

TECHNICAL RESEARCH REPORT

Computation for Nonlinear Balancing

by Andrew J. Newman, P. S. Krishnaprasad

T.R. 98-40



ISR develops, applies and teaches advanced methodologies of design and analysis to solve complex, hierarchical, heterogeneous and dynamic problems of engineering technology and systems for industry and government.

ISR is a permanent institute of the University of Maryland, within the Glenn L. Martin Institute of Technology/A. James Clark School of Engineering. It is a National Science Foundation Engineering Research Center.

Web site <http://www.isr.umd.edu>

Computation for Nonlinear Balancing

Andrew J. Newman ¹

Institute for Systems Research and
Electrical Engineering Department
University of Maryland
College Park, MD 20742
newman@isr.umd.edu

P. S. Krishnaprasad

Institute for Systems Research and
Electrical Engineering Department
University of Maryland
College Park, MD 20742
krishna@isr.umd.edu

Abstract

We illustrate a computational approach to practical nonlinear balancing via the forced damped pendulum example.

1 Introduction

The theory of balancing, introduced by Moore [1], was extended by Scherpen [2] to a class of stable finite-dimensional systems in the nonlinear setting

$$\dot{x} = f(x) + g(x)u \quad (1)$$

$$y = h(x) \quad (2)$$

where $x = (x_1, \dots, x_n)$ are local coordinates for a smooth state manifold, f , g , and h are of class C^∞ , $f(0) = 0$, and $h(0) = 0$.

The existence of efficient computational methods has been crucial in making balancing an effective and important tool for model reduction of linear systems. The balancing change of coordinates can be computed using established matrix equation and decomposition algorithms. In contrast, Scherpen's procedure for nonlinear balancing presents computational difficulties. In this paper we address these difficulties and illustrate computational methods by means of an example.

2 Balancing for the Forced Damped Pendulum

We model a damped pendulum with torque input and measured angular position by

$$\dot{x}_1 = x_2; \quad \dot{x}_2 = -a \sin(x_1) - b x_2 + c u \quad (3)$$

$$y = x_1 \quad (4)$$

where we choose $a = 3$, $b = 1$, and $c = 0.1$.

¹This research was supported by grants from the Northrop Grumman Foundation, the National Science Foundation's Engineering Research Centers Program: NSFD CDR 8803012 and NSF Grant EEC-9527576, and the Army Research Office under the ODDR&E MURI97 Program Grant No. DAAG55-97-1-0114.

The main objects involved in Scherpen's theory are the controllability and observability energy functions, $L_c : \mathbb{R}^n \rightarrow \mathbb{R}$ and $L_o : \mathbb{R}^n \rightarrow \mathbb{R}$, respectively, defined by

$$L_c(x_0) = \min_{\substack{u \in \mathcal{L}_2(-\infty, 0) \\ x(-\infty) = 0 \\ x(0) = x_0}} \frac{1}{2} \int_{-\infty}^0 \|u(t)\|^2 dt \quad (5)$$

and

$$L_o(x_0) = \frac{1}{2} \int_0^\infty \|y(t)\|^2 dt, \quad (6)$$

$$x(0) = x_0, \quad u(t) \equiv 0, \quad 0 \leq t < \infty.$$

Computation of L_c via (5) requires solution of an optimal control problem at each point on a state space grid. One alternative is to numerically solve the Hamilton–Jacobi type partial differential equation (PDE) given by [2] eqn. (12). Both of these methods are computationally intensive (if a numerical solution can be generated at all) and require apriori knowledge of f and g . We offer an empirical approach which could be used in the absence of accurate models.

In the linear case, $L_c(x) = \frac{1}{2} x^T W_c^{-1} x$, is quadratic with W_c denoting the controllability Gramian matrix. Injection of Gaussian white noise at the input terminals produces a Gaussian asymptotic probability density function (PDF) for the state

$$p_\infty(x) = [(2\pi)^n \det(R_\infty)]^{-1/2} \exp(-L_c(x)) \quad (7)$$

where $R_\infty = W_c$ denotes the asymptotic state covariance. Therefore, in the nonlinear setting, we estimate L_c via

$$\widehat{L}_c(x) = -\log(p_\infty(x)) + \text{constant}. \quad (8)$$

This suggests an empirical Monte–Carlo approach in which the system response to Gaussian white noise is recorded for a large number of sample paths, and the data is histogrammed on an appropriate grid to compute a PDF and estimate for L_c . The empirical approach is also used for determination of L_o via numerical integration of (6) where the input signal is null and

the initial condition x_0 takes each value in the state space grid. Figures 1–3 show results for the pendulum system (3–4).

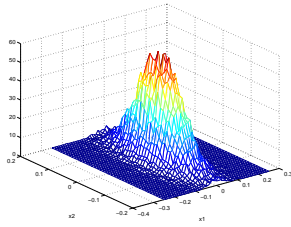


Figure 1: Probability density function for asymptotic state of pendulum system.

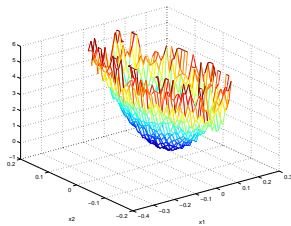


Figure 2: Estimated controllability function for pendulum system.

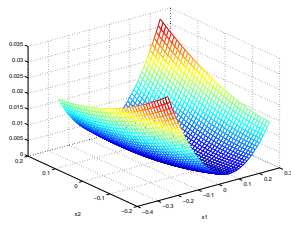


Figure 3: Observability function for pendulum system.

The existence of the balancing transformation is proved using the Morse–Palais lemma (see [3, 4]) which asserts that there exists a change of coordinates under which a function is locally quadratic in a neighborhood \mathcal{U} of a nondegenerate critical point at 0, i.e., $f(x) = \langle A\phi(x), \phi(x) \rangle$, with ϕ a diffeomorphism defined on \mathcal{U} . Computation of ϕ requires the decomposition $f(x) = \langle H(x)x, x \rangle$ (see [3]). It is shown in [4] that $\phi(x) = C(x)x$ where $C(x)^2 = B(x) = H(0)^{-1}H(x)$. Such a C exists on a neighborhood of 0 since $B(0)$ is the identity and a square root function is defined in a neighborhood of the identity operator \mathbb{I} by a convergent power series. The algorithm we use is

$$S_0 = 0, S_{k+1} = \frac{1}{2}[(\mathbb{I} - B) + S_k^2], k = 0, 1, \dots \quad (9)$$

with $C = \mathbb{I} - S_\infty$. It is valid for x such that $\|\mathbb{I} - B(x)\| < 1$ thus providing an estimate of the neighborhood \mathcal{U} .

The relative input–output influence of each state in the balanced realization is measured by the singular

value functions (SVF) (see [2]), which are the point–dependent analogs of the Hankel singular values for a linear system. Figure 4 shows the resulting SVFs for the pendulum system (3–4). We see that in the balanced realization, one state has roughly twice the input–output influence as the other. The values of the singular value functions at the origin are 0.0376 and 0.0193, respectively. As expected, these are in close agreement with the Hankel singular values of the linearized system, which are 0.0384 and 0.0217, respectively.

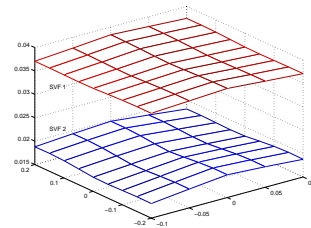


Figure 4: Singular value functions for pendulum system in a small neighborhood of the origin (denoted SVF 1 and SVF 2).

3 Conclusion

We have illustrated a computational approach for practical implementation of Scherpen’s nonlinear balancing procedure via application to the forced damped pendulum system. A Monte–Carlo approach provides an estimate of the controllability energy function. The observability energy function is computed via numerical integration and is also suitable as an empirical method. Numerical implementation of the Morse–Palais lemma is achieved using a successive approximation algorithm for an operator square root which appears in Palais’ proof. The corresponding convergence criterion gives an estimate of the neighborhood in which the given function is locally quadratic. The balanced realization for the pendulum system results in one state having roughly double the input–output influence as the other state.

References

- [1] B. C. Moore, “Principal component analysis in linear systems: Controllability, observability, and model reduction,” *IEEE Transactions on Automatic Control*, vol. 26, pp. 17–32, Feb. 1981.
- [2] J. Scherpen, “Balancing for nonlinear systems,” *Sys. Cont. Lett.*, vol. 21, pp. 143–153, 1993.
- [3] J. Milnor, *Morse Theory*. Princeton University Press, 1963.
- [4] R. S. Palais, “Morse theory on Hilbert manifolds,” *Topology*, vol. 2, pp. 299–340, 1963.