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Absolute Stability Theory, μ Theory, and State-Space Verification of Frequency-Domain Conditions: Connections and Implications for Computation

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Absolute Stability Theory, μ Theory, and State-Space Verification of Frequency-Domain Conditions: Connections and Implications for Computation*

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Abstract

The main contribution of the paper is to show the *equivalence* between the following two approaches for obtaining sufficient conditions for the robust stability of systems with structured uncertainties: (i) apply the classical absolute stability theory with multipliers; (ii) use modern μ theory, specifically, the μ upper bound obtained by Fan, Tits and Doyle [IEEE TAC, Vol. 36, 25-38]. In particular, the relationship between the stability multipliers used in absolute stability theory and the scaling matrices used in the cited reference is explicitly characterized. The development hinges on the derivation of certain properties of a parameterized family of complex LMIs (linear matrix inequalities), a result of independent interest. The derivation also suggests a general computational framework for checking the feasibility of a broad class of frequency-dependent conditions, and in particular, yields a sequence of computable "mixed-μ-norm upper bounds", defined with guaranteed convergence from above to the supremum over frequency of the aforementioned μ upper bound.

1 Introduction

A popular paradigm currently in use for robust control has a nominal finite-dimensional, linear, time-invariant system with the uncertainty Δ in the feedback loop (see Figure 1). Often additional information about the uncertainty is either known or assumed: diagonal or block-diagonal; sector-bounded memoryless, linear time-invariant or parametric, etc. In such cases, the uncertainty is called "structured".

A fundamental question associated with this model is that of robust stability, i.e., "Is the model stable irrespective of the uncertainty Δ, that is, with zero input, do all solutions of the system equations go to zero, irrespective of Δ?" The origins of this question can be found in Russian

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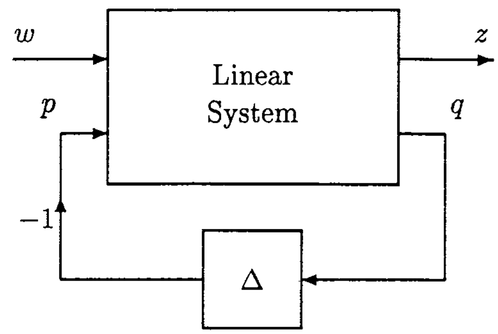

Figure 1: Standard robustness analysis framework.

literature, where stability of the system in Figure 1 was studied for the special case when p and q are scalars, and Δ is required to satisfy additional assumptions. This was known as the *absolute stability problem* [1]. This problem has received considerable attention over the years, and a number of sufficient conditions for stability have been proposed; perhaps the most celebrated of these have been the circle and Popov criteria. These criteria have since then been generalized to multi-input multi-output systems as the small-gain theorem (with loop transformations and scalings) and the passivity theorem (with loop transformations and multipliers). An introduction to these methods can be found in the book by Desoer and Vidyasagar [2].

A second approach to the problem of robust stability of control systems with structured uncertainties is the modern μ (or structured singular value) approach, pioneered by Doyle [3, 4] and Safonov [5, 6]. This approach relies on deriving sufficient conditions for the robust stability of the system in Figure 1 through simple linear-algebraic techniques.

Our main objective in this paper is to show explicitly and rigorously the connections between these two approaches[1]. We illustrate the idea behind our treatment with the special case where Δ, in Figure 1, consists of both unmodeled dynamics and uncertain parameters. A sufficient condition for the robust stability of this system is given in [10], based on the application of the classical passivity theorem with multipliers. In the μ approach, where this problem is called the "mixed-μ problem" (see [11]), a sufficient condition for robust stability is presented in the form of a matrix inequality that should be satisfied at all frequencies.

The condition proposed in [11], which we will refer to as the standard mixed μ upper bound condition, can be recast as an LMI condition that must be satisfied at every boundary point of a "stability" region (the open unit disk or the open left half complex plane) in the complex plane. The passivity-multiplier-based robust stability condition in [10] can be reinterpreted as the same LMI condition, with the additional restriction that the feasible solutions are themselves *functions* of a certain form. Therefore, the first step in our treatment is to establish a general "interpolation" style result for a class of parameterized (by frequencies) family of complex LMIs (many frequency-dependent conditions for stability and robustness [11, 12, 13, 14, 15] belong to this class). This

[1]We must point out that other authors have hinted at the connection [7], [8], [9], without proving equivalence however.

result serves as the basis for all later developments. In particular, this result suggests a state-space method for checking feasibility of any LMI in the class that we study. Besides serving a key stepping stone in our approach, we believe that this result is also of independent interest.

Thus, we show that the standard mixed μ upper bound condition in [11] is mathematically *equivalent* to the passivity-multiplier-based condition in [10]. Concurrently, we explicitly characterize the relationship between the (D, G) scalings used in the standard mixed μ approach, and certain stability multipliers used in the passivity-multiplier-based condition in [10]. Finally, our derivation also suggests a general computational framework for checking the feasibility of a broad class of frequency-dependent conditions, and in particular, yields a sequence of computable "mixed-μ-norm upper bounds", defined with guaranteed convergence to the standard mixed-μ-norm upper bound $\|P\|_{\hat{\mu}}$.

We first carry out the analysis in the discrete-time context, then briefly indicate how it extends to the continuous-time context. The organization of the paper is as follows. Section 2 lists the notation that is used throughout the paper. In Section 3, we present our study of a class of LMIs. Section 4 is concerned with the connections between the standard mixed μ upper bound condition and a passivity-multiplier-based robust stability condition. Implications of the results of Sections 3 and 4 on computation are discussed in Section 5. In Section 6, we present the continuous-time case results. Section 7 contains the conclusions. For clarity of presentation, the proofs of the lemmas, the propositions, and the theorems have been relegated to the appendix.

Preliminary versions of some of the results presented here appeared in [16] and [17].

2 Notation

$\mathbf{N}$	:=	the set of nonnegative integers
$\mathbf{R}$	:=	the set of real numbers
$\mathbf{R}_e$	:=	$\mathbf{R} \cup \{\infty\}$
$\mathbf{C}$	:=	the set of complex numbers
$\mathbf{C}_e$	:=	$\mathbf{C} \cup \{\infty\}$
$\mathbf{C}_-$	:=	$\{z \in \mathbf{C} : \mathrm{Re}\{z\} < 0\}$
$\mathbf{D}$	:=	$\{z \in \mathbf{C} : \|z\| < 1\}$
I_k	:=	the $k \times k$ identity matrix
0_k	:=	the $k \times k$ zero matrix

Given a complex matrix/scalar M,

$\overline{M}$	:=	complex conjugate of M
M^T	:=	transpose of M
M^*	:=	adjoint (conjugate transpose) of M
$\overline{\sigma}(M)$	:=	the largest singular value of M
$\mathrm{He}(M)$	:=	$\frac{1}{2}(M + M^*)$ (Hermitian part of M)
$\mathrm{Sh}(M)$	:=	$\frac{1}{2}(M - M^*)$ (skew Hermitian part of M)

Given a subset Ω of $\mathbf{C}$,

$\mathrm{cl}\Omega$	:=	the closure of Ω in $\mathbf{C}$
$\mathrm{cl_e}\Omega$	:=	the closure of Ω in $\mathbf{C}_\mathrm{e}$ (including ∞ if Ω is unbounded)
$\partial\Omega$	:=	the boundary of Ω in $\mathbf{C}$
$\partial_\mathrm{e}\Omega$	:=	the boundary of Ω in $\mathbf{C}_\mathrm{e}$ (including ∞ if Ω is unbounded)
Ω^c	:=	the complement of Ω in $\mathbf{C}$
$\mathbf{RP}$	:=	the set of real-rational, proper functions
$\mathbf{H}_\infty(\Omega)$	:=	the set of functions which are analytic and bounded in $(\mathrm{cl}\Omega)^\mathrm{c}$
$\mathbf{H}_\infty^{n\times n}(\Omega)$	:=	the set of $n\times n$ transfer function matrices whose entries are in $\mathbf{H}_\infty(\Omega)$
$\mathbf{RH}_\infty(\Omega)$	:=	the set of real-rational functions in $\mathbf{H}_\infty(\Omega)$
$\mathbf{RH}_\infty^{n\times n}(\Omega)$	:=	the set of $n\times n$ transfer function matrices whose entries are in $\mathbf{RH}_\infty(\Omega)$

Given a matrix $M \in \mathbf{C}^{n\times n}$ and three nonnegative integers m_r, m_c, and m_C with $m := m_\mathrm{r} + m_\mathrm{c} + m_\mathrm{C} \le n$, a block structure $\mathcal{K}$ of dimensions $(m_\mathrm{r}, m_\mathrm{c}, m_\mathrm{C})$ is an m-tuple of positive integers, $\mathcal{K} := (k_1^\mathrm{r},\ldots,k_{m_\mathrm{r}}^\mathrm{r}, k_1^\mathrm{c},\ldots,k_{m_\mathrm{c}}^\mathrm{c}, k_1^\mathrm{C},\ldots,k_{m_\mathrm{C}}^\mathrm{C})$, such that $\sum_{i=1}^{m_\mathrm{r}} k_i^\mathrm{r} + \sum_{i=1}^{m_\mathrm{c}} k_i^\mathrm{c} + \sum_{i=1}^{m_\mathrm{C}} k_i^\mathrm{C} = n$. Let $n_\mathrm{c} := \sum_{i=1}^{m_\mathrm{c}} k_i^\mathrm{c}$, and $n_\mathrm{C} := \sum_{i=1}^{m_\mathrm{C}} k_i^\mathrm{C}$. Define

$$
\begin{aligned}
\tilde{\mathcal{D}} &:= \left\{ \begin{array}{r} \mathrm{diag}(D_1^\mathrm{r},\ldots,D_{m_\mathrm{r}}^\mathrm{r}, D_1^\mathrm{c},\ldots,D_{m_\mathrm{c}}^\mathrm{c}, \\ d_1^\mathrm{C} I_{k_1^\mathrm{C}},\ldots,d_{m_\mathrm{C}}^\mathrm{C} I_{k_{m_\mathrm{C}}^\mathrm{C}}) \end{array} : \begin{array}{l} D_i^\mathrm{r} = (D_i^\mathrm{r})^* \in \mathbf{C}^{k_i^\mathrm{r}\times k_i^\mathrm{r}}, \\ D_i^\mathrm{c} = (D_i^\mathrm{c})^* \in \mathbf{C}^{k_i^\mathrm{c}\times k_i^\mathrm{c}}, d_i^\mathrm{C} \in \mathbf{R} \end{array} \right\} \\
\mathcal{D} &:= \left\{ D \in \tilde{\mathcal{D}} : D > 0 \right\} \\
\mathcal{G} &:= \left\{ \mathrm{diag}(G_1^\mathrm{r},\ldots,G_{m_\mathrm{r}}^\mathrm{r}, 0_{k_1^\mathrm{c}},\ldots,0_{k_{m_\mathrm{c}}^\mathrm{c}}, 0_{k_1^\mathrm{C}},\ldots,0_{k_{m_\mathrm{C}}^\mathrm{C}}) : G_i^\mathrm{r} = -(G_i^\mathrm{r})^* \in \mathbf{C}^{k_i^\mathrm{r}\times k_i^\mathrm{r}} \right\} \\
\hat{\mu}(M) &:= \inf_{\beta\ge 0, D\in\mathcal{D}, G\in\mathcal{G}} \left\{ \beta : MDM^* + GM^* - MG - \beta^2 D < 0 \right\}
\end{aligned}
$$

Given $X = \mathbf{C}$ or $\mathbf{RP}$,

$$
\begin{aligned}
\mathcal{S}_\mathrm{r}(X) &:= \left\{ \mathrm{diag}(S_1,\ldots,S_{m_\mathrm{r}}) : S_i \in X^{k_i^\mathrm{r}\times k_i^\mathrm{r}}, i = 1,\ldots,m_\mathrm{r} \right\} \\
\mathcal{S}_\mathrm{c}(X) &:= \left\{ \mathrm{diag}(S_1,\ldots,S_{m_\mathrm{c}}) : S_i \in X^{k_i^\mathrm{c}\times k_i^\mathrm{c}}, i = 1,\ldots,m_\mathrm{c} \right\} \\
\mathcal{S}_\mathrm{C}(X) &:= \left\{ \mathrm{diag}(s_1 I_{k_1^\mathrm{C}},\ldots,s_{m_\mathrm{C}} I_{k_{m_\mathrm{C}}^\mathrm{C}}) : s_i \in X, i = 1,\ldots,m_\mathrm{C} \right\} \\
\mathcal{S}_{\mathrm{c,C}}(X) &:= \left\{ \mathrm{diag}(S_\mathrm{c}, S_\mathrm{C}) : S_\mathrm{c} \in \mathcal{S}_\mathrm{c}(X), S_\mathrm{C} \in \mathcal{S}_\mathrm{C}(X) \right\} \\
\mathcal{S}_{\mathrm{r,c,C}}(X) &:= \left\{ \mathrm{diag}(S_\mathrm{r}, S_\mathrm{c}, S_\mathrm{C}) : S_\mathrm{r} \in \mathcal{S}_\mathrm{r}(X), S_\mathrm{c} \in \mathcal{S}_\mathrm{c}(X), S_\mathrm{C} \in \mathcal{S}_\mathrm{C}(X) \right\} \\
\mathcal{S}_{\mathrm{r,0,0}}(X) &:= \left\{ \mathrm{diag}(S_\mathrm{r}, 0_{n_\mathrm{c}}, 0_{n_\mathrm{C}}) : S_\mathrm{r} \in \mathcal{S}_\mathrm{r}(X) \right\}
\end{aligned}
$$

Given a set Ω and a matrix/scalar function P continuous on $\partial_\mathrm{e}\Omega$,

$\|P\|_{\hat{\mu}} := \sup_{z\in\partial_\mathrm{e}\Omega} \hat{\mu}(P(z))$ (it is not a norm)
$P^\sim$ is defined by $P^\sim(z) := (P(1/\bar{z}))^*$ when $\Omega = \mathbf{D}$, and $(P(-\bar{z}))^*$ when $\Omega = \mathbf{C}_-$
$\overline{P}$ is defined by $\overline{P}(z) := \overline{P(z)}$
$\tilde{P} := (I+P)(I-P)^{-1}$
$\tilde{P}_\alpha := (\alpha I + P)(\alpha I - P)^{-1}$ where α is a positive scalar

3 Framework: Parametrized Families of Complex LMIs

A linear matrix inequality (LMI) is a matrix inequality of the form

$$M(\zeta) \triangleq M_0 + \sum_{i=1}^{\ell} \zeta_i M_i + \left(M_0 + \sum_{i=1}^{\ell} \zeta_i M_i \right)^* > 0, \tag{1}$$

where $\zeta \in \mathbf{C}^{\ell}$ is the variable, and $M_i \in \mathbf{C}^{n\times n}, i = 0,\dots,\ell$ are given data. The inequality symbol in (1) means that $M(\zeta)$ is positive-definite, i.e., $u^*M(\zeta)u > 0$ for all nonzero $u \in \mathbf{C}^n$. LMIs (1) are widely encountered in system and control theory and their applications; see, for example [18].

In this section, we consider a parametrized family of LMIs, every element of which is of the form (1). The variable z parametrizing this family takes on values from the boundary of a subset $\Omega \subseteq \mathbf{C}$ (whose properties we will make precise shortly). Using arguments from the theory of complex variables, we will show that if every member of this parametrized family of LMIs is feasible, then, the feasible variables themselves can be interpolated by complex-valued functions with various properties. We will see later that such statements help establish the equivalence between two classes of numerical procedures for checking (approximately) the feasibility of the parametrized family of LMIs: The first procedure relies on "sampling" along the boundary of Ω; this results in a number of independent LMIs of the form (1). The second procedure involves searches for a function (of the parametrizing variable z) of a certain form, whose values sampled along the boundary of Ω serve as feasible points for the parametrized family of LMIs; this results in a single LMI of the form (1).

We will proceed with the development for a general set Ω; we will soon see that both discrete-time case (the open unit disk) and continuous-time case (the open left half complex plane) fit into this general framework.

The description of the parametrized family of LMIs begins with the definition of a class of sets

$$\Xi := \left\{ \Omega \subset \mathbf{C} : \begin{array}{l} \Omega \text{ is non-empty, open, simply connected, symmetric with respect to the} \\ \text{real axis, and such that there exists a continuous bijection } \psi : \partial_{\mathrm{e}}\Omega \to \partial\mathbf{D} \\ \text{satisfying } \psi(\overline{z}) = \overline{\psi(z)} \text{ for all } z \in \partial_{\mathrm{e}}\Omega \end{array} \right\}.$$

Given a set $\Omega \in \Xi$, we define

$$\mathcal{F}_\Omega := \left\{ F : \partial_{\mathrm{e}}\Omega \to \mathbf{C}^{n\times n} : \ F \text{ is continuous, with } F(\overline{z}) = \overline{F(z)} \right\}.$$

Finally, given complex numbers $x_1,\cdots,x_\ell$ and $n\times n$ complex matrices $M_0,\cdots,M_\ell$, we define the Hermitian matrix

$$L\left(x_1,\cdots,x_\ell,M_0,\cdots,M_\ell\right) := M_0 + \sum_{i=1}^{\ell} x_i M_i + \left(M_0 + \sum_{i=1}^{\ell} x_i M_i\right)^* .$$

Lemma 3.1 below, proved in the appendix, presents different mathematical representations that the solutions of a class of parameterized family of complex LMIs of the form

$$L\left(x_1,\cdots,x_\ell,F_0(z),\cdots,F_\ell(z)\right)) > 0, \tag{2}$$

can take.

Lemma 3.1 *Let $\Omega \in \Xi$. Let $F_i \in \mathcal{F}_\Omega$, $i = 0,\dots,\ell$. Let S be the set of functions s mapping $\mathrm{cl_e}\Omega$ to $\mathbf{C}$, which are continuous on $\mathrm{cl_e}\Omega$, analytic in Ω, and satisfy $s(\overline{z}) = \overline{s(z)}$ for all $z \in \mathrm{cl_e}\Omega$, and let S be equipped with the norm $\|s\|_\infty := \sup_{z\in\mathrm{cl_e}\Omega} |s(z)|$. Let $\{\varphi_i\}$ be a countable basis for S, let*

$$\mathcal{H}_{\mathrm{b}} := \left\{ \sum_{i=0}^{N} \left(a_i\varphi_i + b_i\overline{\varphi_i}\right) : N \in \mathbf{N},\ a_i, b_i \in \mathbf{C} \right\}.$$

and let

$$\mathcal{H}_{\rm p} := \left\{ \sum_{i=0}^{N} \left(a_i z^i + b_i \overline{z}^i \right) : N \in \mathbf{N},\ a_i, b_i \in \mathbf{R} \right\}.$$

The following conditions are equivalent:

(a) *for every $z \in \partial_{\rm e}\Omega$, there exist complex numbers x_i, $i = 1, \ldots, \ell$, such that* (2) *holds;*

(b) *there exist continuous functions $x_i : \partial_{\rm e}\Omega \to \mathbf{C}$, $i = 1, \ldots, \ell$, such that, for all $z \in \partial_{\rm e}\Omega$, $x_i(\overline{z}) = \overline{x_i(z)}$, $i = 1, \ldots, \ell$, and*

$$L\left(x_1(z), \cdots, x_\ell(z), F_0(z), \cdots, F_\ell(z)\right) > 0; \tag{3}$$

(c) *there exist $s_i^j \in S$, $i = 1, \ldots, \ell$, $j = 1, 2$, such that* (3) *holds for all $z \in \partial_{\rm e}\Omega$ with $x_i(z) = s_i^1(z) + \overline{s_i^2(z)}$, for all $z \in \partial_{\rm e}\Omega$, $i = 1, \ldots, \ell$;*

(d) *there exist $x_i \in \mathcal{H}_{\rm b}$, $i = 1, \ldots, \ell$, such that* (3) *holds for all $z \in \partial_{\rm e}\Omega$.*

Moreover, if Ω is bounded, then the above are equivalent to

(e) *there exist $x_i \in \mathcal{H}_{\rm p}$, $i = 1, \ldots, \ell$, such that* (3) *holds for all $z \in \partial\Omega$.*

Lemma 3.1 deals with a very general domain Ω. When Ω is specialized to be the open unit disk $\mathbf{D}$, and the open left half complex plane $\mathbf{C}_-$, respectively, under mild assumptions, more LMI properties ensue. In order to present our results in a compact manner, for the time being, we restrict ourselves to the discrete-time case. The extension to the continuous-time case is carried out in Section 6.

In the discrete-time case, using the fact that for all $z \in \partial\mathbf{D}$, $\overline{z} = z^{-1}$ along with certain approximation results on the closed unit disk, we are able to establish the following propositions. Proposition 3.2 below states that the solution to a parameterized family of complex LMIs

$$L\left(x_1, \cdots, x_\ell, F_0(z), \cdots, F_\ell(z)\right) > 0 \quad \forall z \in \partial\mathbf{D}$$

simultaneously exist in various forms: (i) complex number (at each z), (ii) continuous function in z, or (iii) sums of a polynomial in z and a polynomial in z^{-1}. It is the very result that helps us to clarify the relationships between two large classes of frequency domain conditions, those using scalings and those using multipliers. For notational convenience, let

$$\mathcal{H}_{\rm FIR} := \left\{ \sum_{i=0}^{N} \left(a_i z^i + b_i z^{-i} \right) : N \in \mathbf{N}, a_i, b_i \in \mathbf{R} \right\}.$$

Proposition 3.2 *Let $F_i \in \mathcal{F}_{\mathbf{D}}$, $i = 0, \ldots, \ell$. The following conditions are equivalent:*

(a) *for every $z \in \partial\mathbf{D}$, there exist complex numbers x_i, $i = 1, \ldots, \ell$, such that* (2) *holds;*

(b) *there exist continuous functions $x_i : \partial\mathbf{D} \to \mathbf{C}$, $i = 1, \ldots, \ell$, such that, for all $z \in \partial\mathbf{D}$, $x_i(\overline{z}) = \overline{x_i(z)}$, $i = 1, \ldots, \ell$, and* (3) *holds;*

(c) *there exist $x_i \in \mathcal{H}_{\rm FIR}$, $i = 1, \ldots, \ell$, such that* (3) *holds for all $z \in \partial\mathbf{D}$;*

Proof: Since for all $z \in \partial\mathbf{D}$, $\overline{z} = z^{-1}$, the result follows directly from Lemma 3.1. □

4 Small μ Theorem and Absolute Stability

We now apply the results of §3 to explore the connection between two popular sufficient conditions for the robust stability of discrete-time systems with structured uncertainty. The first is the standard mixed μ upper bound condition, given in [11]. This condition is derived using linear-algebraic methods, and is usually stated as an LMI condition that should be satisfied on the unit circle. The derivation of the second condition involves augmenting the system with multipliers (that are introduced to take advantage of the structure and the nature of the uncertainty), and then applying the classical passivity theorem [10]. The resulting stability condition takes the form of the same LMI as with the mixed μ upper bound condition, with the additional restriction that the LMI variables be values assumed by the multipliers on the unit circle. As the multipliers are typically assumed to belong to a finite basis, this restriction is equivalent to the LMI variables from the mixed μ upper bound condition being interpolated by functions of a certain form. Proposition 3.2 then helps establish the *equivalence* between these two conditions.

To proceed with establishing the equivalence, we need to establish a certain factorization property for a class of strictly positive real, real-rational, proper transfer function matrices. This is known as *canonical factorization* in the literature [19]; a number of sufficient conditions are known for canonical factorization [2, §VI.9.4]. We establish here that strict positive realness of a real-rational, proper transfer function is sufficient for its canonical factorization.

Proposition 4.1 *Suppose that M is an $m \times m$ real-rational biproper transfer function matrix with no poles on the unit circle, and the frequency domain condition*

$$M(e^{j\theta}) + (M(e^{j\theta}))^* > 0, \tag{4}$$

holds for all θ. Then there exist $M_1, M_2 \in \mathbf{RH}_\infty^{m\times m}(\mathbf{D})$, such that $M_1^{-1}, M_2^{-1} \in \mathbf{RH}_\infty^{m\times m}(\mathbf{D})$, and $M = M_1 M_2^\sim$.

Remark: In addition to the hypotheses of Proposition 4.1, suppose further that $M(e^{j\theta})$ is Hermitian for all θ. Then there exists $M_1 \in \mathbf{RH}_\infty^{m\times m}(\mathbf{D})$ such that $M_1^{-1} \in \mathbf{RH}_\infty^{m\times m}(\mathbf{D})$, and $M = M_1 M_1^\sim$. This is the *spectral factorization* of M; it is well-known that condition (4) is both necessary and sufficient for such a factorization (see, for example, [19]). Note however that condition (4) is only sufficient for a canonical factorization of M.

We now state Theorem 4.2, which essentially states that the standard mixed μ upper bound condition is mathematically equivalent to a passivity-multiplier-based stability condition: strict positive realness of $W_2^{-1}\tilde{P}W_1^{-1}$ with W_1 and W_2 belonging to a certain subset of $\mathbf{RH}_\infty^{n\times n}(\mathbf{D})$. In addition, the relationship between the (D, G) scalings used in the mixed μ analysis, and certain stability multipliers W_1 and W_2 used in the absolute stability theory, is explicitly characterized.

Theorem 4.2 *Let $P \in \mathbf{RH}_\infty^{n\times n}(\mathbf{D})$. The following conditions are equivalent:*

(a)

$$\|P\|_{\hat{\mu}} < 1,$$

(b) $(I-P)^{-1}$ *is in* $\mathbf{RH}_\infty^{n\times n}(\mathbf{D})$ *and there exist a positive integer* N*, and* $n\times n$*, real matrices* $Q_i \in \mathcal{S}_{\mathrm{r,c,C}}(\mathbf{C})$*, and* $U_i \in \mathcal{S}_{\mathrm{r,0,0}}(\mathbf{C})$*,* $i=1,\dots,N$*, such that with*

$$D(z) = \sum_{i=0}^{N}(Q_i z^i + Q_i^T z^{-i}), \tag{5}$$

$$G(z) = \sum_{i=0}^{N}(U_i z^i - U_i^T z^{-i}), \tag{6}$$

D and G have no poles on the unit circle and the inequalities

$$D(\mathrm{e}^{\mathrm{j}\theta}) > 0, \tag{7}$$

$$P(\mathrm{e}^{\mathrm{j}\theta})D(\mathrm{e}^{\mathrm{j}\theta})(P(\mathrm{e}^{\mathrm{j}\theta}))^* + G(\mathrm{e}^{\mathrm{j}\theta})(P(\mathrm{e}^{\mathrm{j}\theta}))^* - P(\mathrm{e}^{\mathrm{j}\theta})G(\mathrm{e}^{\mathrm{j}\theta}) - D(\mathrm{e}^{\mathrm{j}\theta}) < 0, \tag{8}$$

hold for all $\theta \in \mathbf{R}$*,*

(c) $(I-P)^{-1}$ *is in* $\mathbf{RH}_\infty^{n\times n}(\mathbf{D})$ *and there exist a positive integer* N*, and* $n\times n$*, real matrices* $R_i, S_i \in \mathcal{S}_{\mathrm{r,c,C}}(\mathbf{C})$*,* $i=1,\dots,N$*, such that with*

$$T(z) = \sum_{i=0}^{N}(R_i z^i + S_i z^{-i}), \tag{9}$$

T has no poles on the unit circle, the lower right $(n_{\mathrm{c}}+n_{\mathrm{C}})\times(n_{\mathrm{c}}+n_{\mathrm{C}})$ *submatrix* $T_{\mathrm{cC}}(\mathrm{e}^{\mathrm{j}\theta})$ *of* $T(\mathrm{e}^{\mathrm{j}\theta})$ *is Hermitian for all* $\theta\in\mathbf{R}$*, and the inequalities*

$$\mathrm{He}(T(\mathrm{e}^{\mathrm{j}\theta})) > 0,$$
$$\mathrm{He}(\tilde{P}(\mathrm{e}^{\mathrm{j}\theta})T(\mathrm{e}^{\mathrm{j}\theta})) > 0,$$

hold for all $\theta \in \mathbf{R}$*,*

(d) $(I-P)^{-1}$ *is in* $\mathbf{RH}_\infty^{n\times n}(\mathbf{D})$ *and there exist transfer function matrices* $W_1 = \mathrm{diag}(W_1^{\mathrm{r}}, W_1^{\mathrm{c}}, W_1^{\mathrm{C}})$ *and* $W_2 = \mathrm{diag}(W_2^{\mathrm{r}}, W_2^{\mathrm{c}}, W_2^{\mathrm{C}})$*, where* W_1^{r} *and* W_2^{r}*,* W_1^{c} *and* W_2^{c}*, and* W_1^{C} *and* W_2^{C}*, are in* $\mathcal{S}_{\mathrm{r}}(\mathbf{RP})$*,* $\mathcal{S}_{\mathrm{c}}(\mathbf{RP})$*, and* $\mathcal{S}_{\mathrm{C}}(\mathbf{RP})$*, respectively, satisfying*

- *(i)* W_1*,* W_2*,* W_1^{-1}*, and* W_2^{-1} *are in* $\mathbf{RH}_\infty^{n\times n}(\mathbf{D})$*,* $W_1^{\mathrm{c}}W_2^{\mathrm{c}} = I$*, and* $W_1^{\mathrm{C}}W_2^{\mathrm{C}} = I$*,*
- *(ii)* W_1W_2 *is strictly positive real,*
- *(iii)* $W_2^{-1}\tilde{P}W_1^{-1}$ *is strictly positive real.*

Moreover, suppose that $(I-P)^{-1}$ *is in* $\mathbf{RH}_\infty^{n\times n}(\mathbf{D})$ *and let* W_1*,* $W_2 \in \mathbf{RH}_\infty^{n\times n}(\mathbf{D})$ *be such that* W_1^{-1}*,* $W_2^{-1} \in \mathbf{RH}_\infty^{n\times n}(\mathbf{D})$*. Let*

$$D(\mathrm{e}^{\mathrm{j}\theta}) = \mathrm{He}(\ (W_1(\mathrm{e}^{\mathrm{j}\theta}))^{-1}(W_2(\mathrm{e}^{\mathrm{j}\theta}))^*),$$
$$G(\mathrm{e}^{\mathrm{j}\theta}) = \mathrm{Sh}(\ (W_1(\mathrm{e}^{\mathrm{j}\theta}))^{-1}(W_2(\mathrm{e}^{\mathrm{j}\theta}))^*).$$

Then $W_2^{-1}\tilde{P}W_1^{-1}$ *is strictly positive real if and only if*

$$P(\mathrm{e}^{\mathrm{j}\theta})D(\mathrm{e}^{\mathrm{j}\theta})(P(\mathrm{e}^{\mathrm{j}\theta}))^* + G(\mathrm{e}^{\mathrm{j}\theta})(P(\mathrm{e}^{\mathrm{j}\theta}))^* - P(\mathrm{e}^{\mathrm{j}\theta})G(\mathrm{e}^{\mathrm{j}\theta}) - D(\mathrm{e}^{\mathrm{j}\theta}) < 0 \quad \forall\theta\in\mathbf{R}.$$

In addition to establishing the precise equivalence between modern mixed μ-theory based sufficient condition for robust staility and the classical passivity-multiplier-based condition, Theorem 4.2 also states that the mutlipliers can be chosen to of a particularly simple form, given in (5) and in (6). This fact has important ramifications for the numerical verification of these robust stability conditions, as we shall see shortly.

5 State-Space Verification of Frequency Domain Conditions

Many frequency-dependent conditions for stability and robustness [11, 12, 13, 14, 15] belong to the class of parameterized families of complex LMIs that we studied in Section 3. From a mathematical viewpoint, these conditions amount to infinite-dimensional convex feasibility problems. Conventionally, there are two methods to handle the infinite dimensionality of these conditions and turn them into finite-dimensional LMIs. One is the so called frequency sampling method, which is to conclude to feasibility of a frequency-dependent condition from feasibility of the condition at a finite number of frequencies. In practice, this is the simplest way to perform the test; however, in theory, there is generally no guarantee that such implication holds (for an exception, see, e.g., [20]). The other method, the basis function method [10], is to select finitely many basis functions and restrict the search to the span of these functions, using a state-space approach to obtain a single LMI. Here, it is not known if and how the choice of basis functions affects the outcome of the stability test, and whether there is a "gap" between the basis function method and the frequency sampling method.

In this section, we discuss the implications of Proposition 3.2 on the numerical solution of infinite-dimensional convex feasibility problems. Specifically, we show that the frequency sampling method and the basis function method are *equivalent* in the limit, as the number of frequency points and the basis elements respectively in each method goes to infinity. In particular, we show that we may choose the basis elements to be of finite impulse response (FIR). Our approach follows a direct application of Proposition 3.2. More precisely, recall that given $n \times n$, real-rational transfer function matrices F_i, $i = 0, \ldots, \ell$, with no poles on the unit circle, an infinite-dimensional convex problem

$$L(x_1, \cdots, x_\ell, F_0(z), \cdots, F_\ell(z)) > 0 \quad \forall z \in \partial\mathbf{D} \tag{10}$$

is feasible if and only if there exist $x_i \in \mathcal{H}_{\text{FIR}}$, $i = 1, \ldots, \ell$, such that

$$L(x_1(z), \cdots, x_\ell(z), F_0(z), \cdots, F_\ell(z)) > 0 \quad \forall z \in \partial\mathbf{D}.$$

When F_i, $i = 0, \ldots, \ell$, are real-rational transfer function matrices (*not necessarily proper*) with no poles on the unit circle, and when $x_i \in \mathcal{H}_{\text{FIR}}$, $i = 1, \ldots, \ell$, an interesting decomposition exists for $L(x_1(z), \cdots, x_\ell(z), F_0(z), \cdots, F_\ell(z))$ when $z \in \partial\mathbf{D}$. Roughly speaking $L(x_1(z), \cdots, x_\ell(z), F_0(z), \cdots, F_\ell(z))$ can in such case be rewritten as the sum of a linear combination of real-rational, *proper* transfer functions and its adjoint. In the statement below, we say that $x \in \mathcal{H}_{\text{FIR}}$ is of order N if $x(z) = \sum_{i=0}^{N}(a_i z^i + b_i z^{-i})$ for some $a_i, b_i \in \mathbf{R}$.

Proposition 5.1 *Let F_i, $i = 0, \ldots, \ell$, be $n \times n$, real-rational transfer function matrices (not necessarily proper) with no poles on the unit circle, let $N_i \in \mathbf{N}$, $i = 1, \ldots, \ell$ and let $t = 2\sum_{i=1}^{\ell}(N_i + 1)$. Then there exist $n \times n$ real-rational, proper transfer function matrices H_k, $k = 0, \ldots, t$, with no poles on the unit circle, such that, for every ℓ-tuple $(x_1, \ldots, x_\ell)$, with $x_i \in \mathcal{H}_{\text{FIR}}$ of order N_i, $i = 1, \ldots, \ell$, there exist scalars $p_k \in \mathbf{R}$, $k = 1, \ldots, t$ such that*

$$L(x_1(z), \cdots, x_\ell(z), F_0(z) \cdots, F_\ell(z)) = H_0(z) + \sum_{k=1}^{t} p_k H_k(z) + \left(H_0(z) + \sum_{k=1}^{t} p_k H_k(z)\right)^* \quad \forall z \in \partial\mathbf{D}.$$

Thus the feasibility condition (10) is equivalent to the existence of nonnegative integers N_i, $i = 1, \ldots, \ell$, and of real numbers p_k's, collected as $\mathbf{p} = (p_1, \ldots, p_t)$, with $t = 2\sum_{i=1}^{\ell}(N_i + 1)$ such that

$$H(e^{j\theta}, N_1, \ldots, N_\ell, \mathbf{p}) + (H(e^{j\theta}, N_1, \ldots, N_\ell, \mathbf{p}))^* > 0 \quad \forall \theta \in [0, 2\pi], \tag{11}$$

where $H(e^{j\theta}, N_1, \ldots, N_\ell, \mathbf{p}) = H_0(e^{j\theta}) + \sum_{k=1}^{t} p_k H_k(e^{j\theta})$. Given $N_1, \ldots, N_\ell$, we can obtain a state-space realization

$$(A, B, C(\mathbf{p}), D(\mathbf{p}))$$

for H, where A and B are constant real matrices, and C are D are real-valued functions affine in $\mathbf{p}$. This enables us to check feasibility of the infinite-dimensional constraint (10) by performing a sequence of finite-dimensional feasibility analyses of (11): if (11) is not feasible in $\mathbf{p}$ for given $N_1, \ldots, N_\ell$, then increase the N_i's. That feasibility of (11) can be ascertained in terms of a finite dimensional LMI follows from the next lemma (see, e.g., [21]), which is a generalization of the classical positive real lemma (see, e.g., [22, 23]).

Lemma 5.2 *Let H be an $n \times n$ real-rational, proper transfer function matrix with no poles on the unit circle, and with state-space realization (A, B, C, D), where A, B, C, and D are real matrices. Then*

$$H(e^{j\theta}) + (H(e^{j\theta}))^* > 0 \quad \forall \theta \in [0, 2\pi],$$

if and only if there exists a real symmetric matrix X satisfying the matrix inequality

$$\begin{pmatrix} A^T X A - X & A^T X B - C^T \\ B^T X A - C & B^T X B - (D + D^T) \end{pmatrix} < 0. \tag{12}$$

Applying this lemma to inequality (11) and noting that C and D are affine in $\mathbf{p}$, thus that the matrix inequality (12) is *affine* in the variables X and $\mathbf{p}$, we end up with a finite-dimensional LMI. Therefore we can conclude that any parameterized LMI feasibility problem of type (10) with the F_i's as in Proposition 5.1 can be approximately solved by solving a sequence of finite-dimensional LMI feasibility problems.

The idea just outlined can be used, for example, to compute the mixed-μ-norm upper bound $\|P\|_{\hat{\mu}}$. Indeed, a byproduct of the equivalence of statements (a), (b), and (c) in Theorem 4.2 is the convergence result stated below. Given $P \in \mathbf{RH}_\infty^{n\times n}(\mathbf{D})$ and given a positive integer N, let $\mathcal{D}_N$, $\mathcal{G}_N$, and $\mathcal{T}_N$ be the sets of functions $D(z)$, $G(z)$, and $T(z)$ of the forms (5), (6), and (9), respectively. Let $M^1_{\mathrm{ub}}(P,N)$ be the infimum of the set of real scalars α such that, for some $D(z) \in \mathcal{D}_N$ and $G(z) \in \mathcal{G}_N$,

$$D(e^{j\theta}) > 0 \quad \forall\theta \in \mathbf{R},$$
$$P(e^{j\theta})D(e^{j\theta})(P(e^{j\theta}))^* + G(e^{j\theta})(P(e^{j\theta}))^* - P(e^{j\theta})G(e^{j\theta}) - \alpha^2 D(e^{j\theta}) < 0 \quad \forall\theta \in \mathbf{R},$$

and let $M^2_{\mathrm{ub}}(P,N)$ be the infimum of the set of scalars α such that, for some $T(z) \in \mathcal{T}_N$

$$\mathrm{He}(T(e^{j\theta})) > 0 \quad \forall\theta \in \mathbf{R},$$
$$\mathrm{He}(\tilde{P}_\alpha(e^{j\theta})T(e^{j\theta})) > 0 \quad \forall\theta \in \mathbf{R}.$$

Then clearly

$$\lim_{N\to\infty} M^1_{\mathrm{ub}}(P,N) = \lim_{N\to\infty} M^2_{\mathrm{ub}}(P,N) = \|P\|_{\hat{\mu}}.$$

Thus $\|P\|_{\hat{\mu}}$ can then be computed as follows, where we now write $M_{\mathrm{ub}}(P,N)$ to denote either $M^1_{\mathrm{ub}}(P,N)$ or $M^2_{\mathrm{ub}}(P,N)$. First let $\alpha_0 = \|P\|_\infty$ (the $\mathbf{H}_\infty$ norm of P), let $\alpha'_0 = 0$ and let $\beta \in (0,1)$. Then, for each positive integer N, generate numbers $\alpha_N > M_{\mathrm{ub}}(P,N)$ and $\alpha'_N \le M_{\mathrm{ub}}(P,N)$ satisfying $\alpha_N \le \alpha_{N-1}$, and $\alpha_N - \alpha'_N \le \beta(\alpha_{N-1} - \alpha'_{N-1})$. It is easy to check that the sequences, $\{\alpha_N\}$ and $\{\alpha'_N\}$, constructed satisfy the relation

$$\alpha'_N \le M_{\mathrm{ub}}(P,N) < \alpha_N \quad \forall N \ge 0.$$

Since $\{\alpha_N - \alpha'_N\}$ and $\{M_{\mathrm{ub}}(P,N)\}$ converge to 0 and $\|P\|_{\hat{\mu}}$, respectively, it follows that both sequences, $\{\alpha_N\}$ and $\{\alpha'_N\}$, converge to $\|P\|_{\hat{\mu}}$. The following algorithm assumes that $\beta = 1/2$ and uses the fact that the sequence $\{M_{\mathrm{ub}}(P,N)\}$ is monotone nonincreasing.

Algorithm

Step 1. Set $N = 0$, $\alpha_0 = \|P\|_\infty$, $\alpha'_0 = 0$.

Step 2. Set $\alpha_{N+1} = \alpha_N$, $\alpha'_{N+1} = \frac{1}{2}(\alpha_N + \alpha'_N)$, $d = \frac{1}{2}(\alpha_N - \alpha'_N)$.

Step 3. While $\alpha'_{N+1} > M_{\mathrm{ub}}(P,N)$, do

set $\alpha'_{N+1} = \alpha'_{N+1} - d$, $\alpha_{N+1} = \alpha_{N+1} - d$.

Step 4. Set $\alpha'_{N+1} = \max(0, \alpha'_{N+1})$.

Step 5. Set $N = N + 1$ and go to Step 2.

6 Synopsis of the Continuous-Time Case

In this section, we state the continuous-time analogue to the results obtained in the previous sections. First, an analogue to Proposition 3.2. Let

$$\mathcal{H}_{\mathrm{UDR}} := \left\{ \sum_{i=0}^{N} \left(\frac{a_i}{(1-z)^i} + \frac{b_i}{(1+z)^i} \right) : N \in \mathbf{N},\ a_i, b_i \in \mathbf{R} \right\}.$$

(The subscript "UDR" stands for unit-decay rate; the two-sided inverse Laplace transforms of the elements of $\mathcal{H}_{\mathrm{UDR}}$ decay exponentially with a unit rate of decay.) We have the following continuous-time representation results for LMI solutions.

Proposition 6.1 *Let $F_i \in \mathcal{F}_{\mathbf{C}_-}$, $i = 0, \ldots, \ell$, The following conditions are equivalent:*

(a) *for every $z \in \partial_e \mathbf{C}_-$, there exist complex numbers x_i, $i = 1, \ldots, \ell$, such that* (2) *holds;*

(b) *there exist continuous functions $x_i : \partial_e \mathbf{C}_- \to \mathbf{C}$, $i = 1, \ldots, \ell$, such that, for all $z \in \partial_e \mathbf{C}_-$, $x_i(\bar{z}) = \overline{x_i(z)}$, $i = 1, \ldots, \ell$, and* (3) *holds;*

(c) *there exist $x_i \in \mathcal{H}_{\mathrm{UDR}}$, $i = 1, \ldots, \ell$, such that,* (3) *holds for all $z \in \partial_e \mathbf{C}_-$;*

We next state Theorem 6.2, our main result for the continuous-time case, which is the continuous-time analogue of Theorem 4.2. It is easily proved by invoking Theorem 4.2, and making use of the bilinear transformation $\phi(z) = \frac{1+z}{1-z}$.

Theorem 6.2 *Let $P \in \mathbf{RH}_\infty^{n\times n}(\mathbf{C}_-)$. The following conditions are equivalent:*

(a)

$$\|P\|_{\hat{\mu}} < 1,$$

(b) *$(I - P)^{-1}$ is in $\mathbf{RH}_\infty^{n\times n}(\mathbf{C}_-)$ and there exist a positive integer N, and $n \times n$, real matrices $Q_i \in \mathcal{S}_{\mathrm{r,c,C}}(\mathbf{C})$, and $U_i \in \mathcal{S}_{\mathrm{r,0,0}}(\mathbf{C})$, $i = 1, \ldots, N$, such that with*

$$D(z) = \sum_{i=0}^{N} \left(Q_i \frac{1}{(1-z)^i} + Q_i^T \frac{1}{(1+z)^i} \right), \tag{13}$$

$$G(z) = \sum_{i=0}^{N} \left(U_i \frac{1}{(1-z)^i} - U_i^T \frac{1}{(1+z)^i} \right), \tag{14}$$

the inequalities

$$\begin{gathered} D(\mathrm{j}\omega) > 0, \\ P(\mathrm{j}\omega)D(\mathrm{j}\omega)(P(\mathrm{j}\omega))^* + G(\mathrm{j}\omega)(P(\mathrm{j}\omega))^* - P(\mathrm{j}\omega)G(\mathrm{j}\omega) - D(\mathrm{j}\omega) < 0, \end{gathered}$$

hold for all $\omega \in \mathbf{R}_e$,

(c) *$(I - P)^{-1}$ is in $\mathbf{RH}_\infty^{n\times n}(\mathbf{C}_-)$ and there exist a positive integer N, and $n \times n$, real matrices $R_i, S_i \in \mathcal{S}_{\mathrm{r,c,C}}(\mathbf{C})$, $i = 1, \ldots, N$, such that with*

$$T(z) = \sum_{i=0}^{N} \left(R_i \frac{1}{(1-z)^i} + S_i \frac{1}{(1+z)^i} \right), \tag{15}$$

the lower right $(n_c + n_C) \times (n_c + n_C)$ submatrix $T_{\mathrm{cC}}(\mathrm{j}\omega)$ of $T(\mathrm{j}\omega)$ is in $\mathcal{S}_{\mathrm{c,C}}(\mathbf{C})$ and is Hermitian for all $\omega \in \mathbf{R}_e$, and the inequalities

$$\begin{gathered} \mathrm{He}(T(\mathrm{j}\omega)) > 0, \\ \mathrm{He}(\tilde{P}(\mathrm{j}\omega)T(\mathrm{j}\omega)) > 0, \end{gathered}$$

hold for all $\omega \in \mathbf{R}_e$,

(d) $(I-P)^{-1}$ *is in* $\mathbf{RH}_\infty^{n\times n}(\mathbf{C}_-)$ *and there exist transfer function matrices* $W_1 = diag(W_1^{\mathrm{r}}, W_1^{\mathrm{c}}, W_1^{\mathrm{C}})$ *and* $W_2 = diag(W_2^{\mathrm{r}}, W_2^{\mathrm{c}}, W_2^{\mathrm{C}})$, *where* W_1^{r} *and* W_2^{r}, W_1^{c} *and* W_2^{c}, *and* W_1^{C} *and* W_2^{C}, *are in* $\mathcal{S}_{\mathrm{r}}(\mathbf{RP})$, $\mathcal{S}_{\mathrm{c}}(\mathbf{RP})$, *and* $\mathcal{S}_{\mathrm{C}}(\mathbf{RP})$, *respectively, satisfying*

  (i) W_1, W_2, W_1^{-1}, *and* W_2^{-1} *are in* $\mathbf{RH}_\infty^{n\times n}(\mathbf{C}_-)$, $W_1^{\mathrm{c}}W_2^{\mathrm{c}} = I$, *and* $W_1^{\mathrm{C}}W_2^{\mathrm{C}} = I$,

  (ii) W_1W_2 *is strictly positive real,*

  (iii) $W_2^{-1}\tilde{P}W_1^{-1}$ *is strictly positive real.*

Moreover, suppose that $(I-P)^{-1}$ *is in* $\mathbf{RH}_\infty^{n\times n}(\mathbf{C}_-)$ *and let* W_1, $W_2 \in \mathbf{RH}_\infty^{n\times n}(\mathbf{C}_-)$ *be such that* W_1^{-1}, $W_2^{-1} \in \mathbf{RH}_\infty^{n\times n}(\mathbf{C}_-)$. *Let*

$$\begin{aligned} D(\omega) &= \mathrm{He}(\ (W_1(\mathrm{j}\omega))^{-1}(W_2(\mathrm{j}\omega))^*), \\ G(\omega) &= \mathrm{Sh}(\ (W_1(\mathrm{j}\omega))^{-1}(W_2(\mathrm{j}\omega))^*). \end{aligned}$$

Then $W_2^{-1}\tilde{P}W_1^{-1}$ *is strictly positive real if and only if*

$$P(\mathrm{j}\omega)D(\mathrm{j}\omega)(P(\mathrm{j}\omega))^* + G(\mathrm{j}\omega)(P(\mathrm{j}\omega))^* - P(\mathrm{j}\omega)G(\mathrm{j}\omega) - D(\mathrm{j}\omega) < 0 \quad \forall\omega \in \mathbf{R}_{\mathrm{e}}.$$

Similar to the discrete-time case, Proposition 6.1 also has implications on computation. Proposition 6.3 below is the analogue of Propostion 5.1. Here we say that $x \in \mathcal{H}_{\mathrm{UDR}}$ is of order N if $x(z) = \sum_{i=0}^N \left(\frac{a_i}{(1-z)^i} + \frac{b_i}{(1+z)^i}\right)$ for some $a_i, b_i \in \mathbf{R}$.

Proposition 6.3 *Let* F_i, $i = 0,\ldots,\ell$, *be* $n\times n$, *real-rational, proper transfer function matrices with no poles on the imaginary axis, and let* $N_i \in \mathbf{N}$, $i = 1,\ldots,\ell$ *and let* $t = 2\sum_{i=1}^{\ell}(N_i+1)$. *Then there exist* $n\times n$ *real-rational, proper transfer function matrices* H_k, $k = 0,\ldots,t$, *with no poles on the imaginary axis, such that, for every* ℓ*-tuple* $(x_1,\ldots,x_\ell)$, *with* $x_i \in \mathcal{H}_{\mathrm{UDR}}$ *of order* N_i, $i = 1,\ldots,\ell$, *there exist scalars* $p_k \in \mathbf{R}$, $k = 1,\ldots,t$ *such that*

$$\begin{aligned} &L(x_1(z),\cdots,x_\ell(z),F_0(z),F_1(z)\cdots,F_\ell(z)) = \\ &\qquad H_0(z) + \textstyle\sum_{k=1}^t p_kH_k(z) + \left(H_0(z) + \sum_{k=1}^t p_kH_k(z)\right)^* \ \forall z \in \partial_{\mathrm{e}}\mathbf{C}_-. \end{aligned}$$

This result, together with a strong version of the continuous-time positive real lemma (again see, e.g., [21]) suggests a scheme to approximately solve parameterized LMI feasibility problems of the type

$$L(x_1,\cdots,x_\ell,F_0(z),\cdots,F_\ell(z)) > 0 \quad \forall z \in \partial_{\mathrm{e}}\mathbf{C}_-, \tag{16}$$

with the F_i's as in Proposition 6.3, by solving a sequence of finite-dimensional LMI feasibility problems. For example, exactly like in the discrete-time case, this can be used to compute a sequence of upper bounds to the continuous-time mixed-μ-norm upper bound $\|P\|_{\hat{\mu}}$ that converges to $\|P\|_{\hat{\mu}}$.

7 Conclusions

A few explicit connections between the modern mixed μ theory and the classical absolute stability theory have been made. It has been shown that the standard mixed μ upper bound condition is mathematically equivalent to a passivity-multiplier-based stability condition in [10]. Concurrently, the relationship between the scaling matrices widely used in mixed μ theory and certain stability multipliers used in absolute stability theory, is explicitly characterized. The establishment of the connections not only clarifies the relationship between two large classes of frequency-domain conditions (those using scaling matrices and those using multipliers) through a careful study of the standard mixed μ upper bound condition, but also offers us an alternative direction, to take advantage of the wealth knowledge of the absolute stability theory, to perform robust stability analysis. Additionally, a sequence of computable mixed-μ-norm upper bounds was defined with guaranteed convergence to the standard mixed-μ-norm upper bound $\|P\|_{\hat{\mu}}$. Moreover, a conceptual algorithm for computing $\|P\|_{\hat{\mu}}$ was given. A state-space method has been developed to check feasibility of a class of general LMIs across frequency (in which the standard mixed μ upper bound condition is a special case). The result has wide applications in checking feasibility of many robust stability, robust performance, and nonlinear stability conditions, as well as in optimal stability margin design problems (e.g., μ synthesis).

A Appendix

A.1 Proof of Lemma 3.1

Before proving Lemma 3.1, we introduce three lemmas which will be used later.

The first lemma, Lemma A.1, extends a result of Packard and Doyle (Theorem 9.10 in [24]), which explores an intrinsic solution property for a class of parameterized family of complex LMIs.

Lemma A.1 *Let $\Omega \in \Xi$. Let $F_i \in \mathcal{F}_\Omega$, $i = 0, \ldots, \ell$. Let $z \in \partial_e\Omega$ be given. Suppose that there exist complex numbers x_i, $i = 1, \ldots, \ell$, such that $L(x_1, \cdots, x_\ell, F_0(z), \cdots, F_\ell(z)) > 0$. Then the inequality also holds with z and x_i, $i = 1, \ldots, \ell$ replaced with $\overline{z}$ and $\overline{x_i}$, respectively. Furthermore, when z is such that $F(z)$ is real (in particular, $z \in \mathbf{R}$ or $z = \infty$), the inequality holds with x_i, $i = 1, \ldots, \ell$, replaced with $\frac{1}{2}(x_i + \overline{x_i})$, $i = 1, \ldots, \ell$.*

Proof: Recall that, given a Hermitian matrix A, $A > 0$ if and only if $y^*Ay > 0$ for all $y \neq 0$. Taking complex conjugates on both sides of the resulting inequality proves the first claim. The second claim is a direct consequence of the first one. □

Lemma A.2 below is a key to prove the implication (b)⇒(c) of Lemma 3.1. With a direct application of the results of the Dirichlet problem on a simply connected set and the Schwarz Reflection Principle, we are able to establish the lemma, which implies the existence of a combination of functions of certain type that matches on $\partial_e\Omega$ a given complex-valued function x satisfying certain properties.

Lemma A.2 *Let $\Omega \in \Xi$. Let $x : \partial_e\Omega \to \mathbf{C}$ be a continuous function satisfying $x(\bar{z}) = \overline{x(z)}$ for all $z \in \partial_e\Omega$. Then there exist $s^i : \mathrm{cl}_e\Omega \to \mathbf{C}$, i=1,2, such that, for i=1,2, s^i is continuous on $\mathrm{cl}_e\Omega$, analytic in Ω, and satisfies $s^i(\bar{z}) = \overline{s^i(z)}$ for all $z \in \mathrm{cl}_e\Omega$, and $x(z) = s^1(z) + \overline{s^2(z)}$ for all $z \in \partial_e\Omega$.*

Proof: We define the sets $\Pi_+ := \{z \in \mathbf{C} : \mathrm{Im}\{z\} > 0\}$ and $\Pi_- := \{z \in \mathbf{C} : \mathrm{Im}\{z\} < 0\}$. It is easy to verify that when $\Omega \in \Xi$, both the sets $\Omega \cap \Pi_+$ and $\Omega \cap \Pi_-$ are simply connected, and the set $\Omega \cap \mathbf{R}$ is an interval (a, b) where $a, b \in \mathbf{R}_e$.

Let $x : \partial_e\Omega \to \mathbf{C}$ be given as assumed. Let $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ denote its real part and imaginary part, respectively. Then both $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are real-valued, continuous functions defined on $\partial_e\Omega$. Define $f_+(z) = \mathrm{Re}\{x\}(z)$ for all $z \in \partial_e\Omega \cap \Pi_+$, and equal to some real-valued continuous function for $z \in [a, b]$. Consider the Dirichlet problem on $\Omega \cap \Pi_+$ with the real-valued, continuous boundary function f_+. It is known that, given a simply connected set X and a real-valued continuous function g defined on $\partial_e X$, there exists a function u which is harmonic inside X and matches g on $\partial_e X$ (see, e.g., Corollary 4.18, pp.274, in [25]). Moreover, for any function u which is harmonic in a simply connected set X, there exists a harmonic conjugate v such that $u + \mathrm{j}v$ is analytic in X (see, e.g., Theorem 2.2(j), pp.202, in [25]). Thus there exists a function analytic in $\Omega \cap \Pi_+$, whose real part matches f_+ on $\partial_e\Omega \cap \Pi_+$. By the Schwarz Reflection Principle (e.g., Theorem 11.14 in [26]), this function extends to a function s_1, continuous on $\mathrm{cl}_e\Omega$, analytic in Ω, with $s_1(\bar{z}) = \overline{s_1(z)}$ for all $z \in \mathrm{cl}_e\Omega$, and $\mathrm{Re}\{s_1\}(z) = \mathrm{Re}\{x\}(z)$ for all $z \in \partial_e\Omega$. Similarly, there exists a function s_2, continuous on $\mathrm{cl}_e\Omega$, analytic in Ω, $s_2(\bar{z}) = \overline{s_2(z)}$ for all $z \in \mathrm{cl}_e\Omega$, and $\mathrm{Im}\{s_2\}(z) = \mathrm{Im}\{x\}(z)$ for all $z \in \partial_e\Omega$. Let $s^1 = \frac{1}{2}(s_1 + s_2)$ and $s^2 = \frac{1}{2}(s_1 - s_2)$. It follows that, for all $z \in \partial_e\Omega$,

$$\begin{aligned} x(z) &= \mathrm{Re}\{x\}(z) + \mathrm{j}\mathrm{Im}\{x\}(z) \\ &= \tfrac{1}{2}(s_1(z) + \overline{s_1(z)}) + \tfrac{1}{2}(s_2(z) - \overline{s_2(z)}) \\ &= \tfrac{1}{2}(s_1(z) + s_2(z) + \overline{s_1(z) - s_2(z)}) \\ &= s^1(z) + \overline{s^2(z)}. \end{aligned}$$

□

Lemma A.3 below, a slight extension of Mergelyan's Theorem (e.g., Theorem 20.5 in [26]), gives conditions for uniform approximation, on a bounded set, of a class of complex-valued functions by polynomials with real coefficients.

Lemma A.3 *Let $\Omega \in \Xi$ be bounded in $\mathbf{C}$. Suppose that $s : \mathrm{cl}\Omega \to \mathbf{C}$ is continuous on $\mathrm{cl}\Omega$ and analytic in Ω. Assume that for all $z \in \mathrm{cl}\Omega$, $s(\bar{z}) = \overline{s(z)}$. Then given $\epsilon > 0$, there exists a polynomial p with real coefficients, such that*

$$\sup_{z \in \mathrm{cl}\Omega} | s(z) - p(z) | < \epsilon.$$

Proof: Given $\epsilon > 0$, by Mergelyan's theorem, there exists a polynomial $p_0 = p + \mathrm{j}q$ where p and q are real-coefficient polynomials, such that

$$\sup_{z \in \mathrm{cl}\Omega} | s(z) - p_0(z) | < \epsilon/2. \tag{17}$$

We show that p is as claimed. First, since for all $z \in \mathrm{cl}\Omega$, $s(\overline{z}) = \overline{s(z)}$, it follows from (17) that, for all $z \in \mathrm{cl}\Omega$,

$$\begin{aligned} |\, \overline{p_0(z)} - p_0(\overline{z})\, | &= |\, \overline{p_0(z)} - s(\overline{z}) + s(\overline{z}) - p_0(\overline{z})\, | \\ &\leq |\, \overline{p_0(z)} - \overline{s(z)}\, | + |\, s(\overline{z}) - p_0(\overline{z})\, | \\ &< \epsilon. \end{aligned}$$

On the other hand, for all $z \in \mathrm{cl}\Omega$,

$$|\, \overline{p_0(z)} - p_0(\overline{z})\, | = |\, \overline{p(z)} - \mathrm{j}\overline{q(z)} - (p(\overline{z}) + \mathrm{j}q(\overline{z}))\, | \,.$$

Since, for any real-coefficient polynomial $m(z)$, $m(\overline{z}) = \overline{m(z)}$ for all z, it follows that

$$|\, \overline{p_0(z)} - p_0(\overline{z})\, | = 2\, |\, q(z)\, | \,.$$

Thus, for all $z \in \mathrm{cl}\Omega$, $|\, q(z)\, | < \epsilon/2$. Again using (17), it follows that, for all $z \in \mathrm{cl}\Omega$,

$$\begin{aligned} |\, s(z) - p(z)\, | &= |\, s(z) - (p(z) + \mathrm{j}q(z)) + \mathrm{j}q(z)\, | \\ &\leq |\, s(z) - p_0(z)\, | + |\, q(z)\, | < \epsilon. \end{aligned}$$

This proves the claim. □

Now we are ready to prove Lemma 3.1.

Proof of Lemma 3.1: We show first that (a)⇒ (b)⇒(c)⇒(d)⇒ (a), and then, under the assumption that Ω is bounded, we show that (c)⇒(e)⇒(a).

To prove the implication (a)⇒(b), with Lemma A.1 in mind, we can obtain statement (b) via the following arguments.

Let $\psi : \partial_{\mathrm{e}}\Omega \to \partial\mathbf{D}$ be a continuous bijection such that $\psi(\overline{z}) = \overline{\psi(z)}$ for all $z \in \partial_{\mathrm{e}}\Omega$ (since $\Omega \in \Xi$, such ψ exists). Thus, for every $z \in \partial_{\mathrm{e}}\Omega$, there exist complex numbers x_i, $i = 1, \ldots, \ell$, such that

$$L(x_1, \cdots, x_\ell, F_0(z), \cdots, F_\ell(z)) > 0$$

if and only if for every $z \in \partial\mathbf{D}$, there exist complex numbers x_i, $i = 1, \ldots, \ell$, such that

$$L(x_1, \cdots, x_\ell, F_0(\psi^{-1}(z)), \cdots, F_\ell(\psi^{-1}(z))) > 0.$$

Furthermore, for $i = 0, \ldots, \ell$, $F_i \circ \psi^{-1} : \partial\mathbf{D} \to \mathbf{C}^{n\times n}$ are continuous, and satisfy $\overline{F_i(\psi^{-1}(z))} = F_i(\psi^{-1}(\overline{z}))$, for all $z \in \partial\mathbf{D}$. Based on this observation, we will construct a continuous function on $\partial\mathbf{D}$ from an appropriate choice of the solutions x_i's. When this is achieved, we change the domain back to Ω, which finally leads to the claim.

In the unit disk case the problem of constructing a continuous function on $\partial\mathbf{D}$ based on data x_i's reduces to that on the closed interval $[-\pi, \pi]$ with same value assigned at the endpoint $-\pi$ and π. The following argument is similar to that in [12].

For each $\theta \in (-\pi, \pi]$, let x_i^θ, $i = 1, \ldots, \ell$, satisfy

$$L(x_1^\theta, \cdots, x_\ell^\theta, F_0(\mathrm{e}^{\mathrm{j}\theta}), \cdots, F_\ell(\mathrm{e}^{\mathrm{j}\theta})) > 0$$

and let $x_i^{-\pi} = x_i^{\pi}$, $i = 1, \ldots, \ell$. By Lemma A.1, the x_i^{θ}'s can be chosen such that, for all $\theta \in [0, \pi]$, $i = 1, \ldots, \ell$, $x_i^{-\theta} = \overline{x_i^{\theta}}$ and, in particular x_i^0 and x_i^{π} are real. Since the F_i's are continuous, for every $\theta \in [0, \pi]$ there exists an open neighborhood $I^{\theta} \subset \mathbf{R}$ of θ such that

$$L(x_1^{\theta}, \cdots, x_{\ell}^{\theta}, F_0(e^{j\theta'}), \cdots, F_{\ell}(e^{j\theta'})) > 0 \quad \forall \theta' \in I^{\theta}.$$

We now show that there exists a finite open cover $\{I_1, \cdots, I_m\} \subset \{I^{\theta}, \theta \in [0, \pi]\}$ with $I_1 = I^0$ and $I_m = I^{\pi}$, where for $i = 1, \ldots, m$, $I_i = (\alpha_i, \alpha_i')$, $\alpha_1 < 0 < \alpha_2 < \cdots < \alpha_m < \alpha_{m-1}' < \pi < \alpha_m'$, and each I_i is not a subset of the union of the other I_j's. Without loss of generality, assume that $[0, \pi] \setminus (I^0 \cup I^{\pi})$ is not empty. Let $\{I_1, \cdots, I_k\} \subset \{I^{\theta}, \theta \in [0, \pi]\}$ be a finite open cover of the compact interval $[0, \pi] \setminus (I^0 \cup I^{\pi})$, and replace each I_i by its intersection with $(0, \pi)$. For $i = 1, \ldots, \ell$, let x_i^j, $j = 1, \ldots, k$, be the associated x_i^{θ}'s. Let $I_0 := I^0$, $I_{k+1} := I^{\pi}$ and, for $i = 1, \ldots, \ell$, $x_i^{k+1} := x_i^{\pi}$. Then $\{I_0, \cdots, I_{k+1}\}$ is a finite open cover of the compact interval $[0, \pi]$. Repeatedly take out any interval I_i which is a subset of the union of the remaining I_j's (clearly I^0 and I^{π} will not be taken out in this process) and denote by m the number of remaining intervals. Finally, relabel the I_i's so that $I_i = (\alpha_i, \alpha_i')$ with α_i's, $i = 1, \ldots, m$ being in increasing order and, for $i = 1, \ldots, \ell$, relabel the x_i^j's accordingly. It is readily checked that the finite cover obtained possesses the desired properties.

Now, for each $i = 1, \ldots, m$, let $V_i := (x_1^i, \cdots, x_{\ell}^i)$. Then V_1 and V_m are real. Define, for $\theta \in [0, \pi]$,

$$V(\theta) := \begin{cases} V_i & \theta \in I_i - \cup_{j \neq i} I_j, i = 1, \ldots, m. \\ \lambda V_i + (1 - \lambda) V_{i+1} & \theta = \lambda \alpha_{i+1} + (1 - \lambda) \alpha_i', \lambda \in (0, 1), i = 1, \ldots, m - 1. \end{cases}$$

Then $V : [0, \pi] \to \mathbf{C}^{\ell}$ is continuous. Note that, since $\alpha_2 > 0$ and $\alpha_{m-1}' < \pi$, $V(0)$ and $V(\pi)$ are real. Extending $V(.)$ by the formula $V(-\theta) = \overline{V(\theta)}$ to the interval $[-\pi, 0]$ results in a continuous function defined on $[-\pi, \pi]$ with the same value $V(\pi)$ assigned at the endpoints $-\pi$ and π. This completes the proof of the implication (a)⇒(b).

The implication (b)⇒(c) is a direct consequence of Lemma A.2.

The implication (c)⇒(d) follows easily from continuity of the F_i's, $i = 1, \ldots, \ell$ over the compact set $\partial_e \Omega$, and uniform approximation of functions $s \in S$ to arbitrarily degree of accuracy by the basis functions $\{\varphi_i\}$. The implication (d)⇒(a) is trivial.

Now we assume that Ω is bounded. To show the implication (c)⇒(e), it suffices to show that given $s \in S$ and any $\epsilon > 0$, there always exists a polynomial p with real coefficients which uniformly approximates the given function s over clΩ within ϵ. This result follows immediately from Lemma A.3.

The implication (e)⇒(a) is obvious. This completes the proof of Lemma 3.1.

A.2 Proof of Proposition 4.1

The proof of Proposition 4.1 makes use of the following two lemmas.

Lemma A.4 *Given a square complex matrix $\tilde{A}$, let $n_-(\tilde{A})$, $n_+(\tilde{A})$, and $n_0(\tilde{A})$ be the numbers of eigenvalues of $\tilde{A}$ inside the open unit disk, outside the closed unit disk, and on the unit circle,* *respectively. Let X be a Hermitian matrix such that*

$$\tilde{A}^* X \tilde{A} - X < 0. \tag{18}$$

Then $n_-(\tilde{A}) = n_+(X)$, $n_+(\tilde{A}) = n_-(X)$, and $n_0(\tilde{A}) = n_0(X) = 0$. Moreover, if $\tilde{A}$ is stable, i.e., $n_+ = n_0 = 0$, $X > 0$, and if $\tilde{A}$ is antistable, i.e., $n_- = n_0 = 0$, $X < 0$.

Proof: This first claim of the lemma is simply the discrete-time version of Theorem 3.3 in [27]. A proof can be given simply by conformally mapping the unit-disk to the right half-plane, and appealing to the proof of Theorem 3.3 in [27]. The second claim is a standard result from Lyapunov theory for finite-dimensional linear time-invariant systems (see, for example, [28, Thm. 23.7]). □

The next lemma plays a key role in the proof of Proposition 4.1. Specifically, we show here that strictly positive-realness of a real-rational proper, transfer function M can be used to derive certain properties that are sufficient for its canonical factorization. In what follows, we let $\chi_-(A)$ be the invariant subspace corresponding to the eigenvalues of a square matrix A inside the open unit disk, and let $\chi_+(A)$ be the invariant subspace corresponding to the eigenvalues of A outside the closed unit disk.

Lemma A.5 *Suppose that M is an $m \times m$ real-rational biproper transfer function matrix with no poles on the unit circle such that the frequency domain condition*

$$M(\mathrm{e}^{\mathrm{j}\theta}) + (M(\mathrm{e}^{\mathrm{j}\theta}))^* > 0, \tag{19}$$

holds for all θ. Then there exits a minimal state-space realization (A, B, C, D) of M, where A, B, C, and D are real matrices, with D invertible, such that $\chi_-(A)$ is spanned by the columns of $T_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}$, and $\chi_+(A - BD^{-1}C)$ is spanned by the columns of a matrix $T_2 = \begin{pmatrix} T_{21} \\ T_{22} \end{pmatrix}$, where both T_{12} and T_{22} are real matrices, and T_{22} is invertible.

Proof: Let (A, B, C, D) be a minimal state-space realization for M, where A, B, C, and D are real matrices. Since M is biproper, D is invertible. Also, since M has no pole on the unit circle, without loss of generality, we may assume that

$$A = \begin{pmatrix} A_- & 0 \\ 0 & A_+ \end{pmatrix}, \tag{20}$$

where $A_- \in \mathbf{R}^{n_- \times n_-}$ and $A_+ \in \mathbf{R}^{n_+ \times n_+}$, and have all eigenvalues inside the open unit disk and outside the closed unit disk, respectively. Then the columns of the matrix

$$T_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}, \tag{21}$$

where $I \in \mathbf{R}^{n_- \times n_-}$, span $\chi_-(A)$.

Since $\mathrm{He}(M(\mathrm{e}^{\mathrm{j}\theta})) > 0$ for all θ, Lemma 5.2 implies that there exists a real symmetric matrix X such that

$$\begin{pmatrix} A^*XA - X & A^*XB - C^* \\ B^*XA - C & B^*XB - (D + D^*) \end{pmatrix} < 0. \tag{22}$$

This implies that

$$A^*XA - X < 0. \tag{23}$$

Partition X as

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix}$$

where $X_{11} \in \mathbf{R}^{n_- \times n_-}$, $X_{12} \in \mathbf{R}^{n_- \times n_+}$, and $X_{22} \in \mathbf{R}^{n_+ \times n_+}$. Since A satisfies (20), inequality (23) implies that

$$A_-^* X_{11} A_- - X_{11} < 0.$$

It follows from Lemma A.4 that $X_{11} > 0$.

On the other hand, premultiplying and postmultiplying inequality (22) by

$$\begin{pmatrix} I & -(D^{-1}C)^* \\ 0 & I \end{pmatrix}$$

and its conjugate transpose respectively yields, with $A^\times$ denoting $A - BD^{-1}C$,

$$\begin{pmatrix} (A^\times)^* X A^\times - X & (A^\times)^* XB + C^* D^{-*} D \\ B^* X A^\times + D^* D^{-1} C & B^* XB - (D + D^*) \end{pmatrix} < 0,$$

which in turn implies that

$$(A^\times)^* X (A^\times) - X < 0. \tag{24}$$

Since inequalities (23) and (24) share the common Hermitian solution X, it follows from Lemma A.4 that A and $A^\times$ have the same *inertia*, that is they both have the same number of eigenvalues inside the open unit disk, outside the closed unit disk, and on the unit circle, respectively. Recall that A can have no eigenvalues on the unit circle, therefore, $A^\times$ has n_+ eigenvalues outside the closed unit disk and n_- eigenvalues inside the open unit disk.

Let T_2 be a real matrix with n_+ columns that span $\chi_+(A^\times)$. This means that $A^\times T_2 = T_2 A_+^\times$, where $A_+^\times$ is a square matrix of size $n_+ \times n_+$ whose eigenvalues are all the eigenvalues of $A^\times$ outside the closed unit disk.

Then multiplying (24) on the left by T_2^* and on the right by T_2, we get

$$T_2^* (A^\times)^* X (A^\times) T_2 - T_2^* X T_2 < 0.$$

Equivalently

$$(A_+^\times)^* T_2^* X T_2 (A_+^\times) - T_2^* X T_2 < 0.$$

Since $A_+^\times$ is anti-stable, it follows from Lemma A.4 that $T_2^* X T_2 < 0$. Partitioning T_2 as

$$T_2 = \begin{pmatrix} T_{21} \\ T_{22} \end{pmatrix}, \tag{25}$$

where $T_{12} \in \mathbf{R}^{n_- \times n_+}$ and $T_{22} \in \mathbf{R}^{n_+ \times n_+}$, we get

$$T_{21}^* X_{11} T_{21} + T_{21}^* X_{12} T_{22} + T_{22}^* X_{12}^* T_{21} + T_{22}^* X_{22} T_{22} < 0.$$

This implies that T_{22} is invertible. Indeed, otherwise, multiplying the above inequality on the right by any nonzero vector v in the null space of T_{22} and on the left by v^* yields

$$v^* T_{21}^* X_{11} T_{21} v < 0,$$

which contradicts the fact that $X_{11} > 0$ (note that $T_{21}v \neq 0$, otherwise T_2 is not of full column rank). The proof is complete. □

Proof of Proposition 4.1:

It turns out that Lemma A.5 yields precisely the conditions required for M to have a canonical factorization [19, Ch. 7, Thm. 1], and hence establishes Proposition 4.1. For convenience, we let $[A, B, C, D]$ denote $C(zI - A)^{-1}B + D$.

Let A, B, C, D, T_{21} and T_{22} be as given by Lemma A.5. Define

$$T = \begin{pmatrix} I & T_{21} \\ 0 & T_{22} \end{pmatrix}.$$

Partition the following matrices as

$$T^{-1}AT = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad T^{-1}B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad CT = \begin{pmatrix} C_1 & C_2 \end{pmatrix},$$

where $A_1 \in \mathbf{R}^{n_- \times n_-}$, $A_2 \in \mathbf{R}^{n_- \times n_+}$, $A_3 \in \mathbf{R}^{n_+ \times n_-}$, $A_4 \in \mathbf{R}^{n_+ \times n_+}$, $B_1 \in \mathbf{R}^{n_- \times m}$, $B_2 \in \mathbf{R}^{n_+ \times m}$, $C_1 \in \mathbf{R}^{m \times n_-}$, $C_2 \in \mathbf{R}^{m \times n_+}$. Define

$$\begin{aligned} M_- &:= [A_1,\ B_1,\ C_1,\ D], \\ M_+ &:= \left[A_4,\ B_2,\ D^{-1}C_2,\ I\right]. \end{aligned}$$

Then

$$\begin{aligned} M_-^{-1} &:= \left[A_1 - B_1 D^{-1} C_1,\ B_1 D^{-1},\ -D^{-1}C_1,\ D^{-1}\right], \\ M_+^{-1} &:= \left[A_4 - B_2 D^{-1} C_2,\ B_2,\ -D^{-1}C_2,\ I\right]. \end{aligned}$$

Then, following [19, Ch. 7, Thm. 1], we have

- $M_-, M_-^{-1}, M_+^{\sim}, (M_+^{-1})^{\sim} \in \mathbf{RH}_\infty^{m\times m}(\mathbf{D})$, and
- $M = M_- M_+$.

Define $M_1 = M_-$ and $M_2 = M_+^{\sim}$. This completes the proof of Proposition 4.1.

A.3 Proof of Theorem 4.2

The proof of Theorem 4.2 makes use of the following lemma.

Lemma A.6 *Let x be in $\mathcal{H}_{\text{FIR}}$. Then given $\epsilon > 0$ there exists $w \in \mathbf{RP}$, with no poles at the origin, such that w uniformly approximates x over $\partial\mathbf{D}$ within ϵ.*

Proof: Let $\epsilon > 0$. Suppose that $x(z) = p(z) + q(z)$, where $p(z) = \sum_0^{l_1} a_i z^i$ and $q(z) = \sum_1^{l_2} b_j z^{-j}$, with $a_i \in \mathbf{R}$, $i = 0, \ldots, l_1$, $a_{l_1} \neq 0$, and $b_j \in \mathbf{R}$, $j = 1, \ldots, l_2$, $b_{l_2} \neq 0$. Define $r_1(z) = \frac{p(z)}{(1+\epsilon_1 z)^{l_1}}$ and $r_2(z) = q(z)\frac{z^{l_2}}{z^{l_2}+\epsilon_2}$, with ϵ_1 and ϵ_2 positive. Then if ϵ_1 and ϵ_2 are small enough, r_1 and r_2 uniformly approximate, within $\frac{1}{2}\epsilon$, p and q, respectively, on $\partial\mathbf{D}$. Thus $w(z) = r_1(z) + r_2(z)$ uniformly approximates, within ϵ, $x(z)$ on $\partial\mathbf{D}$. It is clear that $w(z)$ has no poles at the origin, and belongs to $\mathbf{RP}$. This proves the claim. □

Proof of Theorem 4.2

(a)⇒(b): Suppose that $\|P\|_{\hat{\mu}} < 1$. Since this condition is stronger than the mixed μ condition, it follows from the Small μ Theorem (e.g., [29]) that $(I-P)^{-1} \in \mathbf{RH}_\infty^{n\times n}(\mathbf{D})$. By Proposition 3.2, there exist $\hat{D}(z)$ and $\hat{G}(z)$ whose entries are in $\mathcal{H}_{\text{FIR}}$, such that the inequalities

$$\hat{D}(z) > 0,$$
$$P(z)\hat{D}(z)(P(z))^* + \hat{G}(z)(P(z))^* - P(z)\hat{G}(z) - \hat{D}(z) < 0,$$

hold for all $z \in \partial\mathbf{D}$. Let $D = \hat{D} + \hat{D}^\sim$ and $G = \hat{G} - \hat{G}^\sim$. Since $\bar{z} = z^{-1}$ for all $z \in \partial\mathbf{D}$, it is easy to verify that $D(z)$ and $G(z)$ have the forms as described and satisfy (7) and (8).

(b)⇒(c): Let $D(z)$ and $G(z)$ be as in (b). Then $D(e^{j\theta}) \in \mathcal{D}$, and $G(e^{j\theta}) \in \mathcal{G}$ for all θ. Define $T(z) = D(z) + G(z) = \sum_{i=0}^N ((Q_i + U_i)z^i + (Q_i^T - U_i^T)z^{-i})$ with real matrices $Q_i \in \mathcal{S}_{\text{r,c,C}}(\mathbf{C})$, and $U_i \in \mathcal{S}_{\text{r,0,0}}(\mathbf{C})$. Clearly the submatrix $T_{\text{cC}}(e^{j\theta})$ of $T(e^{j\theta})$ is in $\mathcal{S}_{\text{c,C}}(\mathbf{C})$ and is Hermitian for all θ. Moreover, since for all θ, $D(e^{j\theta})$ is Hermitian and positive definite, and $G(e^{j\theta})$ is skew Hermitian, we have for all θ,

$$\text{He}(T(e^{j\theta})) = D(e^{j\theta}) > 0.$$

On the other hand, replacing D and G by $\frac{1}{2}(T+T^*)$ and $\frac{1}{2}(T-T^*)$, respectively, in the second inequality in (b) yields, for all θ,

$$P(e^{j\theta})(T(e^{j\theta}) + (T(e^{j\theta}))^*)(P(e^{j\theta}))^* + (T(e^{j\theta}) - (T(e^{j\theta}))^*)(P(e^{j\theta}))^* - P(e^{j\theta})(T(e^{j\theta}) - (T(e^{j\theta}))^*)$$
$$-(T(e^{j\theta}) + (T(e^{j\theta}))^*) < 0,$$

i.e.,

$$(I + P(e^{j\theta}))T(e^{j\theta})(I - P(e^{j\theta}))^* + (I - P(e^{j\theta}))(T(e^{j\theta}))^*(I + P(e^{j\theta}))^* > 0. \tag{26}$$

Note that (26) implies that $I - P(e^{j\theta})$ must be nonsingular for all θ. Thus it is equivalent to

$$(I - P(e^{j\theta}))^{-1}(I + P(e^{j\theta}))T(e^{j\theta}) + (T(e^{j\theta}))^*(I + P(e^{j\theta}))^*(I - P(e^{j\theta}))^{-*} > 0,$$

i.e.,

$$\tilde{P}(e^{j\theta})T(\theta) + (\tilde{P}(e^{j\theta})T(e^{j\theta}))^* > 0,$$

i.e.,

$$\text{He}(\tilde{P}(e^{j\theta})T(e^{j\theta})) > 0.$$

This proves the claim.

(c)⇒(d): Suppose that (c) holds.

First, using Lemma A.6 on the entries of T, approximate T on the unit circle with $W = \text{diag}(W^{\text{r}}, W^{\text{c}}, W^{\text{C}}) \in \mathcal{S}_{\text{r,c,C}}(\mathbf{RP})$, with no poles at the origin, closely enough that W:

- like T, has no poles on the unit circle, and
- satisfies the inequalities satisfied by T, namely

$$\mathrm{He}(W(\mathrm{e}^{\mathrm{j}\theta})) > 0, \tag{27}$$

$$\mathrm{He}(\tilde{P}(\mathrm{e}^{\mathrm{j}\theta})W(\mathrm{e}^{\mathrm{j}\theta})) > 0. \tag{28}$$

Then note that, since $T_{\mathrm{cC}}(\mathrm{e}^{\mathrm{j}\theta}) = T_{\mathrm{cC}}^{\sim}(\mathrm{e}^{\mathrm{j}\theta})$ for all $\theta \in \mathbf{R}$, both $W_{\mathrm{cC}} = \mathrm{diag}\left(W^{\mathrm{c}}, W^{\mathrm{C}}\right)$ and $W_{\mathrm{cC}}^{\sim}$ approximate it on the unit circle, and so does $(W_{\mathrm{cC}} + W_{\mathrm{cC}}^{\sim})/2$. Also, since W_{cC} has no poles at the origin, $(W_{\mathrm{cC}} + W_{\mathrm{cC}}^{\sim})/2$, which is Hermitian on the unit circle, is again proper. Thus, there is no loss of generality in assuming that $W^{\mathrm{c}}(\mathrm{e}^{\mathrm{j}\theta})$ and $W^{\mathrm{C}}(\mathrm{e}^{\mathrm{j}\theta})$ are Hermitian for all $\theta \in \mathbf{R}$. Next, without loss of generality, we may assume W has a proper inverse. Indeed, if it does not, it can be replaced by $W + \epsilon I$ with ϵ small enough for (27) and (28) to be preserved. Now note that, under the assumption that (d)*(i)* holds, strict positive realness of $W_1 W_2$ is equivalent to the condition

$$\mathrm{He}\left((W_1(\mathrm{e}^{\mathrm{j}\theta}))^{-1}(W_2(\mathrm{e}^{\mathrm{j}\theta}))^*\right) > 0 \quad \forall\theta$$

and strict positive realness of $W_2^{-1}\tilde{P}W_1^{-1}$ is equivalent to the condition

$$\mathrm{He}\left(\tilde{P}(\mathrm{e}^{\mathrm{j}\theta})(W_1(\mathrm{e}^{\mathrm{j}\theta}))^{-1}(W_2(\mathrm{e}^{\mathrm{j}\theta}))^*\right) > 0 \quad \forall\theta$$

(both of these conditions are obtained via congruence transformations). To complete the proof of the implication (c)⇒(d) it is thus enough to factorize W into $W = W_1^{-1}W_2^{\sim}$ where W_1 and W_2 satisfy (d)*(i)*. The desired factorization of W is possible, in view of Proposition 4.1 and the well-known symmetric factorization result mentioned in the subsequent remark. It remains to apply this result to the block entries of W^{r}, W^{c}, and W^{C} (where W^{r}, W^{c}, and W^{C} all satisfy (4), and both $W^{\mathrm{c}}(\mathrm{e}^{\mathrm{j}\theta})$ and $W^{\mathrm{C}}(\mathrm{e}^{\mathrm{j}\theta})$ are Hermitian for all θ), and define W_1 and W_2 in the obvious way. This completes the proof of the implication (c)⇒(d).

(d) ⇒ (a): Suppose that (d) holds and let W_1, W_2 satisfy *(i)-(iii)*. In particular

$$W_2^{-1}(\mathrm{e}^{\mathrm{j}\theta})\tilde{P}(\mathrm{e}^{\mathrm{j}\theta})W_1^{-1}(\mathrm{e}^{\mathrm{j}\theta}) + (W_1^{-1}(\mathrm{e}^{\mathrm{j}\theta}))^*(\tilde{P}(\mathrm{e}^{\mathrm{j}\theta}))^*(W_2^{-1}(\mathrm{e}^{\mathrm{j}\theta}))^* > 0 \quad \forall\theta \in [0, 2\pi],$$

or, equivalently (via congruence transformation),

$$\tilde{P}(\mathrm{e}^{\mathrm{j}\theta})W_1^{-1}(\mathrm{e}^{\mathrm{j}\theta})(W_2(\mathrm{e}^{\mathrm{j}\theta}))^* + W_2(\mathrm{e}^{\mathrm{j}\theta})(W_1^{-1}(\mathrm{e}^{\mathrm{j}\theta}))^*(\tilde{P}(\mathrm{e}^{\mathrm{j}\theta}))^* > 0 \quad \forall\theta \in [0, 2\pi].$$

Multiplying on the left by $(I-P)$ and on the right by $(I-P)^*$ (another congruence transformation) we get

$$P(\mathrm{e}^{\mathrm{j}\theta})D(\mathrm{e}^{\mathrm{j}\theta})(P(\mathrm{e}^{\mathrm{j}\theta}))^* + G(\mathrm{e}^{\mathrm{j}\theta})(P(\mathrm{e}^{\mathrm{j}\theta}))^* - P(\mathrm{e}^{\mathrm{j}\theta})G(\mathrm{e}^{\mathrm{j}\theta}) - D(\mathrm{e}^{\mathrm{j}\theta}) < 0 \quad \forall\theta \in [0, 2\pi],$$

with $D(\mathrm{e}^{\mathrm{j}\theta}) = \mathrm{He}(\ (W_1(\mathrm{e}^{\mathrm{j}\theta}))^{-1}(W_2(\mathrm{e}^{\mathrm{j}\theta}))^*)$, and $G(\mathrm{e}^{\mathrm{j}\theta}) = \mathrm{Sh}(\ (W_1(\mathrm{e}^{\mathrm{j}\theta}))^{-1}(W_2(\mathrm{e}^{\mathrm{j}\theta}))^*)$. It is easy to check that $D(\mathrm{e}^{\mathrm{j}\theta}) = (D(\mathrm{e}^{\mathrm{j}\theta}))^*$ and $G(\mathrm{e}^{\mathrm{j}\theta}) = -(G(\mathrm{e}^{\mathrm{j}\theta}))^*$ for all θ, and condition *(ii)* implies that

$D(e^{j\theta}) > 0$ for all θ. Since the left-hand side and $D(e^{j\theta})$ are both continuous over the compact set $[0, 2\pi]$, it follows that, for $\alpha \in (0,1)$ close enough to 1 and for all $\theta \in [0, 2\pi]$,

$$P(e^{j\theta})D(e^{j\theta})(P(e^{j\theta}))^* + G(e^{j\theta})(P(e^{j\theta}))^* - P(e^{j\theta})G(e^{j\theta}) - \alpha D(e^{j\theta}) < 0.$$

Since $\alpha \in (0,1)$, this implies that $\|P\|_{\hat{\mu}} < 1$. This completes the proof of the implication (d)⇒(a). The remaining assertion follows directly from the same congruence transformations. The proof of Theorem 4.2 is complete.

A.4 Proof of Proposition 5.1

The proof is by construction. Given any $x_j \in \mathcal{H}_{\text{FIR}}$ of order N_j, for all $z \in \partial\mathbf{D}$

$$\begin{aligned} L(x_1(z), \cdots, x_\ell(z), F_0(z), \cdots, F_\ell(z)) &= F_0(z) + (F_0(z))^* + \textstyle\sum_{j=1}^{\ell}(x_j(z)F_j(z) + (x_j(z))^*(F_j(z))^*) \\ &= R(z) + R^\sim(z) \end{aligned}$$

with $R(z) = F_0(z) + \sum_{j=1}^{\ell}\sum_{i=1}^{N_j}(a_i^j z^i F_j(z) + b_i^j z^{-i} F_j(z))$. Let us decompose $F_0(z)$ and for each (i,j), $z^i F_j(z)$ and $z^{-i}F_j(z)$ as

$$F_0(z) = P_0(z) + G_0(z)$$

and

$$z^i F_j(z) = P_{ij}^+(z) + G_{ij}^+(z)$$

and

$$z^{-i} F_j(z) = P_{ij}^-(z) + G_{ij}^-(z)$$

where the entries of P_0, P_{ij}^+ and P_{ij}^- are polynomials and those of G_0, G_{ij}^+ and G_{ij}^- are proper rational functions, yielding

$$R(z) = P_0(z) + G_0(z) + \sum_{j=1}^{\ell}\sum_{i=1}^{N_j}\left(a_i^j(P_{ij}^+(z) + G_{ij}^+(z)) + b_i^j(P_{ij}^-(z) + G_{ij}^-(z))\right).$$

Note that $P_0^\sim(z)$ and for all i, j, $P_{ij}^{+\sim}(z)$ and $P_{ij}^{-\sim}(z)$, which appear in $R^\sim(z)$, are proper and let

$$S(z) = P_0^\sim(z) + G_0(z) + \sum_{j=1}^{\ell}\sum_{i=1}^{N_j}\left(a_i^j(P_{ij}^{+\sim}(z) + G_{ij}^+(z)) + b_i^j(P_{ij}^{-\sim}(z) + G_{ij}^-(z))\right).$$

Clearly

$$L(x_1(z), \cdots, x_\ell(z), F_0(z), \cdots, F_\ell(z)) = S(z) + S^\sim(z).$$

To complete the proof, simply denote $H_0(z) = P_0^\sim(z) + G_0(z)$ and denote by $H_k(z)$, $k = 1, \ldots, t$ all rational proper functions of the form $P_{ij}^{+\sim}(z) + G_{ij}^+(z)$ and $P_{ij}^{-\sim}(z) + G_{ij}^-(z)$ and by p_k the corresponding coefficients a_i^j and b_i^j.

A.5 Proof of Proposition 6.1

Equivalence of (a) and (b) follows directly from Lemma 3.1. Let $\phi(z) = \frac{1+z}{1-z}$ which is the familiar bilinear transformation that maps $\mathbf{C}_-$ onto $\mathbf{D}$, and its inverse $\phi^{-1}(z) = \frac{z-1}{z+1}$ maps $\mathbf{D}$ back to $\mathbf{C}_-$. The implication (a)⇒(c) follows from the following equivalent statements.

For every $z \in \partial_e \mathbf{C}_-$, there exist complex numbers x_i, $i = 1, \ldots, \ell$, such that $L(x_1, \cdots, x_\ell, F_0(z), \cdots, F_\ell(z)) > 0$,

$\iff$ for every $z^d \in \partial \mathbf{D}$, there exist complex numbers x_i, $i = 1, \ldots, \ell$, such that $L(x_1, \cdots, x_\ell, F_0(\phi^{-1}(z^d)), \cdots, F_\ell(\phi^{-1}(z^d))) > 0$,

$\iff$ there exist $x_i^d \in \mathcal{H}_{\text{FIR}}, i = 1, \ldots, \ell$, such that, for all $z^d \in \partial \mathbf{D}$ $L(x_1^d(z^d), \cdots, x_\ell^d(z^d), F_0(\phi^{-1}(z^d)), \cdots, F_\ell(\phi^{-1}(z^d))) > 0$ (by Proposition 3.2).

Now, we define $z = \phi^{-1}(z^d)$ and $x_i(z) = x_i^d(\phi(z))$, $i = 1, \ldots, \ell$. Since $x_i^d(\phi(z)) \in \mathcal{H}_{\text{UDR}}$, $i = 1, \ldots, \ell$, there exist $x_i \in \mathcal{H}_{\text{UDR}}$, $i = 1, \ldots, \ell$, such that, for all $z \in \partial_e \mathbf{C}_-$

$$L(x_1(z), \cdots, x_\ell(z), F_0(z), \cdots, F_\ell(z)) > 0.$$

The implication (c)⇒(a) is trivial.

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