}_k|^2 \leq M_{\sigma,\sigma_1,\sigma_\gamma}^b M_z k\exp(-2\sigma k) \tag{4.21}$$

In the above derivation, the exponential stability of $\tilde{\Psi}_k\tilde{\Psi}_{j+1}^{-1}$ has been used following (3.10). Also, $K_a, M_{\sigma,\sigma_1}, M_{\sigma,\sigma_1}^a, M_{\sigma,\sigma_1,\sigma_\gamma}, M_{\sigma,\sigma_1,\sigma_\gamma}^b$ are constants independent of k. We have also used the so-called C_r-inequality [25] which states

$$E(\sum_{j=1}^{n}|u_j|)^r \leq n^{r-1}E\sum_{j=1}^{n}|u_j|^r,\ r \geq 1 \tag{4.22}$$

It is clear from the above that as $k \to \infty$, we have $E|\tilde{e}_k|^2 \to 0$ and $E|\hat{x}_k^{NG} - \beta_k^{NG}|^2 \to 0$. Combining these two results, we have (4.15). □

Remark 4.6 One can prove a corollary similar to Corollary 3.1 stating $E|\hat{x}_k^{NG} - \beta_k^G|^2 \to 0$ as $k \to \infty$.

5 Conclusions

In this paper, we investigated the problem of asymptotic forgetting of initial conditions by risk-sensitive filters for linear time-invariant systems. For Gaussian initial conditions, we show that under an asymptotic stability condition satisfied by a state transition matrix associated with the H_∞-like Riccati difference equation, with appropriate stabilizability and detectability condition holding for the linear system under consideration, the optimal risk-sensitive estimate initialized with the true initial covariance matrix approaches a suboptimal risk-sensitive estimate initialized with an incorrect covariance matrix in the mean square sense. For non-Gaussian initial conditions, the analysis is more complex. However, under a certain boundedness condition satisfied by the fourth order absolute moment of the initial state distribution and a slow growth condition satisfied by a certain Radon-Nikodym derivative, we have a similar mean square convergence result.

References

[1] S. Dey and J. Moore, "Risk-sensitive filtering and smoothing for hidden markov models," *Systems and Control Letters*, vol. 25, no. 5, pp. 361–366, 1995.

[2] R. K. Boel, M. R. James, and I. R. Petersen, "Robust filtering," in *Proceedings of the Conference on Decision and Control*, (San Diego), December 1997.

[3] J. Speyer, C. Fan, and R. Banavar, "Optimal stochastic estimation with exponential cost criteria," in *31st Conference on Decision and Control*, vol. 2, pp. 2293–2298, December 1992.

[4] S. Dey and J. Moore, "Risk-sensitive filtering and smoothing via reference probability methods," *IEEE Transaction on Automatic Control*, vol. 42, no. 11, pp. 1587–1591, 1997.

[5] M. James, J. Baras, and R. Elliott, "Risk-sensitive control and dynamic games for partially observed discrete-time nonlinear systems," *IEEE Transaction on Automatic Control*, vol. 39, no. 4, pp. 780–792, 1994.

[6] P. Whittle, "Risk-sensitive linear/quadratic/Gaussian control," *Advances in Applied Probability*, vol. 13, pp. 764–777, 1981.

[7] P. Whittle, "A risk-sensitive maximum principle: The case of imperfect state observations," *IEEE Transactions on Automatic Control*, vol. 36, no. 7, pp. 793–801, 1991.

[8] J. B. Moore, R. J. Elliott, and S. Dey, "Risk-sensitive generalizations of minimum variance estimation and control," in *IFAC Symposium on Nonlinear Control Systems Design*, (Tahoe City, CA), pp. 465–470, June 1995. also see summary in Journal of Mathematical Systems, Estimation and Control, volume 7, no: 1, pages 123-126, 1997.

[9] C. D. Charalambous, "The role of information state and adjoint in relating nonlinear output feedback risk-sensitive control and dynamic games," *IEEE Transactions on Automatic Control*, vol. 42, no. 8, pp. 1163–1170, 1997.

[10] A. M. Makowski, "Filtering formulae for partially observed linear systems with non-gaussian initial conditions," *Stochastics*, vol. 16, pp. 1–24, 1986.

[11] D. Ocone and E. Pardoux, "Asymptotic stability of the optimal filter with respect to its initial conditions," *SIAM Journal Control and Optimization*, vol. 34, no. 1, pp. 226–243, 1996.

[12] R. B. Sowers and A. M. Makowski, "Discrete-time filtering for linear systems with non-gaussian initial conditions: asymptotic behaviour of the difference between the MMSE and LMSE estimates," *IEEE Transaction on Automatic Control*, vol. 37, pp. 114–120, January 1992.

[13] A. Budhiraja and D. Ocone, "Exponential stability of discrete-time filters for bounded observation noise," *System and Control Letters*, vol. 30, pp. 185–194, 1997.

[14] V. Benes, "Exact finite-dimensional filters for certain diffusions with nonlinear drift," *Stochastics*, vol. 5, pp. 65–92, 1981.

[15] D. L. Ocone, "Asymptotic stability of Benes filters." preprint.

[16] B. Deylon and O. Zeitouni, "Lyapunov exponents for filtering problems," in *Applied Stochastic Analysis* (M. Davis and R. Elliott, eds.), (New York), pp. 511–535, Gordon and Breach, 1991.

[17] R. Atar and O. Zeitouni, "Lyapunov exponents for finite-state nonlinear filtering." to appear.

[18] R. Atar and O. Zeitouni, "Exponential stability for nonlinear filtering." preprint, 1996.

[19] C. D. Charalambous, S. Dey, and R. J. Elliott, "New finite-dimensional risk-sensitive filters: Small-noise limits," *IEEE Transactions on Automatic Control, (to appear)*, 1997.

[20] S. Dey, R. J. Elliott, and J. B. Moore, "Finite-dimensional risk-sensitive estimation for continuous-time nonlinear systems," in *Proc. European Control Conference*, (Brussels), July 1997.

[21] S. Dey and C. D. Charalambous, "On asymptotic stability of continuous-time risk-sensitive filters with respect to initial conditions." submitted, 1997.

[22] R. J. Elliott, L. Aggoun, and J. B. Moore, *Hidden Markov Models: Estimation and Control*. Springer-Verlag, 1994.

[23] A. Bensoussan, *Stochastic Control of Partially Observable Systems*. Cambridge, UK: Cambridge University Press, 1992.

[24] R. R. Bitmead, M. Gevers, and V. Wertz, *Adaptive Optimal Control: The Thinking Man's GPC*. Brunswick, Victoria: Prentice Hall of Australia, 1990.

[25] H. F. Chen and L. Guo, *Identification and Stochastic Adaptive Control*. Boston: Birkhauser, 1991.