$$\textit{such that} \quad \begin{cases} lI \leq X_k^{1/2} Y X_k^{1/2} \leq I \\ Y \in \mathcal{Y} \end{cases}$$

Each min/max problem is an LMI convex programming problem and can be solved effectively. We hope that after iterations $1 \leftarrow \mu$ and $l \rightarrow 1$. It is shown in [16] that this simple algorithm does not work very well. In order for the min/max algorithm to work, two properties need to be true

- The sets $\mathcal{X}$ and $\mathcal{Y}$ need to be closed
- For any $X \in \mathcal{X}$, we need $\kappa X \in \mathcal{X}\ \forall\ \kappa > 1$. For any $Y \in \mathcal{Y}$, we need $\kappa Y \in \mathcal{Y}\ \forall\ 0 < \kappa < 1$.

To satisfy these two properties, we need to modify $\mathcal{X}$ and $\mathcal{Y}$ as

$$\mathcal{X} := \{(X, \alpha) \in \mathcal{R}^{n \times n} \times \mathcal{R} : \mathcal{C}_X < -I\} \tag{2.18}$$

$$\mathcal{Y} := \left\{(Y, \beta) \in \mathcal{R}^{n \times n} \times \mathcal{R} : \mathcal{C}_Y < -\beta^2 I\right\} \tag{2.19}$$

where

$$\mathcal{C}_X(X, \alpha) := \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp'} \begin{bmatrix} AX + XA' + \gamma^{-2}\alpha B_1 B_1' & XC_1' \\ C_1 X & -\alpha I \end{bmatrix} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp} \tag{2.20}$$

$$\mathcal{C}_Y(Y, \beta) := \begin{bmatrix} C_3'^{\perp} & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} A'Y + YA + \beta C_1' C_1 & YB \\ B_1' Y & -\gamma^2 \beta I \end{bmatrix} \begin{bmatrix} C_3'^{\perp} & 0 \\ 0 & I \end{bmatrix} \tag{2.21}$$

It can be shown that $\mathcal{X}$ and $\mathcal{Y}$ have the following properties.

Lemma 2.2 *Given the sets shown in (2.18) and (2.19), we have the following properties*

$$(X, \alpha) \in \mathcal{X} \Rightarrow \kappa(X, \alpha) \in \mathcal{X} \quad \forall \quad \kappa > 1 \tag{2.22}$$

$$(Y, \beta) \in \mathcal{Y} \Rightarrow \kappa(Y, \beta) \in \mathcal{Y} \quad \forall \quad 0 < \kappa < 1 \tag{2.23}$$

Proof. Assume that $(X, \alpha) \in \mathcal{X}$. Therefore for any $\kappa > 1$

$$\mathcal{C}_X(\kappa X, \kappa\alpha) = \kappa \mathcal{C}_X(X, \alpha) < -\kappa I < -I$$

This assures that $(\kappa X, \kappa\alpha) \in \mathcal{X}$. Similarly, assume $(Y, \beta) \in \mathcal{Y}$. For any $0 < \kappa < 1$, we have

$$\mathcal{C}_Y(\kappa Y, \kappa\beta) = \kappa \mathcal{C}_Y(Y, \beta) < -\kappa^{-1}(\kappa\beta)^2 I < -(\kappa\beta)^2 I$$

This implies that $(\kappa Y, \kappa\beta) \in \mathcal{Y}$ also. ¶

The significance of Lemma 2.2 means that the min/max algorithm can be used to find a $(P,\ \theta) \in \mathcal{P}$ where

$$\mathcal{P} := \left\{(P,\ \theta) \in \mathcal{R}^{n\times n} \times \mathcal{R}^n : P > 0, 0 < \theta < 1,\ (P^{-1},\ \theta^{-1}) \in \mathcal{X}, (P,\ \theta) \in \mathcal{Y}\right\} \quad (2.24)$$

Algorithm 2.1 The min/max algorithm for solving $\mathcal{H}_\infty$ problem: constant output feedback case

1. *Choose ϵ to be a sufficiently small constant and initialize $Y^{(0)} > 0$ and $\beta^{(0)} > 0$ and k=1.*

2. *Solve the convex optimization problem*

$$(X^{<k>}, \alpha^{<k>}, \mu^{<k>}) := arg \min_{X,\alpha,\mu} \Big\{\mu | I \le Y^{<k-1>^{1/2}} X Y^{<k-1>^{1/2}} \le \mu I, \\ 1 \le \beta^{<k-1>}\alpha \le \mu,\ (X, \alpha) \in \mathcal{X}\Big\} \quad (2.25)$$

3. *Solve the convex optimization problem*

$$(Y^{<k>}, \beta^{<k>}, l^{<k>}) := arg \max_{Y,\beta,l} \Big\{l | lI \le X^{<k>^{1/2}} Y X^{<k>^{1/2}} \le I, \\ l \le \alpha^{<k>}\beta \le 1,\ (Y, \beta) \in \mathcal{Y}\Big\} \quad (2.26)$$

4. *If $\mu^{<k>} - l^{<k>} < \epsilon$, then stop; the algorithm is successful. If $\lambda_{min}(Y^{<k>}) < \epsilon$ or $\beta < \epsilon$, then stop; the algorithm failed. Otherwise let $k = k + 1$ and go to step 2.*

Lemma 2.2 assures that if $\mathcal{P}$ is not empty, we will have the following result as $k \to \infty$.

$$X = Y^{-1},\quad \alpha = \beta^{-1},\quad \mu = l = 1;$$

Therefore $(Y, \beta) \in \mathcal{P}$. It is proved in [16] that the min/max algorithm has the following properties.

$$l^{<k>} < \frac{1}{\mu^{<k+1>}} < l^{<k+1>} < \frac{1}{\mu^{<k+1>}} < \ldots \leq 1 \quad \forall\, k \geq 1,$$
$$Y^{<k>} \geq X^{<k+1>^{-1}} \geq Y^{<k+1>} \geq X^{<k+2>^{-1}} \geq \ldots \quad \forall\, k \geq 1,$$
$$\beta^{<k>} \geq \frac{1}{\alpha^{<k+1>}} \geq \beta^{<k+1>} \geq \frac{1}{\alpha^{<k+2>}} \geq \ldots \quad \forall\, k \geq 1.$$

The two minimization problems in our algorithm are LMI problems, therefore they can be solved effectively. It is obvious that (2.25) is an LMI. It is not so obvious that (2.26) is an LMI because of the β^2 on the right hand side of (2.19). (2.19) can be made into an LMI by applying the Schur complement as

$$\begin{bmatrix} \mathcal{C}_Y & \beta I \\ \beta I & -I \end{bmatrix} < 0$$

So (2.26) is also an LMI optimization problem.

After finding a $(P,\ \theta) \in \mathcal{P}$ using the previous algorithm, we have the following main theorem

Theorem 2.2 *Let $(P,\ \theta) \in \mathcal{P}$, then $\bar{P} := P/\theta$ satisfies the inequalities (2.13) and (2.14).*

Proof. Because $(P,\ \theta) \in \mathcal{P}$, we have $\mathcal{C}_Y(P,\ \theta) < -\theta^2 I$, i.e.

$$\begin{bmatrix} C_3'^{\perp} & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} A'P + PA + \theta C_1' C_1 & \bar{P}B \\ B_1'\bar{P} & -\gamma^2\theta I \end{bmatrix} \begin{bmatrix} C_3'^{\perp} & 0 \\ 0 & I \end{bmatrix} < -\theta^2 I.$$
$$\Rightarrow \begin{bmatrix} C_3'^{\perp} & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} A'\frac{\bar{P}}{\theta} + \frac{\bar{P}}{\theta}A + C_1' C_1 & \frac{\bar{P}}{\theta}B \\ B_1'\frac{\bar{P}}{\theta} & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C_3'^{\perp} & 0 \\ 0 & I \end{bmatrix} < -\theta I.$$
$$\Rightarrow \begin{bmatrix} C_3'^{\perp} & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} A'\bar{P} + \bar{P}A + C_1' C_1 & \bar{P}B \\ B_1'\bar{P} & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C_3'^{\perp} & 0 \\ 0 & I \end{bmatrix} < -\theta I. < 0$$

Thus shows that (2.14) is true. We also have $\mathcal{C}_Y(\bar{P}^{-1},\ \theta^{-1}) < I$, i.e.

$$\begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp\prime} \begin{bmatrix} A\bar{P}^{-1} + \bar{P}^{-1}A' + \gamma^{-2}\theta^{-1}B_1B_1' & \bar{P}^{-1}C_1' \\ C_1\bar{P}^{-1} & -\theta^{-1}I \end{bmatrix} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp} < -I$$

$$\Rightarrow \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp'} \begin{bmatrix} A\theta\bar{P}^{-1} + \theta\bar{P}^{-1}A' + \gamma^{-2}B_1B_1' & \theta\bar{P}^{-1}C_1' \\ C_1\theta\bar{P}^{-1} & -I \end{bmatrix} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp} < -\theta I$$

$$\Rightarrow \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp'} \begin{bmatrix} AP^{-1} + P^{-1}A' + \gamma^{-2}B_1B_1' & P^{-1}C_1' \\ C_1P^{-1} & -I \end{bmatrix} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp} < -\theta I < 0$$

This shows that (2.13) is true. Therefore $\bar{P}$ solves problem (2.6) for some G having the form in (2.9). ¶

The G obtained from (2.9) will make the $\mathcal{H}_\infty$ norm of the closed-loop static output feedback system

$$\|T_{zw}(G)\|_\infty < \gamma$$

2.3 Example 2.1

In this section we will present an example to demonstrate the min/max algorithm for finding a static output feedback controller such that the closed-loop system has $\mathcal{H}_\infty$ norm smaller than γ. The system matrices are as follows.

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.456 \\ 0.0482 & -1.01 & 0.0024 & -4.02 \\ 0.1 & 0.368 & -0.707 & 1.42 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.442 & 0.176 \\ 3.54 & -7.59 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, C_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

Here we are looking for a static output feedback G so that the closed-loop $\mathcal{H}_\infty$ norm is less than $\gamma = 13$.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$\mu^{<k>}$	6.096	1.866	1.035	1.003	1
$\alpha^{<k>}$	6.096	50.16	51.93	52.07	52.09
$l^{<k>}$	0.2267	0.825	0.9908	0.9992	0.9999
$\beta^{<k>}$	0.03719	0.01993	0.01926	0.0192	0.0192

Table 2.1: Result of the first five iterations of the min/max algorithm applied to Example 2.1

Using the min/max algorithm starting from $Y^{(0)} = I$ and $\beta^{(0)} = 1$ and $\epsilon = 1e^{-6}$, we achieve convergence in 9 iterations. Table 2.1 shows that at $k = 4$ the μ and l are already very close to 1, and $\beta \approx \alpha^{-1}$ and $Y \approx X^{-1}$.

$$Y^{(4)} = \begin{bmatrix} 0.196 & -0.0549 & -0.0886 & -0.0735 \\ -0.0549 & 0.114 & 0.0362 & 0.065 \\ -0.0886 & 0.0362 & 0.0968 & 0.044 \\ -0.0735 & 0.065 & 0.044 & 0.116 \end{bmatrix}$$

$$X^{(4)^{-1}} = \begin{bmatrix} 0.196 & -0.0549 & -0.0886 & -0.0735 \\ -0.0549 & 0.114 & 0.0362 & 0.065 \\ -0.0886 & 0.0362 & 0.0969 & 0.044 \\ -0.0735 & 0.065 & 0.044 & 0.116 \end{bmatrix}$$

Eventually, we obtain Y as

$$Y = \begin{bmatrix} 0.196 & -0.0549 & -0.0886 & -0.0735 \\ -0.0549 & 0.114 & 0.0362 & 0.065 \\ -0.0886 & 0.0362 & 0.0968 & 0.044 \\ -0.0735 & 0.065 & 0.044 & 0.116 \end{bmatrix},$$

With $\rho = 1$ and $L = 0$, we obtained G from (2.9) as

$$G = \begin{bmatrix} 1.06 \\ 1.44 \end{bmatrix}$$

We have $\|T_{zw}(G)\|_\infty = 12.4751 < \gamma$.

2.4 The Gradient of the γ-contour

Consider the following system with state feedback

$$\begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_1 u \\ u = Kx \end{cases} \tag{2.27}$$

Definition 2.1 *γ-contour*

The γ-contour is defined as as the subset of the set of stabilizing state feedbacks K, such that $\|T_{zw}(K)\|_\infty = \gamma$. We denote it as

$$\mathcal{K}_\gamma = \{K | \|T_{zw}(K)\|_\infty = \gamma\} \tag{2.28}$$

Lemma 1.4 showed that all the optimal solutions of the $\mathcal{H}_2/\mathcal{H}_\infty$ will lie on the boundary of the $\mathcal{H}_\infty$ constraint. For any K such that $\|T_{zw}(K)\|_\infty = \gamma$, the eigenvalues of the associated Hamiltonian matrix

$$H := \begin{bmatrix} A + B_2 K & \gamma^{-2} B_1 B_1' \\ -(C_1 + D_1 K)'(C_1 + D_1 K) & -(A + B_2 K)' \end{bmatrix} \tag{2.29}$$

will include pure imaginary defective eigenvalues. It is of interest to study the local behavior of these pure imaginary eigenvalues along the γ-contour $\mathcal{K}_\gamma$.

We need to discuss the properties of the eigenvalue derivative of a repeated eigenvalue first.

2.4.1 The Derivative of Repeated Eigenvalues/Eigenvectors

The local behavior of an eigenvalue has strong connection with the derivative of the eigenvalue and its corresponding eigenvector derivative. Usually the eigenvalue derivative can be computed easily. However difficulties will occur if a matrix has repeated eigenvalues. It becomes even worse when the repeated eigenvalue is defective and the eigenvalue is on the imaginary axis. In this section, we will discuss only the eigenvalue

behavior of a real matrix. The eigenvalue λ of a real matrix is defective if the Jordan block of the matrix contains the sub-block

$$\begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots \\ & & \vdots & & \\ 0 & \dots & 0 & \lambda & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{bmatrix}$$

The eigenspace corresponding to the defective eigenvalue is one-dimensional.

The eigenvalue λ is non-defective if the corresponding Jordan sub-block is

$$\begin{bmatrix} \lambda & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & 0 & \dots \\ & & \vdots & & \\ 0 & \dots & 0 & \lambda & 0 \\ 0 & \dots & \dots & 0 & \lambda \end{bmatrix}$$

The eigenspace corresponding to the non-defective eigenvalue is m-dimensional if the multiplicity of λ is m.

We summarize this property as follows.

- If a one parameter non-defective matrix has repeated eigenvalues, the repeated eigenvalues are not differentiable with respect to that parameter. However the derivative in the neighborhood of the repeated eigenvalue does exist. Or we can say that the eigenvalue derivative at the repeated eigenvalues has a multiple derivative.

- If a one parameter matrix has defective repeated eigenvalues, these eigenvalues are not differentiable. The derivative in the neighborhood of the repeated eigenvalue approaches ∞ as it gets closer to the defective repeated eigenvalue.

Derivative of a Repeated Eigenvalue [5, 7, 18, 28, 40, 41]

Given a one parameter(p) eigenvalue problem with repeated eigenvalues of order m at $p = 0$.

$$(H(p) - \lambda(p)I)\, u(p) = 0 \tag{2.30}$$

where $H(p)$ is an $n \times n$ matrix depending on the scalar parameter p, $\lambda(p)$ is a repeated eigenvalue of $H(p)$ of order m, and $u(p)$ is the eigenvector of $H(p)$ corresponding to $\lambda(p)$. We know that the dimension of the invariant space corresponding to the repeated eigenvalue will be m. Let U_1 and V_1 be $n \times m$ matrices that span the right and left invariant space respectively. If these repeated eigenvalues are non-defective at $p = 0$, U_1 and V_1 can be chosen such that $V_1'U_1 = I$. Any vector in the right invariant space can be represented as

$$u(p) = U_1(p)c(p) = U_1(p)\begin{bmatrix} c_1(p) \\ \vdots \\ c_m(p) \end{bmatrix} \tag{2.31}$$

So (2.30) can be rewritten as

$$(H(p) - \lambda(p)I)\, U_1(p)c(p) = 0 \tag{2.32}$$

Taking the first derivative of (2.32) gives

$$\left(\frac{\partial H}{\partial p} - \frac{\partial \lambda}{\partial p}I\right) U_1 c + (H - \lambda I)\frac{\partial U_1 c}{\partial p} = 0 \tag{2.33}$$

Multiplying (2.33) in front by V_1', we have

$$(V_1'\frac{dH(p)}{dp}U_1 - \frac{d\lambda}{dp}V_1'U_1)c = (V_1'\frac{dH(p)}{dp}U_1 - \frac{d\lambda}{dp}I)c = 0. \tag{2.34}$$

(2.34) is an eigenvalue problem of size m. The m eigenvalues of (2.34) are the eigenvalue derivatives at the repeated eigenvalues λ when $p = 0$. There are m derivatives because when parameter p of the matrix is perturbed, the repeated eigenvalues at $p = 0$ will split into m distinct eigenvalues. Each eigenvalue derivative we get from (2.34)

evaluated at $p = 0$ describes how each eigenvalue split. When the eigenvalues split, the eigenspaces will also split into m distinct 1-dimensional eigenspaces corresponding to each eigenvalue.

The reason that a defective repeated eigenvalue has an ∞ derivative, is that the left and right invariant spaces must be orthogonal. One simple example reveals this. Given a matrix

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

The matrix has right and left eigenspaces spanned by U_1,V_1 where

$$U_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad V_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

So (2.34) can not be true unless $d\lambda/dp$ is ∞ because $V_1'U_1 = 0$.

One interesting observation for defective and non-defective repeated eigenvalues is shown in Figure 2.1. For non-defective repeated eigenvalues, the perturbed eigenvalues will cross each other at the repeated eigenvalue. For defective repeated eigenvalues, the perturbed eigenvalues will move face to face and meet at the repeated eigenvalue then changed direction abruptly in the opposite way.

Derivative of a Repeated Eigenvector

Next we would like to obtain the derivative of the eigenvector U_1c corresponding to the repeated eigenvalue where c is one of the eigenvectors of (2.34). Assume the following normalizing condition is satisfied, i.e. we let U_1c be a unit vector.

$$c'U_1'U_1c = 1 \tag{2.35}$$

The eigenvector derivative can be expressed as $d(U_1c)/dp = \hat{U}\xi$, where $\hat{U} = [\hat{U}_1, \hat{U}_2]$, $\hat{U}_1 = U_1(0)$ and $\hat{U}_2 = U_2(0)$. $\hat{U}_1$ is the basis of the eigenspaces evaluated at $p = 0$

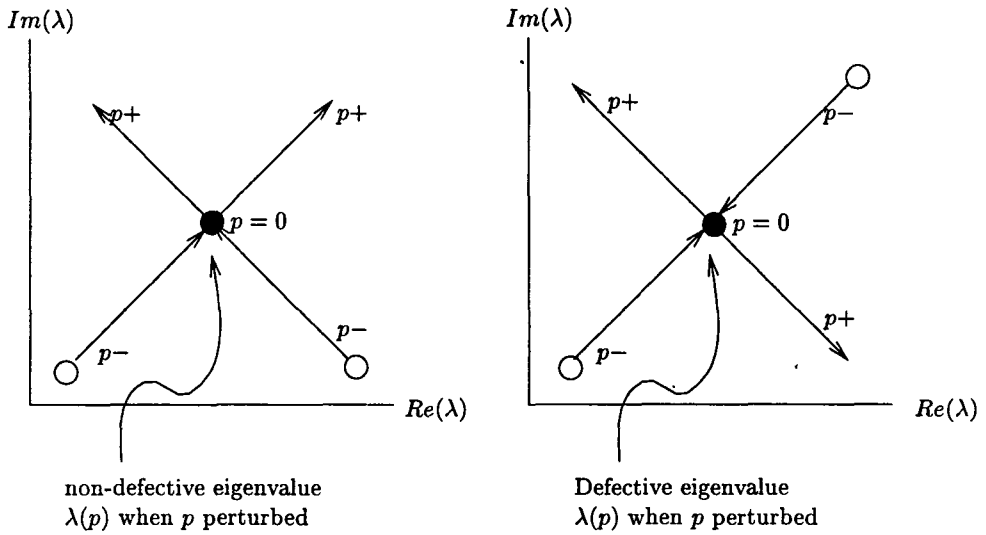

Figure 2.1: Repeated eigenvalues(defective/non-defective) movement when perturbed

corresponding to $\lambda(0)$. and ξ is an $n \times 1$ vector. To determine ξ is equivalent to computing the eigenvector derivative.

First we know that $H - \lambda I$ is of rank $(n - m)$ and that (2.33) has the structure

$$\left(\frac{\partial H}{\partial p} - \frac{\partial \lambda}{\partial p} I\right) \hat{U}_1 c + [\,(H - \lambda I)\hat{U}_1 \quad (H - \lambda I)\hat{U}_2\,] \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \\ \xi_{m+1} \\ \vdots \\ \xi_n \end{bmatrix}$$

$$= \left(\frac{\partial H}{\partial p} - \frac{\partial \lambda}{\partial p} I\right) \hat{U}_1 c + [\,0_{n\times m} \quad (H - \lambda I)\hat{U}_2\,] \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \\ \xi_{m+1} \\ \vdots \\ \xi_n \end{bmatrix} = 0 \qquad (2.36)$$

It is obvious that one can determine $\xi_{m+1}, \cdots, \xi_n$ by solving equation (2.36) because $(H - \lambda I)\hat{U}_2$ is of full column rank, $n - m$. $\hat{U}_2[\xi_{m+1}, \cdots, \xi_n]'$ is sometimes called the

homogeneous solution of the eigenvector derivative. The following derivation shows how we can obtain the particular solution of the eigenvector derivative.

To determine $\xi_1, \cdots, \xi_m$ one has to take the second derivative of our eigenvalue equation to obtain

$$\left(\frac{\partial^2 H}{\partial p^2} - \frac{\partial^2 \lambda}{\partial p^2} I\right) U_1 c + 2\left(\frac{\partial H}{\partial p} - \frac{\partial \lambda}{\partial p} I\right) \frac{\partial (U_1 c)}{\partial p} + (H - \lambda I) \frac{\partial^2 (U_1 c)}{\partial p^2} = 0 \qquad (2.37)$$

Let $\hat{V} = [\hat{V}_1 \ \hat{V}_2]$ be the span of the left eigenspace with $\hat{V}_1$ corresponding to the repeated eigenvalues evaluated at $p = 0$, i.e. $\hat{V}_1'(H(0) - \lambda(0)I) = 0$. Multiply $\hat{V}_1'$ in front of (2.37), we have

$$\hat{V}_1'\left(\frac{\partial^2 H}{\partial p^2} - \frac{\partial^2 \lambda}{\partial p^2} I\right) \hat{U}_1 c + 2\hat{V}_1'\left(\frac{\partial H}{\partial p} - \frac{\partial \lambda}{\partial p} I\right) \hat{U}_1 \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix} + 2\hat{V}_1'\left(\frac{\partial H}{\partial p} - \frac{\partial \lambda}{\partial p} I\right) \hat{U}_2 \begin{bmatrix} \xi_{m+1} \\ \vdots \\ \xi_n \end{bmatrix} = 0 \qquad (2.38)$$

Observe that (2.38) is m equations. We have already determined $d\lambda/dp$, c, $\xi_{m+1}, \cdots, \xi_n$. We have $\xi_1, \cdots, \xi_m$ and $\frac{\partial^2 \lambda}{\partial p^2}$ to be determined. We need an additional equation to be able to solve for the total of $m+1$ unknowns. This will come from taking the derivative of the normalizing equation (2.35) and evaluated at $p = 0$. It gives

$$c'\hat{U}_1'\hat{U}_2 \begin{bmatrix} \xi_{m+1} \\ \vdots \\ \xi_n \end{bmatrix} + c'\hat{U}_1'\hat{U}_1 \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix} = 0 \qquad (2.39)$$

Combining (2.38) and (2.39) we can solve for $\xi_1, \cdots, \xi_m$ and $\frac{\partial^2 \lambda}{\partial p^2}$ in a quadratic form. We will omit the details of solving them. Or one can go to the listed references. One other point is that this algorithm can be extended to suit the case when the repeated eigenvalue is complex.

2.4.2 The Gradient of the γ-Contour With Respect to K

In this subsection we concentrate on the case where H has two pairs of complex conjugate pure imaginary eigenvalues. We will treat the case where H has two zero

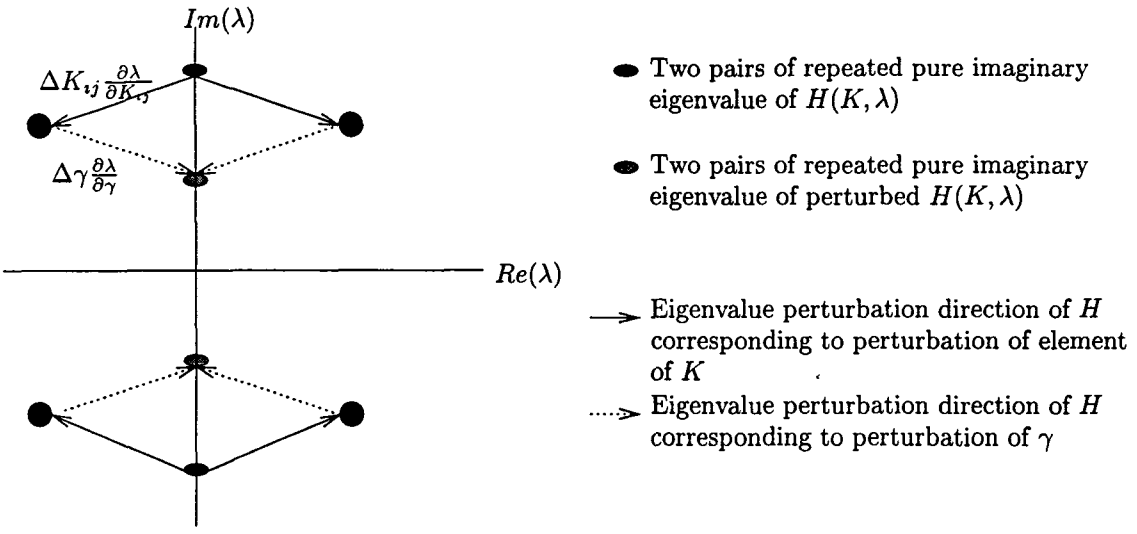

Figure 2.2: Perturbed imaginary eigenvalues of H corresponding to the perturbation of γ and element of K.

eigenvalues later. Our primary goal in this section is to obtain the perturbation of γ corresponding to perturbation of each element of K. These perturbations form the gradient of γ with respect to K. We will define the gradient of the γ-contour with respect to K as $[d\gamma/dk_{ij}]$ where k_{ij} is the (i,j) entry of K. $[d\gamma/dk_{ij}]$ is a column vector with entries $d\gamma/dk_{ij}$ in lexicographic order.

The gradient gives us a tangent plane along which to search for better performance without drifting too far away from the $\mathcal{H}_\infty$ constraint. The idea behind this is the simple fact that the Hamiltonian matrix for an arbitrary K and its corresponding γ always has at least two pairs of pure imaginary eigenvalues. But we will assume that the Hamiltonian matrices have exactly two pairs of pure imaginary eigenvalues for all stabilizing K.

So when we perturb K along the γ-contour, the associated Hamiltonian matrix must also be perturbed in such a way that the perturbed H continues to have two pairs of pure imaginary eigenvalues as illustrated in Figure 2.2. It is easy to see from

Figure 2.2 that

$$Re(\Delta k_{ij}\frac{d\lambda}{dk_{ij}} + \Delta\gamma\frac{d\lambda}{d\gamma}) = 0 \tag{2.40}$$

where λ is the pure imaginary eigenvalue of H. Here we make use of the eigenvalue derivative discussed in previous subsection.

Given a nominal pair (K,γ), consider the following eigenfunction equation.

$$(H(K,\gamma) - \lambda I)u = 0; \tag{2.41}$$

A perturbation of Δk_{ij}, $\Delta\gamma$ will lead to the following first-order approximation.

$$\left(H(K,\gamma) + \frac{dH}{dk_{ij}}\Delta k_{ij} + \frac{dH}{d\gamma}\Delta\gamma - (\lambda + \frac{d\lambda}{dk_{ij}}\Delta k_{ij} + \frac{d\lambda}{d\gamma}\Delta\gamma)I\right) \left(u + \frac{du}{dk_{ij}}\Delta k_{ij} + \frac{du}{d\gamma}\Delta\gamma\right) = 0 \tag{2.42}$$

We choose $\lambda = j\omega$ be the pure imaginary eigenvalue of H. In order to remain on the imaginary axis, (2.40) must be true. This in turn gives a simple representation of the perturbations of γ vs k_{ij} as

$$\frac{\Delta\gamma}{\Delta k_{ij}} = \frac{-Re(d\lambda/dk_{ij})}{Re(d\lambda/d\gamma)} \tag{2.43}$$

(2.43) requires us to compute the eigenvalue derivative of λ with respect to k_{ij} and to γ. As we mentioned, the pure imaginary eigenvalues of H are defective. So we have a problem in obtaining the eigenvalue derivative of a defective repeated eigenvalue because the derivative of this particular eigenvalue is not defined.

Because of these defective eigenvalues, (2.34) can not be directly applied in order to obtain the eigenvalue derivative. The derivative of the eigenvalue blows up. However the eigenvalue derivative in the neighborhood of the defective repeated eigenvalue still exists. And our interest is to find $d\gamma/dk_{ij}$. So we investigate the derivative in the neighborhood of the repeated defective eigenvalue with respect to γ and k_{ij} separately. (2.34) gives us

$$\frac{d\lambda}{d\gamma} = \frac{V_1'\frac{dH}{d\gamma}U_1}{V_1'U_1} \tag{2.44}$$

$$\frac{d\lambda}{dk_{ij}} = \frac{V_1' \frac{dH}{dk_{ij}} U_1}{V_1' U_1} \tag{2.45}$$

Let $V_1(p)$ and $U_1(p)$ be denoted as the neighborhood of V_1 and U_1 respectively. Then

$$V_1(p)' U_1(p) \neq 0 \quad \lim_{p \to 0} V_1(p)' U_1(p) = 0; \tag{2.46}$$

Putting (2.44) (2.45) into (2.43) and taking the limit, we have the following lemma.

Lemma 2.3 *Given a system as in (2.27), a stabilizing K which makes the norm of the closed-loop system $\|T_{zw}(K)\|_\infty = \gamma$ and a Hamiltonian matrix H defined as in (2.29). Then*

$$\frac{d\gamma}{dK} = [\, \tfrac{d\gamma}{dk_{ij}} \,] \tag{2.47}$$

$$\frac{d\gamma}{dk_{ij}} = -\frac{Re\left(V_1' \frac{dH}{dk_{ij}} U_1\right)}{Re\left(V_1' \frac{dH}{d\gamma} U_1\right)} \tag{2.48}$$

where U_1 and V_1 are the right and left eigenvectors corresponding to the defective repeated pure imaginary eigenvalue of H and $Re(.)$ denotes the real part of the argument.

Proof. The proof is done by means of the previous construction ¶

This Lemma gives an efficient way to compute the gradient of γ with respect to K. In Chapter 5, we use it to provide a direction for perturbing the K so that the closed-loop $\mathcal{H}_\infty$ norm will be strictly smaller than γ.

Chapter 3

Solving the $\mathcal{H}_2/\mathcal{H}_\infty$ Problem Using a BMI

Our original approach for solving the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem was to solve the highly coupled Riccati and Lyapunov type equations using a continuation method as described in Chapter 1. Due to the difficulties in applying the continuation method, we use the bilinear matrix inequality approach in this chapter to solve the $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem. To obtain the corresponding BMIs, we need to first introduce the following concepts.

3.1 Dissipative Systems and the Riccati Inequality

In [47, 50], Trentelman and Willems showed that the concept of dissipative system for a linear, time-invariant, finite-dimensional system with a quadratic supply rate and storage function gives a natural interpretation for studying the Riccati inequality for many control problems. It leads to the solvability of the Riccati inequality and the associated linear matrix inequality. We will briefly review the subject.

Dissipative System [47]

Consider a time invariant dynamic system Σ. Let z represent all accessible entries of Σ. In other words, z is all the measurable inputs/outputs of the dynamic system. The $s(z)$ is called a supply rate, as a function of z it represents the "performance" of Σ

such as power rate or energy consumption of the system. $s(z)$ is not limited to those performance measures, it can be any function that is of interest regarding the system. A precise definition of supply rate is complicated and unnecessary here because we will always use quadratic supply rates. A definition can be found in [47]. We define (Σ, s) to be dissipative if

$$\int_0^{t_1} s(z(t))dt \geq 0 \tag{3.1}$$

for any periodic $z(t)$ with period t_1.

Internal Dissipative Systems [47]

Given a time invariant dynamic system in state space form Σ_s with supply rate $s(z)$ and a function $V(x)$(function of state x), (Σ_s, s, V) is called internal dissipative if for all $t_1 > 0$

$$V(x(0)) + \int_0^{t_1} s(z(t))dt \geq V(x(t_1)) \tag{3.2}$$

Any function $V(x)$ which satisfies the inequality (3.2) is called a **storage function**.

From the definition of (3.2), it is clear that if a V is a storage function then $V+C$ is also a storage function for any $C \in \mathcal{R}$. It is convenient to define a normalized storage function at some particular x^* such that $V(x^*) = 0$.

Assumed that we normalize the storage function such that $V(0) = 0$, there exist two special storage functions that are important in optimization theory.

$$V_a(x) = \sup_z \left(-\int_0^{t_1} s(z(t))dt | t_1 \geq 0, \forall(z,x), x(0) = x, x(t_1) = 0 \right) \tag{3.3}$$

$$V_r(x) = \inf_z \left(\int_0^{t_1} s(z(t))dt | t_1 \geq 0, \forall(z,x), x(0) = 0, x(t_1) = x \right) \tag{3.4}$$

V_a is the *available storage* of Σ_s. V_r is called required supply Σ_s.

Two important properties of V [47]

- $V_a(x) \leq V(x) \leq V_r(x)$ where V is any normalized storage function.

- The set of all normalized storage functions $\mathcal{V}$ is convex.

Existence Condition of V for L.T.I Systems [47]

Given a linear time invariant controllable system

$$\Sigma_{DV} := \begin{cases} \dot{x} = Ax + Bw \\ z = Cx + Dw \end{cases} \tag{3.5}$$

Let $s = z'Mz$ be a quadratic supply rate, where $M = M'$. It can be shown that if there exist a storage function such that the system is internal dissipative then there exists at least one quadratic storage function of the form $V = x'Qx$. The following Lemma describes the details

Lemma 3.1 [47] *Given the system Σ_{DV} and supply rate s, the following statements are equivalent:*

1. *There exists $V : \mathcal{R}^n \to \mathcal{R}$ such that (Σ_{DV}, s, V) is internally dissipative.*
2. *There exists a symmetric $Q \in \mathcal{R}^{n\times n}$ such that if we define $V(x) := x'Qx$ then (Σ_{DV}, s, V) is internally dissipative.*
3. *For all (z, x) and for all $T \geq 0$ such that $x(0) = x(T) = 0$ we have $\int_0^T s(z(t))dt \geq 0$*
4. *If, in addition, Σ_{DV} is minimal then (Σ_{DV}, s) is dissipative.*

Proof. Please refer to [47] ¶

The set of all quadratic storage functions $V = x'Qx$ such that (Σ_{DV}, s, V) is internal dissipative can be characterized as the symmetric solutions of a linear matrix inequality by the following derivation. From the definition of internal dissipative system, we have

$$\int_0^{t_1} \left(-\frac{d}{dt}(x'Qx) + z'Mz\right) dt \geq 0 \quad or \tag{3.6}$$

$$\int_0^{t_1} \begin{bmatrix} x \\ z \end{bmatrix}' \begin{bmatrix} -A'Q - QA + C'MC & -QB + C'MD \\ -B'Q + D'MC & D'MD \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} dt \geq 0 \tag{3.7}$$

Because x, z are arbitrary, (3.7) implies

$$\begin{bmatrix} -A'Q - QA + C'MC & -QB + C'MD \\ -B'Q + D'MC & D'MD \end{bmatrix} \geq 0 \tag{3.8}$$

Let $G(s) = D + C(sI - A)^{-1}B$, then internal dissipativeness of (Σ_{DV}, s, V) is equivalent to

$$G(j\omega)^{\sim} M G(j\omega) \geq 0, \ \forall \omega \ such \ that \ j\omega \ \notin \sigma(A) \tag{3.9}$$

where $G(jw)^{\sim}$ is the conjugate transpose of $G(j\omega)$ and $\sigma(A)$ is the set of all eigenvalues of A.

The existence condition of a nonnegative storage function is described by the following lemma.

Lemma 3.2 [47] *Given the LTI system (3.5) and a predefined quadratic supply rate $s = z'Mz$ where M is symmetric matrix, the following statements are equivalent.*

1. *There exists a storage function $V \geq 0$ for (Σ, s). In particular, a quadratic storage function $V = x'Qx \geq 0$ where $Q \in \mathcal{R}^{n \times n}$ is symmetric and positive semi-definite.*

2. *Given $x(0) = 0$ and $T > 0$, $\int_0^T s(z(t))dt \geq 0 \ \forall w \in \mathcal{L}_2[0, T]$*

3. *There exists a $Q \geq 0$ such that the following LMI holds*

$$\begin{bmatrix} -A'Q - QA + C'MC & -QB + C'MD \\ -B'Q + D'MC & D'MD \end{bmatrix} \geq 0$$

4. $G(s)^{\sim} M G(s) \geq 0 \ \forall s, s \notin \sigma(A), Re(s) \geq 0$

Proof. Please refer to [47] ¶

Relation between LMI's and Riccati equations [47]

Assuming $D'MD > 0$(regularity assumption), and letting F denote the matrix on the left hand side of the LMI (3.8).

- Due to the regularity assumption, the Schur complement formula guarantees that there exists a symmetric matrix Q such that $F \geq 0$ if and only if the following Riccati inequality is true.

$$R(Q) := -A'Q - QA + C'MC - (-QB + C'MD)(D'MD)^{-1}(-B'Q + D'MC) \geq 0 \tag{3.10}$$

- Let Q_+ and Q_- be the maximum and minimum solutions of the Riccati equation $R(Q) = 0$. Corresponding to the quadratic supply rate $x'Mx$, the required supply V_r and the available storage V_a of the LTI system Σ_{DV} are

$$V_r = x'Q_+x, \quad V_a = x'Q_-x. \tag{3.11}$$

3.2 Application

Using the concept of dissipative system, several famous lemmas in control theory can be described easily by properly choosing the supply rate function s and applying the lemma in the previous section.

3.2.1 Positive Real Lemma

Consider a single-input/single-output linear time-invariant system

$$\Sigma_p := \begin{cases} \dot{x} = Ax + Bu \\ y = C_0x + D_0u \end{cases} \tag{3.12}$$

with the regularity assumption that $D_0' + D_0 > 0$. Let $G_0(s) = D_0 + C_0(sI - A)^{-1}B$ be the transfer function of Σ_p. The Positive Real Lemma gives the conditions for a

lossy system, i.e. $\int_0^\infty u'ydt \geq 0$ with $x(0) = 0$. By defining

$$z = Cx + Du = \begin{bmatrix} 0 \\ C_0 \end{bmatrix} x + \begin{bmatrix} I \\ D_0 \end{bmatrix} u \tag{3.13}$$

a supply rate $s = u'y$ can be rewritten as

$$s = z'Mz = z'(\frac{1}{2}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix})z \tag{3.14}$$

Using Lemma 3.2 , the following statements are equivalent.

1. There exists a quadratic storage function $V \geq 0$ such that (Σ_p, s, V) is internal dissipative.

2. Let (z, x) of Σ_p, $\forall u \in \mathcal{L}_2[0,T]$ such that $x(0) = 0$ and $T > 0$, $\int_0^T u'zdt \geq 0$.

3. The following LMI has a real symmetric solution $Q \geq 0$

$$\begin{bmatrix} -A'Q - QA & -QB + C_0' \\ -B'Q + C_0 & D_0' + D_0 \end{bmatrix} \geq 0 \tag{3.15}$$

4. $G_0^\sim(s) + G_0(s) \geq 0 \;\; \forall s \textit{ such that } s \notin \sigma(A),\; Re(s) \geq 0$ because

$$G^\sim(s)MG(s) = \begin{bmatrix} I \\ G_0(s) \end{bmatrix}^\sim \frac{1}{2}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} I \\ G_0(s) \end{bmatrix} = (G_0^\sim(s) + G_0(s)) \geq 0$$

5. The following Riccati inequality has a symmetric solution $Q \geq 0$

$$-A'Q - QA - (-B'Q + C_0)'(D_0' + D_0)^{-1}(-B'Q + C_0) \geq 0 \tag{3.16}$$

Items 1, 2, 3 and 4 are obvious by choosing M as in (3.14). Item 5 is equivalent to item 2 through the Schur complement formula and regularity assumption.

3.2.2 Bounded Real Lemma

Consider the following linear time-invariant system

$$\Sigma_p := \begin{cases} \dot{x} = Ax + Bw \\ y = C_0x + D_0w \end{cases} \tag{3.17}$$

with the assumption $\gamma^2 I - D_0' D_0 > 0$. Let $G_0(s) = D_0 + C_0(sI - A)^{-1}B$ denote the corresponding transfer function . By choosing

$$z = Cx + Dw = \begin{bmatrix} 0 \\ C_0 \end{bmatrix} x + \begin{bmatrix} I \\ D_0 \end{bmatrix} w \tag{3.18}$$

A supply rate $s = \gamma^2 w'w - y'y$ can be rewritten as

$$s = z'Mz = z'(\begin{bmatrix} \gamma^2 I_m & 0 \\ 0 & -I_p \end{bmatrix})z \tag{3.19}$$

Again, using Lemma 3.2, the following statements are equivalent.

1. There exists a quadratic storage function $V \geq 0$ such that (Σ_p, s, V) is internal dissipative.

2. Let (z, x) of Σ_p, $\forall u \in \mathcal{L}_2[0,T]$ such that $x(0) = 0$ and $T > 0$, $\int_0^T (\gamma^2 w'w - z'z)dt \geq 0$.

3. The following LMI has a real symmetric solution $Q \geq 0$

$$\begin{bmatrix} -A'Q - QA - C_0'C_0 & QB - C_0'D_0 \\ B'Q - D_0'C_0 & \gamma^2 I - D_0'D_0 \end{bmatrix} \geq 0 \tag{3.20}$$

4. $\gamma^2 I - G_0^{\sim}(s)G_0(s) \geq 0 \;\; \forall s \textit{ such that } s \notin \sigma(A), \; Re(s) \geq 0$

5. The following Riccati inequality has a real symmetric solution $Q \geq 0$.

$$-A'Q - QA - C_0'C_0 - (B'Q - D_0'C_0)'(\gamma^2 I - D_0'D_0)^{-1}(B'Q - D_0'C_0) \geq 0 \tag{3.21}$$

Item 1, 2 and 3 are again from the definition of internal dissipative system. Item 2 describes the requirement of the Bounded Real Lemma, i.e. $\gamma^2\|w\|_2^2 - \|y\|_2^2 \geq 0$ and $x(0) = 0$. Item 4 can be derived as

$$G^{\sim}(s)MG(s) = \begin{bmatrix} I \\ G_0(s) \end{bmatrix}^{\sim} \begin{bmatrix} \gamma^2 I_m & 0 \\ 0 & -I_p \end{bmatrix} \begin{bmatrix} I \\ G_0(s) \end{bmatrix} = \gamma^2 I - G_0^{\sim}(s)G_0(s) \geq 0$$

Item 5 is from item 3 through the Schur complement formula and regularity assumption.

3.2.3 $\mathcal{H}_\infty$ Control Problem

The H_∞ problem can also be explained within the concept of dissipative system and solved by the standard LMI solver. Both the state feedback and dynamic case can be easily explained. The dynamic controller case can be seen in [11]. The parameter dependent case can be seen in [10]. We will discuss only the state feedback case.

This problem is equivalent to the synthesis problem of the bounded real lemma. Given a standard H_∞ state feedback setup

$$\begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z_1 = C_1 x + D_1 u \\ y = x \end{cases} \tag{3.22}$$

with the following assumptions

Assumption 3.1

a. (A, B_2) *is stabilizable.*

b. D_1 *is of full column rank.*

The closed-loop system using state feedback is

$$\begin{cases} \dot{x} = (A + B_2 K)x + B_1 w \\ z_1 = (C_1 + D_1 K)x \end{cases} \tag{3.23}$$

This is a closed-loop system without the direct feedforward term from disturbance w. So the regularity assumption is satisfied automatically. We seek a K such that

$$\gamma^2 \|w\|_2^2 - \|z_1\|_2^2 \geq 0 \tag{3.24}$$

where γ is a predetermined constant. The problem can be interpreted as finding a control K such that

$$\frac{\|z_1\|_2}{\|w\|_2} \leq \gamma$$

$\forall w \in \mathcal{H}_2$.

Applying the bounded real lemma previously discussed, (3.21), without the direct feedforward D_0, we are looking for $Q \geq 0$ and K such that

$$(A + B_2K)'Q + Q(A + B_2K) + \gamma^{-2}QB_1B_1'Q + (C_1 + D_1K)'(C_1 + D_1K) \leq 0 \quad (3.25)$$

Completing the quadratic term in K and choosing $K = -\tilde{D}^{-1}(D_1'C_1 + B_2'Q)$, gives the familiar Riccati inequality

$$\tilde{A}'Q + Q\tilde{A} + Q(\gamma^{-2}B_1B_1' - B_2\tilde{D}^{-1}B_2')Q + C_1'(I - D_1\tilde{D}^{-1}D_1')C_1 \leq 0 \qquad (3.26)$$

where $\tilde{D} = D_1'D_1$ and $\tilde{A} = A - B_2\tilde{D}^{-1}D_1'C_1$. However (3.26) cannot be transformed into an LMI because $(\gamma^{-2}B_1B_1' - B_2\tilde{D}^{-1}B_2')$ is not non-negative. For convenience, let $D_1^{\perp}$ be a full column rank matrix such that $D_1^{\perp}D_1^{\perp\prime} = (I - D_1\tilde{D}^{-1}D_1')$ and $D_1^{\perp\prime}D_1 = 0$. As will be discussed in Chapter 4, if the size of z_1 is larger than the size of u then Q is guaranteed to be positive. If we assume that $Q > 0$ then defining $P = Q^{-1}$ gives the Riccati inequality

$$P\tilde{A}' + \tilde{A}P + (\gamma^{-2}B_1B_1' - B_2\tilde{D}^{-1}B_2') + PC_1'D_1^{\perp}D_1^{\perp\prime}C_1P \leq 0 \qquad (3.27)$$

(3.27) is equivalent to

$$\begin{bmatrix} \tilde{A}P + P\tilde{A}' + (\gamma^{-2}B_1B_1' - B_2\tilde{D}^{-1}B_2') & PC_1'D_1^{\perp} \\ D_1^{\perp\prime}C_1P & -I \end{bmatrix} \leq 0 \qquad (3.28)$$

Given each $P > 0$ such that (3.28) holds, there exists a $K = -\tilde{D}^{-1}(D_1'C_1 + B_2'P^{-1}) = -\tilde{D}^{-1}(D_1'C_1 + B_2'Q)$ such that (3.24) is satisfied. This means the $\mathcal{H}_\infty$ norm of the closed-loop triple $(A + B_2K, B_1, C_1 + D_1K)$ will be less than γ.

$\mathcal{H}_\infty$ central solution

The solution of the Riccati inequality (3.26) may not be unique. One special solution of the above $\mathcal{H}_\infty$ control problem is obtained by minimizing the objective $Tr(B_1B_1'Q)$.

$$\min Tr(B_1B_1'Q) \qquad (3.29)$$

where Q satisfies (3.26). This is equivalent to

$$\min Tr(B_1 B_1' P^{-1}) \tag{3.30}$$

where P satisfies (3.27). Using the Schur complement, we have the following LMI eigenvalue problem.

$$\begin{array}{ll} & \min(Tr(\Lambda)) \\ \textit{subject to} & \begin{bmatrix} \Lambda & B_1' \\ B_1 & P \end{bmatrix} \geq 0 \\ \textit{and} & \begin{bmatrix} \tilde{A}P + P\tilde{A}' + (\gamma^{-2}B_1B_1' - B_2\tilde{D}^{-1}B_2') & PC_1'D_1^{\perp} \\ D_1^{\perp\prime}C_1P & -I \end{bmatrix} \leq 0 \\ \textit{and} & P \geq 0 \end{array} \tag{3.31}$$

Instead of solving this LMI eigenvalue problem numerically, we can also solve the optimization problem through analyzing the Riccati inequality (3.26). It turns out that the minimum solution is the $Q \geq 0$ satisfying the the Riccati equation

$$\tilde{A}'Q + Q\tilde{A} + Q(\gamma^{-2}B_1B_1' - B_2\tilde{D}^{-1}B_2')Q + C_1'D_1^{\perp}D_1^{\perp\prime}C_1 = 0 \tag{3.32}$$

This is called the central controller in [15]. It also coincides with the solution of the minimum entropy problem [29].

3.2.4 Solving the LQ Problem

The state feedback LQ problem is to find a constant feedback control of the form $u = Kx$ such that the energy of the output is minimized given an initial condition x_0 and with no disturbance input, i.e. $w = 0$, for the following system

$$\begin{cases} \dot{x} = Ax + B_0w + B_2u \\ z_0 = C_0x + D_0u \end{cases} \tag{3.33}$$

subject to the following assumption

Assumption 3.2

a. *D_0 is of full column rank.*

b. *(A, B_2) is stabilizable.*

If we close the loop with state feedback K, the problem becomes

$$\begin{aligned} & \min_K \|z_0\|_2^2 \\ \textit{such that} \quad & \begin{cases} \dot{x} = (A + B_2K)x \\ z_0 = (C_0 + D_0K)x \end{cases} \\ \textit{and} \quad & x(0) = x_0 \end{aligned} \tag{3.34}$$

We temporarily remove B_0w from the closed-loop system because it plays no role under the condition that $w = 0$. It is known that there exists a quadratic storage function(Lyapunov function) for any observable system such that the closed-loop system is internal dissipative. Here we choose $z = z_0$ and the supply rate $s = \|z_0\|^2 = z_0'Iz_0$ with $M = I$. Using this concept, the LQ problem is equivalent to finding the minimal positive storage(Lyapunov) function over K that bounds $\|z_0\|_2^2$. In fact, $\|z_0\|_2^2$ is exactly the available storage $V_a(x_0) = x_0'Q^-x_0$ with opposite sign. That is, the LQ problem is equivalent to

$$\begin{aligned} & \min \ x_0'Qx_0 \\ \textit{such that} \quad & Q \geq 0 \\ \textit{and} \quad & x_0'Qx_0 \geq \|z_0\|_2^2 \end{aligned} \tag{3.35}$$

where z_0 is as in (3.34). Equation (3.35) can be rewritten as

$$\begin{aligned} & \int_0^\infty \dot{x}'Qx + x'Q\dot{x} + z_0'z_0 \ dt \\ = & \int_0^\infty x'[(A + B_2K)'Q + Q(A + B_2K) + (C_0 + D_0K)'(C_0 + D_0K)]x \ dt \\ \leq & \ 0 \end{aligned}$$

This can be true if and only if

$$(A + B_2K)'Q + Q(A + B_2K) + (C_0 + D_0K)'(C_0 + D_0K) \leq 0 \tag{3.36}$$

This gives us an inequality version of the problem of obtaining a minimal quadratic cost. If one is interested in obtaining the $\mathcal{H}_2$ norm of the output instead of a quadratic cost, we can solve the following minimization problem.

$$\begin{aligned} & \min\ Tr(B_0'QB_0) \\ \textit{such that} \quad & Q \geq 0 \\ \textit{and} \quad & (A+B_2K)'Q + Q(A+B_2K) + (C_0+D_0K)'(C_0+D_0K) \leq 0 \end{aligned} \tag{3.37}$$

This is because that $\mathcal{H}_2$ norm of a system is the sum of the quadratic costs corresponding to each initial condition generated by an impulse on each separate input.

(3.37) is a matrix inequality with coupled K and Q terms, and a quadratic term in K. If there is no other constraint, the solution of this optimization problem is straight forward. Rearrange (3.37), so that K appears in the quadratic terms as

$$\begin{aligned} & \tilde{A}'Q + Q\tilde{A} + C_0'D_0^{\perp}D_0^{\perp\prime}C_0 - QB_2\tilde{D}_0B_2'Q \\ + \quad & \left(K + D_0^{+}C_0 + \tilde{D}_0B_2'Q\right)'\tilde{D}_0^{-1}\left(K + D_0^{+}C_0 + \tilde{D}_0B_2'Q\right) \leq 0 \end{aligned} \tag{3.38}$$

where $(D_0^{\perp}D_0^{\perp\prime}) = I - D_0(D_0'D_0)^{-1}D_0'$, $\tilde{D}_0 = (D_0'D_0)^{-1}$, $D_0^{+} = \tilde{D}_0D_0'$ and $\tilde{A} = (A - B_2'D_0^{+}C_0)$. Define the Riccati operator $\mathcal{R}(Q)$ as

$$\mathcal{R}(Q) := \tilde{A}'Q + Q\tilde{A} + C_0'D_0^{\perp}D_0^{\perp\prime}C_0 - QB_2\tilde{D}_0B_2'Q \tag{3.39}$$

and

$$N := \left(K + D_0^{+}C_0 + (D_0'D_0)^{-1}B_2'Q\right)'\tilde{D}_0^{-1}\left(K + D_0^{+}C_0 + (D_0'D_0)^{-1}B_2'Q\right)$$

(3.38) can be written as

$$\mathcal{R}(Q) + N + M = 0 \tag{3.40}$$

for arbitrary symmetric $M \geq 0$. (3.40) remains a Riccati equation but differs from $\mathcal{R}(Q) = 0$ only in the constant terms. It is well known that the nonnegative solution Q is a monotone increasing function of the constant term of (3.40) [23]. That is K

can be chosen as $K = -\tilde{D}_0(D_0'C_0 + B_2'Q)$ to make $N = 0$ thereby produce a smaller nonnegative Q. The LQ problem is thereby transformed into

$$\begin{aligned} & \min x_0'Qx_0 \\ \textit{such that} \quad & Q \geq 0 \\ \textit{and} \quad & \mathcal{R}(Q) \leq 0 \ (\textit{or } \mathcal{R}(Q) + M = 0) \end{aligned} \tag{3.41}$$

Using the same argument one can also choose $M = 0$. The solution for the problem (3.41) is therefore the $Q \geq 0$ such that

$$\mathcal{R}(Q) = 0 \tag{3.42}$$

We have obtained the familiar LQ solution (3.42) from the concept of dissipative system.

The solution of the optimization problem (3.41) can also be computed by an LMI solver. Defining $P := Q^{-1}$, we transform $\mathcal{R}(Q) \leq 0$ into an LMI by applying the Schur complement. We have the following optimization problem.

$$\begin{aligned} & \min \lambda \\ s.t. \quad & \begin{cases} \begin{bmatrix} \lambda & x_0' \\ x_0 & P \end{bmatrix} \geq 0 \\ P \geq 0 \\ \begin{bmatrix} P\tilde{A}' + \tilde{A}P + B_2\tilde{D}B_2' & PC_0'D_0^{\perp} \\ D_0^{\perp\prime}C_0P & -I \end{bmatrix} \leq 0 \end{cases} \end{aligned}$$

The trick of the previous derivation is that we can choose K to eliminate the coupling between K and Q. However, in case there is another constraint containing K, choosing such a K can complicate the additional constraint considerably. Our $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem is one such example.

3.3 Bilinear Matrix Inequality

Bilinear matrix inequalities are important to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem because of the following facts. It is not possible to eliminate the controller parameters K from the matrix inequality derived from the Bounded Real Lemma while trying to obtain an inequality in LMI form because K is coupled with equations arising from the second criterion. If K can not be eliminated, then there will always be coupling between the controller parameters K and the Lyapunov function parameters. Therefore, it is not possible to find a way to transform the original equation into LMI form.

If we give up pursuing an LMI form and allow coupling between any two variables but do not allow quadratic terms in the variables, it is possible to derive a general form for the controller synthesis problem. This general from is usually called a Bilinear Matrix Inequality(BMI). The BMI is generally not a convex function. However it does provide an attractive property in that it is a convex function for each variable when the other variables are fixed. So far there is no good algorithm to solve the BMI problem.

Several simple algorithms [13] have been tried such as (a) fix some variables and use an LMI solver to solve the convex subproblems alternately, (b) use the eigenvalue derivative to make the smallest eigenvalue of the BMI greater than 0, (c) use an interior point algorithm similar to the algorithm that solves the LMI problem, and (d) branch and bound method. The first and second algorithms have proven to be not so useful. The third algorithm is able to find a local solution but suffer from discontinuity problem under certain conditions. The fourth algorithm is attractive for finding a global solution but requires a massive amount of computation.

In [19, 25], it is shown that many robust control synthesis problems may be reduced to BMI feasibility problems. We will formulate our mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem into a BMI problem and use the interior point algorithm to solve it.

Definition 3.1 The Biaffine Matrix Inequality, BMI feasibility problem

Given matrices $F_{ij} = F'_{ij} \in R_{m\times m}$ *for* $i \in \{0,\ldots,n_x\}$, $j \in \{0,\ldots,n_y\}$ *Define the biaffine function* $\tilde{F} : R^{n_x} \times R^{n_y} \to R^{m\times m}$:

$$F(x,y) := F_{00} + \sum_{i=1}^{n_x} x_i F_{i0} + \sum_{j=1}^{n_y} y_i F_{0j} + \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} x_i y_j F_{ij} \tag{3.43}$$

Find, if it exists,$(x,y) \in R^{n_x} \times R^{n_y}$ *such that*

$$\tilde{F}(x,y) \geq 0 \tag{3.44}$$

Definition 3.2 The Bilinear Matrix Inequality feasibility problem

Given matrices $F_{ij} = F'_{ij} \in R_{m\times m}$ *for* $i \in \{1,\ldots,n_x\}$, $j \in \{1,\ldots,n_y\}$ *Define the bilinear function* $F : R^{n_x} \times R^{n_y} \to R_{m\times m}$:

$$\tilde{F}(x,y) := \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} x_i y_j F_{ij} \tag{3.45}$$

Find, if it exists, $(x,y) \in R^{n_x} \times R^{n_y}$ *such that*

$$\tilde{F}(x,y) \geq 0 \tag{3.46}$$

The matrix inequality feasibility problem is usually solved by finding an (x,y) such that the smallest eigenvalue of $F(x,y)$ is larger than 0. Defining a function $\Lambda(x,y) = -\underline{\lambda}(F(x,y))$, an (x,y) is feasible for $F(x,y) \geq 0$ if and only if

$$0 \geq min(\Lambda(x,y))$$

is nonempty. $\Lambda(x,y)$ exhibits an interesting property of biconvexity [19]. That is, given a BMI $\Lambda(x,y) \geq 0$, if $\Lambda(x_1,y_1) \geq 0$ and $\Lambda(x_2,y_2) \geq 0$ we have

$$\Lambda(x_\alpha,y_\beta) \leq (1-\alpha)(1-\beta)\Lambda(x_1,y_1)+(1-\alpha)\beta\Lambda(x_1,y_2)+\alpha(1-\beta)\Lambda(x_2,y_1)+\alpha\beta\Lambda(x_2,y_2)$$

for $\alpha,\beta \in [0,1]$ and $(x_\alpha,y_\beta) = ((1-\alpha)x_1 + \alpha x_2, (1-\beta)y_1 + \beta y_2)$. However the biconvexity doesn't imply that one can obtain a global solution. As a matter of fact, the feasible domain may not even be connected. For example the feasible domain of $xy \leq -1$ is not connected. One other problem of using the smallest eigenvalue $\Lambda(x,y)$ to determine the positiveness of F is that $\Lambda(x,y)$ may not be continuous.

3.3.1 Interior Point Methods

We will attempt to solve the BMI using interior point methods which are commonly used in many constrained optimization problems, e.g. the LMI eigenvalue problem. Given a BMI as defined in (3.43), a feasible set X is defined as

$$X := \{(x, y) \in R^{n_x+n_y} | F(x, y) \geq 0\} \tag{3.47}$$

First we have to find a barrier function such that the function is defined and analytic for all $x \in X$ and unbounded for $x \notin X$. There are many way to choose the barrier function. The most suitable choice for a matrix inequality is defining the barrier function as

$$\phi(x, y) := \begin{cases} \log det(F(x, y))^{-1} & (x, y) \in X \\ \infty & (x, y) \notin X \end{cases} \tag{3.48}$$

The function ϕ is finite if and only if $(x, y) \in int\ X$; otherwise ϕ will be unbounded. So the feasibility problem of finding an (x, y) such that (3.47) holds can be solved by solving the following problem

$$(x^*, y^*) = arg \min_{x,y}(\phi(x, y)) \tag{3.49}$$

For a BMI problem, ϕ is not a convex function. However if

- the set X is nonempty,
- ϕ is continuous function and bounded below

then a local minimum is guaranteed. Many search algorithms can be used to obtain such a local minimum. We use a modified Newton's method to perform the optimization because the first and second derivatives can be easily computed.

One can compute the first and second derivatives of ϕ using the facts that

$$\frac{\partial \phi}{\partial x_i} = Tr(F^{-1}\frac{\partial F}{\partial x_i}) = Tr\left(F^{-1}(F_{i0} + \sum_j F_{ij}y_j)\right) \tag{3.50}$$

$$\frac{\partial \phi}{\partial y_j} = Tr(F^{-1}\frac{\partial F}{\partial y_j}) = Tr\left(F^{-1}(F_{0j} + \sum_i F_{ij}x_j)\right) \tag{3.51}$$

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = Tr(F^{-1}\frac{\partial F}{\partial x_i}F^{-1}\frac{\partial F}{\partial x_j}) \tag{3.52}$$

$$\frac{\partial^2 \phi}{\partial y_i \partial y_j} = Tr(F^{-1}\frac{\partial F}{\partial y_i}F^{-1}\frac{\partial F}{\partial y_j}) \tag{3.53}$$

$$\frac{\partial^2 \phi}{\partial x_i \partial y_j} = Tr(F^{-1}\frac{\partial F}{\partial x_i}F^{-1}\frac{\partial F}{\partial y_j}) + Tr(F^{-1}\frac{\partial^2 F}{\partial x_i \partial y_j}) \tag{3.54}$$

For many BMI problems, the function ϕ will not be bounded below or continuous. So there are still difficulties when using Newton's method as well as search algorithms. Our $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem has this problem. We will discuss our treatment of this problem in Chapters 4 and 5.

3.3.2 BMI eigenvalue problem

If we are given a minimization problem with linear objective and BMI constraint, we will show that an algorithm similar to that which solves the feasibility problem can be applied. This problem is also called the BMI eigenvalue problem and has the following form

$$\begin{aligned} &\min_{x,y} \quad c_x' x + c_y' y \\ s.t. \quad & F(x,y) \geq 0 \end{aligned} \tag{3.55}$$

where $F(x,y)$ is defined as in (3.43). $c_x \in R^{n_x}, c_y \in R^{n_y}$ are constant vectors. The problem (3.55) is equivalent to

$$\begin{aligned} &\min \lambda \\ s.t. \quad & \begin{cases} \lambda - c_x' x - c_y' y \geq 0 \\ F(x,y) \geq 0 \end{cases} \end{aligned} \tag{3.56}$$

The simplest way to find a minimum of the optimization problem (3.56) is to employ the Method of Centers as in the LMI eigenvalue problem [4]. Define ϕ as

$$\phi(x,y,\lambda) := \begin{cases} \log\ det(F(x,y))^{-1} - \log(\lambda - c_x'x - c_y'y) & \\ \qquad\qquad\qquad\qquad if\ (x,y) \in X\ and\ \lambda > c_x'x + c_y'y \\ \infty \qquad\qquad\qquad\qquad otherwise \end{cases} \tag{3.57}$$

Note that ϕ is the barrier function as we defined earlier corresponding to the inequality

$$\tilde{F} = \begin{bmatrix} F & 0 \\ 0 & \lambda - c_x'x - c_y'y \end{bmatrix} \geq 0$$

Algorithm 3.1 Method of centers: For solving BMI eigenvalue problem

1. *Initialize* $\lambda^{<0>}$, $x^{<0>}$, $y^{<0>}$ *such that the constraints are satisfied. Set* $k = 1$.

2. *Find the center of* $\phi(x,y,\lambda^{<k-1>})$ *with* $\lambda^{<k-1>}$ *fixed by solving the optimization problem*

$$(x^{<k>}, y^{<k>}) = arg \min_{x,y} \phi(x,y,\lambda^{<k-1>})$$

3. *Let*

$$\lambda^{<k>} = (1-\theta)(c_x'x^{<k>} + c_y'y^{<k>}) + \theta\lambda^{<k-1>} \tag{3.58}$$

for some $0 < \theta < 1$.

4. *If* $\|\lambda^{<k-1>} - \lambda^{<k>}\|$ *is sufficiently small then stop. Else set* $k = k+1$ *and repeat from step 2.*

Because in step 3 we have $\lambda^{<k-1>} > c_x'x^{<k>} + c_y'y^{<k>}$ and $0 < \theta < 1$. We have

$$\lambda^{<0>} > \ldots > \lambda^{<k-1>} > \lambda^{<k>} > \ldots$$

The optimization can be achieved if a limit point of $\lambda^{<k>}$ exists as $k \to \infty$. In [9], it is shown that if $\Lambda(x,y)$ is bounded below, then the limit point $\lambda^{<k>}$ exists.

3.4 $\mathcal{H}_2/\mathcal{H}_\infty$ problem

Now we are ready to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem using the BMI solver. Consider the static state feedback problem of a one-input/two-output system.

$$\begin{cases} \dot{x} = Ax + B_0 w_0 + B_1 w_1 + B_2 u \\ z_0 = C_0 x + D_0 u \\ z_1 = C_1 x + D_1 u \\ y = x \end{cases} \tag{3.59}$$

and the closed-loop system with a static feedback K

$$\begin{cases} \dot{x} = (A + B_2 K)x + B_0 w_0 + B_1 w_1 \\ z_0 = (C_0 + D_0 K)x \\ z_1 = (C_1 + D_1 K)x \end{cases} \tag{3.60}$$

We will use the following notation occasionally for abbreviation.

$$A_K = A + B_2 K, \quad C_{1K} = C_1 + D_1 K, \quad C_{2K} = C_0 + D_0 K,$$

The $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem is to find a stabilizing K such that

$$\begin{aligned} & \min_K \|T_{z_0 w_0}\|_2^2 \\ s.t. \quad & \|T_{z_1 w_1}\|_\infty \leq \gamma \end{aligned} \tag{3.61}$$

By applying the Bounded Real Lemma (3.20) without the direct feed forward term and replacing A, B, C_0 an D_0 in (3.20) with $(A + B_2 K)$, B_1, $(C_1 + D_1 K)$ and 0 respectively, the following inequality has to be satisfied for $P_1 \geq 0$.

$$\begin{pmatrix} -(A + B_2 K)' P_1 - P_1 (A + B_2 K) - (C_1 + D_1 K)'(C_1 + D_1 K) & P_1 B \\ B' P_1 & \gamma^2 I \end{pmatrix} \geq 0 \tag{3.62}$$

Using the reverse Schur complement we have

$$A_K' P_1 + P_1 A_K + P_1 B_1 (\gamma^2 I)^{-1} B_1 P_1 + C_{1K}' C_{1K} \leq 0 \tag{3.63}$$

If we define $Q_1 := P_1^{-1}$ we also have

$$Q_1 A_K' + A_K Q_1 + B_1(\gamma^2 I)^{-1} B_1 + Q_1 C_{1K}' C_{1K} Q_1 \leq 0 \tag{3.64}$$

(3.63) and (3.64) can be transformed into BMIs as

$$\begin{bmatrix} -(A+B_2K)'P_1 - P_1(A+B_2K) & P_1B & (C_1+D_1K)' \\ B'P_1 & \gamma^2 I & 0 \\ (C_1+D_1K & 0 & I \end{bmatrix} \geq 0 \tag{3.65}$$

$$\begin{bmatrix} Q_1A_K' + A_KQ_1 + B_1(\gamma^2 I)^{-1}B_1 & Q_1C_{1K}' \\ C_{1K}Q_1 & -I \end{bmatrix} \leq 0 \tag{3.66}$$

Both (3.65) and (3.66) do not contain any purely quadratic term but do contain cross coupled terms.

As shown earlier, computing the $\mathcal{H}_2$ norm of the closed-loop system can also be transformed into the optimization problem of finding $P_0 \geq 0$ such that

$$\begin{aligned} &\min\ Tr(B_0'P_0B_0) \\ &(A+B_2K)'P_0 + P_0(A+B_2K) + (C_0+D_0K)'(C_0+D_0K) \leq 0 \end{aligned} \tag{3.67}$$

We can also transform (3.67) into a BMI optimization problem as

$$\begin{aligned} &\min\ Tr(B_0'P_0B_0) \\ &\begin{bmatrix} (A+B_2K)'P_0 + P_0(A+B_2K) & (C_0+D_0K)' \\ (C_0+D_0K) & -I \end{bmatrix} \leq 0 \end{aligned} \tag{3.68}$$

We can also choose $Q_0 := P_0^{-1}$ to have a BMI optimization problem in terms of Q_0 as

$$\begin{aligned} &\min\ Tr(\Lambda) \\ &\begin{bmatrix} \Lambda & B_0' \\ B_0 & Q_0 \end{bmatrix} \geq 0 \\ &\begin{bmatrix} Q_0(A+B_2K)' + (A+B_2K)Q_0 & Q_0(C_0+D_0K)' \\ (C_0+D_0K)Q_0 & -I \end{bmatrix} \leq 0 \end{aligned} \tag{3.69}$$

For the $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem, we can add either constraint (3.63) or (3.64) to the optimization problem (3.68) or (3.69). As a result we have four different

ways to transform the $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem into a BMI eigenvalue problem. Specifically, choosing P_1, P_0 we have

$$\begin{aligned}
&\min\ Tr(B_0' P_0 B_0) \\
&\begin{bmatrix} (A+B_2K)'P_0 + P_0(A+B_2K) & (C_0+D_0K)' \\ (C_0+D_0K) & -I \end{bmatrix} \leq 0 \\
&\begin{bmatrix} -(A+B_2K)'P_1 - P_1(A+B_2K) & P_1B & (C_1+D_1K)' \\ B'P_1 & \gamma^2 I & 0 \\ (C_1+D_1K) & 0 & I \end{bmatrix} \geq 0 \\
&P_1 \geq 0, \quad P_0 \geq 0
\end{aligned} \tag{3.70}$$

Choosing Q_1, P_0 we have

$$\begin{aligned}
&\min\ Tr(B_0' P_0 B_0) \\
&\begin{bmatrix} (A+B_2K)'P_0 + P_0(A+B_2K) & (C_0+D_0K)' \\ (C_0+D_0K) & -I \end{bmatrix} \leq 0 \\
&\begin{bmatrix} Q_1A_K' + A_KQ_1 + B_1(\gamma^2 I)^{-1}B_1 & Q_1C_{1K}' \\ C_{1K}Q_1 & -I \end{bmatrix} \leq 0 \\
&Q_1 \geq 0, \quad P_0 \geq 0
\end{aligned} \tag{3.71}$$

Choosing P_1, Q_0 we have

$$\begin{aligned}
&\min\ Tr(\Lambda) \\
&\begin{bmatrix} \Lambda & B_0' \\ B_0 & Q_0 \end{bmatrix} \geq 0 \\
&\begin{bmatrix} Q_0(A+B_2K)' + (A+B_2K)Q_0 & Q_0(C_0+D_0K)' \\ (C_0+D_0K)Q_0 & -I \end{bmatrix} \leq 0 \\
&\begin{bmatrix} -(A+B_2K)'P_1 - P_1(A+B_2K) & P_1B & (C_1+D_1K)' \\ B'P_1 & \gamma^2 I & 0 \\ (C_1+D_1K & 0 & I \end{bmatrix} \geq 0 \\
&P_1 \geq 0, \quad Q_0 \geq 0
\end{aligned} \tag{3.72}$$

Choosing Q_1, Q_0 we have

$$\min\ Tr(\Lambda)$$

$$
\begin{aligned}
&\begin{bmatrix} \Lambda & B_0' \\ B_0 & Q_0 \end{bmatrix} \geq 0 \\
&\begin{bmatrix} Q_0(A+B_2K)' + (A+B_2K)Q_0 & Q_0(C_0+D_0K)' \\ (C_0+D_0K)Q_0 & -I \end{bmatrix} \leq 0 \\
&\begin{bmatrix} Q_1A_K' + A_KQ_1 + B_1(\gamma^2 I)^{-1}B_1 & Q_1C_{1K}' \\ C_{1K}Q_1 & -I \end{bmatrix} \leq 0 \\
&Q_1 \geq 0, \quad Q_0 \geq 0
\end{aligned} \tag{3.73}
$$

The variable K cross-couples with two different variables in any of the four optimization problems. The removal of K by completion of the square as in the pure LQ problem is not possible. So we have to apply a BMI solver to solve the BMI eigenvalue problem.

We summarize the BMI setup as follows. Here we choose the variables Q_1 and Q_0 as in (3.73)

- $\min \lambda$
- $F_1 := -\begin{bmatrix} Q_0(A+B_2K)' + (A+B_2K)Q_0 & Q_0(C_0+D_0K)' \\ (C_0+D_0K)Q_0 & -I \end{bmatrix} \geq 0$
- $F_2 := \begin{bmatrix} \Lambda & B_0' \\ B_0 & Q_0 \end{bmatrix} \geq 0$
- $F_3 := -\begin{bmatrix} Q_1A_K' + A_KQ_1 + B_1(\gamma^2 I)^{-1}B_1 & Q_1C_{1K}' \\ C_{1K}Q_1 & -I \end{bmatrix} \geq 0$
- $F_4 := Q_1 \geq 0$
- $F_5 := \lambda - Tr(\Lambda) \geq 0$

Define ϕ as

$$
\phi = \begin{cases} -\log det(F_1) - \log det(F_2) - \log det(F_3) - \log det(F_4) - \log det(F_5) \\ \qquad if F_1, F_2, F_3, F_4, F_5 \geq 0 \\ \infty \qquad otherwise \end{cases} \tag{3.74}
$$

To solve this problem, the following algorithm similar to the interior point method of centers in the LMI eigenvalue problem is used.

Algorithm 3.2 Method of centers: Solving $\mathcal{H}_2/\mathcal{H}_\infty$ BMI eigenvalue problem

1. *Initialize* $\lambda^{<0>}, \Lambda^{<0>}, Q_1^{<0>}, Q_0^{<0>}$ *to satisfy the constraint. Set k=1.*

2. *Let* $(\Lambda^{<k>}, Q_1^{<k>}, Q_0^{<k>}, K^{<k>}) = arg \min \phi(\lambda^{<k-1>})$.

3. *Let* $\lambda^{<k>} = (1-\theta)Tr(\Lambda^{<k-1>}) + \theta\lambda^{<k-1>}$ *for some* $0 < \theta < 1$.

4. *Set* $k = k+1$. *Repeat step 2*

It has been proposed in [25] that if the minimizer in step 2 is global, then the algorithm is also global. The function ϕ is a convex function if all the F_n's are LMI's. However in our case the F_n's are BMI's. Therefore the global minimizer of $\phi(\lambda^{<k>})$ in step 2 using Newton's method is generally not guaranteed. But if the set of all feasible Q_1 and Q_0 is bounded and closed, we can obtain a local minimum. The following theorem gives the conditions such that Algorithm 3.2 will at least converge to a local minimum.

Theorem 3.1 *The function* ϕ *defined in (3.74) with fixed* λ *is bounded below if* (A_K, C_{0K}) *and* (A_K, C_{1K}) *is observable for all stabilizing* K.

Proof. It suffices to prove ϕ is bounded below by showing F_i $i = 1, \ldots, 5$ are bounded above for all feasible K, Q_0, Q_1 and Λ with λ fixed.

The $\mathcal{H}_2$ norm of the closed-loop system needs to be bounded above by the fixed λ. The full column rank of D_0 guarantees the boundedness of all feasible K, i.e. $\|K\| < \xi < \infty$ for some sufficient large ξ for all K satisfies F_i's.

Because (A_K, C_{0K}) and (A_K, C_{1K}) is observable, it is well known that $P_0 > 0$ and $P_1 > 0$. Moreover, we can find a sufficient small positive ϵ_0 and ϵ_1 such that $P_0 > \epsilon_0 I$ and $P_1 > \epsilon_1 I$ because K is bounded.

- F_4 is bounded above because $Q_1 = P_1^{-1} < \epsilon_1^{-1} I < \infty$.

- F_5 is bounded above because $F_2 \geq 0$ assures that $\Lambda \geq 0$ and $\lambda - Tr(\Lambda) < \infty$.

- $F_2 < \infty$ is equivalent to $\Lambda - B_0' Q_0^{-1} B_0 < \infty$ and $Q_0 < \infty$. The former is due to $F_5 \geq 0$ and $Q_0^{-1} > 0$ The latter is because $Q_0 = P_0^{-1} < \epsilon_0^{-1} I < \infty$.

- $F_1 < \infty$ can be assured because both Q_0 and K is bounded.

- $F_3 < \infty$ can be assured because both Q_1 and K is bounded.

¶

We end this section by summarizing the solvability of the BMI optimization problem (3.70)-(3.73) using the method of centers. They can be proved similarly to Theorem 3.1.

- (3.70) is solvable if both (A_K, B_0) and (A_K, B_1) are controllable for all stabilizing K.

- (3.71) is solvable if (A_K, B_0) is controllable and (A_K, C_{1K}) is observable for all stabilizing K.

- (3.72) is solvable if (A_K, B_1) is controllable and (A_K, C_{0K}) is observable for all stabilizing K.

- (3.73) is solvable if both (A_K, C_{0K}) and (A_K, C_{1K}) are observable for all stabilizing K.

3.4.1 Numerical Example

Example 3.1

We will demonstrate an example with $p_2 = 2$ and $p_1 = 2$ and the number of control inputs $m_2 = 1$. As will become clear in Chapter 4 in this case, because $p_1 > m_2$ and

	K_2	K_∞	K_0	K^*
$\|T_{z_0w}\|_2^2$	13.0072	17.4275	72.8759	14.7106
$\|T_{z_1w}\|_\infty$	4.3593	2.7779	2.1940	3

Table 3.1: Performance comparison of Example 3.1 with $\gamma = 3$

$p_2 > m_2$, we will not have an unobservable subspace on either output z_1 or z_0. Our BMI solver is straight forward. We have

$$\begin{cases} \dot{x} = \begin{bmatrix} -0.0825 & 0.089 & 0 \\ 0.187 & 0.43 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.935 \\ 0.384 \\ 1 \end{bmatrix} w + \begin{bmatrix} 0.519 \\ 0.831 \\ 1 \end{bmatrix} u \\ z_1 = \begin{bmatrix} 0.167 & 0.487 & 1 \\ 0.179 & 0.435 & -0.116 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u \\ z_0 = \begin{bmatrix} -0.946 & 0.324 & -1 \\ 0.0194 & 0.331 & -0.465 \end{bmatrix} x \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \end{cases} \tag{3.75}$$

We have the $\mathcal{H}_2$ optimal solution and the $\mathcal{H}_\infty$ controller for $\gamma = 3$ as

$$K_2 = [0.82 \quad 3.22 \quad -6.2] \quad K_\infty = [2.36 \quad 7.93 \quad -13.6]$$

In practice, we will use K_∞ as the initial solution for our BMI solver. However we choose to use an initial solution as far as possible from the optimum to demonstrate our algorithm. We choose

$$K_0 = [16.3 \quad 48.4 \quad -79.9]$$

Our algorithm gives an optimal solution at

$$K^* = [1.78 \quad 5.26 \quad -9.64]$$

The result in Table 3.1 shows that our optimal solution K^* represents a relatively good trade-off between K_2 and K_∞. It has $\mathcal{H}_2$ norm relatively closer to the LQ result. At the same time it fulfills the constraint that its $\mathcal{H}_\infty$ norm be less than or equal to γ.

k	$\lambda^{<k>}$	$\|T_{z_0w}(K^{<k>})\|_2^2$	$\|T_{z_1w}(K^{<k>})\|_\infty$	Newton's iterations
1	72.6828	22.0692	2.49803	49
2	30.0294	18.6622	2.60147	8
3	21.2932	16.78	2.71415	10
4	17.6106	15.7044	2.82579	10
5	15.8932	15.1327	2.91517	10
6	15.2904	14.9241	2.95653	9
7	14.8319	14.7664	2.99199	12
8	14.7485	14.7361	2.99849	18
9	14.7436	14.7319	2.99858	25
10	14.7403	14.7253	2.99824	33
11	14.7355	14.7212	2.99834	22
12	14.7299	14.7181	2.99864	16
13	14.7248	14.7158	2.99897	17
14	14.719	14.7135	2.99938	13
15	14.7149	14.712	2.99967	11
16	14.7125	14.7112	2.99985	13
17	14.7112	14.7108	2.99995	9
18	14.7108	14.7106	2.99998	10
19	14.7106	14.7105	3	17

Table 3.2: Numerical iterations of BMI solver of Example 5.1

The details of the interior point algorithm of this problem are shown in Table 3.2. This numerical experiment takes approximate 35 seconds using Matlab 4.2c running on a Sun Sparc-20 workstation. The second column is the value of $\lambda^{<i>}$ as computed by our algorithm. The third is the actual $\mathcal{H}_2$ norm of the closed-loop system of T_{z_0w} given $K^{<i>}$. The fourth column is the $\mathcal{H}_\infty$ norm of the closed-loop system given $K^{<i>}$. The fifth column is the number of inner iterations for minimizing ϕ of a given λ Our BMI solver behaves relatively well as indicated in Table 3.2. It takes 19 outer iterations to converge to the optimal solution. However the solution is very close to optimal after iteration 7. Each outer iteration takes an average of 10 inner iterations except for the first iteration. The reason the first iteration takes more inner iterations is because of the way we initialize Q_1 and Q_0 which makes the second and third inequality of the problem (3.73) very close to the boundary. The interior point algorithms is slow in minimizing ϕ because of the poor conditioning due to the heavy penalty for nearing the boundary.

3.4.2 Numerical Difficulties Due To Unobservability

We will show the difficulties of our BMI algorithm caused by the unobservability of the closed-loop system. Consider (A, B, C) be a stable unobservable triplet, in particular

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; C = [\, C_1 \quad 0 \,], \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let (A, B, C) have $\mathcal{H}_\infty$ norm $< \gamma$. Because x_2 is unobservable from output, it doesn't play any role in computing the $\mathcal{H}_\infty$ norm. There exists a $P_{11} > 0$ satisfying the Riccati inequality of Bounded Real Lemma of the subsystem corresponding to x_1, i.e.

$$A_{11}'P_{11} + P_{11}11 + \gamma^{-2}P_{11}'B_1B_1'P_{11} + C_1'C_1 \leq 0$$

It can be verified easily that $P = \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix}$ is feasible to the inequality

$$\begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix}' \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix} + \begin{bmatrix} C_1'C_1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$+\gamma^{-2} \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1B_1' & B_1B_2' \\ B_2B_1' & B_2B_2' \end{bmatrix} \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix} \leq 0$$

i.e P satisfy the Riccati inequality

$$A'P + PA + \gamma^{-2}PBB'P + C'C \leq 0$$

or there exists a $Q = P^{-1}$ with infinity eigenvalue satisfying

$$QA' + AQ + \gamma^{-2}BB' + QC'CQ \leq 0$$

Similar arguments apply to the Lyapunov inequality.

In our numerical experiment, we had the problem that either Q_1 or Q_0 would become unbounded for our interior point algorithm if the dimension of z_1 or z_0 was larger than the dimension of u. The reason that Q_1 becomes unbounded can be explained using the previous argument.

Given a state feedback K such that $(A + B_2K, C_1 + D_1K)$ is unobservable, there exists a feasible Q_1 to the $\mathcal{H}_\infty$ inequality that has at least one eigenvalue at infinity. Because the barrier function ϕ contains a term $-\log det(Q_1)$, therefore $\phi = -\infty$ when evaluated at this particular unobservable closed-loop system. With an ϕ that is not bounded below, the minimizer of our interior point algorithm tends to be attracted to the K that will make the closed-loop system unobservable. This will cause our algorithm to fail.

Q_0 is unbounded if there exists a K such that $(A + B_2K, C_0 + D_0K)$ becomes unobservable. We will have a feasible Q_0 that contains at least one eigenvalue at infinity. Because the barrier function ϕ contains the term $-\log det(Q_0)$, therefore $\phi = -\infty$ when evaluated at this particular unobservable closed-loop system.

Similarly, we have feasible and unbounded P_1 or P_0 if there exists K such that $(A + B_2K, B_1)$ is uncontrollable or $(A + B_2K, B_0)$ is uncontrollable.

In the next chapter, we will show the conditions such that the closed-loop system is observable or controllable for all possible K. Under this conditions, the boundedness of P_1, P_0, Q_1 or Q_0 is guaranteed. We will present a solution when these condition are not met.

Chapter 4

Invariant/Unobservable Space

In the previous chapter, we discussed the difficulty of solving the BMI eigenvalue problem. The unobservability (or uncontrollability) caused by some state feedback is the reason that our interior point algorithm sometimes failed. One way to remedy this problem is to parameterize, in advance, each particular subset of state feedback that will result in unobservability (uncontrollability) of the closed-loop system. The unobservable (uncontrollable) closed-loop system can be represented by a lower order state space system by ignoring the unobservable (uncontrollable) part of the system. The $\mathcal{H}_2/\mathcal{H}_\infty$ optimization on this subset of state feedbacks can be solved using the interior point algorithm without any problem provided that the variables involved are bounded. The solution of the lower order problem can be easily perturbed into a larger sub-problem as an initial solution. The new sub-problem is larger in the sense that we allow a larger state feedback parameterization that may produce an unobservable (uncontrollable) subspace containing in the previous problem. Because we obtained the initial solution of the larger problem from the optimal solution of a problem corresponding to this particular subspace we can expect that the interior point algorithm will not run into that unobservable (uncontrollable) subspace. If we gradually enlarge the sub-problems, we can eventually obtain a solution for the full problem.

However this algorithm is not applicable if the the unobservable (uncontrollable) subspace is not finite. In order to achieve the parameterization of state feedback for those finite unobservable or uncontrollable systems, we have to study the geometric theory first. The topic was pioneered by Wonham [51] and most recently discussed by Basile and Marro [2]. Many of the algorithms and concepts in this chapter can be found in more detail from the materials cited above. We will also make use of the algorithms and MATLAB code developed by the latter authors to obtain the special structure for the state feedback linear system of our $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

4.1 Invariants

In this section, we review some basic concepts of the geometric theory of linear systems due to Wonham, Basile and Marro. Studying the various invariants allows the elimination of some of the coordinate dependency of linear systems. We will start with the most basic A-Invariant which is crucial to the forming of finite unobservable controlled invariants.

Definition 4.1 [2, 51] A-Invariant

Consider a linear transformation $A : \mathcal{X} \to \mathcal{X}$ with $\mathcal{X} := R^n$. A subspace $\mathcal{V} \subseteq \mathcal{X}$ is called A-invariant if

$$A\mathcal{V} \subseteq \mathcal{V} \tag{4.1}$$

From the definition above, an A-invariant subspace $\mathcal{V}$ has the following property. There exists a matrix Λ such that

$$AV = V\Lambda \tag{4.2}$$

where V contains the vectors that span $\mathcal{V}$, i.e. $\mathcal{V} = im(V)$. Moreover, let $T = [V \quad V^\perp]$ where $V^\perp$ is the matrix containing the vectors that span the orthogonal

subspace to $\mathcal{V}$. A similarity transformation using T has the following form

$$T^{-1}AT = \begin{bmatrix} \Lambda & A_{12} \\ 0 & A_{22} \end{bmatrix} \tag{4.3}$$

From the previous property of an A-invariant subspace, we can also conclude that given a state equation

$$\dot{x} = Ax \tag{4.4}$$

and that $\mathcal{V}$ is A-invariant, then any state trajectory with nonzero initial condition $x_0 \in \mathcal{V}$ will remain in $\mathcal{V}$.

Intuitively, we know that the largest invariant subspace of $A \in \mathcal{R}^{n\times n}$ is $\mathcal{R}^{n\times n}$. We can also define the smallest invariant subspace of A as $\emptyset$. To study the invariant subspaces in between these two extremes, we need the following concepts.

Definition 4.2 Partial Ordering

Given a set $\mathcal{X}$, a binary relation $\subseteq$ in $\mathcal{X}$ is a partial ordering if

- *it is reflexive, i.e. $x \subseteq x$ for all $x \in \mathcal{X}$;*
- *it is antisymmetric, i.e. $x \subseteq y$, $x \neq y \Rightarrow y \not\subseteq x$;*
- *it is transitive, i.e. $x \subseteq y$ and $y \subseteq z \Rightarrow x \subseteq z$;*

Definition 4.3 Lattice

A lattice $\mathcal{L}$ is a partially ordered set in which for every pair $x, y \in \mathcal{L}$ there exists a least upper bound and greatest lower bound.

Throughout this thesis, we will use the following convention

- $\subseteq$ is the partial order relation;
- $\cup$: the smallest upper bound;
- $\cap$: the largest lower bound;

With these definitions, we can define the lattice of all A-invariant subspaces. Let A have the following similarity transformation

$$P^{-1}AP = \begin{bmatrix} \Lambda_1 & \dots & 0 & \dots & 0 \\ & \ddots & & \ddots & \\ 0 & \dots & \Lambda_i & \dots & 0 \\ & \ddots & & \ddots & \\ 0 & \dots & 0 & \dots & \Lambda_\mu \end{bmatrix} \tag{4.5}$$

$$P = [V_1 \quad \dots \quad V_i \quad \dots \quad V_\mu] \tag{4.6}$$

where Λ_i is the i_{th} Jordan block of A with V_i its corresponding eigenvector or generalized eigenvectors and $\mu \leq n$. Here Λ_i can be

$$[\lambda] \quad \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \quad or \quad \begin{bmatrix} \lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{bmatrix}$$

and V_i can have column rank 1,2 or more.

Lemma 4.1 *Any space spanned by V_i or V_i's is an A-invariant subspace. Moreover, all the A-invariant subspaces and $\emptyset$ form a lattice with $\emptyset$ as the universal lower bound and $\mathcal{R}^{n\times n}$ as the universal upper bound and V_i $i = 1, \dots, \mu$ as the elements.*

Figure 4.1(a) shows such a lattice with $\mu = 3$. Let a A-invariant subspace $\{V_i\}$ be corresponding to a defective eigenvalue block, with the eigenvector and generalized eigenvectors defined as v_{i0}, v_{i1}, ..., v_{ik-1}, such that

$$Av_{i0} = v_{i0}\lambda$$

$$Av_{ij} = v_{ij}\lambda + v_{ij-1} \quad j = 1, \dots, k-1$$

$\{v_{i0}\}$, $\{v_{i0}\ v_{i1}\} \dots \{v_{i0} \dots v_{ik-1}\}$ form a chain of A-invariant subspace within $\{V_i\}$ as shown in Figure 4.1(b). Any element within the chain can form a larger invariant subspace with others to be part of the lattice. For the case associated with a pair

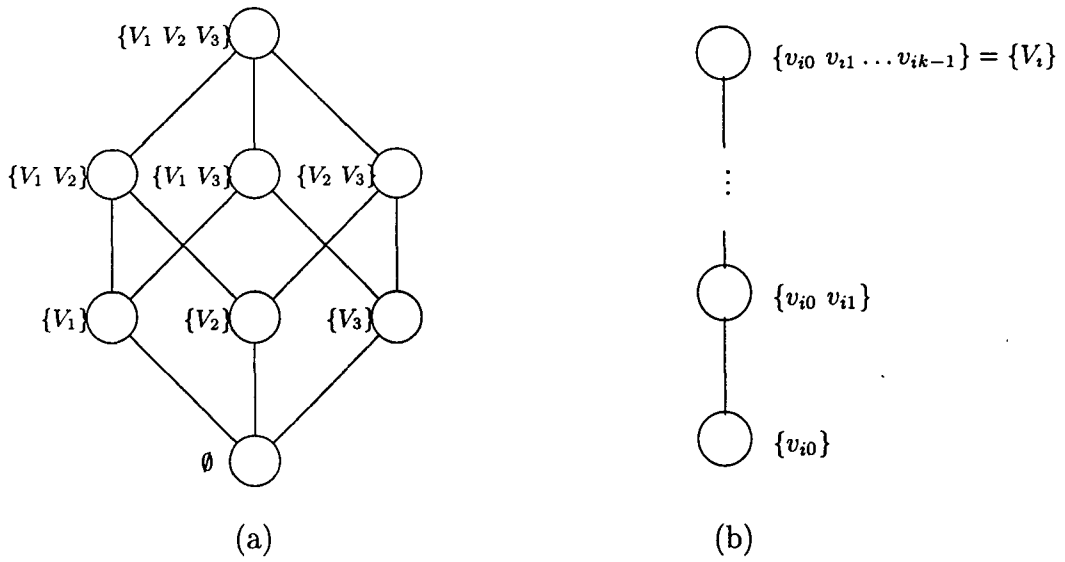

Figure 4.1: (a)Diagram describing the lattice of A-invariant subspaces with $\mu = 3$. (b)A-invariant subspaces chain of a defective eigenvalue block

complex conjugate eigenvalues, the elementary invariant space is the space spanned by the real and imaginary part of their eigenvectors.

Knowing the definition of A-invariant subspace, it is often necessary for us to know whether these A-invariant subspaces are contained in or contain another subspace of $\mathcal{X}$.

4.1.1 Minimal A-Invariant Containing $\mathcal{K}$

Let $\mathcal{K} \subseteq \mathcal{X}$ be a subspace of $\mathcal{X}$. We will use the symbol $\mathcal{T}(A;\mathcal{K})$ to denote an A-invariant containing $\mathcal{K}$. Obviously, the $\max \mathcal{T}(A;\mathcal{K})$ is $\mathcal{X}$. To find the $\min \mathcal{T}(A;\mathcal{K})$, the following algorithm can be used.

Algorithm 4.1 [2]Finding the minimal A-invariant containing $\mathcal{K}$, denoted $\min \mathcal{T}(A;\mathcal{K})$.

1. $\mathcal{Z}_0 := \mathcal{K}$; $i = 0$.

2. $\mathcal{Z}_i := \mathcal{K} \cup A\mathcal{Z}_{i-1}$.

3. $i = i + 1$ and repeat step 2 until $\mathcal{Z}_i = \mathcal{Z}_{i-1}$

Suppose the algorithm stops at $\mathcal{Z}_k$. It is easy to see that $\mathcal{Z}_k$ contains $\mathcal{K}$. Also because of the stopping condition, we have

$$\mathcal{Z}_k := \mathcal{K} \cup A\mathcal{Z}_k \Rightarrow A\mathcal{Z}_k \subseteq \mathcal{Z}_k. \tag{4.7}$$

So $\mathcal{Z}_k$ is an A-invariant that contains $\mathcal{K}$. The minimality can be proved by assuming that there is an A-invariant space $\mathcal{Y}$ that contains $\mathcal{K}$. We have $\mathcal{Y} \supseteq \mathcal{K} = \mathcal{Z}_0$. By induction, if $\mathcal{Y} \supseteq \mathcal{Z}_{i-1}$, we have

$$\mathcal{Y} \supseteq \mathcal{K} + A\mathcal{Y} \supseteq \mathcal{K} + A\mathcal{Z}_{i-1} = \mathcal{Z}_i$$

for all i. This means $\mathcal{Y} \supseteq \mathcal{Z}_k$ and shows that $\mathcal{Z}_k$ is minimum.

We can also study the controllability of the pair (A, B) from the concept of A-invariant.

Lemma 4.2 *The pair (A, B) is controllable if and only if*

$$\min \mathcal{T}(A; \mathcal{B}) = \mathcal{X} \tag{4.8}$$

where $\mathcal{B} := im(B)$ denotes the image space spanned by the columns of B.

Proof. **[if part]**Assume (A, B) is controllable. If $im(V) := \min \mathcal{T}(A; im(B)) \neq \mathcal{X}$ there exists a matrix $T = [V \ V^{\perp}]$ such that

$$\hat{A} := T^{-1}AT = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \quad \hat{B} := T^{-1}B = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}.$$

The 0 in $\hat{A}$ is because $im(V)$ is A-invariant. The 0 in $\hat{B}$ is due to $V^{\perp} \not\subset im(B)$. It is easy to see $(\hat{A}, \hat{B})$ is not controllable which contradicts the assumption.

[only if part]Assume (4.8) is true. From the algorithm 4.1,

$$\mathcal{B} + A\mathcal{B} + A^2\mathcal{B} + \ldots A^{n-1}\mathcal{B} := \mathcal{X} \tag{4.9}$$

This is equivalent to the rank of matrix $[B \quad AB \quad \ldots \quad A^{n-1}B]$ being equal to the dimension of $\mathcal{X}$, which is n. So (A, B) is controllable. ¶

4.1.2 Maximal A-Invariant Contained in $\mathcal{L}$

$\mathcal{L}$ is any subspace of $\mathcal{X}$. We will denote an A-Invariant contained in $\mathcal{L}$ as $\mathcal{S}(A;\mathcal{L})$. $\min \mathcal{S}(A;\mathcal{L})$ is clearly $\emptyset$. To obtain $\max \mathcal{S}(A;\mathcal{L})$, we can use the following algorithm.

Algorithm 4.2 [2] Finding the maximal A-invariant contained in $\mathcal{L}$, *denoted* $\max \mathcal{S}(A;\mathcal{L})$, *when A is invertible*

1. $\mathcal{Z}_0 := \mathcal{L}$; $i = 0$.

2. $\mathcal{Z}_i := \mathcal{L} \cap A^{-1}\mathcal{Z}_{i-1}$.

3. $i = i + 1$ *and repeat step 2 until* $\mathcal{Z}_i = \mathcal{Z}_{i-1}$

Alternatively, using the facts

$$A\mathcal{V} \subseteq \mathcal{V} \quad \Leftrightarrow \quad A'\mathcal{V}^{\perp} \supseteq \mathcal{V}^{\perp}$$

$$\mathcal{L} \supseteq \mathcal{V} \quad \Leftrightarrow \quad \mathcal{L}^{\perp} \subseteq \mathcal{V}^{\perp}$$

the $\max \mathcal{S}(A;\mathcal{L})$ can also be derived from the algorithm that computes $\min \mathcal{T}(;)$ as

$$\max \mathcal{S}(A;\mathcal{L}) = (\min \mathcal{T}(A';\mathcal{L}^{\perp}))^{\perp} \tag{4.10}$$

The duality is most useful when A is not invertible.

We can use the concept of maximal A-invariant to define the observability of a linear system.

Lemma 4.3 *The pair (A, C) is observable if and only if*

$$\max \mathcal{S}(A; ker(C)) = \emptyset \tag{4.11}$$

Proof. **[if part]**Assume that $\mathcal{V} = \max \mathcal{S}(A; ker(C))$ is not empty. There exists a transformation matrix $T = [V \ V^{\perp}]$ where $\mathcal{V} = im(V)$ such that

$$T^{-1}AT = \hat{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad CT = \hat{C} = [0 \quad C_2] \tag{4.12}$$

The 0 in the $\hat{A}$ matrix is due to the fact that V spans the A-invariant. The 0 in $\hat{C}$ is due to $\mathcal{V}$ being contained in $ker(C)$. Obviously, $(\hat{A}, \hat{C})$ form an unobservable pair similar to the pair (A, C). This contradicts the assumption that (A, C) is observable.

[only if part]Assume $\max \mathcal{S}(A; ker(C)) = \emptyset$. From (4.10), this implies

$$\min \mathcal{T}(A'; ker(C)^{\perp}) = \min \mathcal{T}(A'; im(C')) = \mathcal{X}.$$

From the construction of $min\mathcal{T}(;)$ in algorithm 4.1, this implies the rank of matrix $[C' \quad A'C' \quad \ldots \quad (A')^{n-1}C']$ is the dimension of $\mathcal{X}$, which is n. This implies that (A, C) is observable. ¶

One of the important applications of $\max \mathcal{S}(;)$ and $\min \mathcal{T}(;)$ is to investigate the controllability and observability of a linear time-invariant system. Given

$$\begin{cases} \dot{x} = Ax + Bu \\ z = Cx \end{cases} \tag{4.13}$$

Let $\mathcal{Q} = \max \mathcal{S}(A; ker(C))$ and $\mathcal{R} = \min \mathcal{T}(A; im(B))$. We can choose a similarity transformation matrix $T := [T_1 \; T_2 \; T_3 \; T_4]$ satisfying $im(T_1) = \mathcal{Q} \cap \mathcal{R}$, $im([T_1 \; T_3]) = \mathcal{R}$ and $im([T_1 \; T_2]) = \mathcal{Q}$. Applying this transformation gives a new system matrix showing the controllability and observability of (4.13) completely. The new system matrix triple $(\hat{A}, \hat{B}, \hat{C})$ has the following form.

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & \hat{A}_{14} \\ 0 & \hat{A}_{22} & 0 & \hat{A}_{24} \\ 0 & 0 & \hat{A}_{33} & \hat{A}_{34} \\ 0 & 0 & 0 & \hat{A}_{44} \end{bmatrix} \qquad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ 0 \\ \hat{B}_3 \\ 0 \end{bmatrix} \tag{4.14}$$

$$\hat{C} = [0 \quad 0 \quad \hat{C}_3 \quad \hat{C}_4] \tag{4.15}$$

This is usually called the Kalman canonical decomposition. The above result can be checked by expanding $AT = T\hat{A}$, $T\hat{B} = B$ and $\hat{C} = CT$ directly. The 0's in $\hat{A}$ are because $im([T_1 \; T_2 \; T_3])$ is an invariant of A. The 0's in $\hat{C}$ are because $im([T_1 \; T_3])$ is in $ker(C)$. The 0's in $\hat{B}$ are because $im([T_3 \; T_4])$ is not in imB.

A further simplification of the Kalman canonical decomposition can be achieved. But we need the following preliminary results.

4.1.3 Complementability of an Invariant

Let $A : \mathcal{X} \leftrightarrow \mathcal{X}$ be a linear map and let $\mathcal{V} \subseteq \mathcal{X}$ be an A-invariant subspace. A subspace $\mathcal{U} \subseteq \mathcal{X}$ is the complement of $\mathcal{V}$ if $\mathcal{V} \oplus \mathcal{U} = \mathcal{X}$ and $\mathcal{U}$ is also A-invariant. We will denote the complement of $\mathcal{V}$ as $\mathcal{V}^{\square}$.

The similarity transformation of A using the matrix $[V\ U]$, where V, U are the bases of $\mathcal{V}$ and $\mathcal{U}$ respectively, possesses the following form

$$[V\ U]^{-1} A [V\ U] = \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & \tilde{A}_{22} \end{bmatrix}$$

Lemma 4.4 *Let $A : \mathcal{X} \leftrightarrow \mathcal{X}$ be a linear map. Let V be a basis of the A-invariant subspace $\mathcal{V}$. Let $T = [V\ T_2]$ and*

$$T^{-1} A T = \hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}. \tag{4.16}$$

Then $\mathcal{U} = im(U)$, which is the complement of $\mathcal{V}$, exists if and only if the Sylvester equation

$$\hat{A}_{11} X - X \hat{A}_{22} = -\hat{A}_{12} \tag{4.17}$$

admits a solution X. Moreover $U = VX + T_2$.

Proof. Let

$$\hat{T} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$$

Applying similarity transformation on $\hat{A}$ with $\hat{T}$, in order for the $(1,2)$ sub-matrix to be zero, (4.17) has to be true. ¶

Note that it is well known that the Sylvester equation (4.17) has a unique solution if and only $\hat{A}_{11}$ and $\hat{A}_{22}$ have no common eigenvalue.

Using the above facts, it is possible to further reduce the Kalman canonical form by choosing the new coordinate $T = [T_1, T_2, T_3, T_4]$ where $[T_1, T_2]$ span $\max \mathcal{S}(A; ker(C))$, $[T_1, T_3]$ span $\min \mathcal{S}(A; im(B))$, $T_1 = \max \mathcal{S}(A; ker(C)) \cap \min \mathcal{T}(A; im(B))$, T_2 spans a subspace that is A-invariant and contained in $ker(C)$, T_3 spans a subspace that is A-invariant and containing $im(B)$. T_1, T_2, T_3 and T_4 are complement to each other with respect to A matrix. In other words the subspaces spanned by T_1, T_2, T_3 or T_4 are all A-invariant. We will have

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & 0 & 0 & 0 \\ 0 & \tilde{A}_{22} & 0 & 0 \\ 0 & 0 & \tilde{A}_{33} & 0 \\ 0 & 0 & 0 & \tilde{A}_{44} \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \\ \tilde{B}_3 \\ 0 \end{bmatrix} \tag{4.18}$$

$$\tilde{C} = [0 \quad 0 \quad \tilde{C}_3 \quad \tilde{C}_4] \tag{4.19}$$

provided that $\hat{A}_{11}$, $\hat{A}_{22}$, $\hat{A}_{33}$ and $\hat{A}_{44}$ in (4.15) mutually satisfy the solvability conditions of the corresponding Sylvester equations.

This particular form will be used later in our BMI algorithm for solving mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems.

4.2 (A, B)-Controlled Invariant

The concept of A-invariant can be extended to the so-called controlled and conditioned invariant. They are useful tools for linear system synthesis problems. We will be mainly interested in the concept of controlled invariant. However, the conditioned invariant can be derived in a dual manner.

Given a linear system $\dot{x} = Ax + Bu$ and a subspace $\mathcal{V}$, in order to confine the state trajectory $x(t) \in \mathcal{V}$ we must have $\dot{x}(t) \in \mathcal{V}$ also. From the state equation, this means that whenever $x(t) \in \mathcal{V}$, $\dot{x}(t) = Ax(t) + Bu(t) = v(t) \in \mathcal{V}$, i.e. $Ax(t) = v(t) - Bu(t)$ for any $u(t)$ and $v(t) \in \mathcal{V}$. This implies $A\mathcal{V} \subseteq \mathcal{V} + im(B)$. The reverse is also true. For each $\mathcal{V}$ having this property, we have the following definition.

THESIS REPORT

Ph.D.

Mixed H_2/H_∞ Optimization: A BMI Solution

by S.D. Yen
Advisors: W.S. Levine

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Abstract

Title of Dissertation: Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Optimization: A BMI Solution

Shih Don Yen, Doctor of Philosophy, 1996

Dissertation directed by: William S. Levine
Department of Electrical Engineering

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem has received lots of attention since $\mathcal{H}_\infty$ control theory was developed. This problem occurs when trying to achieve both good and robust performance for a LTI centralize system. Using the extra degrees of freedom of an $\mathcal{H}_\infty$ solution, one may obtain a solution that is as close to the "pure" $\mathcal{H}_2$ solution as possible. In other words, this is an $\mathcal{H}_2$ optimization problem with $\mathcal{H}_\infty$ norm constraint.

However, an optimal solution for this multi-criteria problem involving different performance measures is difficult to obtain. Only sub-optimal (upper bound) solution can be computed efficiently. This study concentrated on obtaining an effective computation algorithm to find the optimal solution for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

It was found that the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem can be transformed into BMI (Bilinear Matrix Inequality) optimization problem. An interior point algorithm is used to solve this BMI problem. However, the interior point algorithm does not always work because of the discontinuity nature of BMI. The conditions on which the interior point algorithm will work successfully were provided. Furthermore, it was found that the discontinuity is due to the unobservability of the closed-loop system. By analyzing

the unobservable controlled invariants, the BMI problem can be transformed into a lattice of BMI subproblems without the discontinuity problem. By traversing through the lattice, the original BMI optimization problem can be solved.

The algorithm is applied to find an $\mathcal{H}_2/\mathcal{H}_\infty$ solution for a robust F-14 flight control problem. An solution can be computed efficiently. Using this $\mathcal{H}_2/\mathcal{H}_\infty$ controller achieves the expected performance of good vertical wind gust suppression and

This algorithm has the potential to solve other optimization problems involving coupled Lyapunov inequality or Riccati inequality which can not be reduced into an LMI convex programming problem.

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Optimization: A BMI Solution

by

Shih Don Yen

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland at College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1996

Advisory Committee:

William S. Levine, Chairman/Advisor
Professor Gilmer L. Blankenship
Professor Andre L. Tits
Professor Steven I. Marcus
Professor Jewel B. Barlow

Dedication

To my parents, ShuJing, Eric and Ashley

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Chapter 1

Introduction

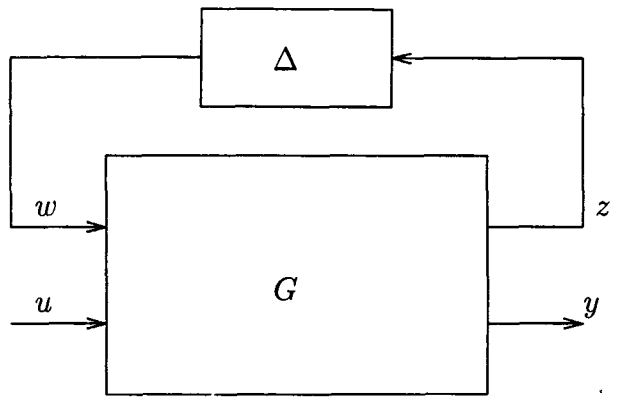
1.1 Motivation

Two of the most important measures of system performance in optimal control theory are the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. Given a stable multi-input multi-output transfer function matrix $G(s)$, they are defined as

$$\begin{aligned} \|G\|_2 &:= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} Trace[G(j\omega)^{\sim}G(j\omega)]d\omega\right)^{1/2} \\ \|G\|_\infty &:= \sup_{\omega} \sigma_{max}[G(j\omega)] \end{aligned}$$

where $\sim$ denotes the complex conjugate transpose operator and σ_{max} is the maximum singular value.

In the context of $\mathcal{H}_\infty$ control theory, it is well known [8, 54] that, using the small gain theorem, the robustness of the system can be guaranteed to a certain degree, either in the sense of system variation or under exogenous inputs as long as the perturbation is under a certain bound. Let $\gamma = \|G_{zw}\|_\infty$, where G_{zw} is the transfer function from w to z as in Figure 1.1 with $\Delta = 0$. The system will remain stable as long as $\|\Delta\|_\infty < \gamma^{-1}$. This is the so-called worst-case stability. So the smaller γ is, the more tolerant to the disturbance. In classical SISO(single input single output) control this is comparable to obtaining a larger stability margin in terms of phase/gain margin. It is intuitive that something has to be compromised in order to obtain a larger



$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = G \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} G_{zw} & G_{zu} \\ G_{yw} & G_{yu} \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}$$

Figure 1.1: Illustration of the use of the small gain theorem for robustness

stability margin. Therefore, it is also well known that $\mathcal{H}_\infty$ control designed is generally conservative. That is, in designing a controller that makes $\|G_{zw}\|_\infty$ small one may drive the performance of other input/output pairs out of the performance specification. In practice, the performance of the system can be conveniently expressed in terms of its $\mathcal{H}_2$ norm as in LQG design. Therefore we have the so-called mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control, the idea of which is to trade some performance in terms of the $\mathcal{H}_2$ norm for some robustness in terms of the $\mathcal{H}_\infty$ norm. This can be interpreted as designing the control to produce the fastest response for the nominal plant while maintaining the stability of the system as long as the perturbation of the system or the noise input is limited to a certain extent. At first glance, these two objectives may seem to be contradictory. However, as shown in [15] the solution of the $\mathcal{H}_\infty$ bound problem is not unique. All solutions of the $\mathcal{H}_\infty$ bound problem can be parameterized as $LFT(K_0, Q)$ where K_0 is a stabilizing controller satisfying the bound, LFT denotes Linear Fractional Transformation and Q is an arbitrary $\mathcal{H}_\infty$ transfer function. This gives us an extra degree of freedom to make the $\mathcal{H}_2$ objective as small as possible.

The two different performances can appear in different block diagrams as shown in

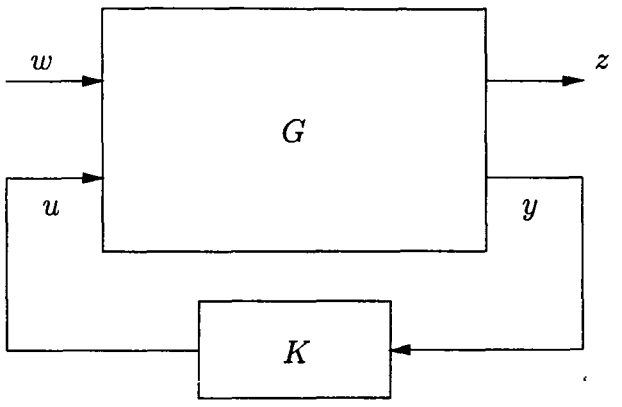


Figure 1.2: The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem where both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ objectives appear on the same input-output pair; The problem is to $\min_K \|T_{zw}\|_2^2$, s.t. $\|T_{zw}\|_\infty \leq \gamma$.

Figures 1.2-1.4 where the plants are assumed to be centralized. A similar scheme can be seen in the μ-synthesis problem which is also a multiple objective problem design except that all the objectives involved are in the $\mathcal{H}_\infty$ norm.

1.2 Background

Several different mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems have been studied during the past few years. Bernstein and Haddad [3] pioneered this topic by considering the system in Figure 1.3 with output feedback dynamic control. They reduced the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem to a suboptimal control problem by replacing the $\mathcal{H}_2$ norm objective with an upper bound on the $\mathcal{H}_2$ norm. Using the Lagrange multiplier method, they found the solution of two coupled Riccati equations to be a necessary condition for suboptimality. The coupled equations can be further reduced into four smaller coupled equations (two Riccati type and two Lyapunov type) similar to the optimal projection equations used in the reduced order LQG problem [12]. A continuation method [35] was suggested to solve these coupled equations. In [53], it was proved that the above coupled equations are indeed a necessary and sufficient condition for obtaining solution

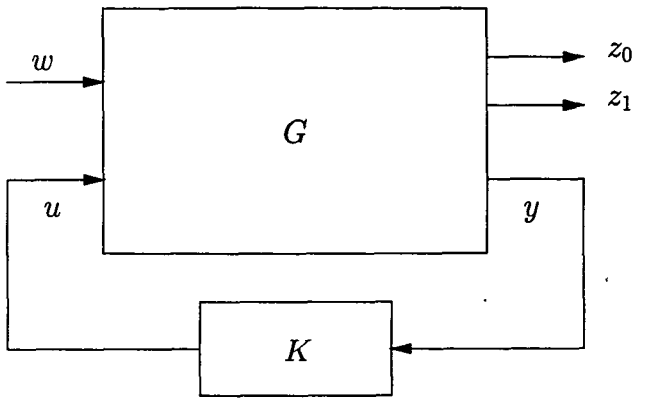


Figure 1.3: The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem where $\mathcal{H}_2$ and $\mathcal{H}_\infty$ objectives appear on different outputs but share the same input; The problem is to $\min_K \|T_{z_0 w}\|_2^2$, s.t. $\|T_{z_1 w}\|_\infty \leq \gamma$.

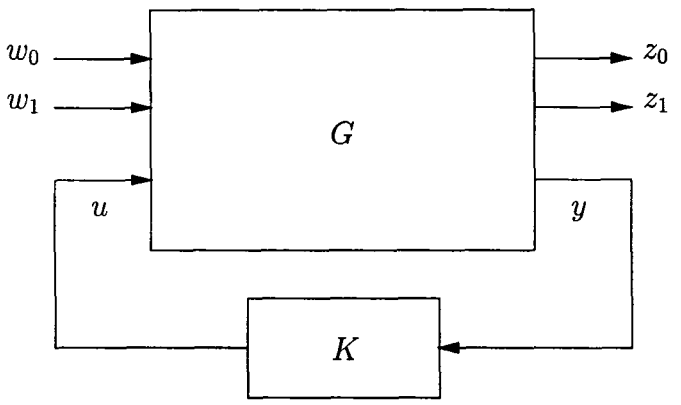


Figure 1.4: The general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem with different exogenous inputs and different outputs for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ objectives; The problem is to $\min_K \|T_{z_0 w_0}\|_2^2$ subject to $\|T_{z_1 w_1}\|_\infty \leq \gamma$.

of the suboptimal mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

In Mustafa and Glover [29], the solution for the minimum entropy problem of the system in Figure 1.2 was proved to be equivalent to the solution for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ suboptimal problem. This solution is usually called the central solution of the $\mathcal{H}_\infty$ problem and can be easily derived using the four-Riccati equations approach that appeared in [15].

Rotea [21] found that under certain conditions the state feedback case of the system in Figure 1.4 can achieve simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control by means of a dynamic controller. The simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ optimality means that its $\mathcal{H}_2$ performance is as good as that obtained using the LQ optimal gain matrix. At the same time, its $\mathcal{H}_\infty$ norm constraint is met.

In [22], Rotea and Khargonekar gave a convex optimization method for solving the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ suboptimal problem for a system with one input/two outputs as shown in Figure 1.3. They showed that for a state feedback case, the problem can be rewritten as a problem with a convex constraint set and a convex objective function which can be solved efficiently by many existing algorithms. They also show that the dynamic output feedback case can be solved by transforming it into a state feedback type of problem.

For the one-input/one-output problem, the $\mathcal{H}_2/\mathcal{H}_\infty$ suboptimal controller gives a relatively good result. For example, in [37] Rotea concludes that by properly choosing the γ, more specifically choosing $\gamma = \sqrt{2}\gamma_\infty$, where γ_∞ is the smallest $\mathcal{H}_\infty$ norm the closed-loop system can achieve, the $\mathcal{H}_2$ norm of the closed-loop system using the suboptimal solution will be no worse than $\sqrt{2}$ times the respective LQ optimal values for the one-input/one-output problem. However the numerical result in [33] shows that the $\mathcal{H}_2$ norm gap between the optimal and suboptimal solutions can be significant for the one-input/two-output problem or two-input/two-output general problem. This shows the importance of looking for an optimal solution for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$

problem.

We attempted to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem by modifying the objective of the problem into the sum of an optimal objective and a suboptimal objective weighted with λ and $1-\lambda$ respectively, where $0 \leq \lambda \leq 1$ is an artificial parameter. Using the Lagrange multiplier method, we obtained 4 coupled equations similar to (1.8), (1.13), (1.15) and (1.16) with an additional parameter λ. When $\lambda = 0$, the problem is the suboptimal $\mathcal{H}_2/\mathcal{H}_\infty$ problem which has an analytic solution. When $\lambda = 1$ we have the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem. Applying the continuation method by varying λ from 0 to 1, we move the solution from the suboptimal solution to the optimal solution. The first problem with this method is that the solution is not guaranteed to be global. A second problem is that the continuation method is relatively expensive computationally. Another problem is that for the general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem, even the suboptimal solutions require solving the coupled Riccati equations. So the advantage of obtaining the initial solution by solving a Riccati equation for a one-input/one-output system or a convex programming algorithm for a one-input/two-output system is lost. We have had limited success for the state feedback case when the suboptimal solution is not too "far" away from the optimal solution. However for the fixed order dynamic controller case the continuation method has difficulty to move from $\lambda = 0$ to $\lambda = 1$. In [6, 36], Ridgely also used a similar algorithm to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

One of the most significant findings is due to Megretski in [27]. He showed that the optimal solution for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem for the dynamic control case does not have a fixed order solution like the LQG problem. The optimal controller has infinite order and does not have a stabilizing state space representation.

At the same time Sznaier in [44, 45] obtained a procedure via convex programming that can approximate and converge to this infinite dimensional controller to arbitrarily large order for both SISO and MIMO discrete time problems.

Other developments on the topic include the singular control problem, reduced order suboptimal problem, discrete time suboptimal problem, and decentralized mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem, all of which are only slight variants of our original problem. Noticeably, Stoorvogel [43] studied another type of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem in which one tries to minimize the $\mathcal{H}_2$ norm of the input/output pair among all the possible perturbation constraints. In other words, one intends to find the control which will produce the best $\mathcal{H}_2$ norm with all possible perturbations present. This contrasts with the original case where one tries to minimize the $\mathcal{H}_2$ norm of the nominal plant.

Many of the problems mentioned above involve solving coupled Riccati/Lyapunov equations. Other than the continuation method, there was not any efficient way to solve these equations. Recently, the linear matrix inequality (LMI) approach, as in [4] has been found useful in solving many control problems. Gahinet developed an LMI toolbox which further makes the LMI approach attractive. Although the LMI approach is able to solve some multi-criteria optimization problems, we were not able to reformulate the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem into LMI form. The best that can be done is to make our problem into a biaffine matrix inequality (BMI) as proposed by Goh and Safonov et al in [19]. They discussed the bi-convex property of BMI and its geometrical interpretation. They also suggested solving BMI's by alternating solution of 2 LMI problems similar to the D-K iteration for solving the μ-synthesis problem. However, the alternating LMI method doesn't always give a solution. The main result of our research is an algorithm that can solve the BMI eigenvalue problem of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

The recent trends in mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization can be categorized in two broad directions. The first direction is due to the maturation of the techniques for solving an LMI problem. Many attempts were made in previous years to transform different varieties of the upper bound mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems into LMIs, especially for the dynamic controller case which can not be solved easily [22]. Another direction

is to extend the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem to linear time-varying systems and linear parametrically varying systems.

In [32], Paganini obtained a condition for the so-called Robust $\mathcal{H}_2$ performance problem by extending a condition from the Robust $\mathcal{H}_\infty$ performance problem of a one-input/one-output system as in Figure 1.2 with structured uncertainty. This was done by approximating the white noise signal with a set of finite energy signals. The Robust $\mathcal{H}_2$ performance problem then can be obtained using the Robust $\mathcal{H}_\infty$ performance condition by finding the worst case among that particular set. The result is an LMI that depends on frequency ω and an energy bound constraint. A test can be performed by gridding the ω-axis to produce a set of LMIs. However for the synthesis problem a D-K iteration type of solution is suggested.

For $\mathcal{H}_2$ or $\mathcal{H}_\infty$ state feedback synthesis problems, a common practice for obtaining an LMI is to eliminate the coupling between the state feedback and Lyapunov/Riccati variable. This can be done by substituting the state feedback K with VQ^{-1}. Scherer [38] shows that for a fixed order dynamic controller case, i.e finding an (A_c, B_c, C_c, D_c), there exists a similar transformation of variables that can eliminate the coupling between the control variables and Lyapunov/Riccati variables. For an upper bound mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem, these transformations result in a set of LMIs that can be solved efficiently.

The linear time-varying (LTV), linear parametrically-varying (LPV) version of the upper bound mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem is studied also by Scherer in [39]. The LTV problem can be easily transformed into a differential linear matrix inequality (DLMI), e.g. using the concept of internal dissipative system. For the DLMI problem, he proposed solving an initial value problem of a purturbed Riccati differential equation to test the solvability of the DLMI. For the LPV problem, the solvability by means of LMI's depends on the complexity of the parameter space. If the parameter space is smooth and of moderate dimension, the parameter dependent LMI can be scheduled

and solved as in the LTI case. This approach is especially useful if the parameter space is convex and has finite extremes.

The fundamental difference between their developments of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ theory and ours is that we are pursuing the non-conservative solution instead of an upper bound. There is no known way that will yield an LMI optimization problem for the non-conservative case. However, their approaches for the dynamic controller case may be useful when we extend our solution of the current constant output feedback case. Furthermore, their treatments for the LTV and LPV system offered a good mathematical insight in terms of the proof of sufficiency when a simplification or specialization is assumed. This is valuable to us if a similar extension is to be made.

1.3 Introduction to the Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problem: State Feedback Case

A finite-dimensional linear time-invariant system, such as the system G shown in Figure 1.4, can be described by the following state-space equation.

$$\begin{cases} \dot{x} = Ax + B_0 w_0 + B_1 w_1 + B_2 u \\ z_0 = C_0 x + D_0 u \\ z_1 = C_1 x + D_1 u \\ y = x \end{cases} \tag{1.1}$$

where all the matrices in (1.1) are constant real matrices of compatible dimensions, i.e.

$$A \in \mathcal{R}^{n\times n}, \quad B_0 \in \mathcal{R}^{n\times m_0}, \quad B_1 \in \mathcal{R}^{n\times m_1}, \quad B_2 \in \mathcal{R}^{n\times m_2},$$
$$C_0 \in \mathcal{R}^{p_0\times n}, \quad C_1 \in \mathcal{R}^{p_1\times n}, \quad D_0 \in \mathcal{R}^{p_0\times m_0}, \quad D_1 \in \mathcal{R}^{p_1\times m_1}$$

u is the control input and w_0 and w_1 are unmeasurable disturbances. For simplicity, we will assume $w_1 \in \mathcal{L}_2[0,\infty]$ and w_0 is white noise.

The following standard assumptions [15] are made.

Assumption 1.1

a. $(A,\ B_2)$ *is stabilizable.*

b. D_0 *and* D_1 *are of full column rank. We will occasionally use the stronger assumptions* $D_0'D_0 = I$, $D_0'C_0 = 0$, $D_1'D_1 = I$ *and* $D_1'C_1 = 0$ *to simplify the computations.*

c. $\begin{bmatrix} A - j\omega I & B_2 \\ C_0 & D_0 \end{bmatrix}$ *has full column rank for all* $\omega \in \mathcal{R}$.

d. $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_1 \end{bmatrix}$ *has full column rank for all* $\omega \in \mathcal{R}$.

Assumption 1.1a assures that there exists a stabilizing controller. Assumption 1.1b means the control weighting is nonsingular and implies that the optimal controls are finite. Another implication of assumption 1.1b is that the number of outputs must be greater than or equal to the number of control inputs u, i.e. $p_0 \geq m_0$ and $p_1 \geq m_1$. The relaxation of assumption 1.1b leads to a singular control problem and is discussed in [42]. Assumptions 1.1c and 1.1d guarantee the solutions of the Riccati equations associated with $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problem exist.

For the static case, we want to find a constant matrix gain $K \in \mathcal{R}^{m_2 \times n}$, i.e. $u = Kx$ such that K solves the following constrained optimization problem

Problem $(\mathcal{P}_s)$:

$$\begin{aligned} &\min_K \|T_{z_0w_0}\|_2^2 \\ \text{subject to} \quad &\|T_{z_1w_1}\|_\infty \leq \gamma \end{aligned} \tag{1.2}$$

where $T_{z_0w_0}$ is the closed-loop transfer function matrix from w_0 to z_0 and $T_{z_1w_1}$ is the closed-loop transfer function matrix from w_1 to z_1.

Let K_{LQ} be the LQ optimal control for $T_{z_1w_1}$ and $\gamma_2 = \|T_{z_0,w_0}(K_{LQ})\|_\infty$. Let K_∞ be the $\mathcal{H}_\infty$ optimal control for $T_{z_0w_0}$ with $\gamma_\infty = \|T_{z_0,w_0}(K_\infty)\|_\infty$. If the $\mathcal{H}_\infty$ norm

constraint parameter γ is chosen to be larger than γ_2 then K_{LQ} is also the optimal solution of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem. If γ is chosen to be smaller than γ_∞ then we will have no solution for our problem. To avoid these trivial results we will assume that

Assumption 1.2

$$\gamma_\infty \leq \gamma \leq \gamma_2 \tag{1.3}$$

In the following discussion, we will consider the problem as shown in Figure 1.3. It is the special case of the general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem with $B_0 := B_1$ and $w := w_0 := w_1$.

By applying constant state feedback $u = Kx$, the closed-loop system in Figure 1.3 becomes

$$\begin{cases} \dot{x} = (A + B_2K)x + B_1w \\ z_0 = (C_0 + D_0K)x \\ z_1 = (C_1 + D_1K)x \end{cases} \tag{1.4}$$

We need to formulate the $\mathcal{H}_2$ norm and $\mathcal{H}_\infty$ constraint first before proceeding. The definition of the $\mathcal{H}_2$ norm of a strictly proper stable system $T_{zw} := (A, B, C)$ can also be expressed in the time domain as

$$\|T_{zw}\|_2^2 = \sum_i \left(\int_0^\infty h_i(t)'h_i(t)dt \right)$$

where $h_i(t)$ is the response of the system to an impulse on the i_{th} input (i.e. $w_i(t) = \delta(t)$ and $w_j(t) = 0$ $j \neq i$). We know that an impulse input at w_i gives the system states an initial condition $x_0 = b_i$ where b_i is the i_{th} column of B. So we have

$$\|T_{zw}\|_2^2 = \sum_i \left(\int_0^\infty b_i' e^{A't} C'Ce^{At} b_i dt \right) \tag{1.5}$$

It is well known [20] that for stable A

$$Y = \int_0^\infty e^{A't} C'Ce^{At} dt$$

where $Y \geq 0$ is the observability grammian satisfying the following Lyapunov equation

$$L(Y) := A'Y + YA + C'C = 0. \tag{1.6}$$

So the $\mathcal{H}_2$ norm shown in (1.5) can be expressed as

$$\|T_{zw}\|_2^2 = Tr(BB'Y)$$

Using the above fact, the stable triple $(A + B_2K, B_1, C_1 + D_1K)$ in (1.4) has the $\mathcal{H}_2$ norm

$$\|T_{z_1w}(K)\|_2^2 = Tr(B_1B_1'Y) \tag{1.7}$$

where $Y \geq 0$ and satisfies the following Lyapunov equation

$$(A + B_2K)'Y + Y(A + B_2K) + (C_1 + D_1K)'(C_1 + D_1K) = 0 \tag{1.8}$$

To satisfy the constraint (1.2), we need the bounded real lemma of H_∞ theory. The version given below is a modification of Lemma 4 in [15].

Lemma 1.1 *Given a strictly proper stable transfer function matrix $G(s) = C(sI - A)^{-1}B$ and $\gamma > 0$, define the Hamiltonian matrix*

$$H := \begin{bmatrix} A & \gamma^{-2}BB' \\ -C'C & -A' \end{bmatrix} \tag{1.9}$$

The following conditions are equivalent

1. $\|G\|_\infty \leq \gamma$.
2. *There exists a nonnegative(positive if (A, B) is controllable) definite symmetric solution X for the following equation*

$$A'X + XA + \gamma^{-2}XBB'X + C'C = 0. \tag{1.10}$$

Moreover, $A + \gamma^{-2}BB'X$ has all its eigenvalues in the closed left half plane and has at most one conjugate pair of pure imaginary eigenvalue.

Proof. The strict inequality part of this lemma has been shown in [15] where it is shown that $\|G\|_\infty < \gamma$ if and only if $X \geq 0$ solves (1.10) and the eigenvalues of $A+\gamma^{-2}BB'X$ are in the open left half plane. We need to prove that $\|G\|_\infty = \gamma$ if and only if $X \geq 0$ solves (1.10) and $A+\gamma^{-2}BB'X$ has exactly one conjugate pair of pure imaginary eigenvalue.

Let $G(s)$ be the transfer function of a stable system (A, B, C, D) such that $\|G\|_\infty = \gamma$. It is easy to check that the transfer function $(\gamma^2 I - G^\sim G)^{-1}$ has the following representation.

$$(\gamma^2 I - G^\sim G)^{-1}(s) = \left(\begin{array}{cc|c} A & \gamma^{-2}BB' & B \\ -C'C & -A' & 0 \\ \hline 0 & B' & \gamma^2 I \end{array}\right) \tag{1.11}$$

where the notation $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is shorthand for the transfer function $D+C(sI-A)^{-1}B$ of a linear system and the A-matrix of $(\gamma^2 I - G^\sim G)$ is the Hamiltonian matrix H defined in (1.9). $\|G\|_\infty = \gamma$ means $\sigma_{max}(G(j\omega)) = \gamma$ at a finite number of ω's, say w_i, $i = 1, 2, \ldots m$ and $\sigma_{max}(G(j\omega)) < \gamma$ for all other ω. This in turn implies that $(\gamma^2 I - G^\sim G)^{-1}(s)$ has poles at $j\omega_i$. Therefore, the eigenvalues of H contain $j\omega_i$. We also know that there exists a symmetric $X \geq 0$ that solves the Riccati equation (1.10) associated with the Hamiltonian matrix H as long as (A, B) is stabilizable. Applying a similarity transformation to H as

$$\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} H \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = \begin{bmatrix} A+\gamma^{-2}BB'X & \gamma^{-2}BB' \\ 0 & -(A+\gamma^{-2}BB'X)' \end{bmatrix} \tag{1.12}$$

we conclude that the eigenvalues of $A+\gamma^{-2}BB'X$ are in the closed left plane. Among these eigenvalues, $j\omega_i$ must be on the imaginary axis.

The proof of $2 \to 1$ can be done in reverse fashion. One more point worth mentioning is that because of the particular structure of the Hamiltonian matrix H, any eigenvalue $j\omega_i$ must be of even multiplicity and defective. ¶

Using Lemma 1.1, we know that $\|T_{z_0w}(K)\|_\infty \leq \gamma$ if and only if the following Riccati equation

$$(A+B_2K)'X + X(A+B_2K) + \gamma^{-2}XB_1B_1'X + (C_0+D_0K)'(C_0+D_0K) = 0 \quad (1.13)$$

has a non-negative symmetric solution X.

To optimize (1.7) subject to the constraints (1.8) (1.13), form the Lagrangian

$$\hat{J}(K,P,Q,X,Y) = Tr\,(B_1B_1'Y + (1.8)\times P + (1.13)\times Q) \quad (1.14)$$

where the Lagrange multipliers P and Q are nonnegative symmetric matrices which are not both zero. Taking partial derivatives of $\hat{J}$ with respect to K, X and Y, yields

$$\begin{aligned}
\frac{\partial \hat{J}}{\partial X} &= (A+B_2K+\gamma^{-2}B_1B_1'X)Q + Q(A+B_2K+\gamma^{-2}B_1B_1'X)' = 0 \quad (1.15)\\
\frac{\partial \hat{J}}{\partial Y} &= (A+B_2K)P + P(A+B_2K)' + B_1B_1' = 0 \quad (1.16)\\
\frac{\partial \hat{J}}{\partial K} &\rightarrow -B_2'(XQ+YP)(P+Q)^{-1} = K \quad (1.17)
\end{aligned}$$

It can be readily seen that 5 coupled equations (1.8), (1.13), (1.15), (1.16) and (1.17) have to be solved in order to solve the optimization problem. An additional difficulty may be seen from the fact that equation (1.15) implies that $(A+B_2K+\gamma^{-2}B_1B_1'X)$ is singular when K makes the $\mathcal{H}_\infty$ norm of the closed-loop system equal to γ, i.e. has an eigenvalue on the imaginary axis. To summarize, we have the following Lemmas.

Lemma 1.2 *If K is an optimal solution for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem ($\mathcal{P}_s$), there exist nonnegative symmetric matrices X, Y, P and Q that solve (1.8), (1.13), (1.15), 1.16) and (1.17)*

Proof. The Lagrange multipliers method used in the previous derivation assures that (1.8), (1.13), (1.15), 1.16) and (1.17), are the necessary conditions for the optimality of Problem ($\mathcal{P}_s$). ¶

Lemma 1.3 *If $K, X, Y, P \geq 0, Q \geq 0$ solve (1.8), (1.13), (1.15), 1.16) and (1.17), then $(A + B_2K + \gamma^{-2}B_1B_1'X)$ has all its eigenvalues in the closed left half plane. Moreover, at least one of the eigenvalues is on the imaginary axis.*

Proof. Define $\tilde{A} := (A + B_2K + \gamma^{-2}B_1B_1'X)$. X being the stabilizing solution of (1.13) guarantees $\tilde{A}$ has all its eigenvalues in the closed left half plane. (1.15) can be vectorized as

$$(\tilde{A} \otimes I + I \otimes \tilde{A}')\underline{q} = 0; \tag{1.18}$$

where $\underline{q} = vec[Q]$ is the vectorization of Q by piling each column of Q into a big column vector. (1.18) implies that $(\tilde{A} \otimes I + I \otimes \tilde{A}')$ must have at least one zero eigenvalue. It is known that the eigenvalues of $(\tilde{A} \otimes I + I \otimes \tilde{A}')$ are $\lambda_i + \lambda_j$ where λ_i is the i_{th} eigenvalue of $\tilde{A}$. At least one combination of $\lambda_i + \lambda_j$ must be equal to zero. Because $Re(\lambda_i) \leq 0 \ \forall i$, this is only possible when at least one of the eigenvalues of $\tilde{A}$ is 0 or one pair of complex eigenvalues of $\tilde{A}$ are pure imaginary. ¶

Lemma 1.4 *If $K, X, Y, P \geq 0, Q \geq 0$ solve (1.8), (1.13), (1.15), (1.16) and (1.17), then $\|T_{z_1 w_0}(K)\|_\infty = \gamma$.*

Proof. From Lemma 1.3, we know that eigenvalues of $(A + B_2K + \gamma^{-2}B_1B_1'X)$ are in the closed left half plane. The nonnegativeness of the solution, P, of Lyapunov equation (1.16) assures the stability of $A + B_2K$. Lemma 1.1 shows that the $\mathcal{H}_\infty$ norm of $(A + B_2K, B_1, C_0 + D_0K)$ is equal to γ. ¶

Lemma 1.4 says that the optimal solution derived from solving the 5 coupled equations should be at the boundary of the constraint $\|T_{zw}\|_\infty \leq \gamma$. The even multiplicity of the imaginary eigenvalues of the Hamiltonian matrix H provides the crucial algebraic property of this particular boundary.

In the next section, a continuation algorithm is proposed to solve 5 coupled equations

1.4 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Optimal Control:Continuation Method

For simplicity, we will work on the one-input/one-output system shown in Figure 1.2, i.e. $B_0 = B_1$, $C_0 = C_1$ and $D_0 = D_1$. We also assume that D_1 has been normalized, i.e. $D_1'D_1 = I$ and $D_1'C_1 = 0$. The results can be easily extended to the one-input/two-output case. One advantage to working on the one-input/one-output case is that we have a suboptimal solution as the initial solution for our numerical procedure immediately by choosing the $\mathcal{H}_\infty$ central solution. For the one-input/two-output case we will need another procedure as in [3, 22] to obtain the suboptimal solution. Our initial approach to solve the optimal problem is to use a continuation method to solve the coupled Riccati equations.

1.4.1 State Feedback Case

A simple system with state feedback will be used to demonstrate the procedure for obtaining a static/dynamic controller for the mixed H_2/H_∞ problem. Difficulties with the algorithm will be pointed out as a topic for further research.

In Lemma 1.2, we have derived five coupled equations as the necessary conditions for solving the Problem $(\mathcal{P}_s)$. To solve these five equations, Newton's method usually doesn't work because these equations are highly coupled. If the initial solution is out of the contractive region, Newton's method will not converge to a solution. To improve the chance that the initial solution will be contractive, a continuation method with suitable choice of continuation parameter is usually useful.

Consider the system

$$\begin{cases} \dot{x} = Ax + B_1 w_1 + B_2 u \\ z_1 = C_1 x + D_1 u \\ y = x \end{cases} \tag{1.19}$$

with the closed-loop system when applying constant state feedback $u = Kx$ as

$$\begin{cases} \dot{x} = (A + B_2K)x + B_1w_1 \\ z_1 = (C_1 + D_1K)x \end{cases} \tag{1.20}$$

A continuation method can be used to solve the equations by modifying the objective of problem $\mathcal{P}_s$ as

$$J_2(K, X, Y, \lambda) = \lambda Tr(B_1B_1'Y) + (1 - \lambda)Tr(B_1B_1'X) \tag{1.21}$$

where Y is the nonnegative symmetric solution of

$$(A + B_2K)'Y + Y(A + B_2K) + (C_1 + D_1K)'(C_1 + D_1K) = 0 \tag{1.22}$$

and X is the nonnegative symmetric solution of the Riccati equation.

$$(A + B_2K)'X + X(A + B_2K) + \gamma^{-2}XB_1B_1'X + (C_1 + D_1K)'(C_1 + D_1K) = 0 \tag{1.23}$$

Then (1.15),(1.16) and 1.17) will be modified as

$$\begin{aligned} \frac{\partial J_2}{\partial X} &= (A + B_2K + \gamma^{-2}B_1B_1'X)Q \\ &\quad +Q(A + B_2K + \gamma^{-2}B_1B_1'X)' + (1 - \lambda)B_1B_1' = 0 && (1.24) \\ \frac{\partial J_2}{\partial Y} &= (A + B_2K)P + P(A + B_2K)' + \lambda B_1B_1' = 0 && (1.25) \\ \frac{\partial J_2}{\partial K} &\rightarrow -B_2'(XQ + YP)(P + Q)^{-1} = K && (1.26) \end{aligned}$$

This defines a new optimization problem for a fixed λ

Problem $(\mathcal{P}_{s\lambda})$:

$$\min_{K,X,Y} J_2(K, X, Y, \lambda) \tag{1.27}$$

where K, X and Y satisfy (1.22), (1.23), (1.24),(1.25) and (1.26).

It has been stated in [3] that

$$Tr(B_1B_1'X) \geq Tr(B_1B_1'Y). \tag{1.28}$$

We can prove the above statement by subtracting (1.22) from (1.23) to obtain

$$(A + B_2K)'(X - Y) + (X - Y)(A + B_2K) + \gamma^{-2}XB_1B_1'X = 0$$

This is a Lyapunov equation in $(X - Y)$ with stable $(A + B_2K)$ and nonnegative $\gamma^{-2}XB_1B_1'X$. It guarantees that $(X - Y)$ is nonnegative.

When $\lambda = 0$ the problem $(\mathcal{P}_{s\lambda})$ reduces to a performance bound problem. The Lyapunov equations for Y and P (1.22) and (1.25) can be ignored. The reduced solution for the performance bound problem can be computed analytically as

$$K_o = -B_2'X_o \tag{1.29}$$

where X_o is the solution for the following equation.

$$A'X + XA + X(\gamma^{-2}B_1B_1' - B_2B_2')X + C_1'C_1 = 0 \tag{1.30}$$

This solution is exactly the so-called central controller for H_∞ problem for a perfect state observation(full information) system [15]. It has been proven in [30] that the central controller described in [15] is a solution for the performance bound problem for the suboptimal problem.

Having derived the coupled equations for the problem $(\mathcal{P}_{s\lambda})$:, we are ready to solve our problem by applying continuation methods, by varying λ from 0 to 1, using (1.29) as the initial solution. A primitive algorithm is described below

Algorithm 1.1

1. *Choose* $\lambda = 0$, $P^{<0>} = 0$; *Solve for* $Q^{<0>}$ *and* $Y^{<0>}$ *using* $K^{<0>}$ *in (1.29); Set* $l = 1$;

2. *Choose* $\Delta\lambda > 0$ *and* $\lambda = \lambda + \Delta\lambda$; *Stop the procedure if* $\|\lambda^{<l>} - \lambda^{<l-1>}\| < \epsilon$ *where* ϵ *is the intended precision;*

3. *Solve for* $Y^{<l>}$ *and* $P^{<l>}$ *with fixed* $K^{<l-1>}$ *from (1.22) and (1.25) respectively;*

4. *Obtain $X^{<l>}$ by solving the Riccati equation*

$$(A + B_2K^{<l-1>} + \gamma^{-2}B_1B_1'X^{<l-1>})'X + X(A + B_2K^{<l-1>} + \gamma^{-2}B_1B_1'X^{<l-1>})$$
$$-\gamma^{-2}XB_1B_1'X + (C_1 + D_1K^{<l-1>})'(C_1 + D_1K^{<l-1>}) = 0 \qquad (1.31)$$

5. *Solve for $Q^{<l>}$ from*

$$\begin{aligned}(A + B_2K^{<l-1>} + \gamma^{-2}B_1B_1'X^{<l>})Q \quad &+ \quad Q(A + B_2K^{<l-1>} \\ &+ \quad \gamma^{-2}B_1B_1'X^{<l>})' + (1-\lambda)B_1B_1' = 0\end{aligned}$$

6. *Compute $K^{<l>}$*

$$K^{<l>} = -B_2'(X^{<l>}Q^{<l>} + Y^{<l>}P^{<l>})(P^{<l>} + Q^{<l>})^{-1}$$

7. *Let $\Delta K^{<l>} = |K^{<l>} - K^{<l-1>}|$*

   - *If the sequence $\Delta K^{<l>} \geq \xi$ for the intended precision ξ but is improving then set $l = l + 1$ and go to step 3;*
   - *If the sequence $\Delta K^{<l>} \geq \xi$ but is not converging, reduce λ; Set $l = 1$ and go to step 3;*
   - *If $\Delta K^{<l>} < \xi$, set $X^{<0>} = X^{<l+1>}$, $l = 1$, and go to step 2;*

The loop between step 3 and step 7 is to solve the problem $\mathcal{P}_{s\lambda}$ and is sometimes called the corrector stage of the continuation method. The corrector stage solves the five coupled equations with λ fixed. The outer loop starting at step 2 is called the predictor stage in which we increase λ to a degree that the corrector stage will converge. A more sophisticated projection method is actually used to predict the best step size $\Delta\lambda$ and the initial values of the other variables in this numerical experiment.

Several difficulties with this algorithm will be summarized. They are as follows

- Overall performance can be described as inefficient especially when λ is close to 1. However the convergence is guaranteed provided that the continuation step size, $\Delta\lambda$, is small enough.

- The Riccati equation (1.31) in step 4 of the algorithm guarantees a non-negative solution, $X^{<l>}$ only when $A + B_2K^{<l>} + \gamma^{-2}B_1B_1'X^{<l>}$ is a stable matrix [50]. However, as we mentioned earlier, the solution will fall on the boundary of the H_∞ constraint. This means that $A + B_2K^{<l>} + \gamma^{-2}B_1B_1'X^{<l>}$ will have imaginary axis eigenvalue(s). Therefore, during the corrector stage of the continuation method, the stability will not be assured. And the iteration may fail because the Riccati equation does not have a non-negative solution.

- It is suspected that the homotopy path will have a turning point at $\lambda = 1$. When the continuation approaches a turning point, the Jacobian of the equations(operator) will become singular which poses difficulties during the corrector stage of the continuation method. This means that only small increments of the continuation parameter λ are allowed when it approaches the critical point and the continuation eventually fails. Our equation has to be augmented [31, 34] in order to have smooth progress.

1.4.2 Dynamic Controller Case

Having found a method for the solution of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem using static feedback, we are ready to pursue further the dynamic controller case because, in general, full state information is not available. It is also worth mentioning that our approach (to be) works as well for a reduced order controller which can be an alternative for an expensive full order controller. The reduced order controller gives up a small amount of performance.

Now let us consider the following system.

$$\begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_1 u \\ y = C_2 x + D_2 w \end{cases}$$

Following the standard assumptions for LQG problems and by suitable transformation, we are able to normalize D_1 and D_2 as follows

$$D_1' D_1 = I \quad D_1' C_1 = 0$$

$$D_2 D_2' = I \quad D_2' C_2 = 0$$

The purpose of the problem is to design a controller K_c of order n_c

$$\begin{cases} \dot{x}_c = A_c x_c + B_c y \\ u = C_c x_c \end{cases}$$

to achieve the following objectives.

Problem $(\mathcal{P}_K)$:

$$\min_{K_c} \|T_{zw}\|_2 \tag{1.32}$$

$$s.t. \qquad \|T_{zw}\|_\infty \leq \gamma \tag{1.33}$$

where T_{zw} is the closed-loop transfer function matrix from w to z. As introduced previously, several works have been published on performance bounds of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

In our approach, the $\mathcal{H}_\infty$ dynamic central controller defined in [15] will be used as the initial solution. The continuation method is used for the problem $\mathcal{P}_K$. When the continuation starts, the formulation for the performance bound is used. The formulation will be given later. Before we do that, the closed-loop transfer function

matrix will be given. Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ form the (A,B,C) triplet of the closed-loop system.

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A & B_2C_c \\ B_cC_2 & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} B_1 \\ B_cD_2 \end{bmatrix} w \\ z = [C_1 \quad D_1C_c] \begin{bmatrix} x \\ x_c \end{bmatrix} \end{cases} \tag{1.34}$$

Similar to the state feedback case in (1.8), the H_2 performance can be written as

$$J(A_c, B_c, C_c) := \|T_{zw}\|_2^2 = Tr(\tilde{B}\tilde{B}'Y) \tag{1.35}$$

where Y is the observability grammian of $\tilde{A}$, $\tilde{C}$, i.e. $Y \geq 0$ satisfies the following equation

$$\tilde{A}'Y + Y\tilde{A} + \tilde{C}'\tilde{C} = 0 \tag{1.36}$$

To assure that $\|T_{zw}\|_\infty$ satisfies the constraint γ, there should exist a non-negative X that is the solution of the following Riccati equation.

$$\tilde{A}'X + X\tilde{A} + \gamma^{-2}X\tilde{B}\tilde{B}'X + \tilde{C}'\tilde{C} = 0 \tag{1.37}$$

As in (1.28), $Tr(\tilde{B}\tilde{B}'X) \geq Tr(\tilde{B}\tilde{B}'Y)$, therefore $Tr(\tilde{B}\tilde{B}'X)$ is an upper bound for $J(A_c, B_c, C_c)$. An analytic solution for the LQG bound problem is also given in [30]. This gives us the motivation to modify the performance (1.35) as

$$J_1(A_c, B_c, C_c, \lambda) = \lambda Tr(\tilde{B}\tilde{B}'Y) + (1-\lambda)Tr(\tilde{B}\tilde{B}'X) \tag{1.38}$$

Problem $(\mathcal{P}_{K\lambda})$:

$$\min_{A_c, B_c, C_c} J_1(A_c, B_c, C_c, \lambda) \tag{1.39}$$

$$s.t. \qquad \|T_{zw}\|_\infty \leq \gamma \tag{1.40}$$

Then the solution in [30], i.e. the central solution of $\mathcal{H}_\infty$ problem, solves the problem $\mathcal{P}_{K\lambda}$ when $\lambda = 0$. $\mathcal{P}_{K\lambda}$ is equivalent to $\mathcal{P}_K$ when $\lambda = 1$. Using the continuation method by varying λ from 0 to 1, one may be able to solve the problem $\mathcal{P}_K$.

To solve the problem $\mathcal{P}_{K\lambda}$ using a Lagrange multiplier method similar to the one used in [3], we modify the performance by adding two non-negative symmetric multipliers P and Q. Then derivatives are taken with respect to X, Y, A_c, B_c and C_c. Then make the derivatives equal to 0. The following equations result after some straightforward calculations.

$$\begin{aligned} J_2(A_c, B_c, C_c, \lambda) &= Tr[\lambda\tilde{B}\tilde{B}'Y + (1-\lambda)\tilde{B}\tilde{B}'X) \\ &\quad +(1.36) \times P + (1.37) \times Q] \end{aligned} \tag{1.41}$$

$$\frac{\partial J_2}{\partial Y} = \tilde{A}P + P\tilde{A}' + \lambda\tilde{B}\tilde{B}' = 0 \tag{1.42}$$

$$\frac{\partial J_2}{\partial X} = (\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}'X)Q + Q(\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}'X)' + (1-\lambda)\tilde{B}\tilde{B}' = 0 \tag{1.43}$$

$$\frac{\partial J_2}{\partial A_c} \rightarrow P_{12}'Y_{12} + P_{22}Y_{22} + Q_{12}'X_{12} + Q_{22}X_{22} = 0 \tag{1.44}$$

$$\begin{aligned} \frac{\partial J_2}{\partial B_c} \rightarrow\ & B_c'(Y_{22} + X_{12}'Q_{11}X_{12} + X_{22}Q_{12}'X_{12} + X_{12}'Q_{12}X_{22} + X_{22}Q_{22}X_{22}) \\ & +C_2(P_{11}Y_{12} + P_{12}Y_{22} + Q_{11}X_{12} + Q_{12}X_{22}) + D_2B_1'(\lambda Y_{12} + (1-\lambda)X_{12}) \\ & +D_2B_1'(X_{11}Q_{11}X_{12} + X_{12}Q_{12}'X_{12} + X_{11}Q_{12}X_{22} + X_{12}Q_{22}X_{22}) = 0 \end{aligned} \tag{1.45}$$

$$\begin{aligned} \frac{\partial J_2}{\partial C_c} \rightarrow\ & (P_{12}'Y_{11} + P_{22}Y_{12}' + Q_{12}'X_{11} + Q_{22}X_{12}')B_2 \\ & (P_{12}' + Q_{12}')C_1'D_1 + (P_{22} + Q_{22})C_c' = 0 \end{aligned} \tag{1.46}$$

Note that A_c, B_c and C_c are contained in $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ respectively as in (1.34). In the above equations, we partition X, Y, P and Q accordingly with order n and n_c, i.e.

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}' & X_{22} \end{bmatrix}; Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}' & Y_{22} \end{bmatrix}; Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}; P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix}$$

From (1.45) (1.46) B_c and C_c can be calculated if P, Q, X and Y are known. To solve for A_c in terms of X, Y, Q, P, we do the following mathematics on portions of (1.36) and (1.37) as

$$\begin{aligned} & P_{12}' \times (1,2) \text{ of } (1.36) + P_{22} \times (2,2) \text{ of } (1.36) \\ + \quad & Q_{12}' \times (1,2) \text{ of } (1.37) + Q_{22} \times (2,2) \text{ of } (1.37) \end{aligned}$$

The A_c will have the following form provided X_{22} and P_{22} are invertible.

$$X_{22}^{-1}Y_{22}A_c + A_cQ_{22}P_{22}^{-1} + W = 0 \tag{1.47}$$

where

$$\begin{aligned} W = -X_{22}^{-1}\Big(P_{12}'(A'Y_{12} + C_2'B_c'Y_{22} + Y_{11}B_2C_c) &+ P_{22}(C_c'B_2'Y_{12} + Y_{12}'B_2C_c + C_c'C_c) + \\ Q_{12}'(A'X_{12} + C_2'B_c'X_{22} + X_{11}B_2C_c) &+ Q_{22}(C_c'B_2'X_{12} + X_{12}'B_2C_c + C_c'C_c) + \\ \gamma^{-2}(Q_{12}'(X_{11}B_1B_1'X_{12} + X_{12}B_cB_c'X_{22}) &+ \\ Q_{22}(X_{12}'B_1B_1'X_{12} + X_{22}B_cB_c'X_{22}))\Big)'P_{22}^{-1} \end{aligned} \tag{1.48}$$

A_c can be solved using a Kronecker product.

It may appear that the above setup for solving the problem is complicated. A lot of computation is needed for the numerical solution to determine A_c, B_c and C_c. However, two simple facts will reduce the problem considerably. One can observe that the input-output behavior of the controller is invariant under similarity transformation. That is, given a controller A_c, B_c and C_c that solves the mixed H_2/H_∞ optimization problem $\mathcal{P}_K$, the controller $R^{-1}A_cR, R^{-1}B_c$ and C_cR also solves the same problem. This suggests that we find a canonical form to describe the structure of the controllers. We choose a canonical form introduced in [48] for fixed order compensators as follows.

$$A_c := \begin{bmatrix} A_{c1} & \dots & 0 \\ & \ddots & \\ 0 & \dots & A_{c\frac{n_c}{2}} \end{bmatrix}, \quad B_c := \begin{bmatrix} B_{c1} \\ \vdots \\ B_{c\frac{n_c}{2}} \end{bmatrix}, \quad C_c := [\, C_{c1} \quad \dots \quad C_{c\frac{n_c}{2}} \,] \tag{1.49}$$

where

$$A_{ci} := \begin{bmatrix} 0 & 1 \\ x & x \end{bmatrix}, \quad B_{ci} := \textit{full matrix}, \quad C_{ci} := \begin{bmatrix} 1 & x & \dots & x \\ x & x & \dots & x \end{bmatrix}. \tag{1.50}$$

A_c is formed by putting A_{ci} $i = 1, 2, \dots, n_c/2$ on its diagonal, B_c is a full matrix and $C_c = [\, C_{c1} \quad \dots \quad C_{ci} \quad \dots \,]$ $i = 1, 2, \dots, n_c/2$. x is a scalar parameter to be determined by the optimization procedure. If n_c is an odd number, an additional one dimensional

block will be added, i.e.

$$A_{c(\frac{n_c}{2}+1)} := [\,x\,], \quad B_{c(\frac{n_c}{2}+1)} := [\,x\,], \quad C_{c(\frac{n_c}{2}+1)} := [\,1\,]. \tag{1.51}$$

The number of parameters of the controller to be determined is $n_c + (p_c \times n_c - \frac{n_c}{2}) + n_c \times m_c$.

There are problems when we use a fixed structure for the controller triplet (A_c, B_c, C_c). If we use the fixed point algorithm, i.e. to solve for A_c, B_c, C_c and P, Q, X, Y alternatively during the corrector stage, the new A_c, B_c and C_c will be of different structure. Therefore it is likely that the fixed-point algorithm will not work. So we choose to use Newton's method to solve the minimization problem on J_2 as in (1.41). We solve the following zero finding problem

$$F(x, \lambda) = 0; \tag{1.52}$$

where

$$F := vec[\frac{\partial J_2}{\partial A_c}, \frac{\partial J_2}{\partial B_c}, \frac{\partial J_2}{\partial C_c}, \frac{\partial J_2}{\partial Q}, \frac{\partial J_2}{\partial P}, \frac{\partial J_2}{\partial X}, \frac{\partial J_2}{\partial Y}]$$

$$x = vec[A_c, B_c, C_c, Q, P, X, Y].$$

The continuation algorithm is basically parallel to what we used for the static controller case. Unfortunately, we also have similar difficulties while we doing the numerical experiment. The attention then turn to other alternatives such as eigenvalue derivatives method and BMI eigenvalue problem.

1.5 Thesis outline

In Chapter 2, we will present several tools that will be needed in finding the solution for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem. One of the tools is to find a solution for the $\mathcal{H}_\infty$ control problem with constant output feedback. We need this solution as an initial solution for our BMI algorithm for solving the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem after the state

feedback parameterization. An alternating LMI algorithm will be employed to solve this problem.

Another tool is the gradient of the $\mathcal{H}_\infty$ norm with respect to the controller parameters. Using the fact that Hamiltonian matrices are related to Riccati equations by the Bounded Real Lemma, one can derive this gradient effectively. The positive definite solution of the Riccati equation is related to the eigenspace corresponding to the stable eigenvalues of the Hamiltonian Matrix. In $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problems, it is proved that the optimal solution for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems will occur at the boundary of the $\mathcal{H}_\infty$ constraint. It can also be proved that the Hamiltonian matrix will have dual defective pure imaginary eigenvalues when a control is on the boundary of the $\mathcal{H}_\infty$ constraint. We will discuss the properties of the eigenvalue derivative near the imaginary axis of a Hamiltonian Matrix [24]

In Chapter 3 we will formulate mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems as BMI eigenvalue problems using the concept of dissipative system. We will apply an interior point algorithm to solve the BMI eigenvalue problem. We also discuss the problem with this interior point algorithm caused by the unobservability/uncontrollability of the closed-loop system.

The concept of controlled invariant of an LTI system [51] will be discussed in Chapter 4. We will discuss how to obtain unobservable and uncontrollable subspaces. We will parameterize the set of state feedbacks that cause the closed-loop system to become unobservable/uncontrollable. We also discuss the existence condition for these unobservable/uncontrollable subspaces and how these finite subspaces form a lattice structure. A dual lattice structure which we need to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem will be studied.

In Chapter 5, the BMI problem that appeared in chapter 3 will be partitioned into a lattice of constant output feedback $\mathcal{H}_2/\mathcal{H}_\infty$ sub-problems according to the dual lattice structure of the unobservable spaces for each output. Modifications of the BMI

eigenvalue problem will be made for the reduced sub-problem. We also discuss how to obtain the complete mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem by traversing through the dual lattice of sub-problems from the top down. We will show how the algorithm of Chapter 2 helps us eliminate many of the sub-problems or gives us an initial solution for the sub-problems or perturbs the optimal solution of one subproblem as an initial solution of its child subproblem. The constant output feedback case of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem will also be presented. Several simple examples using this algorithm will be presented. We will also comment on the globality of the sub-problem.

In chapter 6, we apply our algorithm to solve a F-14 flight control problem. Finally conclusions and recommendations for further research are given in Chapter 7.

1.6 Contributions

The major contributions of this work are considered to be:

- Formulation of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem into a BMI optimization problem for both constant state feedback and constant output feedback cases. An interior point algorithm is proposed to solve the BMI eigenvalue problem. The condition such that the interior point algorithm will converge to at least a local minima is given.

- Finding that the difficulty with the BMI algorithm is due to the unobservability of the closed-loop system when the dimension of the output is equal to the dimension of the control input.

- Construction of the lattices of unobservable (A, B)-controlled invariants and corresponding feedback parameterizations.

- An algorithm to traverse through the lattice of sub-problems corresponding to the lattice of feedback parameterizations in order to obtain the solution of the

full $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem and avoid the difficulty caused by the unobservability of the closed-loop system when the dimension of the output is equal to the dimension of the control input.

- An efficient algorithm to solve the constant output feedback $\mathcal{H}_\infty$ control problem using alternating LMI.
- Demonstration of the advantages of an $\mathcal{H}_2/\mathcal{H}_\infty$ controller which provides a good compromise between two competing criteria through a robust F-14 flight control problem.

1.7 Notation

$\mathcal{R}^{m\times n}$:	real matrix of size $m \times n$
$\mathcal{L}_2[t_1, t_2]$:	square integrable function on $[t_1, t_2]$
γ_∞	smallest possible $\mathcal{H}_\infty$ norm of a given system.
γ_2:	$\mathcal{H}_\infty$ norm of the system using $\mathcal{H}_2$ controller.
$\|G\|_2$:	$\mathcal{H}_2$ norm of G; $\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} Trace[G(j\omega)^{\sim}G(j\omega)]d\omega\right)^{1/2}$
$\|G\|_\infty$:	$\sup_\omega \sigma_{max}[G(j\omega)]$
$\sigma_{max}(M)$:	largest singular value of matrix M
$\sigma(A, \mathcal{V})$:	eigenvalues restricted to the linear eigenspace $\mathcal{V}$
$\left(\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right)$:	$D + C(sI - A)^{-1}B$, transfer function of system matrices (A, B, C, D)
$\lambda_i(M)$:	ith eigenvalue of matrix M
T_{zw}:	transfer function from w to z
$LFT(G, K)$:	Linear Fractional Transformation; given $G := \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ $LFT(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$
$Tr(M)$:	trace of the matrix M

$M > 0$:	M is positive definite matrix
$M \geq 0$:	M is nonnegative definite matrix
$M > N$:	$M - N$ is positive definite
$vec[M]$:	make matrix M into vector
$vec[M]_k$:	kth element of $vec(M)$
$im(V)$:	image space spanned by vectors V
$\mathcal{V}^{\perp}$:	null space of $\mathcal{V}^{\perp}$, $\{y :< y, x >= 0, \forall x \in \mathcal{V}\}$
$V^{\perp}$:	the matrix containing the vectors that span $im(V)^{\perp}$
$\mathcal{V}^{\square}(A)$:	complement vector space of $\mathcal{V}$ with respect to A
$V^{\square}(A)$:	the matrix containing the vectors that span $\mathcal{V}^{\square}(A)$
V^{+}:	$V^{+} = (V'V)^{-1}V'$ for a full column rank matrix V
$A \otimes B$:	direct product of A and B
$\begin{bmatrix} Q & * \\ S & R \end{bmatrix}$:	For a symmetric matrix, * is short for S'.

Chapter 2

$\mathcal{H}_\infty$ Sub-Optimal Problem with Static Output Feedback

In most cases it is not possible to observe all the internal states of a system. Usually the observations consist of combinations of subsets of states. In the case that the internal states can not be fully measured, it is possible to obtain an estimate of the states through a dynamic filter. However, if the feedback is restricted to have constant gains, we have a static output feedback problem.

Many control problems for systems with complete state feedback, such as the linear quadratic problem, the stabilizability problem, the $\mathcal{H}_\infty$ control problem or the sub-optimal $\mathcal{H}_2/\mathcal{H}_\infty$ problem are convex problems. They can be solved effectively using existing algorithms such as LMI or by solving a Riccati equation.

Unfortunately, for the static output feedback problem it is not possible to obtain such a convex setup. In [49], necessary conditions for solving the static output feedback LQR were obtained through coupled Lyapunov equations. Since then little research [1] on the necessary and sufficient conditions for optimality of static output feedback problems has been done. Until Skelton, et al. in [13, 14, 16], there was no efficient algorithm for solving these problems. In particular, in [14], the LQ suboptimal control problem for static output feedback feedback systems was solved by solving two LMIs

alternatively using a min/max algorithm.

We found that a similar algorithm can be applied to solve the suboptimal $\mathcal{H}_\infty$ version of the static output feedback problem. The reasons that we want to study this particular problem are

- We will partition the original $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem into a lattice of sub-problems. Each subsystem is a system with partial state feedback which is a special case of a static output feedback system. Obtaining the $\mathcal{H}_\infty$ static output feedback suboptimal solution will give us an initial solution for the $\mathcal{H}_2/\mathcal{H}_\infty$ sub-problem.

- A partial state feedback $\mathcal{H}_2/\mathcal{H}_\infty$ sub-problem can be dropped from the optimization procedure if it does not have an $\mathcal{H}_\infty$ sub-optimal solution. The min/max algorithm is not only useful in finding a sub-optimal $\mathcal{H}_\infty$ solution, but also indicates that no sub-optimal $\mathcal{H}_\infty$ solution exists whenever the algorithm fails. With this property, many of the sub-problems can be eliminated. This greatly reduces the amount of effort for the complete procedure.

- Our $\mathcal{H}_2/\mathcal{H}_\infty$ static state feedback optimization can also be applied to solve the $\mathcal{H}_2/\mathcal{H}_\infty$ static output feedback optimization. Such problems can also be partitioned into a lattice of static output feedback problems.

- An $\mathcal{H}_2/\mathcal{H}_\infty$ dynamic feedback problem can be transformed into a static output feedback problem with a special system structure. Knowing the $\mathcal{H}_\infty$ suboptimal solution can also help in solving the dynamic controller problem.

2.1 Problem Setup

Consider the following system

$$\begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_2 u \\ y = C_3 x \end{cases} \tag{2.1}$$

The $\mathcal{H}_\infty$ static output feedback problem is to seek a gain matrix G, i.e.

$$u = Gy \tag{2.2}$$

such that the $\mathcal{H}_\infty$ norm of the closed-loop system is smaller than a pre-specified value γ, i.e.

Problem $(\mathcal{P}_{of})$:

$$\|T_{zw}(G)\|_\infty < \gamma \tag{2.3}$$

From the Bounded Real Lemma, we know that when applying (2.2) to the system, (2.3) implies the following Riccati inequality has to be true.

$$(A + B_2GC_3)'P + P(A + B_2GC_3) + \gamma^{-2}PB_1B_1'P + (C_1 + D_1GC_3)'(C_1 + D_1GC_3) < 0 \tag{2.4}$$

Or using the Schur complement formula, we have

$$\begin{bmatrix} (A + B_2GC_3)'P + P(A + B_2GC_3) & (C_1 + D_1GC_3)' & PB_1 \\ (C_1 + D_1GC_3) & -I & 0 \\ B_1'P & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{2.5}$$

We will make use of the following lemma to obtain the main theorem.

Lemma 2.1 (Iwasaki and Skelton [46]). *Let $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{p\times n}$ and $\Theta = \Theta' \in \mathcal{R}^{n\times n}$ where $m < n$ and $p < n$. Then there exists a matrix $G \in \mathcal{R}^{m\times p}$ satisfying*

$$BGC + (BGC)' + \Theta < 0 \tag{2.6}$$

if and only if the matrices B, C and Θ satisfy

$$B^{\perp\prime}\Theta B^{\perp} < 0, \tag{2.7}$$

$$C'^{\perp\prime}\Theta C'^{\perp} < 0 \tag{2.8}$$

If (2.7) and (2.8) hold true, the set of all G that satisfy (2.6) can be expressed as

$$G = -\rho B'\Phi C'(C\Phi C')^{-1} + \rho S^{1/2}L(C\Phi C')^{-1/2}, \tag{2.9}$$

where L is an arbitrary matrix such that $\|L\| < 1$, ρ is an arbitrary positive scalar such that

$$\rho > \rho_{min}, \qquad \rho_{min} := \lambda_{max}[B^{+}(\Theta - \Theta B^{\perp}(B^{\perp\prime}\Theta B^{\perp})^{-1}B^{\perp\prime}\Theta)B^{+\prime}], \tag{2.10}$$

and positive definite matrices Φ and S are defined by

$$\Phi := (\rho BB' - \Theta)^{-1}, \qquad S := \rho I - B'[\Phi - \Phi C'(C\Phi C')^{-1}C\Phi]B \tag{2.11}$$

Proof. Please refer to [46]. ¶

To apply Lemma 2.1, we need to rewrite (2.5) in the form of (2.6) by the following derivation

$$\begin{aligned}
&\begin{bmatrix} (A+B_2GC_3)'P + P(A+B_2GC_3) & (C_1+D_1GC_3)' & PB_1 \\ (C_1+D_1GC_3) & -I & 0 \\ B_1'P & 0 & -\gamma^2 I \end{bmatrix} \\
= &\begin{bmatrix} PB_2GC_3 & 0 & 0 \\ D_1GC_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} PB_2GC_3 & 0 & 0 \\ D_1GC_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}' + \begin{bmatrix} A'P+PA & C_1' & PB_1 \\ C_1 & -I & 0 \\ B_1'P & 0 & -\gamma^2 I \end{bmatrix} \\
= &\begin{bmatrix} PB_2 \\ D_1 \\ 0 \end{bmatrix} G\,[C_3 \quad 0 \quad 0] + \left(\begin{bmatrix} PB_2 \\ D_1 \\ 0 \end{bmatrix} G\,[C_3 \quad 0 \quad 0]\right)' + \begin{bmatrix} A'P+PA & C_1' & PB_1 \\ C_1 & -I & 0 \\ B_1'P & 0 & -\gamma^2 I \end{bmatrix} \\
< &\ 0.
\end{aligned}$$

So we have

$$B = \begin{bmatrix} PB_2 \\ D_1 \\ 0 \end{bmatrix}, \quad C = [C_3 \quad 0 \quad 0]$$

$$\Theta = \begin{bmatrix} A'P + PA & C_1' & PB_1 \\ C_1 & -I & 0 \\ B_1'P & 0 & -\gamma^2 I \end{bmatrix} \tag{2.12}$$

Now we are ready to state the main theorem.

Theorem 2.1 *Given the system in (2.1) with the feedback as in (2.2). The following 3 statements are equivalent*

1. $\|T_{zw}(G)\|_\infty < \gamma$

2. *There exists a P and its inverse P^{-1} such that*

$$\begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp'} \begin{bmatrix} AP^{-1} + P^{-1}A' + \gamma^{-2}B_1B_1' & P^{-1}C_1' \\ C_1P^{-1} & -I \end{bmatrix} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp} < 0 \tag{2.13}$$

$$\begin{bmatrix} C_3'^{\perp} & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} A'P + PA + C_1'C_1 & PB \\ B_1'P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C_3'^{\perp} & 0 \\ 0 & I \end{bmatrix} < 0 \tag{2.14}$$

3. $\mathcal{P} \neq \emptyset$ *where*

$$\begin{aligned} \mathcal{P} &:= \left\{ P \in \mathcal{R}^{n\times n} : P > 0,\ P^{-1} \in \mathcal{X}, P \in \mathcal{Y} \right\} \\ \mathcal{X} &:= \left\{ X \in \mathcal{R}^{n\times n} : \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp'} \begin{bmatrix} AX + XA' + \gamma^{-2}B_1B_1' & XC_1' \\ C_1X & -I \end{bmatrix} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp} < 0 \right\} \\ \mathcal{Y} &:= \left\{ Y \in \mathcal{R}^{n\times n} : \begin{bmatrix} C_3'^{\perp} & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} A'Y + YA + C_1'C_1 & YB \\ B_1'Y & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C_3'^{\perp} & 0 \\ 0 & I \end{bmatrix} < 0 \right\} \end{aligned}$$

Proof. The proof of the equivalence between 1 and 2 can be derived directly from (2.6) by applying Lemma 2.1. From the facts that

$$[C_3 \quad 0 \quad 0]'^{\perp} = \begin{bmatrix} C_3'^{\perp} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{2.15}$$

and

$$\begin{bmatrix} PB_2 \\ D_1 \\ 0 \end{bmatrix}^{\perp} = \begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} B_2 \\ D_1 \\ 0 \end{bmatrix}^{\perp} \tag{2.16}$$

(2.16) can be checked by the following derivation.

$$\{\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} B_2 \\ D_1 \\ 0 \end{bmatrix}^{\perp}\}' \begin{bmatrix} PB_2 \\ D_1 \\ 0 \end{bmatrix} = \begin{bmatrix} B_2 \\ D_1 \\ 0 \end{bmatrix}^{\perp'} \begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} PB_2 \\ D_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} B_2 \\ D_1 \\ 0 \end{bmatrix}^{\perp'} \begin{bmatrix} B_2 \\ D_1 \\ 0 \end{bmatrix} = 0$$

Also we have

$$\begin{bmatrix} B_2 \\ D_1 \\ 0 \end{bmatrix}^{\perp} = \begin{bmatrix} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp} & 0 \\ 0 & I \end{bmatrix}. \tag{2.17}$$

So (2.7) is the same as

$$\begin{bmatrix} B_2 \\ D_1 \\ 0 \end{bmatrix}^{\perp'} \begin{bmatrix} P^{-1}A' + AP^{-1} & P^{-1}C_1' & B_1 \\ C_1P^{-1} & -I & 0 \\ B_1' & 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} B_2 \\ D_1 \\ 0 \end{bmatrix}^{\perp}$$

$$= \begin{bmatrix} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp} & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} P^{-1}A' + AP^{-1} & P^{-1}C_1' & B_1 \\ C_1P^{-1} & -I & 0 \\ B_1' & 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp} & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp'} \begin{bmatrix} AP^{-1} + P^{-1}A' & P^{-1}C_1' \\ C_1P^{-1} & -I \end{bmatrix} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp} & \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp'} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\ [B_1' \quad 0] \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp} & -\gamma^2 I \end{bmatrix} < 0.$$

The reverse Schur complement formula gives

$$\begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp'} \begin{bmatrix} AP^{-1} + P^{-1}A' & P^{-1}C_1' \\ C_1P^{-1} & -I \end{bmatrix} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp'} + \gamma^{-2} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp'} \begin{bmatrix} B_1B_1' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp}$$

$$= \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp'} \begin{bmatrix} AP^{-1} + P^{-1}A' + \gamma^{-2}B_1B_1' & P^{-1}C_1' \\ C_1P^{-1} & -I \end{bmatrix} \begin{bmatrix} B_2 \\ D_1 \end{bmatrix}^{\perp} < 0$$

This proves (2.13). (2.14) can be obtained in similar way. 3 is just another way of stating 2. ¶

We know that the sets $\mathcal{X}$ and $\mathcal{Y}$ are convex because the inequalities are LMIs. However, the set $\mathcal{P}$ is not convex. Our problem is to find an $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ such that $X = Y^{-1}$. The algorithm in the next section provides a way to find such an X, Y.

2.2 Algorithm

We will make use of the min/max algorithm proposed in [16, 17]. The min/max algorithm finds a positive definite matrix P such that P belongs to an open convex set $\mathcal{X}$ and P^{-1} belongs to another open convex set $\mathcal{Y}$.

The idea of this min/max algorithm is to augment each set $\mathcal{X}$ and $\mathcal{Y}$ with additional LMI constraints and a minimization criterion and then use the LMI solver to solve the minimization problem. For each iteration we modify the augmented constraint and perform the minimization repeatedly. At the end of the algorithm X will approach Y^{-1}, i.e.

$$\lim_{k\to\infty} \|X^{<k>} - (Y^{<k>})^{-1}\| = 0$$

.

One possible choice is to solve the following min/max problems alternatively

$$\min_{X,\mu} \mu$$

$$\textit{such that} \quad \begin{cases} I \le Y_{k-1}^{1/2} X Y_{k-1}^{1/2} \le \mu I \\ X \in \mathcal{X} \end{cases}$$

$$\max_{Y,l} l$$

Definition 4.4 $(A,B)-$Controlled Invariant

Consider a pair (A,B). A subspace $\mathcal{V} \subseteq \mathcal{X}$ is said to be an (A,B)-controlled invariant if

$$A\mathcal{V} \subseteq \mathcal{V} + \mathcal{B} \quad with \quad \mathcal{B} := im(B). \tag{4.20}$$

A dual of controlled invariant on the pair (A,C) can be defined as followings.

Definition 4.5 (A,C)-Conditioned Invariant

Consider a pair (A,C). A subspace $\mathcal{W}$ is said to be an (A,C)-conditioned invariant if

$$A(\mathcal{W} \cap \mathcal{C}) \subseteq \mathcal{W} \quad with \quad \mathcal{C} := ker(C); \tag{4.21}$$

The (A,B)-controlled invariant will be denoted as $\mathcal{V}(A,B)$. The existence of $\mathcal{V}(A,B)$ implies that there exists a state feedback F such that $\mathcal{V}$ is $A+BF$ invariant, i.e.

Lemma 4.5 [2] *Let $\mathcal{V} \in \mathcal{X}$. Then there exists a matrix $F \in \mathcal{R}^{m\times n}$ such that*

$$(A+BF)\mathcal{V} \subseteq \mathcal{V} \tag{4.22}$$

if and only if $\mathcal{V}$ is an (A,B)-controlled invariant.

Proof. **[if part]**We will prove this by constructing such an F. Suppose that we already found $\mathcal{V}(A,B)$ with V as a unitary basis. To find the desired F, we first observe that (4.20) is equivalent to the existence of Λ and U such that

$$AV = V\Lambda + BU = [\,V \quad B\,]\begin{bmatrix}\Lambda \\ U\end{bmatrix} \tag{4.23}$$

For any Λ and U that satisfy (4.23), we can obtain F as

$$F = -U(V'V)^{-1}V' \tag{4.24}$$

It is easy to see that $\mathcal{V}$ is $(A+BF)$ invariant with this choice of F.

$$(A+BF)V = (A - BU(V'V)^{-1}V')V = AV - BU = V\Lambda \tag{4.25}$$

This shows that $\mathcal{V}$ is $(A+BF)$ invariant.

[only if part]If there exists an F such that $(A+BF)\mathcal{V} \subseteq \mathcal{V}$, there exists a Λ such that $(A+BF)V = V\Lambda$. This implies

$$AV = V\Lambda - BFV = V\Lambda + BU \tag{4.26}$$

by letting $U = -FV$. Eqn. (4.26) shows that $\mathcal{V}$ is an (A, B)-controlled invariant. ¶

In the proof of the previous lemma, for each controlled invariant $\mathcal{V}$, there exists Λ and U such that Eqn. (4.23) holds. Such a U will give a state feedback F such that $\mathcal{V}$ is $(A+BF)$- invariant. From (4.25) we also conclude that Λ has the eigenvalues of $\lambda(A+BF, \mathcal{V})$. Here we use $\lambda(A+BF, \mathcal{V})$ to denote the eigenvalues of $A+BF$ restricted to $\mathcal{V}$.

However for each $\mathcal{V}$, Λ and U are not necessary unique. If $[V \quad B]$ is square and invertible, we can solve (4.23) by

$$\begin{bmatrix} \Lambda \\ U \end{bmatrix} = [V \quad B]^{-1} AV \tag{4.27}$$

If $[V \quad B]$ is a wide matrix and of full row rank ,

$$\begin{bmatrix} \Lambda \\ U \end{bmatrix} = \begin{bmatrix} V & B \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} AV \\ x_3 \end{bmatrix} \tag{4.28}$$

where x_1,x_2 and x_3 are free variables that one can choose to fit the need. For example, one can choose the x_i so that Λ is stable. This property is very useful in the pole assignment application.

If $[V \quad B]$ is a long matrix, it is not likely to have an F such that $\mathcal{V}$ is $(A+BF)$ invariant. The only possible condition is that the row rank of AV is less than or equal to the column rank of $[V \quad B]$. The details will be given later for this situation.

4.2.1 Parameterization of State Feedback F

Given an (A, B)-controlled invariant space $\mathcal{V}$, we will show a way to parameterize F such that $\mathcal{V}$ is $(A+BF)$ invariant. We let V denote an orthogonal basis of $\mathcal{V}$ and $V^{\perp}$

be its orthogonal complement,i.e.

$$[V \quad V^{\perp}]'[V \quad V^{\perp}] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Here A, B and V are real matrices of size $n \times n$, $n \times m$ and $n \times q$ respectively.

We first perform the singular value decomposition of matrix $[V \ B]$ in (4.23) as

$$uS_2v' = svd([V \ B]). \tag{4.29}$$

(4.23) becomes

$$AV = uS_2v' \begin{bmatrix} \Lambda \\ U \end{bmatrix} \tag{4.30}$$

or

$$u'AV = S_2 \left(v' \begin{bmatrix} \Lambda \\ U \end{bmatrix} \right) \tag{4.31}$$

It can be proved that if V is a controlled invariant, then S_2 and $u'AV$ will have the following form

$$S_2 = \begin{bmatrix} \hat{S}_2 & 0 \\ 0 & 0 \end{bmatrix} \quad u'AV = \begin{bmatrix} \eta \\ 0 \end{bmatrix} \tag{4.32}$$

where $\hat{S}_2$ and η are of full row rank of size $k \times k$ and $k \times q$ respectively. Then the solution can be derived as

$$v' \begin{bmatrix} \Lambda \\ U \end{bmatrix} = \begin{bmatrix} \hat{S}_2^{-1}\eta \\ W \end{bmatrix} \tag{4.33}$$

where W is an arbitrary $(m + q - k) \times q$ matrix.

$$\begin{bmatrix} \Lambda \\ U \end{bmatrix} = v \begin{bmatrix} \hat{S}_2^{-1}\eta \\ W \end{bmatrix} \tag{4.34}$$

Let v be partitioned as an $(m, q) \times (k, m + q - k)$ matrix

$$v = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \tag{4.35}$$

We have the concise form

$$\Lambda = v_{11}\hat{S}_2^{-1}\eta + v_{12}W \tag{4.36}$$

$$U = v_{21}\hat{S}_2^{-1}\eta + v_{22}W \tag{4.37}$$

Using (4.24) F has the form

$$F = -(v_{21}\hat{S}_2^{-1}\eta + v_{22}W)(V'V)^{-1}V' = -(v_{21}\hat{S}_2^{-1}\eta + v_{22}W)V' \tag{4.38}$$

Here, we have derived a set of F's parameterized by W such that $\mathcal{V}$ is $A+BF$ invariant, i.e.

$$(A + BF)V = V\Lambda. \tag{4.39}$$

However this set of F's is not the only such set that makes $\mathcal{V}$ be $A + BF$ invariant. We classify this set of F's as a particular solution. We will classify the other set of F's as a trivial solution. It can be obtained by the following derivation.

Given any particular solution F_1 which makes $\mathcal{V}$ be $A + BF_1$ invariant,

$$F = F_1 + YV^{\perp'} \tag{4.40}$$

will also make $\mathcal{V}$ be $A + BF$ invariant, because

$$(A + B(F_1 + YV^{\perp'}))V = (A + BF_1)V + BYV^{\perp'}V = (A + BF_1)V = V\Lambda$$

where Y is an arbitrary matrix of the size $m \times (n - q)$. We have the following Lemma

Lemma 4.6 *Let $im(V) = \mathcal{V}$ be an (A, B)-controlled invariant space. The set of $F \in \mathcal{F}$ such that $\mathcal{V}$ is $A + BF$ invariant can be parameterized as*

$$\mathcal{F} := \left\{F : F = -(v_{21}\hat{S}_2^{-1}\eta + v_{22}W)V' + YV^{\perp'} \ \forall W \in R^{(m+q-k)\times q},\ Y \in R^{m\times(n-q)}\right\} \tag{4.41}$$

with all the matrices defined as above and W and Y the free parameters. Moreover, if $m + q - k = 0$, then we will not have the free parameter W any more.

Proof. by construction as above. ¶

Lemma 4.6 is based on prior knowledge of an (A, B)-controlled invariant. It will not be useful unless we have an effective algorithm to compute $\mathcal{V}(A, B)$

We have discussed the properties of a controlled invariant and how to obtain its corresponding state feedback parameterization. We will proceed to discuss an algorithm [2] for obtaining a controlled invariant.

Definition 4.6 (A, B)-controlled invariant contained in $\mathcal{K}$

An (A, B)-controlled invariant contained in $\mathcal{K}$ will be denoted by $\mathcal{V}(A, B; \mathcal{K})$ and mean

$$\mathcal{V} \subseteq \mathcal{K} \quad \textit{and} \quad A\mathcal{V} \subseteq \mathcal{V} + \mathcal{B} \tag{4.42}$$

where $\mathcal{B} := im(B)$.

There exists a maximal (A, B)-controlled invariant contained in $\mathcal{K}$ denoted by $\max \mathcal{V}(A, B; \mathcal{K})$. Assuming that A is invertible, it can be computed by

Algorithm 4.3 [2] Computing the maximal (A, B)-controlled invariant contained in $\mathcal{K}$, denoted $\max \mathcal{V}(A, B; \mathcal{K})$

1. $\mathcal{V}_0 := \mathcal{K}$; $i = 0$.

2. $\mathcal{V}_i := \mathcal{K} \cap A^{-1}(\mathcal{V}_{i-1} + \mathcal{B})$.

3. $i = i + 1$ *and repeat step 2 until* $\mathcal{V}_k = \mathcal{V}_{k+1}$

Proof. First we need to prove that $\mathcal{V}_i$ is a decreasing subspace. It can be proved by induction. Let $\hat{\mathcal{V}}_i$ be defined as

$$\hat{\mathcal{V}}_i = \hat{\mathcal{V}}_{i-1} \cap A^{-1}(\hat{\mathcal{V}}_{i-1} + \mathcal{B}) \text{ and } \hat{\mathcal{V}}_0 = \mathcal{K}$$

From the definition we know that $\hat{\mathcal{V}}_i$ is decreasing. Also Note that $\hat{\mathcal{V}}_0 = \mathcal{V}_0$ and assume that $\hat{\mathcal{V}}_{i-1} = \mathcal{V}_{i-1}$. We have

$$\begin{aligned}
\hat{\mathcal{V}}_i &= \mathcal{V}_{i-1} \cap A^{-1}(\mathcal{V}_{i-1} + \mathcal{B}) \\
&= \mathcal{K} \cap A^{-1}(\mathcal{V}_{i-2} + \mathcal{B}) \cap A^{-1}(\mathcal{V}_{i-1} + \mathcal{B}) \\
&= \mathcal{K} \cap A^{-1}(\mathcal{V}_{i-1} + \mathcal{B}) = \mathcal{V}_i
\end{aligned}$$

This shows that $\hat{\mathcal{V}}_i$ is equivalent to $\mathcal{V}_i$ $\forall$ i. Therefore $\mathcal{V}_i$ is also decreasing. At the termination condition $\mathcal{V}_k = \mathcal{V}_{k+1}$, we have $\mathcal{V}_k \subseteq \mathcal{K}$ and $\mathcal{V}_k \subseteq A^{-1}(\mathcal{V}_k + \mathcal{B})$ and it implies $A\mathcal{V}_k \subseteq \mathcal{V}_k + \mathcal{B}$. Because $\mathcal{V}_i$ is decreasing, the controlled invariant can be achieved in a finite number of steps. In fact it can be achieved in at most $n-1$ steps because $\mathcal{V}_i$ is a finite dimensional decreasing subspace and $\mathcal{K}$ is at least one dimensional.

To show that $\mathcal{V}_k$ is indeed maximal, let $\mathcal{S}$ be another controlled invariant contained in $\mathcal{K}$ we have $\mathcal{S} \subseteq \mathcal{V}_0$ and assume that $\mathcal{S} \subseteq \mathcal{V}_{i-1}$

$$\mathcal{S} \subseteq \mathcal{K} \cap A^{-1}(\mathcal{S} + \mathcal{B}) \subseteq \mathcal{K} \cap A^{-1}(\mathcal{V}_{i-1} + \mathcal{B}) = \mathcal{V}_i$$

The above equation is also true when $i = k$. Therefore $\mathcal{S} \subseteq \mathcal{V}_k$ proves that $\mathcal{V}_k$ is maximal for all controlled invariants that are contained in $\mathcal{K}$. ¶

Lemma 4.7 *If the pair (A, B) is controllable and A is invertible, then* $\max \mathcal{V}(A, B) = \mathcal{X}$ *and* $\max \mathcal{V}(A, B; \mathcal{K}) = \mathcal{K}$.

Proof. $\max \mathcal{V}(A, B)$ is in fact $\max \mathcal{V}(A, B; \mathcal{X})$. Using Algorithm 4.3 we know that in at most $n-1$ steps we can achieve the maximum. Also if $\mathcal{K} = \mathcal{X}$ we will have

$$\mathcal{V}_n = A^{-1}(\mathcal{V}_{n-1} + \mathcal{B}) \text{ or } A\mathcal{V}_n = \mathcal{V}_{n-1} + \mathcal{B}$$

Using the recursive algorithm from Lemma 4.3

$$\begin{aligned} & A^n \mathcal{V}_n = A^{n-1}\mathcal{B} + A^{n-2}\mathcal{B} + \ldots + \mathcal{B} \\ \text{or} \quad & \mathcal{V}_n = A^{-n}(A^{n-1}\mathcal{B} + A^{n-2}\mathcal{B} + \ldots + \mathcal{B}) = \mathcal{X} \end{aligned}$$

The last equality is from the definition of controllability. $\max \mathcal{V}(A, B; \mathcal{K}) = \mathcal{K}$ is the direct consequence of the above result for any arbitrary $\mathcal{K}$. ¶

Next, we discuss an important application of the maximum controlled invariant. It is called the almost disturbance rejection problem and is essential to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

4.3 Almost Disturbance Rejection Problem Without a D Matrix

We consider the following linear system without a D matrix.

$$\begin{cases} \dot{x} = Ax + Bu \\ z = Cx \end{cases} \tag{4.43}$$

The almost disturbance rejection problem is to find a state feedback control F such that as many states as possible can not be observed from the output. In other words, any non-zero initial condition on these unobservable states will result in zero output. This means that the closed-loop triple $(A + BF, B, C)$ has a similarity transformation of the following form

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \qquad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} \tag{4.44}$$

$$\hat{C} = [0 \quad \hat{C}_2] \tag{4.45}$$

To obtain such an F, one can compute

$$\mathcal{V}^* = \max \mathcal{V}(A, B; ker(C)). \tag{4.46}$$

Once we obtain $\mathcal{V}^*$, Lemma 4.6 will give the complete set of F's.

However, not all the triplets (A, B, C) will have an unobservable (A, B)-controlled invariant. The following theorem gives the condition for not having such an invariant.

Theorem 4.1 *Consider a system triplet (A, B, C) with (A, B) controllable and (A, C) observable. Let p be the row rank of C, m the column rank of B and A be $n \times n$. B and C are assumed to be of full rank. If $m < p < n$ and $ker(C) \cap im(B) = \emptyset$ then*

$$\max \mathcal{V}(A, B; ker(C)) = \emptyset. \tag{4.47}$$

Proof. Assume there exists an (A, B)-controlled invariant contained in $ker(C)$, namely $\mathcal{V}$ with the basis V. Let q be the column rank of V. We have $q \leq n - p$

because $ker(C)$ has the rank $n-p$. One can safely choose $T = [V\ B\ [V\ B]^{\perp}]$ because $V \cap B = \emptyset$ and $q+m \leq n-p+m < n$. Using T as a new basis, we can transform the triplet (A, B, C) into

$$\hat{A} = T^{-1}AT = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} \\ 0 & \hat{A}_{22} & \hat{A}_{23} \\ 0 & \hat{A}_{32} & \hat{A}_{33} \end{bmatrix},$$

$$\hat{B} = T^{-1}B = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}, \quad \hat{C} = CT = [0 \quad C_1 \quad C_2], \quad \hat{V} = T^{-1}V = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}$$

$\hat{V}$ remains an $(\hat{A}, \hat{B})$-controlled invariant contained in $ker(\hat{C})$. So there exists a nonzero solution (Λ, U) for the equation

$$\hat{A}\hat{V} = [\hat{V} \quad \hat{B}] \begin{bmatrix} \Lambda \\ U \end{bmatrix} \tag{4.48}$$

That is

$$\begin{bmatrix} \hat{A}_{11} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Lambda \\ U \end{bmatrix} \tag{4.49}$$

must have a solution. This is a contradiction to the assumption that (A, C) is observable and U must be nonzero to make the state unobservable from the output. ¶

This theorem says that if the number of outputs is larger than the number of inputs, i.e. $p > m$ and B and C are of full column rank, then it is not possible to find a state feedback that will make the closed-loop system unobservable. This also means that $\max \mathcal{V}(A, B, ker(C)) = \emptyset$.

In the case that the number of outputs is equal to the number of inputs, i.e. $p = m$, we also have the following essential lemma.

Lemma 4.8 *Consider the linear system triplet (A, B, C) with (A, B) controllable. Let p be the row rank of C, m be the column rank of B and A be $n \times n$. B and C are assumed*

to be of full rank. If $m = p < n$ and $ker(C) \cap im(B) = \emptyset$, then $\max \mathcal{V}(A, B, ker(C)) = ker(C)$

Proof. We can prove this Lemma by showing that there exists a nonzero solution (Λ, U) for (4.23) for $V = ker(C)$. By the assumption $ker(C) \cap im(B) = \emptyset$ and $p = m$,

$$Aker(C) = [\, ker(C) \quad B \,] \begin{bmatrix} \Lambda \\ U \end{bmatrix}$$

admits a solution because $[\, ker(C) \quad B \,]$ is square and invertible. ¶

4.3.1 Lattice of Unobservable (A, B)-Controlled Invariants

In the previous section, we have demonstrated the almost disturbance rejection problem. As discussed earlier, the problem is actually to maximize the number of unobservable states. However the unobservable states are exactly the cause of the problems we encountered with our BMI solver in Chapter 3. We need to investigate under what conditions, by what control and on which state, a given LTI system will be made unobservable. Theorem 4.1 shows that when $p > m$ there exists no state feedback that will make the output unobservable. In the case that $p < m$ the set of unobservable controlled invariants is not finite. For the case that $p = m$ there exists a finite number of unobservable (A, B) controlled invariants. Moreover the finite invariants form a lattice relation. This is the main topic in this section.

Theorem 4.2 *Assume that (A, B, C) are minimal. Let p be the row rank of C, m be the column rank of B and A be $n \times n$. B and C are assumed to be of full rank. If $m = p < n$ and $ker(C) \cap im(B) = \emptyset$ then the set of all unobservable (A, B)-controlled invariants $\mathcal{V}(A, B, ker(C))$ and $\emptyset$ form a lattice structure with the $\subseteq$, $\cap$ and $\cup$ relationship.*

Proof. Lemma 4.8 shows that $ker(C) = \max \mathcal{V}(A, B; ker(C))$. Then there exists a state feedback F_0 from Lemma 4.6 such that $(A + BF_0, B, C)$ has the following

structure under the new coordinates $T = [ker(C)\ (ker(C))^{\perp}]$.

$$T^{-1}(A+BF_0)T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CT = [0 \quad I],$$

Let $S = [S_1, S_1^{\square}]$ where

$$S_1 = \begin{bmatrix} S_{11} \\ 0 \end{bmatrix}, \quad S_{11}^{-1} A_{11} S_{11} = \Lambda$$

Λ is the Jordan form of A_{11} and $S_1^{\square}$ is the complement of S_1 with respect to $(A+BF_0)$ matrix. Under the new coordinates S we have

$$\hat{A} = \begin{bmatrix} \Lambda & 0 \\ 0 & \hat{A}_{22} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \quad \hat{C} = [0, \hat{C}_2] \tag{4.50}$$

$\hat{C}_2$ is an invertible $p \times p$ square matrix. $\hat{B}_2$ is an invertible $p \times m$ square matrix because $p = m$. Under the new coordinates $V^* = ker(\hat{C}) = \max \mathcal{V}(\hat{A}, \hat{B}; ker(\hat{C}))$ is

$$V^* = \begin{bmatrix} I_{(n-p)\times(n-p)} \\ 0 \end{bmatrix}$$

Because

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots \\ & \ddots & \\ 0 & \dots & \lambda_\mu \end{bmatrix}$$

where the λ_i's are elementary Jordan blocks. For simplicity, we demonstrate only the case where the λ_i's are simple blocks. Let e_i be the unit basis corresponding to λ_i. We have $e_i \in ker(\hat{C})\ \ i = 1, \dots \mu$. We can prove that e_i is also (A, B)-controlled invariant because (4.23) admits a solution $(\lambda_i, 0)$ as

$$\begin{bmatrix} \lambda_1 & 0 & \dots & \\ & \ddots & & 0 \\ 0 & \dots & \lambda_\mu & \\ & 0 & & \hat{A}_{22} \end{bmatrix} e_i = e_i \lambda_i + \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} 0$$

The 0 associated with $\hat{B}$ has to be zero because $\hat{B}_2$ is invertible. We prove that $e_i, i = 1, \dots \mu$ is an (A, B)-controlled invariant contained in $ker(\hat{C})$.

The space spanned by a linear combination of $e_i s$, i.e. $\sum_i \epsilon_i e_i$, with more than one nonzero $\epsilon_i' s$ can not be a (A, B)-controlled invariant because there will be no solution for (4.23), i.e

$$\hat{A} \sum_i \epsilon_i e_i \neq \sum_i \epsilon_i e_i \lambda + \hat{B} U$$

for any nonzero (λ, U).

However any space spanned by $e_i, i = 1, \ldots \mu$ is (A, B)-controlled invariant because (4.23) admits solution as, e.g.

$$\begin{bmatrix} \lambda_1 & 0 & \ldots & \\ & \ddots & & 0 \\ 0 & \ldots & \lambda_\mu & \\ & 0 & & \hat{A}_{22} \end{bmatrix} [e_1 \ e_2] = [e_1 \ e_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} 0$$

There exist a total of $2^\mu - 1$ (A, B)-controlled invariants contained in $\ker(\hat{C})$. Together with $\emptyset$, these $\mathcal{V}(\hat{A}, \hat{B}, ker(\hat{C}))$ form a lattice $\mathcal{L}$ with e_i as the elements of the lattice, $\emptyset$ as the root and $[e_1 \ e_2 \ \ldots \ e_\mu]$ as its top.

The merit of this proof lies in the fact that because $p = m$, $\hat{B}_2$ has to be square and invertible. In order to satisfy (4.23) the U in (4.23) has to be zero. $[e_1 \ \ldots \ e_\mu]$ is already an $\hat{A}$ invariant so it can accept that $U = 0$. The lattice of this (A, B)-controlled invariant is exactly identical to the lattice of A-invariant shown in Figure 4.1. ¶

We just showed how to obtain the lattice of unobservable (A, B)-controlled invariants. As a matter of fact, there are two other lattices involved. One is the lattice formed by the combination of unobservable modes Λ_i corresponding to each e_i. The other lattice is formed by the set of state feedbacks that will make e_i an unobservable (A, B)-controlled invariant. the set of state feedback F that will

Theorem 4.3 *With the assumptions as in Theorem 4.2, the set of state feedback parameterizations corresponding to each $\mathcal{V}(A, B, ker(C))$ in the lattice forms a lattice itself.*

Proof. We follow the coordinate transformation to get the $(\hat{A}, \hat{B}, \hat{C})$ as in Theorem 4.2. Because $U = 0$ for each $\mathcal{V} \in \mathcal{L}$ with q as its the column rank, the state feedback parameterization is simply

$$\mathcal{F}(V) = \left\{F | F = YV^{\perp'} \quad \forall\, Y \in R^{m\times(n-q)}\right\} \tag{4.51}$$

where $\mathcal{V} = im(V)$. If $\mathcal{V}_1 \subseteq \mathcal{V}_2$, then $\mathcal{F}(V_1) \supseteq \mathcal{F}(V_2)$. This means that $\mathcal{F}(V)$ form a lattice of the same structure as $\mathcal{L}$ with reverse partial ordering relationship. ¶

We recapitulate here the method to obtain the lattices of unobservable (A, B)-controlled invariant, unobservable modes and their corresponding state feedback parameterizations for the case that $p = m$.

1. Obtain $ker(C) = \max \mathcal{V}(A, B; ker(C))$.

2. Obtain one state feedback F_0 from the state feedback parameterization as in Lemma 4.6.

3. Transform the triplet $(A + BF_0, B, C)$ into (4.50) using the similarity transformation matrix T, S as in Theorem 4.2, i.e $\hat{A} = S^{-1}T^{-1}(A + BF_0)TS$, $\hat{B} = S^{-1}T^{-1}B$ and $\hat{C} = CTS$.

4. Obtain the elements $e_i, i = 1, \ldots \mu$ of the lattice $\mathcal{L}(e_i)$. In the original coordinates, the elements of the lattice can be computed as TSe_i. We can also form the lattice of $\mathcal{L}(\Lambda_i)$ at the same time. Be aware that e_i can be two-dimensional for complex Λ_i and more for larger Jordan blocks.

5. Let $V \in \mathcal{L}(e_i)$ with column rank q. The state feedback parameterization corresponding to V is $\mathcal{F}(V)$ as in (4.51). In the original coordinates before the F_0 is added to the system it is

$$\mathcal{F}(VS^{-1}T^{-1}) = \left\{F | F = F_0 + Y(VS^{-1}T^{-1})^{\perp'} \quad \forall\, Y \in R^{m\times(n-q)}\right\} \tag{4.52}$$

Example 4.1

Consider the following triplet

$$A = \begin{bmatrix} 4 & -2 & -3 & -5 & 2 \\ 2 & 1 & 4 & 1 & -1 \\ -3 & 2 & 2 & 3 & -3 \\ -5 & 4 & 2 & -3 & -3 \\ 2 & -2 & 1 & -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} -4 & 4 & -1 \\ -1 & 0 & -3 \\ 3 & 0 & -5 \\ 4 & -2 & 4 \\ -5 & 4 & -5 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 4 & 4 & 2 & -5 \\ -2 & 0 & -5 & 3 & 1 \\ -3 & -1 & 2 & -4 & 3 \end{bmatrix}$$

Using the algorithm 4.3 we can compute $V^* = ker(C)$ as

$$V^* = \begin{bmatrix} -0.145 & 0.295 \\ -0.548 & 0.591 \\ 0.481 & 0.161 \\ 0.638 & 0.233 \\ 0.202 & 0.695 \end{bmatrix}$$

Lemma 4.6 gives the set of state feedback parameterizations corresponding to $ker(C)$ as

$$\begin{aligned} F &= F_0 + YV^{*\perp'} \\ &= \begin{bmatrix} -0.24 & -0.728 & 0.41 & 0.536 & -0.0406 \\ -0.329 & -1.15 & 0.886 & 1.17 & 0.259 \\ -0.116 & -0.448 & 0.406 & 0.538 & 0.182 \end{bmatrix} \\ &\quad + Y_{3\times 3} \begin{bmatrix} -0.311 & 0.357 & 0.793 & -0.28 & -0.262 \\ -0.433 & 0.466 & -0.278 & 0.624 & -0.358 \\ -0.779 & -0.0762 & -0.191 & -0.266 & 0.529 \end{bmatrix} \end{aligned}$$

The closed-loop system $(A + BF_0, C)$ under the unitary basis $T = [V^* \ V^{*\perp}]$, i.e.

$(T^{-1}(A+BF)T, T^{-1}B, CT)$, has the following form

$$T'(A+BF_0)T = \begin{bmatrix} 1.35 & -3.68 & 4.45 & 4.64 & 2.53 \\ -2.05 & -0.624 & 2.83 & -1.99 & -3.08 \\ 0 & 0 & 2.83 & 5.01 & -1.65 \\ 0 & 0 & 5.74 & 3.31 & 1.04 \\ 0 & 0 & 0.36 & 1.81 & -1.55 \end{bmatrix}$$

$$CT = \begin{bmatrix} 0 & 0 & 5.35 & 3.79 & -4.24 \\ 0 & 0 & -4.44 & 3.77 & 2.25 \\ 0 & 0 & 2.5 & -3.29 & 4.68 \end{bmatrix}$$

The top-left two by two subblock of $T^{-1}(A+BF)T$ is equal to Λ. If we transform $T^{-1}(A+BF)T$ into Jordan form by matrix S we have

$$S^{-1}T^{-1}(A+BF_0)TS = \begin{bmatrix} 3.28 & 0 & 0 & 0 & 0 \\ 0 & -2.56 & 0 & 0 & 0 \\ 0 & 0 & 8.36 & 0 & 0 \\ 0 & 0 & 0 & -0.403 & 0 \\ 0 & 0 & 0 & 0 & -3.37 \end{bmatrix},$$

$$CTS = \begin{bmatrix} 0 & 0 & 3.68 & -1.43 & -0.407 \\ 0 & 0 & 0.253 & 2.34 & -1.55 \\ 0 & 0 & -0.094 & 0.135 & 2.28 \end{bmatrix}$$

In the new coordinates $[e_1\ e_2]$, $[e_1]$, $[e_2]$ and $\emptyset$ form the lattice of unobservable (A,B)-controlled invariants where $e_1 = [1\ 0\ 0\ 0\ 0]'$ and $e_2 = [0\ 1\ 0\ 0\ 0]'$. And the lattice of state feedback parameterizations is $\mathcal{F}([e_1\ e_2]) = Y_{3\times3}[e_1\ e_2]^{\perp}$, $\mathcal{F}([e_1]) = Y_{3\times4}[e_1]^{\perp}$, $\mathcal{F}([e_2]) = Y_{3\times4}[e_2]^{\perp}$ and $Y_{3\times5}$ respectively. Let V_1 and V_2 be the e_1 and e_2 under the original coordinates,

$$[V_1\ V_2],\ V_1 = \begin{bmatrix} -0.266 \\ -0.76 \\ 0.351 \\ 0.456 \\ -0.145 \end{bmatrix},\ V_2 = \begin{bmatrix} -0.115 \\ -0.0549 \\ -0.447 \\ -0.607 \\ -0.644 \end{bmatrix},\ \emptyset$$

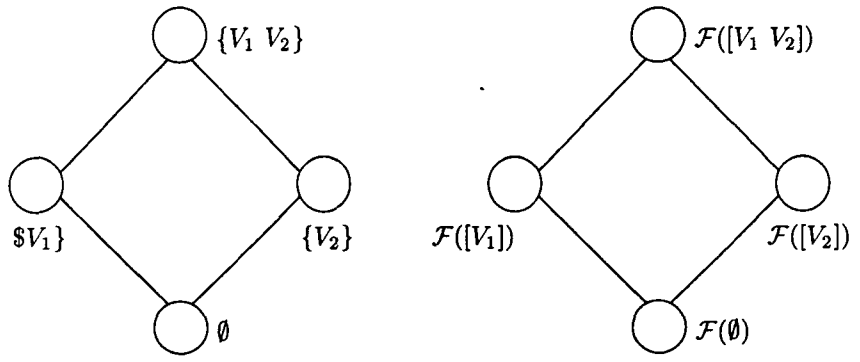


Figure 4.2: Lattices of (A, B)-controlled invariants contained in $ker(C)$ and their corresponding state feedback parameterizations of Example 4.1

form the lattice of unobservable (A, B)-controlled invariants. And their corresponding state feedback parameterizations before the F_0 is added are $\mathcal{F}([V_1\ V_2]) = F_0 + Y_{3\times3}[V_1\ V_2]^{\perp}$, $\mathcal{F}([V_1]) = F_0 + Y_{3\times4}[V_1]^{\perp}$, $\mathcal{F}([V_2]) = F_0 + Y_{3\times4}[V_2]^{\perp}$ and $\mathcal{F}(\emptyset) = F_0 + Y_{3\times5}$ respectively.

Figure 4.2 shows the lattice of all controlled invariants contained in $ker(C)$ under the operation $\subseteq \cup \cap$ and the lattice of their corresponding state feedback parameterizations under the operation $\supseteq \cup \cap$.

4.3.2 Uncontrollable (A, B)-Controlled Invariant

Uncontrollable (A, B)-controlled invariant is a related problem to its unobservable counterpart, Consider the following system with both (A, B_1) and (A, B_2) controllable

$$\dot{x} = Ax + B_1 w + B_2 u \tag{4.53}$$

where w is an unmeasurable disturbance. The problem can be viewed as finding a state feedback $u = Kx$ so that disturbance w will affect the closed-loop system as little as possible. Similar to the relationship between disturbance rejection and the unobservability of the closed-loop system, the problem is related to the uncontrollability of the closed-loop system. It can be interpreted as finding the minimal (A, B_2)

controlled invariant containing $im(B_1)$. Although the controllable (A, B_2)-controlled invariant will form a lattice similar to the unobservable $(A.B)$-controlled invariant, the uncontrollable (A, B_2)-controlled invariant does not have this property. However, in our application, we only need the following lemma.

Lemma 4.9 *Consider the system in (4.53) with (A, B_1) and (A, B_2) controllable. Let m_1 be the column rank of B_1, m_2 the column rank of B_2 and A be $n \times n$. B_1 and B_2 are assumed to be of full rank. If $m_2 + m_1 < n$ and $im(B_1) \cap im(B_2) = \emptyset$ then*

$$\min \mathcal{V}(A, B_2; im(B_1)) = \mathcal{X}. \tag{4.54}$$

Proof. Without loss of generalities, we assume $B_1 = [I\ 0]'$ and partition the closed-loop system matrices accordingly as

$$A + B_2K = \begin{bmatrix} A_{11} + B_{21}K_1 & A_{12} + B_{21}K_2 \\ A_{21} + B_{22}K_1 & A_{22} + B_{22}K_2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}$$

In order for $A + B_2K$ to be uncontrollable from w, there exists K_1 such that

$$A_{21} + B_{22}K_1 = 0$$

This equation is solvable only if $n - m_2 \leq m_1$, i.e. B_{22} is a long matrix. We conclude that if $m_1 + m_2 < n$ it is not possible to make $(A + B_2K, B_1)$ uncontrollable, i.e. $\min \mathcal{V}(A, B_2; im(B_1)) = \mathcal{X}$. ¶

4.4 Disturbance Rejection With D Matrix

In the case that a direct feed forward matrix D appears in the system matrix as

$$\begin{cases} \dot{x} = Ax + Bu \\ z = Cx + Du \end{cases} \tag{4.55}$$

the almost disturbance rejection problem is to find a maximal $(A + BF)$-invariant contained in $ker(C+DF)$. Because of the unknown term F appearing in $ker(C+DF)$

this problem has to be solved by introducing a new state variable y such that $\dot{y} = z$. We have the augmented system

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} u = \hat{A}\hat{x} + \hat{B}u \\ \hat{z} = [0 \quad I] \begin{bmatrix} x \\ y \end{bmatrix} = \hat{C}\hat{x} \end{cases} \tag{4.56}$$

Lemma 4.10 *Apply $u = Kx$ to both (4.55) and (4.56). The closed-loop system in (4.55) is unobservable if and only if the closed-loop system in (4.56) is unobservable. The closed-loop system in (4.55) is observable if and only if the closed-loop system in (4.56) is observable.*

Proof. The closed-loop system of (4.55) has the pair $(A + BK, C + DK)$ and the closed-loop system of (4.56) has the pair

$$(\begin{bmatrix} A + BK & 0 \\ C + DK & 0 \end{bmatrix}, [0 \; I])$$

The observability matrices of (4.55) and (4.56) are

$$\begin{bmatrix} C + DK \\ (C + DK)(A + BK) \\ \vdots \\ (C + DK)(A + BK)^{n-1} \end{bmatrix}, \begin{bmatrix} 0 & I \\ C + DK & 0 \\ (C + DK)(A + BK) & 0 \\ \vdots & \\ (C + DK)(A + BK)^{n+p-1} & 0 \end{bmatrix}$$

respectively. Obviously, if the first matrix is of full column rank n, the second matrix is of full column rank $n + p$ and vice versa. If the first matrix loses column rank, the second matrix will also lose column rank. ¶

Lemma 4.10 shows that y inherits the observability condition of z. So the problem becomes finding

$$\hat{\mathcal{V}}^* = \max \mathcal{V} \left(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \begin{bmatrix} B \\ D \end{bmatrix}, ker([0 \quad I]) \right). \tag{4.57}$$

After finding $\hat{\mathcal{V}}^*$ and a corresponding $\hat{F}$ using the procedure discussed earlier. we have

$$(\hat{A} + \hat{B}\hat{F})\hat{\mathcal{V}}^* \subseteq \hat{\mathcal{V}}^* \tag{4.58}$$

$$\hat{\mathcal{V}}^* \subseteq ker(\hat{C}) \tag{4.59}$$

Because $ker([0 \quad I])$ contains only the space spanned by x , i.e. the y part must be 0, the $(\hat{A}, \hat{B})$ controlled invariant $\hat{\mathcal{V}}$ contained in $ker(\hat{C})$ must have the following form

$$\hat{V}^* = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathcal{V}^* \right\} \tag{4.60}$$

One particular $\hat{F}$ can be rewritten as

$$\begin{aligned} \hat{F} &= -\hat{U}(\hat{V}^{*\prime}\hat{V}^*)^{-1}\hat{V}^{*\prime} \\ &= -\hat{U}(V^{*\prime}V^*)^{-1}[V^{*\prime} \quad 0] \\ &= [-\hat{U}(V^{*\prime}V^*)^{-1}V^{*\prime} \quad 0] \\ &= [F \quad 0] \end{aligned} \tag{4.61}$$

where F is the real state feedback that we are seeking. Also from the definition of controlled invariant,

$$\hat{A}\hat{\mathcal{V}}^* \subseteq \hat{\mathcal{V}}^* + \hat{\mathcal{B}}. \tag{4.62}$$

or equivalently

$$\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} V^* \\ 0 \end{bmatrix} = \begin{bmatrix} V^* \\ 0 \end{bmatrix} \hat{\Lambda} + \begin{bmatrix} B \\ D \end{bmatrix} \hat{U} \tag{4.63}$$

Putting (4.60) into (4.62) implies

$$A\mathcal{V}^* \subseteq \mathcal{V}^* + \mathcal{B} \tag{4.64}$$

$$C\mathcal{V}^* \subseteq im(D) \tag{4.65}$$

Applying F to the system and using (4.61) and (4.64), we have

$$(A + BF)\mathcal{V}^* \subseteq \mathcal{V}^*. \tag{4.66}$$

Using (4.63), (4.64) and (4.65), we have

$$(C + DF)V^* = CV^* - D\hat{U} = 0; \tag{4.67}$$

Using the augmented system, we can conclude with the follow theorems.

Theorem 4.4 *Consider a system (A, B, C, D) with (A, B) controllable and (A, C) observable. Let p be the row rank of C, m the column rank of B and A be $n \times n$. B, C and D are assumed to be of full rank. If $m < p < n$ and $ker(C) \cap im(B) = \emptyset$ then it is not possible to find a state feedback that makes the closed-loop system unobservable.*

Proof. It can be proved using Theorem 4.1 on the augmented system. ¶

Lemma 4.11 *Consider a linear system (A, B, C, D) with (A, B) controllable and (A, C) observable. Let p be the row rank of C, m be the column rank of B and A be $n \times n$. B, C and D are assumed to be of full rank. If $m = p < n$ and $ker(C) \cap im(B) = \emptyset$, then the largest unobservable (A, B) controlled invariant is $\mathcal{X}$. The corresponding state feedback is $F = D^{-1}C$.*

Proof. It can be proved using Lemma 4.8 on the augmented system. ¶

Theorem 4.5 *Assume that (A, B, C, D) are minimal. Let p be the row rank of C, m be the column rank of B and A be $n \times n$. B, C and D are assumed to be of full rank. If $m = p < n$ then the set of all unobservable (A, B) controlled invariants is finite and forms a lattice structure with the $\subseteq$, $\cap$ and $\cup$ relationship.*

The elements of the lattice are the spaces spanned by $V_1, \ldots, V_\mu$ where

$$(A + BD^{-1}C)V_i = V_i\Lambda_i \tag{4.68}$$

and Λ_i is the elementary Jordan block of $(A + BD^{-1}C)$.

Any q dimensional space $\mathcal{V}$ spanned by the combination of V_i is an unobservable (A, B) controlled invariant.

Moreover the set of state feedbacks that will make V unobservable can be parameterized as

$$\mathcal{F}(V) := \left\{K | K = F_{v0} + YV^{\perp'},\ \forall\, Y \in \mathcal{R}^{m\times(n-q)}\right\} \tag{4.69}$$

where F_{v0} is obtained from (4.61).

Proof. It can be proved using Theorem 4.2 on the augmented system. ¶

Example 5.2

Using the system matrices in example 1 with additional $D = I_3$ matrix, we can compute the V^*

$$V^* = \begin{bmatrix} -0.457 & -0.441 & 0.158 & -0.851 & 0.305 \\ -0.0971 & -0.255 & -0.913 & -0.0212 & -0.12 \\ 0.445 & -0.42 & 0.235 & -0.168 & 0.462 \\ 0.393 & 0.41 & 0.163 & 0.287 & 0.593 \\ -0.655 & -0.629 & -0.245 & -0.407 & 0.573 \end{bmatrix}$$

V^* is actually the eigenvectors of $A + BF_0$ where $F_0 = D^{-1}C$. We can also obtain F_0 from (4.61).

$$F_0 = \begin{bmatrix} 0 & -4 & -4 & -2 & 5 \\ 2 & 0 & 5 & -3 & -1 \\ 3 & 1 & -2 & 4 & -3 \end{bmatrix}$$

The closed-loop system under the basis V^* happens to be

$$V^{*-1}(A + BF_0)V* = \begin{bmatrix} -38.4 & 0 & 0 & 0 & 0 \\ 0 & 34.8 & 0 & 0 & 0 \\ 0 & 0 & 2.02 & 0 & 0 \\ 0 & 0 & 0 & 11.5 & 0 \\ 0 & 0 & 0 & 0 & -4.03 \end{bmatrix}$$

$$(C + DF)V^* = 0_{3\times 5}$$

Moreover each column of V^* is itself an invariant so any combination of columns of V^* is also. For example, Let V_1 be the first column of V^*. Then we can find the following set of F's

$$F = \begin{bmatrix} 2.49 & 0.529 & -2.42 & -2.14 & 3.57 \\ -0.359 & -0.0762 & 0.349 & 0.309 & -0.515 \\ -0.54 & -0.115 & 0.525 & 0.465 & -0.775 \end{bmatrix}$$
$$+Y_{3\times 4} \begin{bmatrix} -0.89 & 0.0499 & -0.228 & -0.202 & 0.337 \\ 0 & 0.5 & 0.775 & -0.199 & 0.332 \\ 0 & 0.442 & -0.199 & 0.824 & 0.294 \\ 0 & -0.737 & 0.332 & 0.294 & 0.511 \end{bmatrix}$$

In particular with $Y = 0_{3\times 4}$ the closed-loop system under the basis V^* has the form

$$V^{*-1}(A+BF)V^* = \begin{bmatrix} -38.4 & -25.2 & -18.3 & -26 & 3.06 \\ 0 & -1.48 & -2.69 & 1.1 & -6.84 \\ 0 & 3.93 & 1.62 & 2.78 & -0.325 \\ 0 & 1.2 & 0.0218 & 0.701 & 3.97 \\ 0 & -1.55 & -3.64 & 0.586 & -4.83 \end{bmatrix}$$

$$(C+DF)V^* = \begin{bmatrix} 0 & -2.08 & -3.04 & -1.94 & 0.0432 \\ 0 & 4.07 & -0.974 & 3.54 & -0.619 \\ 0 & -2.06 & -0.0699 & 0.691 & -0.599 \end{bmatrix}$$

The set of F's indeed makes V_1 unobservable from the output.

In this chapter we obtained the conditions that make some subspaces unobservable from the output using the concept of (A, B) controlled invariant. We also obtained parameterizations of the state feedbacks that make each particular subspace unobservable. In particular when $p > m$ there exists no unobservable subspace. When $p = m$, the unobservable subspaces are finite and form a lattice structure. We will discuss how to apply these properties to solve our mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem using BMI when $p = m$ in next chapter.

Chapter 5

Unobservable Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problem: BMI Solution

In the previous chapter we obtained a parameterization of all state feedbacks that will result in unobservability at the output when the number of outputs is equal to the number of control inputs. And the set of all unobservable subspaces forms a lattice structure with the relation $\cap$, $\cup$ and $\subseteq$. The state feedback parameterization corresponding to each unobservable subspace forms a lattice structure with the relation $\cap$, $\cup$ and $\supseteq$. The unobservable modes of the closed-loop system also form a lattice structure. All of them have the same structure.

As we will see, each state feedback parameterization corresponds to an $\mathcal{H}_2/\mathcal{H}_\infty$ constant output feedback problem. It may seem that that in order to solve our $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem, we need to solve a lattice of sub-problems and this is not efficient.

However, in control design, we require that the system be internally stable. So we can ignore the part of the lattice that corresponds to any unstable unobservable mode. Another reason for discarding the unstable unobservable space is due to the nature of the interior point algorithm. We use it to solve the BMI eigenvalue problem. Any solution obtained is guaranteed to be a stable closed-loop system even if the

system becomes unobservable. In other words it is neither necessary nor allowed to compute a subproblem with unstable unobservable mode. It is easy to see that the set of all stable unobservable modes of the closed-loop system is a sub-lattice of the original lattice as are the corresponding stable unobservable subspace and stable state feedback parameterization. This fact may reduce the number of sub-problems we need to solve considerably.

Some of these problems will not be feasible, i.e. the $\mathcal{H}_\infty$ norm of the output is larger than the constraint γ for all possible feedbacks in each particular state feedback parameterization. The state feedback parameterization may not even produce a stable closed-loop system. This means that we don't have to solve these $\mathcal{H}_2/\mathcal{H}_\infty$ subproblems at all. The algorithm we discussed in Chapter 2, for obtaining an $\mathcal{H}_\infty$ solution for the constant output feedback system case can identify these infeasible problems at a relatively efficient cost. This further reduces the number of $\mathcal{H}_2/\mathcal{H}_\infty$ output feedback subproblems that we need to solve.

5.1 Dual Unobservable Subspace and Its State Feedback Parameterization

In our $\mathcal{H}_2/\mathcal{H}_\infty$ problem, there are two outputs involved. The unobservable problem we encounter while solving the BMI problem may occur in either output. Therefore we have to take unobservable subspaces of both outputs into account.

Consider the following system

$$\begin{cases} \dot{x} = Ax + B_0 w_0 + B_1 w_1 + B_2 u \\ z_0 = C_0 x + D_0 u \\ z_1 = C_1 x + D_1 u \end{cases} \tag{5.1}$$

Let n, m_0, m_1, m_2, p_1 and p_0 be the dimensions of x, w_0, w_1, u, z_1 and z_0 respectively. From Theorem 4.1, if $p_1 > m_2$ and $p_0 > m_2$ both z_1 and z_0 will not become

unobservable for any state feedback. Therefore, we will not have an unobservability problem. The algorithm in Chapter 3 is sufficient to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem under these conditions. We will assume that $p_1 = m_2$ and $p_0 = m_2$ in this chapter.

From the previous chapter let V_s^* be the basis of the largest stable unobservable subspace $\mathcal{V}_s^*$ of the output z_1 with dimension q_v^*. V_s^* contains the elementary stable unobservable vectors $V_s^* = [V_1 \quad \dots \quad V_l]$ where q_{vi} is the dimension of V_i and $\sum_{i=1}^{l} q_{vi} = q_v^*$. Each V_i corresponds to an unobservable stable mode Λ_{vi} that is in basic Jordan form. We also know that given any space $\mathcal{V}$, spanned by some elementary stable unobservable vectors V_i's, is also a stable unobservable subspace and $\mathcal{V} \subseteq \mathcal{V}_s^*$, Let the dimension of V be q_v. Each V has a corresponding state feedback parameterization of the form

$$\mathcal{F}(V) = \left\{F | F = F_{v0} + XV^{\perp\prime} \quad \forall\, X \in R^{m_2 \times (n-q_v)}\right\} \tag{5.2}$$

Let U_s^* be a basis of the largest stable unobservable subspace $\mathcal{U}_s^*$ of the output z_0 with the dimension q_u^*. U_s^* contains the elementary stable unobservable vectors, $U_s^* = [U_1 \quad \dots \quad U_h]$ where q_{ui} is the dimension of U_i and $\sum_{i=1}^{h} q_{ui} = q_u^*$. Each U_i corresponds to an unobservable stable mode Λ_{ui} that is in basic Jordan form. We also know that given any space $\mathcal{U}$ spanned by some elementary stable unobservable vectors U_i is also a stable unobservable subspace and $\mathcal{U} \subseteq \mathcal{U}_s^*$. Let the dimension of U be q_u. Each U has a corresponding state feedback parameterization of the form

$$\mathcal{F}(U) = \left\{F | F = F_{u0} + YU^{\perp\prime} \quad \forall\, Y \in R^{m_2 \times (n-q_u)}\right\} \tag{5.3}$$

For convenience, we will define the lattice structure as

- $\mathcal{L}_V$: lattice of stable unobservable subspaces of z_1
- $\mathcal{L}_{F_V}$: lattice of stable unobservable state feedback parameterizations for z_1
- $\mathcal{L}_{\Lambda_V}$: lattice of stable unobservable modes for z_1

- $\mathcal{L}_U$: lattice of stable unobservable subspaces of z_0
- $\mathcal{L}_{F_U}$: lattice of stable unobservable state feedback parameterizations for z_0
- $\mathcal{L}_{\Lambda_U}$: lattice of stable unobservable modes for z_0

Now given that $\mathcal{V} \in \mathcal{L}_V$ and $\mathcal{V} \subseteq \mathcal{V}_s^*$ with corresponding state feedback $\mathcal{F}(V)$ and $\mathcal{U} \in \mathcal{L}_U$ and $\mathcal{U} \subseteq \mathcal{U}_s^*$ with corresponding state feedback $\mathcal{F}(U)$, we want to further parameterize the state feedback so that the closed-loop system for z_1 has the unobservable space $\mathcal{V}$ and z_0 has the unobservable space $\mathcal{U}$. The parameterization can be easily done by re-parameterizing the equation

$$F_{V0} + XV^{\perp'} = F_{U0} + YU^{\perp'} \tag{5.4}$$

Let q_v be the number of columns of V and q_u be the number of columns of U. By rearranging (5.4), we have

$$F_{V0} - F_{U0} = [X \quad Y] \begin{bmatrix} -V^{\perp'} \\ U^{\perp'} \end{bmatrix} \tag{5.5}$$

$\begin{bmatrix} -V^{\perp'} \\ U^{\perp'} \end{bmatrix}$ is of the size $n \times (n - q_v + n - q_u)$. In order for (5.5) to have a solution (right invertible), $n - qv + n - q_u$ must be larger than or equal to n, i.e. $q_v + q_u \leq n$. Using singular value decomposition one can obtain a parameterization for X, Y as

$$X = J_V + ZH_V; \quad Y = J_U + ZH_U; \tag{5.6}$$

where Z is a free parameter of size $m_2 \times (n - q_v - q_u)$ and J_V, J_U and H_V and H_U are constant matrices of proper size. We conclude that the set of all state feedbacks that will make $\mathcal{V}$ unobservable from z_1 and $\mathcal{U}$ from z_0 can be expressed as

$$\begin{aligned} F &= (F_{V0} + J_V V^{\perp'}) + ZH_V V^{\perp'} \quad or \\ &= (F_{U0} + J_U U^{\perp'}) + ZH_U U^{\perp'} \end{aligned} \tag{5.7}$$

(5.7) also shows two identities that we may need later

$$(F_{V0} + J_V V^{\perp'}) = (F_{U0} + J_U U^{\perp'}) \tag{5.8}$$

$$H_V V^{\perp'} = H_U U^{\perp'} \tag{5.9}$$

We will define this set of state feedback parameterizations as

$$\begin{aligned} \mathcal{K}(V,U) &:= \left\{F | F = (F_{V0} + J_V V^{\perp'}) + Z H_V V^{\perp'} \ \ \forall Z \in \mathcal{R}^{m_2 \times (n-q_v-q_u)}\right\} \ or \\ &:= \left\{F | F = (F_{U0} + J_U U^{\perp'}) + Z H_U U^{\perp'} \ \ \forall Z \in \mathcal{R}^{m_2 \times (n-q_v-q_u)}\right\} \end{aligned} \tag{5.10}$$

We will also denote the set $\mathcal{K}(V,U)$ as $\mathcal{K}(Z)$ when needed.

We just obtained the state feedback parameterization that will make $\mathcal{U}$ unobservable on z_0 and $\mathcal{V}$ unobservable on z_1 at the same time. Let $V_s^* = [\,V_1 \ \ \ldots \ \ V_l\,]$ and $U_s^* = [\,U_1 \ \ \ldots \ \ U_h\,]$ as defined above. We can define a lattice $\mathcal{L}_{VU}$ whose elements contain $[\,V_1 \ \ \ldots \ \ V_l \ \ U_1 \ \ \ldots \ \ U_h\,]$. The corresponding state feedback parameterization also form a lattice $\mathcal{L}_{\mathcal{K}_{UV}}$ of compatible structure subject to the constraint of $q_v + q_u \leq n$. We summarize the definition as

- $\mathcal{L}_{VU}$: lattice of unobservable space formed by (V_s^*, U_s^*)
- $\mathcal{L}_{F_{UV}}$: lattice of state feedback parameterization for $\mathcal{L}_{VU}$.

Figure 5.1 shows an example with $q_v^* = 2$, $q_u^* = 1$ and $n = 3$. $\mathcal{L}_V$ and its corresponding state feedback parameterization is a two-element lattice. $\mathcal{L}_U$ and its corresponding state feedback parameterization is a one-element lattice. The combination of $\mathcal{L}_V$ and $\mathcal{L}_U$ makes a three-element lattice. However, because $n = 3$ it is not possible to make three different elementary vector spaces unobservable, two for z_1 and one for z_0, at the same time so the lattice of $\mathcal{L}_{F_{UV}}$ is without the top node.

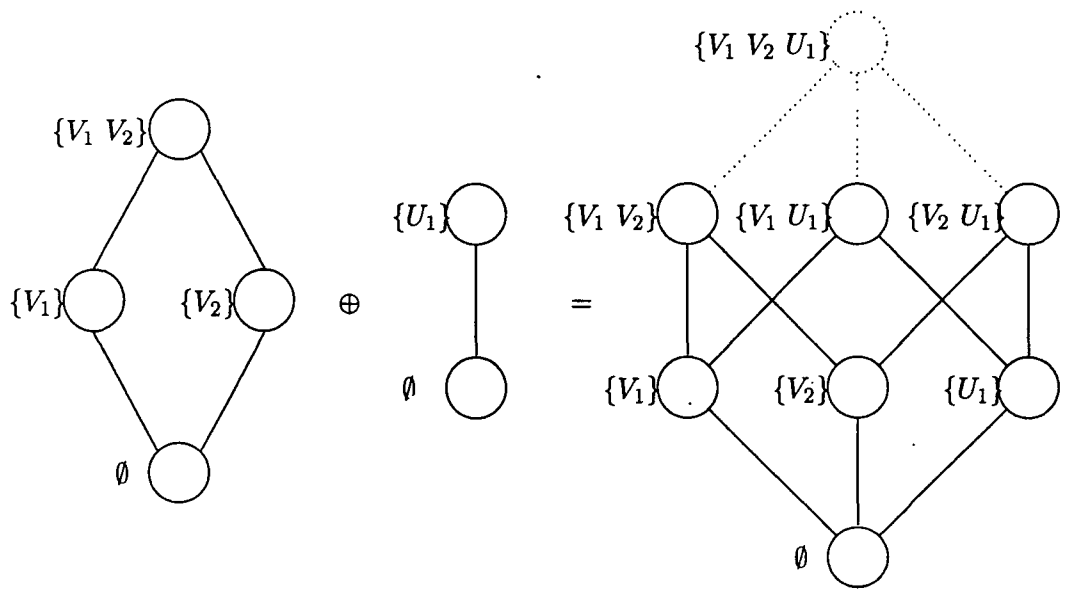


Figure 5.1: Forming the lattice of dual unobservable space $q_v^* = 2$, $q_u^* = 1$ and $n = 3$

5.2 Reduction into Sub-Problems

Now assume we are given the previous set of state feedbacks that will makes $\mathcal{V}$ unobservable from z_1 and $\mathcal{U}$ unobservable from z_0 assuming that $\mathcal{V} \cap \mathcal{U} = \emptyset$.

Applying the set of state feedbacks $F \in \mathcal{K}(V, U)$ to the system, we will have the closed-loop system of z_1

$$\begin{aligned} \dot{x} &= (A + B_2 F)x + B_0 w_0 + B_1 w_1 \\ z_1 &= (C_1 + D_1 F)x \end{aligned}$$

Substitute the parameterization in terms of Z to obtain

$$\begin{aligned} \dot{x} &= (A + B_2 F_{V0} + B_2 J_V V^{\perp'})x + B_2 Z H_V V^{\perp'} x + B_0 w_0 + B_1 w_1 \\ z_1 &= (C_1 + D_1 F_{V0} + D_1 J_V V^{\perp'})x + D_1 Z H_V V^{\perp'} x \end{aligned}$$

Under the new coordinates $T_V = [\,V \quad U \quad [V \quad U]^{\square}\,]$ such that

$$T_V^{-1}(A + B_2F_{V0} + B_2J_VV^{\perp\prime})T_V = \begin{bmatrix} \Lambda_V & 0 & 0 \\ 0 & \Lambda_U & 0 \\ 0 & 0 & A_3 \end{bmatrix}$$

$$(C_1 + D_1F_{V0} + D_1J_VV^{\perp\prime})T_V = [\,0 \quad \hat{C}_{12} \quad \hat{C}_{13}\,]$$

where Λ_V is the matrix of unobservable closed-loop modes corresponding to V and Λ_U is the matrix of invariant closed-loop modes corresponding to U. Here $[\,V \quad U\,]^{\square}$ is the complementary matrix of $[\,V \quad U\,]$ with respect to the matrix $(A+B_2F_{V0}+B_2J_VV^{\perp\prime})$.

The first column of the transformed A matrix results because the basis V is $(A + B_2F_{V0} + B_2J_VV^{\perp\prime})$ invariant. The second column results because the basis U is $(A + B_2F_{U0} + B_2J_VU^{\perp\prime})$ invariant so it is also $(A + B_2F_{V0} + B_2J_VV^{\perp\prime})$ invariant due to (5.4). The third column of A is because we choose the last part of the basis as the complement of $[\,V \quad U\,]$.

The first column of the transformed C matrix is zero because V is not only $(A + B_2F_{V0}+B_2J_VV^{\perp\prime})$ invariant but is also contained in the $ker(C_1+D_1F_{V0}+D_1J_VV^{\perp\prime})$. The term that contains the parameter Z can be also simplified as follows.

$$\begin{aligned}
& T_V^{-1}B_2ZH_VV^{\perp\prime}T_V \\
= \; & T_V^{-1}B_2Z\,[\,0 \quad H_VV^{\perp\prime}U \quad H_VV^{\perp\prime}[V\ U]^{\square}\,] \\
= \; & T_V^{-1}B_2Z\,[\,0 \quad H_UU^{\perp\prime}U \quad H_VV^{\perp\prime}[V\ U]^{\square}\,] \\
= \; & [\,0 \quad 0 \quad T_V^{-1}B_2ZH_VV^{\perp\prime}[V\ U]^{\square}\,] \\
= \; & [\,0 \quad 0 \quad T_V^{-1}B_2\tilde{Z}\,] = [\,0 \quad 0 \quad \hat{B}_2\tilde{Z}\,]
\end{aligned}$$

The second equality is from (5.9). It is easy to confirm that $H_VV^{\perp\prime}[V\ U]^{\square}$ is of the size $(n - q_v - q_u) \times (n - q_v - q_u)$ and is invertible, so we have the fourth equality by defining $\tilde{Z} = ZH_VV^{\perp\prime}[VU]^{\perp}$. Similarly we have

$$D_1ZH_VV^{\perp\prime}T_V = [\,0 \quad 0 \quad D_1\tilde{Z}\,]$$

We will drop the $\tilde{Z}$ and use Z because both are parameters of the same size. So the complete closed-loop system of z_1 can be transformed to

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} \Lambda_V & 0 & 0 \\ 0 & \Lambda_U & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} \hat{B}_{11} \\ \hat{B}_{12} \\ \hat{B}_{13} \end{bmatrix} w + \begin{bmatrix} \hat{B}_{21} \\ \hat{B}_{22} \\ \hat{B}_{23} \end{bmatrix} [0 \quad 0 \quad Z] \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} \tag{5.11}$$

$$z_1 = [0 \quad \hat{C}_{12} \quad \hat{C}_{13}] \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + D_1 [0 \quad 0 \quad Z] \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} \tag{5.12}$$

Using a similar derivation on the z_0 system,

$$\dot{x} = (A + B_2 F)x + +B_0 w_0 + B_1 w_1 \tag{5.13}$$

$$z_0 = (C_0 + D_0 F)x \tag{5.14}$$

under the new basis $T_U := [U \quad V \quad [V \quad U]^{\square}]$ where $[V \quad U]^{\square}$ is chosen as the complementary basis of $[V \quad U]$ with respect to $A + B_2 F_{U0}$, we obtain the transformed system

$$\begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_1 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} \Lambda_U & 0 & 0 \\ 0 & \Lambda_V & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_1 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} \hat{B}_{02} \\ \hat{B}_{01} \\ \hat{B}_{03} \end{bmatrix} w_0 + \begin{bmatrix} \hat{B}_{12} \\ \hat{B}_{11} \\ \hat{B}_{13} \end{bmatrix} w_1 + \begin{bmatrix} \hat{B}_{22} \\ \hat{B}_{21} \\ \hat{B}_{23} \end{bmatrix} [0 \quad 0 \quad Z] \begin{bmatrix} \hat{x}_2 \\ \hat{x}_1 \\ \hat{x}_3 \end{bmatrix} \tag{5.15}$$

$$z_0 = [0 \quad \hat{C}_{01} \quad \hat{C}_{03}] \begin{bmatrix} \hat{x}_2 \\ \hat{x}_1 \\ \hat{x}_3 \end{bmatrix} + D_0 [0 \quad 0 \quad Z] \begin{bmatrix} \hat{x}_2 \\ \hat{x}_1 \\ \hat{x}_3 \end{bmatrix} \tag{5.16}$$

where Λ_U is the matrix of unobservable closed-loop modes corresponding to U.

We can see that $\hat{x}_1$ will not affect z_1 and $\hat{x}_2$ will not affect z_0. The modes of $\hat{x}_1$, Λ_V, will remain unchanged. The modes of $\hat{x}_2$, Λ_U, will remain unchanged. Both are stable modes because we chose $\mathcal{V} \subseteq \mathcal{V}_s$ and $\mathcal{V}_2 \subseteq \mathcal{U}_s$. Now the parameter Z can

be treated as a partial state feedback on $\hat{x}_3$. In dealing with the $\mathcal{H}_\infty$ problem on z_1, $\hat{x}_1$ can be ignored. In dealing with the $\mathcal{H}_2$ problem on z_0, $\hat{x}_1$ can be ignored. Therefore, system (5.1) with state feedback contained in $\mathcal{K}(U,V)$ can be reduced into the following partial state feedback system.

$$\begin{cases} \dot{\hat{x}}_1 = \Lambda_V \hat{x}_1 + \hat{B}_{01} w_0 + \hat{B}_{11} w_1 + \hat{B}_{21} v \\ \dot{\hat{x}}_2 = \Lambda_U \hat{x}_2 + \hat{B}_{01} w_0 + \hat{B}_{12} w_1 + \hat{B}_{22} v \\ \dot{\hat{x}}_3 = A_{33} \hat{x}_3 + \hat{B}_{01} w_0 + \hat{B}_{13} w_1 + \hat{B}_{23} v \\ z_0 = \hat{C}_{21} \hat{x}_1 + \hat{C}_{23} \hat{x}_3 + D_0 v \\ z_1 = \hat{C}_{12} \hat{x}_2 + \hat{C}_{13} \hat{x}_3 + D_1 v \\ y = x_3 v = Zy \end{cases} \tag{5.17}$$

So the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem of system (5.17) can be reduced to

$$\min_{Q_0, Q_1, Z} Tr\left(\begin{bmatrix} \hat{B}_{01} \\ \hat{B}_{03} \end{bmatrix}' Q_0^{-1} \begin{bmatrix} \hat{B}_{01} \\ \hat{B}_{03} \end{bmatrix} \right) \tag{5.18}$$

$$A_U Q_0 + Q_0 A_U' + Q_0 [\hat{C}_{01} \quad \hat{C}_{03} + D_0 Z]' [\hat{C}_{01} \quad \hat{C}_{03} + D_0 Z] Q_0 \leq 0 \tag{5.19}$$

$$Q_0 \geq 0 \tag{5.20}$$

$$A_V Q_1 + Q_1 A_V' + \gamma^{-2} \begin{bmatrix} \hat{B}_{12} \\ \hat{B}_{13} \end{bmatrix} \begin{bmatrix} \hat{B}_{12} \\ \hat{B}_{13} \end{bmatrix}' + Q_1 [\hat{C}_{12} \quad \hat{C}_{13} + D_1 Z]' [\hat{C}_{12} \quad \hat{C}_{13} + D_1 Z] Q_1 \leq 0 \tag{5.21}$$

$$Q_1 \geq 0 \tag{5.22}$$

where

$$A_U = \begin{bmatrix} \Lambda_V & \hat{B}_{21} Z \\ 0 & A_3 + \hat{B}_{23} Z \end{bmatrix} \quad A_V = \begin{bmatrix} \Lambda_U & \hat{B}_{22} Z \\ 0 & A_3 + \hat{B}_{23} Z \end{bmatrix}$$

(5.18)-(5.20) is the inequality form for obtaining $\|T_{z_0 w_0}\|_2^2$. (5.21)-(5.22) are obtained from the standard $\mathcal{H}_\infty$ constraint inequality associated with the Bounded Real Lemma by ignoring state $\hat{x}_1$. The equations can be transformed into BMI form as

$$\min \lambda \tag{5.23}$$

$$\lambda - Tr(E) \geq 0 \tag{5.24}$$

$$\begin{bmatrix} E & [\hat{B}'_{01} \quad \hat{B}'_{03}] \\ \begin{bmatrix} \hat{B}_{01} \\ \hat{B}_{03} \end{bmatrix} & Q_0 \end{bmatrix} \geq 0 \tag{5.25}$$

$$\begin{bmatrix} A_U Q_0 + Q_0 A'_U & Q_0 \begin{bmatrix} \hat{C}'_{01} \\ \hat{C}'_{03} + Z' D'_0 \end{bmatrix} \\ [\hat{C}_{01} \quad \hat{C}_{03} + D_0 Z] Q_0 & -I \end{bmatrix} \leq 0 \tag{5.26}$$

$$Q_0 \geq 0 \tag{5.27}$$

$$\begin{bmatrix} A_V Q_1 + Q_1 A'_V + \gamma^{-2} \begin{bmatrix} \hat{B}_{12} \\ \hat{B}_{13} \end{bmatrix} \begin{bmatrix} \hat{B}_{12} \\ \hat{B}_{13} \end{bmatrix}' & Q_1 \begin{bmatrix} \hat{C}'_{12} \\ \hat{C}'_{13} + Z' D'_1 \end{bmatrix} \\ [\hat{C}_{12} \quad \hat{C}_{13} + D_1 Z] Q_1 & -I \end{bmatrix} < 0 \tag{5.28}$$

$$Q_1 \geq 0 \tag{5.29}$$

Due to the fact that state feedback Z only affects the modes of x_3, different Q_0 can give the same $\mathcal{H}_2$ norm for z_0. This means that the optimal solution is not unique, This poses a difficulty for our BMI solver using interior point algorithm. We need to find a way to reduce the Q_0 to solve the problem. Let $P_0 = Q_0^{-1}$ and partition both into

$$Q_0 = \begin{bmatrix} Q_{011} & Q_{012} \\ Q'_{012} & Q_{022} \end{bmatrix} \quad P_0 = \begin{bmatrix} P_{011} & P_{012} \\ P'_{012} & P_{022} \end{bmatrix}$$

The $\mathcal{H}_2$ norm of z_0 can be expressed as

$$\begin{aligned} \min tr(\hat{B}'_{11} P_{011} \hat{B}_{11}) + tr(\hat{B}'_{13} P'_{012} \hat{B}_{11}) \\ + tr(\hat{B}'_{11} P_{012} \hat{B}_{13}) + tr(\hat{B}'_{13} P_{022} \hat{B}_{13}) \end{aligned} \tag{5.30}$$

$$\begin{aligned} \begin{bmatrix} P_{011} & P_{012} \\ P'_{012} & P_{022} \end{bmatrix} \begin{bmatrix} \Lambda_V & \hat{B}_{21} Z \\ 0 & A_3 + \hat{B}_{23} Z \end{bmatrix} + \begin{bmatrix} \Lambda_V & \hat{B}_{21} Z \\ 0 & A_3 + \hat{B}_{23} Z \end{bmatrix}' \begin{bmatrix} P_{011} & P_{012} \\ P'_{012} & P_{022} \end{bmatrix} \\ + \begin{bmatrix} \hat{C}'_{01} \hat{C}_{01} & \hat{C}'_{01} (\hat{C}_{03} + D_0 Z) \\ (\hat{C}_{03} + D_0 Z)' \hat{C}_{01} & (\hat{C}_{03} + D_0 Z)' (\hat{C}_{03} + D_0 Z) \end{bmatrix} \leq 0 \end{aligned} \tag{5.31}$$

$$P_0 \geq 0 \tag{5.32}$$

From this equation, one can immediately determine the value of $P_{011} = N$ as the

solution of the Lyapunov equation

$$N\Lambda_V + \Lambda_V' N + \hat{C}_{01}'\hat{C}_{01} = 0; \tag{5.33}$$

Because the $(1,1)$ block of (5.31) is 0, in order for the inequality in (5.31) to hold, the $(1,2)$ and $(2,1)$ blocks have to be 0, i.e.

$$N\hat{B}_{11}Z + P_{012}(A_3 + \hat{B}_{23}Z) + \Lambda_V' P_{012} + \hat{C}_{01}'(\hat{C}_{03} + D_0 Z) = 0 \tag{5.34}$$

$$\begin{aligned}[P_{012}' \quad P_{022}]\begin{bmatrix}\hat{B}_{22}Z \\ A_3 + \hat{B}_{23}Z\end{bmatrix} + [(\hat{B}_{22}Z)' \quad (A_3 + \hat{B}_{23}Z)']\begin{bmatrix}P_{012} \\ P_{022}\end{bmatrix} \\ +(\hat{C}_{03} + D_0 Z)'(\hat{C}_{03} + D_0 Z) \leq 0\end{aligned} \tag{5.35}$$

After some manipulation using the identity $P_0 = Q_0^{-1}$ and defining $\hat{Q}_{012} = Q_{012}Q_{022}^{-1}$ we obtain an alternative form for (5.34),(5.35)

$$-N^{-1}\Lambda_V' N\hat{Q}_{012} - \hat{Q}_{012}(A_3 + \hat{B}_{23}Z) + (\hat{B}_{11}Z + N^{-1}\hat{C}_{01}'(\hat{C}_{03} + D_0 Z)) = 0 \tag{5.36}$$

$$\begin{aligned}&(A_3 + \hat{B}_{23}Z)Q_{022} + Q_{022}(A_3 + \hat{B}_{23}Z)' \\ + \quad & Q_{022}(\hat{Q}_{012}'\hat{C}_{01}' + (\hat{C}_{03} + D_0 Z)')(\hat{C}_{01}\hat{Q}_{012} + (\hat{C}_{03} + D_0 Z))Q_{022} \leq 0\end{aligned} \tag{5.37}$$

(5.30) can also be expressed as

$$\begin{aligned}\min Tr(\hat{B}_{11}'N\hat{B}_{11}) + Tr((\hat{B}_{11} - \hat{Q}_{012}\hat{B}_{13})'N^{-1}(\hat{B}_{11} - \hat{Q}_{012}\hat{B}_{13})) \\ +Tr(\hat{B}_{13}'Q_{022}^{-1}\hat{B}_{13})\end{aligned} \tag{5.38}$$

From this reduction of Q_0 we can express the BMI equation as

$$\min \lambda \tag{5.39}$$

$$\lambda - Tr(E) \geq 0 \tag{5.40}$$

$$\begin{bmatrix} E & (\hat{B}_{11} - \hat{Q}_{012}\hat{B}_{13})' & \hat{B}_{13}' \\ (\hat{B}_{11} - \hat{Q}_{012}\hat{B}_{13}) & N & 0 \\ \hat{B}_{13} & 0 & Q_{022} \end{bmatrix} \geq 0 \tag{5.41}$$

$$-\begin{bmatrix} (A_3+\hat{B}_{23}Z)Q_{022}+Q_{022}(A_3+\hat{B}_{23}Z)' & * \\ (\hat{C}_{01}\hat{Q}_{012}+\hat{C}_{03}+D_0Z)Q_{022} & -I \end{bmatrix} \geq 0 \tag{5.42}$$

$$Q_{022} \geq 0 \tag{5.43}$$

$$-N^{-1}\Lambda_V' N\hat{Q}_{012} - \hat{Q}_{012}(A_3+\hat{B}_{23}Z) + (\hat{B}_{11}Z + N^{-1}\hat{C}_{01}'(\hat{C}_{03}+D_0Z)) = 0 \tag{5.44}$$

$$-\begin{bmatrix} A_VQ_1+Q_1A_V'+\gamma^{-2}\begin{bmatrix}\hat{B}_{12}\\ \hat{B}_{13}\end{bmatrix}\begin{bmatrix}\hat{B}_{12}\\ \hat{B}_{13}\end{bmatrix}' & * \\ [\hat{C}_{12} \quad \hat{C}_{13}+D_1Z]Q_1 & -I \end{bmatrix} \geq 0 \tag{5.45}$$

$$Q_1 \geq 0 \tag{5.46}$$

We will denote the objective of the above sub-problem as

$$\lambda^* = \mathcal{P}(V,U) \tag{5.47}$$

with the optimal solution $Z^* = arg\ \mathcal{P}(V,U)$.

Problem $\mathcal{P}(V,U)$ is similar to the BMI problem in Chapter 3 except we have an additional equality (5.44). We will again apply the interior point algorithm to solve the problem. First we fix λ. Then apply Newton's method to minimize a barrier function from the inequality constraint. Finally improve the value of λ and repeat the minimization procedure of the barrier function until λ converges. We will need the first and second derivative of the barrier functions from (5.40), (5.43), (5.46), (5.41) and (5.42). The inequalities (5.40), (5.43), (5.46) are LMIs. The inequality (5.45) is a BMI. The derivative can be derived easily using (3.54).

The only problem is the variable Q_{012} in equality (5.44) and its involvement in (5.41) and (5.42). Fortunately, given a Z, (5.44) is a Sylvester equation for variable Q_{012}. Q_{012} can be computed easily as long as Λ_V and $A_3+\hat{B}_{23}Z$ share no common eigenvalue by means of

$$vec[Q_{012}] = (I\otimes N^{-1}\Lambda_V'N + (A_3+\hat{B}_{23}Z)'\otimes I)^{-1} vec[\hat{B}_{11}Z + N^{-1}\hat{C}_{01}'(\hat{C}_{03}+D_0Z)] \tag{5.48}$$

Therefore, for a given Z, we can obtain the derivative using the matrix derivative

formula. Let $q_{10} = vec[Q_{012}]$, $z = vec[Z]$, $M(z) = (I \otimes N^{-1}\Lambda_V' N + (A_3 + \hat{B}_{23}Z)' \otimes I)$ and $l(z) = vec[\hat{B}_{11}Z + N^{-1}\hat{C}_{01}'(\hat{C}_{03} + D_0 Z)]$ we have

$$\begin{aligned}
q_{10} &= M^{-1}(z)l(z) \\
\frac{\partial q_{10}}{\partial z_j} &= M^{-1}\frac{\partial l}{\partial z_j} - M^{-1}\frac{\partial M}{\partial z_j}M^{-1}l \\
\frac{\partial^2 q_{10}}{\partial z_j \partial z_i} &= M^{-1}\left(\frac{\partial M}{\partial z_j}M^{-1}\frac{\partial M}{\partial z_i} + \frac{\partial M}{\partial z_i}M^{-1}\frac{\partial M}{\partial z_j}\right)M^{-1}l - M^{-1}\frac{\partial M}{\partial z_j}M^{-1}\frac{\partial l}{\partial z_i}
\end{aligned}$$

given that $\partial^2 M/\partial z_j \partial z_i = 0$ and $\partial^2 l/\partial z_j \partial z_i = 0$ in (5.48) due to both M and l being linear functions of Z.

E, Q_{022}, Q_1 and Z are chosen as the variables of our interior point algorithm. Let

$$e = vec[E], q_0 = vec[Q_{022}], q_1 = vec[Q_1], z = vec[Z]; \tag{5.49}$$

$$x = vec[e, q_0, q_1, z] \tag{5.50}$$

Define F_1, F_2, F_3, F_4 and F_5 as the functions on the left hand side of (5.40), (5.41), (5.42), (5.45) and (5.46) respectively. We ignore (5.43) because the $(3,3)$ block of (5.41) guarantees $Q_{022} \geq 0$ already. We will obtain Q_{012} explicitly in terms of Z in each iteration by solving the equality (5.43). Let the barrier function be defined as

$$\phi = log(det(F_1)) + log(det(F_2)) + log(det(F_3)) + log(det(F_4)) + log(det(F_5)) \tag{5.51}$$

Given λ, the method of centers is to solve

$$\min_x \phi \tag{5.52}$$

To solve (5.52) using Newton's method we will need to compute the first and second derivatives of the objective function ϕ. The only problem in obtaining the derivatives is the Q_{012} terms appearing in F_2 and F_3. Fortunately, Q_{012} in F_2 is linear and Q_{012} appearing in F_3 is bilinear with Q_{022}. So we have

$$\begin{aligned}
\frac{\partial \phi}{\partial x_i} &= \sum_k \frac{\partial log(det(F_k))}{\partial x_i} \\
\frac{\partial^2 \phi}{\partial x_i \partial x_j} &= \sum_k \frac{\partial^2 log(det(F_k))}{\partial x_i \partial x_j}
\end{aligned}$$

or in more detail the first derivatives are

$$
\begin{aligned}
\frac{\partial \phi}{\partial e_i} &= \frac{\partial log(det(F_1))}{\partial e_i} + \frac{\partial log(det(F_2))}{\partial e_i} \\
\frac{\partial \phi}{\partial q_{02i}} &= \frac{\partial log(det(F_2))}{\partial q_{02i}} + \frac{\partial log(det(F_3))}{\partial q_{02i}} \\
\frac{\partial \phi}{\partial q_{11i}} &= \frac{\partial log(det(F_4))}{\partial q_{11i}} + \frac{\partial log(det(F_5))}{\partial q_{11i}} \\
\frac{\partial \phi}{\partial z_i} &= \frac{\partial log(det(F_3))}{\partial z_i} + \frac{\partial log(det(F_5))}{\partial z_i} \\
&\quad + \sum_l \left(\frac{\partial log(det(F_2))}{\partial q_{12l}} \frac{\partial q_{12l}}{\partial z_i}\right) + \sum_l \left(\frac{\partial log(det(F_3))}{\partial q_{12l}} \frac{\partial q_{12l}}{\partial z_i}\right)
\end{aligned}
$$

The second derivatives are

$$
\begin{aligned}
\frac{\partial^2 \phi}{\partial e_i \partial e_j} &= \frac{\partial^2 log(det(F_1))}{\partial e_i \partial e_j} + \frac{\partial^2 log(det(F_2))}{\partial e_i \partial e_j} \\
\frac{\partial^2 \phi}{\partial e_i \partial q_{02j}} &= \frac{\partial^2 log(det(F_2))}{\partial e_i \partial q_{02j}} \\
\frac{\partial^2 \phi}{\partial e_i \partial q_{11j}} &= 0 \\
\frac{\partial^2 \phi}{\partial e_i \partial z_j} &= \sum_l \frac{\partial^2 log(det(F_2))}{\partial e_i \partial q_{12l}} \frac{\partial q_{12l}}{\partial z_j} \\
\frac{\partial^2 \phi}{\partial q_{02i} \partial q_{02j}} &= \frac{\partial^2 log(det(F_2))}{\partial q_{02i} \partial q_{02j}} + \frac{\partial^2 log(det(F_3))}{\partial q_{02i} \partial q_{02j}} \\
\frac{\partial^2 \phi}{\partial q_{02i} \partial q_{11j}} &= 0 \\
\frac{\partial^2 \phi}{\partial q_{02i} \partial z_j} &= \sum_l \frac{\partial^2 log(det(F_2))}{\partial q_{02i} \partial q_{12l}} \frac{\partial q_{12l}}{\partial z_j} + \sum_l \frac{\partial^2 log(det(F_3))}{\partial q_{02i} \partial q_{12l}} \frac{\partial q_{12l}}{\partial z_j} \\
&\quad + \frac{\partial^2 log(det(F_3))}{\partial q_{02i} \partial z_j} \\
\frac{\partial^2 \phi}{\partial q_{11i} \partial q_{11j}} &= \frac{\partial^2 log(det(F_4))}{\partial q_{11i} \partial q_{11j}} + \frac{\partial^2 log(det(F_5))}{\partial q_{11i} \partial q_{11j}} \\
\frac{\partial^2 \phi}{\partial q_{11i} \partial z_j} &= \sum_l \frac{\partial^2 log(det(F_4))}{\partial q_{11i} \partial q_{12l}} \frac{\partial q_{12l}}{\partial z_j} \\
\frac{\partial^2 \phi}{\partial z_i \partial z_j} &= \frac{\partial^2 log(det(F_3))}{\partial z_i \partial z_j} + \frac{\partial^2 log(det(F_4))}{\partial z_i \partial z_j} \\
&\quad + \sum_l \frac{\partial^2 log(det(F_3))}{\partial z_i \partial q_{12l}} \frac{\partial q_{12l}}{\partial z_j}
\end{aligned}
$$

$$+\sum_l \sum_k \frac{\partial^2 log(det(F_2))}{\partial q_{12l} \partial q_{12k}} \frac{\partial q_{12l}}{\partial z_i} \frac{\partial q_{12k}}{\partial z_j}$$
$$+\sum_l \sum_k \frac{\partial^2 log(det(F_3))}{\partial q_{12l} \partial q_{12k}} \frac{\partial q_{12l}}{\partial z_i} \frac{\partial q_{12k}}{\partial z_j}$$

After we obtain the derivatives, the method of centers can be obtained using the following algorithm

Algorithm 5.1 Interior point algorithm: finding the center of ϕ

1. *Fix λ and $x^{<0>}$ and i=0.*

2. *Compute $\phi(\lambda, x^{<i>})$, $\frac{\partial \phi}{\partial x}(\lambda, x^{<i>})$ and $\frac{\partial^2 \phi}{\partial x^2}(\lambda, x^{<i>})$.*

3. $\Delta x = -(\frac{\partial^2 \phi}{\partial x^2})^{-1} \frac{\partial \phi}{\partial x}$

4. *Choose s such that $1 \geq s \geq 0$ and $x^{<i>} + s\Delta x$ is feasible. This means that we have to choose s as close to 1 as possible such that λ, $x + s\Delta x$ satisfies the constraints (5.40), (5.41), (5.42), (5.44), (5.45) and (5.46).*

5. *If $\|\frac{\partial \phi}{\partial x}(\lambda, x^{<i>})\|_2 < 1e-6$ stop else let $x^{<i+1>} = x^{<i>} + s\Delta x$, i=i+1 and repeat step 2.*

We will denote the pass of center procedure as

$$x^* = arg\ \mathcal{M}(\lambda, x) \tag{5.53}$$

Now we can present the complete solution for $\mathcal{P}(V, U)$.

Algorithm 5.2 Method of centers for solving $\mathcal{P}(V, U)$

1. *Given U, V and its corresponding state feedback parameterization (5.10), compute the system matrix (5.17) and from the sub-problem $\mathcal{P}(V, U)$.*

2. *Initialize $\lambda^{<0>}$ and $x^{<0>}$, i=0.*

3. *Solve the problem*

$$x^{<i>*} = arg\mathcal{M}(\lambda^{<i>}, x^{<i>}).$$

4. *Compute*

$$H = [\, \frac{\partial^2 \phi}{\partial x^2} \quad \frac{\partial^2 \phi}{\partial x \partial \lambda} \,] \,|_{\lambda^{<i>}, x^{<i>*}}$$

5. *Compute* $\xi = ker(H)$.

6. *Find the largest* t *such that* $[x^{<i>*'},\ \lambda^{<i>}]' + t\xi$ *is feasible. and* λ *is decreasing.*

7. *If* $t \leq 1e-6$ *then stop else*

$$\begin{bmatrix} x^{<i+1>} \\ \lambda^{<i+1>} \end{bmatrix} = \begin{bmatrix} x^{<i*>} \\ \lambda^{<i>} \end{bmatrix} + t\xi \tag{5.54}$$

and $i = i + 1$ *and repeat step 3*

In steps 4 and 5, we use the prediction algorithm commonly used in the continuation method to obtain an update of $\lambda^{<i>}$ and $x^{<i>}$. H is an $m \times (m+1)$ matrix where m is the number of parameters. The kernel ξ of H provides a predictor direction to reduce λ. In step 6, we choose t as large as possible to maximize the improvement of λ. But due to the interior point constraint, we need to check the feasibility of the prediction. When the algorithm stops $\lambda^{<i>}$ is the optimal objective of $\mathcal{P}(V, U)$ and $x^{<i>*}$ and its corresponding E, Q_{012}, Q_{022}, Q_1 , Z are the optimal solution of $\mathcal{P}(V, U)$.

5.3 Perturbation into Larger Problem

Once we obtain an optimal solution of the $\mathcal{H}_2/\mathcal{H}_\infty$ sub-problem of a given $\mathcal{V} = im(V)$, $\mathcal{U} = im(U)$ associated with state feedback parameterization

$$\begin{aligned} \mathcal{K}(U, V) : &= \left\{ K | K = F_{V0} + ZV^{\perp'} \right\} \ or \\ : &= \left\{ K | K = F_{U0} + ZU^{\perp'} \right\}. \end{aligned}$$

we are ready to solve the larger problem by relaxing either $\mathcal{V}$ or $\mathcal{U}$. Relaxing either $\mathcal{V}$ or $\mathcal{U}$ means we can remove one elementary unobservable space from either $\mathcal{V}$ or $\mathcal{U}$ and obtain a new set of state feedback parameterizations. By relaxing the unobservable space following the lattice, our final goal is to solve the full problem, i.e. when $\mathcal{V} = \emptyset$ and $\mathcal{U} = \emptyset$.

Assume that we are relaxing $\mathcal{V}$ to $\mathcal{V}_-$ by removing one elementary unobservable subspace V_1. Now we have

$$\mathcal{V}_- \subset \mathcal{V} \tag{5.55}$$

and

$$\mathcal{K}(V,U) \subset \mathcal{K}(V_-,U) \tag{5.56}$$

It is obvious that

$$\mathcal{P}(V_-,U) \leq \mathcal{P}(V,U) \tag{5.57}$$

If we solve the sub-problem $\mathcal{P}(V_-,U)$ from an arbitrary initial solution, it is still possible that the solution might run into some unobservable space.

If we start from the optimal solution of sub-problem $\mathcal{P}(V,U)$, i.e. $K^*(V,U)$, with a slight perturbation in the direction of improving the $\mathcal{H}_2$ norm, it is not possible to make (V,U) unobservable again. This is because while we solve $\mathcal{P}(V_-,U)$ λ is always less that $\lambda^{<0>}$. Therefore,

$$\begin{aligned} \lambda &< \lambda^{<0>} = \mathcal{P}(V,U) - \delta \\ &< \mathcal{P}(V,U) = \|T_{z_0w_0}(K^*(U,V))\|_2^2 \\ &< \|T_{z_0w_0}(\mathcal{K}(U,V))\|_2^2 \end{aligned}$$

where δ is the improvement of the $\mathcal{H}_2$ norm after the perturbation. So every K corresponding to λ has the property

$$K \notin \mathcal{K}(U,V). \tag{5.58}$$

This assures that K will not run into $\mathcal{K}(V,U)$. Therefore it will not make (U,V) unobservable.

However while we perturb in the direction of improving the $\mathcal{H}_2$ norm, the $\mathcal{H}_\infty$ norm constraint might be violated. So we choose the perturbation in a direction that improves the $\mathcal{H}_2$ norm and reduces the $\mathcal{H}_\infty$ norm at the same time. Both the $\mathcal{H}_2$ norm and $\mathcal{H}_\infty$ norm are locally differentiable functions with respect to Z, which is the parameter of $\mathcal{K}(V_-,U)$. The gradients of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms with respect to $z = vec[Z]$ are

$$\Delta h_0 = -\frac{d\|T_{z_0w_0}\|_2^2}{dz} = -\nabla \|T_{z_0w_0}\|_2^2(z) \qquad \Delta h_\infty = -\frac{d\|T_{z_1w_1}\|_\infty}{dz} = -\nabla \|T_{z_1w_1}\|_\infty(z) \tag{5.59}$$

where ∇ is the gradient operator. Let $\Delta h = \Delta h_0 + \Delta h_\infty$, then for sufficiently small positive s

$$\|T_{z_0w_0}\|_2^2(z + s\Delta h) < \|T_{z_0w_0}\|_2^2(z) \tag{5.60}$$

$$\|T_{z_1w_1}\|_\infty(z + s\Delta h) < \|T_{z_1w_1}\|_\infty(z) \tag{5.61}$$

So we can choose

$$\lambda^{<0>} = \|T_{z_0w_0}\|_2^2|_{z+s\Delta h}$$
$$z^{<0>} = z + s\Delta h$$

The corresponding $q_1^{<0>}$, $q_0^{<0>}$ and $q_{10}^{<0>}$ can also be obtained by solving the Riccati equation and Lyapunov equation with $Z^{<0>}$ fixed. We have a set of feasible $[\lambda^{<0>}, q_1^{<0>}, q_0^{<0>}, z^{<0>}]$ and are ready to solve $\mathcal{P}(V_-,U)$.

Figure 5.2 shows the way we perturb from the optimal solution of a sub-problem indicated by dashed line. The direction Δh gives decreasing $\mathcal{H}_2$ norm and decreasing $\mathcal{H}_\infty$ norm.

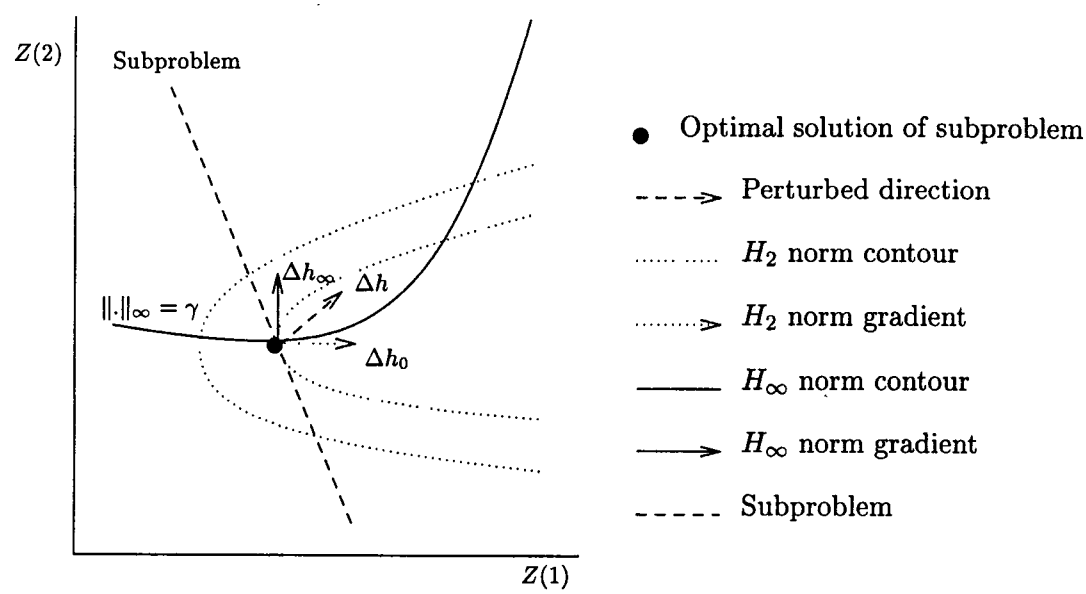


Figure 5.2: Illustration for obtaining perturbation direction

5.4 Solving the full problem

As we mentioned earlier our final goal is to solve the problem

$$\mathcal{P}(\emptyset, \emptyset)$$

However in order to prevent running into any unobservable space that will stall our BMI solver we need to solve the sub-problems that are its immediate ancestors in the lattice structure $\mathcal{L}_{VU}$ first. And in order to solve any sub-problem $\mathcal{P}(V, U)$ we need to solve the $\mathcal{P}(V_i, U_i)$ such that (V_i, U_i) are the immediate ancestors of (U, V) in the lattice structure $\mathcal{L}_{VU}$. This means we need to solve the sub-problem from the top level of the lattice which has no unobservability problem first. We propose the following algorithm.

Algorithm 5.3 Solving the full problem

- *If $p_1 > m_2$ and $p_0 > m_2$ use Algorithm 3.2 described in Chapter 3 to solve the problem.*

- *If $m_1 + m_2 < n$ and $m_0 + m_2 < n$ solve BMI problem (3.73) described in Chapter 3 to solve the problem. This is due to Corollary 4.9*

- *If $p_1 = m_2$ or $p_0 = m_2$ follow the following procedure.*

1. *Find the maximum stable unobservable subspace $\mathcal{V}_s^*$ for z_1 and $\mathcal{U}_s^*$ for z_0 and their elementary stable unobservable subspaces. We also have to find the state feedback parameterization for each unobservable subspace.*

2. *Create the lattice structure of all possible stable unobservable subspaces from z_1/z_0 and the lattice of their state feedback parameterization. Remember that the lattice structure can be a semi-lattice because of the constraint $q_v + q_u < n$.*

3. *Repeat from the top level down. Consider a $(V, U) \in \mathcal{L}_{UV}$*

   (a) *Make sure the sub-problems corresponding to all the immediate ancestors of (V, U) are solved if there are any.*

   (b) *If the optimal solution exists for its ancestors*

      - *Pick one solution K that produces the least closed-loop $\mathcal{H}_2$ norm on z_0 from the optimal solution of all its ancestral sub-problems.*

      - *Perturb the K within the current state feedback parameterization in the direction of reducing the $\mathcal{H}_2$ norm on z_0 while the $\mathcal{H}_\infty$ norm on z_1 satisfies the constraint.*

   (c) *Else pick a K that satisfies the $\mathcal{H}_\infty$ norm constraint on z_1. We can make use of the algorithm for finding a suboptimal $\mathcal{H}_\infty$ problem with constant output feedback in Chapter 2. If one can not find a K that is feasible, then skip this sub-problem because this sub-problem will not have an optimal solution.*

4. *Solve the sub-problem $\mathcal{P}(V, U)$ with the initial solution just chosen.*

5. *If $U = \emptyset$ and $V = \emptyset$ then stop else repeat from step 3 with its neighbor on the same level. If all the neighbor sub-problems have been solved go to the next level.*

We will present a few examples to demonstrate the algorithm. The first example shows how to traverse through the lattice of unobservable subspaces to solve the full problem. The second example is a simple example with state feedback K of size 1×2 in order to show all the sub-problems of our algorithm on the contour plot of the K-plane.

5.4.1 Example 5.1

Consider the following system

$$\begin{cases} \dot{x} = \begin{bmatrix} -0.0825 & 0.089 & 0 \\ 0.187 & 0.43 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.935 \\ 0.384 \\ 1 \end{bmatrix} w_0 + \begin{bmatrix} 0.935 \\ 0.384 \\ 1 \end{bmatrix} w_1 + \begin{bmatrix} 0.519 \\ 0.831 \\ 1 \end{bmatrix} u \\ z_0 = [-0.946 \quad 0.324 \quad -1]\, x - 1u \\ z_1 = [0.167 \quad 0.487 \quad 1]\, x + 1u \end{cases} \tag{5.62}$$

where $m_2 = p_1 = p_0 = 1$. The problem is to find a state feedback that $\min_K \|T_{z_0 w_0}\|_2^2$ subject to $\|T_{z_1 w_1}\|_\infty \leq \gamma$ where we choose $\gamma = 2$.

First we have to find the maximum unobservable space for both output z_1 and z_0.

The Maximum Unobservable Subspace

It is easy to find the maximum unobservable subspace. For z_1

$$V^* = \begin{bmatrix} 0.659 & -0.946 & -0.217 \\ 0.542 & 0.324 & -0.771 \\ 0.522 & -0.000762 & 0.599 \end{bmatrix} \quad \Lambda_V = \begin{bmatrix} -0.715 & 0 & 0 \\ 0 & -0.113 & 0 \\ 0 & 0 & 0.686 \end{bmatrix}$$

where V^* spans the maximum unobservable subspace for z_1. Each column of V^* or combinations of columns can span an unobservable subspace by itself with the

corresponding unobservable modes as shown in Λ_V. The same applied to output z_0, gives

$$U^* = \begin{bmatrix} -0.636 & 0.0694 & 0.207 \\ -0.556 & 0.717 & 0.57 \\ -0.536 & 0.297 & -0.155 \end{bmatrix} \quad \Lambda_U = \begin{bmatrix} -0.787 & 0 & 0 \\ 0 & 0.456 & 0.2 \\ 0 & -0.2 & 0.456 \end{bmatrix}$$

where U^* spans the maximum unobservable subspace for z_0 and Λ_U are the corresponding unobservable mode.

Stable Maximum Unobservable Subspace

Some of the columns of V^* and U^* have unstable unobservable modes. So we can ignore these subspaces because they will result in an internally unstable system. So we choose for z_1

$$V_s^* = \begin{bmatrix} 0.659 & -0.946 \\ 0.542 & 0.324 \\ 0.522 & -0.000762 \end{bmatrix} \quad \Lambda_{Vs} = \begin{bmatrix} -0.715 & 0 \\ 0 & -0.113 \end{bmatrix}$$

$$U_s^* = \begin{bmatrix} -0.636 \\ -0.556 \\ -0.536 \end{bmatrix} \quad \Lambda_{Us} = -0.7868$$

Form the lattice

Let

$$V_1 = \begin{bmatrix} 0.659 \\ 0.542 \\ 0.522 \end{bmatrix} \quad V_2 = \begin{bmatrix} -0.946 \\ 0.324 \\ -0.000762 \end{bmatrix} \quad U_1 = \begin{bmatrix} -0.636 \\ -0.556 \\ -0.536 \end{bmatrix}$$

where V_1 and V_2 are from V_s^* and U_1 is from U_s^*. Vectors V_1, V_2 and U_1 form the lattice as in Figure 5.3 At the same time, the state feedback corresponding to each unobservable subspace can be derived as

$$\mathcal{K}_{V_1 V_2 U_1} \quad := \quad \{K | K = [0.927 \quad 2.7 \quad -5.69]\}$$

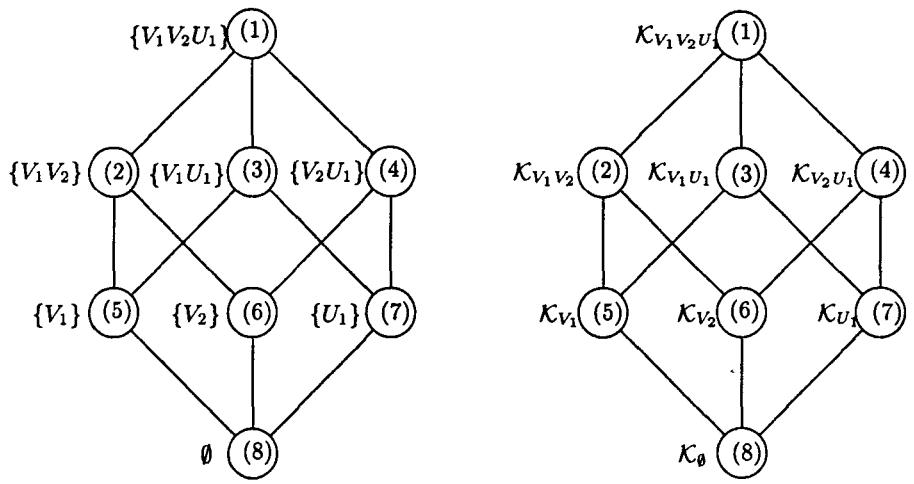


Figure 5.3: Lattice structure of Example 5.2

$$
\begin{aligned}
\mathcal{K}_{V_1U_1} &:= \{K|K = [0.961 \quad -1.45 \quad -1.42] + Z_{1\times1}[0.00409 \quad -0.493 \quad 0.507]\} \\
\mathcal{K}_{V_2U_1} &:= \{K|K = [-0.276 \quad -0.806 \quad -0.623] + Z_{1\times1}[-0.136 \quad -0.395 \quad 0.571]\} \\
\mathcal{K}_{V_1V_2} &:= \{K|K = [-0.264 \quad -0.769 \quad -0.584] + Z_{1\times1}[-0.189 \quad -0.552 \quad 0.812]\} \\
\mathcal{K}_{V_1} &:= \{K|K = [-0.59 \quad -0.485 \quad -0.467] + Z_{1\times2}\begin{bmatrix} -0.542 & 0.823 & -0.171 \\ -0.522 & -0.171 & 0.836 \end{bmatrix}\} \\
\mathcal{K}_{V_2} &:= \{K|K = [-8.03\ 10^{-4} \quad 2.75\ 10^{-4} \quad -6.47\ 10^{-7}] \\
&\qquad +Z_{1\times2}\begin{bmatrix} 0.324 & 0.946 & 1.27\ 10^{-4} \\ -7.62\ 10^{-4} & 1.27\ 10^{-4} & 1 \end{bmatrix}\} \\
\mathcal{K}_{U_1} &:= \{K|K = [-0.608 \quad -0.532 \quad -0.513] + Z_{1\times2}\begin{bmatrix} -0.556 & 0.811 & -0.182 \\ -0.536 & -0.182 & 0.825 \end{bmatrix}\} \\
\mathcal{K}_{\emptyset} &:= \{K|K = Z_{1\times3}\}
\end{aligned}
$$

The set of parameterizations for each unobservable subspace lattice forms a lattice of the same structure.

Solving the Sub-Problems

After obtaining the parameterization of state feedbacks, we are ready to solve the $\mathcal{H}_2/\mathcal{H}_\infty$ sub-problem with each set of parameterizations. We solve the sub-problems from the top of the lattice and then traverse through to the bottom.

	$\{U_1\}$		$\emptyset$	
	(1)	-	(2)	$K^{<0>} = [3.1 \quad 9.04 \quad -15]$
$\{V_1V_2\}$	-	-	9.1287	$K^* = [2.83 \quad 8.23 \quad -13.8]$
	(3)	-	(5)	$K^{<0>} = [2.83 \quad 8.23 \quad -13.8]$
$\{V_1\}$	-	-	9.1208	$K^* = [2.76 \quad 8.16 \quad -13.7]$
	(4)	$K^{<0>} = [2.76 \quad 8.16 \quad -13.7]$	(6)	$K^{<0>} = [2.65 \quad 7.73 \quad -12.9]$
$\{V_2\}$	8.7943	$K^* = [2.65 \quad 7.71 \quad -12.9]$	8.7769	$K^* = [2.65 \quad 7.71 \quad -12.9]$
	(7)	$K^{<0>} = [2.65 \quad 7.71 \quad -12.9]$	(8)	$K^{<0>} = [2.65 \quad 7.71 \quad -12.9]$
$\emptyset$	8.7903	$K^* = [2.64 \quad 7.69 \quad -12.9]$	8.7538	$K^* = [2.64 \quad 7.65 \quad -12.8]$

Table 5.1: $K^{<0>}$, K^* and λ^* for each sub-problem of Example 5.1 for $\gamma = 2$

As we mentioned earlier, each sub-problem requires an initial solution. If any sub-problem of the immediate ancestry of the current problem has an optimal solution, we have to perturb the best optimal solution of the ancestral sub-problems to get the initial solution for the current sub-problem. Otherwise we can pick any initial solution that satisfies the $\mathcal{H}_\infty$ constraint.

Here $\mathcal{K}_{V_1V_2U_1}$ and $\mathcal{K}_{V_1U_1}$ are not feasible, i.e. for all $\mathcal{K}_{V_1V_2U_1}$ and $\mathcal{K}_{V_1U_1}$ the $\mathcal{H}_\infty$ norm is larger than γ.

We show the details of solving each sub-problem as follows.

1. It can be skipped because $\mathcal{K}_{V_1V_2U_1}$ results in a closed-loop system $\mathcal{H}_\infty$ norm larger than γ.

2. Because the sub-problem (1) does not have an optimal solution, any initial solution satisfies the $\mathcal{H}_\infty$ norm constraint can be used. While solving this sub-problem it is guaranteed that the solution will not run into the previous unobservable subspace. So we choose to start from $K^{<0>} = [3.1 \quad 9.04 \quad -15]$. We obtain the optimal solution $K^* = [2.83 \quad 8.23 \quad -13.8]$ with closed-loop system

$\mathcal{H}_2$ norm 9.1281.

3. This sub-problem again can be skipped because all the state feedback parameterizations are not feasible.

4. This sub-problem is similar to sub-problem (2). We can choose an arbitrary initial solution. We choose to start from $K^{<0>} = [\,2.76 \quad 8.16 \quad -13.7\,]$ to obtain $K^* = [\,2.65 \quad 7.71 \quad -12.9\,]$ which gives an optimal closed-loop system $\mathcal{H}_2$ norm 8.7943.

5. We need an initial solution from either sub-problem (2) or sub-problem (3). Because sub-problem (3) has no solution we choose the optimal solution of sub-problem (2) as the initial solution of this sub-problem. We also need to perturb the $K^*_{V_1V_2}$ slightly to be away from the unobservable space in the direction that will reduce the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. Reducing the $\mathcal{H}_2$ norm guarantees that the solution won't go to the unobservable subspace. Reducing the $\mathcal{H}_\infty$ norm guarantees the initial solution is feasible.

6. We need an initial solution from either sub-problem (2) or sub-problem (4). Because the optimal solution of sub-problem (4) is smaller than the optimal solution of sub-problem (2) we choose it as our initial solution of this sub-problem. We also need to perturb the solution slightly to guarantee it is feasible and won't go back to the sub-problem (2).

7. We need an initial solution from either sub-problem (3) or sub-problem (4). Sub-problem (3) has no solution. So we use the optimal solution of sub-problem (4) as the initial solution of this sub-problem. Perturbation is also required.

8. This is the general problem. The initial solution can be obtained from sub-problem (5),(6) or (7). We choose the the optimal solution of sub-problem (6)

	K_2	K_∞	K^*
$\|T_{z_0w_0}\|_2^2$	6.5926	10.4738	8.7538
$\|T_{z_1w_1}\|_\infty$	3.1884	1.8983	2

Table 5.2: Performance comparison of Example 5.1 with $\gamma = 2$

because it produces the least closed-loop $\mathcal{H}_2$ norm. Similarly perturbation is required to steer away from the unobservable subspace.

Result

Table 5.2 again shows a reasonable trade off between the $\mathcal{H}_2$ optimal result and the $\mathcal{H}_2/\mathcal{H}_\infty$ suboptimal result. However, due to the parameterization of state feedback, the computing effort increased dramatically.

5.4.2 Example 5.2

In this example, we present an example where our $\mathcal{H}_2/\mathcal{H}_\infty$ optimal solution provides a significant improvement in $\mathcal{H}_2$ norm over the $\mathcal{H}_2/\mathcal{H}_\infty$ suboptimal solution. Consider the following system

$$\begin{cases} \dot{x} = \begin{bmatrix} -0.0825 & 0.089 \\ 0.187 & 0.43 \end{bmatrix} x + + \begin{bmatrix} 0.935 \\ 0.384 \end{bmatrix} w_0 + \begin{bmatrix} 0.935 \\ 0.384 \end{bmatrix} w_1 + \begin{bmatrix} 0.519 \\ 0.831 \end{bmatrix} u \\ z_0 = [\,-0.946 \quad 0.324\,]\, x - 1u \\ z_1 = [\,0.167 \quad 0.487\,]\, x + 1u \end{cases} \tag{5.63}$$

The problem is to find a state feedback that $\min_K \|T_{z_0w_0}\|_2^2$ subject to $\|T_{z_1w_1}\|_\infty \leq \gamma$ where we choose $\gamma = 0.5$.

The lattice of unobservable subspaces has the same structure as in the previous example. The maximum stable unobservable subspaces for z_1 and z_0 and their corre-

	$\{U_1\}$		$\emptyset$	
	(1)	-	(2)	$K^{<0>} = [-0.167 \quad -0.487]$
$\{V_1 V_2\}$	-	-	25.2854	$K^* = [-0.167 \quad -0.487]$
	(3)	-	(5)	$K^{<0>} = [-0.167 \quad -0.487]$
$\{V_1\}$	-	-	2.8207	$K^* = [-0.193 \quad -0.563]$
	(4)	-	(6)	$K^{<0>} = [-0.167 \quad -0.487]$
$\{V_2\}$	-	-	1.4350	$K^* = [-0.254 \quad -0.591]$
	(7)	-	(8)	$K^{<0>} = [-0.254 \quad -0.591]$
$\emptyset$	-	-	1.3630	$K^* = [-0.262 \quad -0.610]$

Table 5.3: $K^{<0>}$, K^*, λ^* for each sub-problem of Example 5.2 for $\gamma = 0.5$

sponding unobservable stable modes are

$$V_s^* = \begin{bmatrix} -0.943 & -0.766 \\ 0.332 & 0.643 \end{bmatrix} \quad \Lambda_{sV} = \begin{bmatrix} -0.111 & 0 \\ 0 & -0.0315 \end{bmatrix}$$

$$U_s^* = \begin{bmatrix} -0.885 \\ -0.466 \end{bmatrix} \quad \Lambda_{sU} = [-0.4385]$$

respectively.

1. This sub-problem can be skipped because the set of state feedback parameterizations is $\emptyset$. This can also be seen from $q_v + q_u = 3$ which is larger than n=2.

2. This sub-problem need not be solved by our BMI solver because $q_v + q_u = 2$. The free parameter $Z \in \mathcal{R}^{0\times 2}$ means that

$$\mathcal{K}([V_1 \ V_2], \emptyset) = \{K | K = [-0.167 \quad -0.487]\}.$$

$\mathcal{K}([V_1 \ V_2], \emptyset)$ contains no free parameter. It happens that the $T_{z_1 w_1}$ using the feedback $[-0.167 \quad -0.487]$ has $\mathcal{H}_\infty$ norm less than γ. So $[-0.167 \quad -0.487]$ is a feasible solution. It is also an optimal solution for this sub-problem because $\mathcal{K}([V_1 \ V_2], \emptyset)$ has no free parameter.

	K_2	K_∞	K^*
$\|T_{z_0w_0}\|_2^2$	0.0430	3.8251	1.3021
$\|T_{z_1w_1}\|_\infty$	0.6411	0.4321	0.5

Table 5.4: Performance comparison of Example 5.2 with $\gamma = 0.5$

3. This sub-problem need not be solved by our BMI solver because the set of state feedback parameterizations $\mathcal{K}(V_1, U_1)$ contains a single element. It has a closed-loop system $\mathcal{H}_\infty$ norm larger than γ. So it is not feasible.

4. It is the same as (3).

5. Use the optimal solution from (2) with slight perturbation as the initial solution of this sub-problem.

6. It is the same as (5).

7. The state feedback parameterization of this sub-problem is not feasible.

8. Use the smallest optimal solution among the sub-problem (5),(6) or (7) , which is (5), with slight perturbation as the initial solution.

Result

In Table 5.4, K_2 is the LQ solution, K_∞ is an $\mathcal{H}_\infty$ solution obtained by solving the upper bound $\mathcal{H}_2/\mathcal{H}_\infty$ problem [22] and K^* is our $\mathcal{H}_2/\mathcal{H}_\infty$ solution. The closed loop $\mathcal{H}_\infty$ norm with K_2 is 0.6411 which is larger than the specification $\gamma = 0.5$. K_∞ produces a closed loop $\mathcal{H}_\infty$ norm only slightly smaller than γ. However the closed loop $\mathcal{H}_2$ norm is relatively large. K^* has a closed loop $\mathcal{H}_2$ norm only 1/3 of the $\mathcal{H}_2$ norm produced by K_∞ without trading off too much on the $\mathcal{H}_\infty$ norm. This demonstrates the value of finding a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ solution.

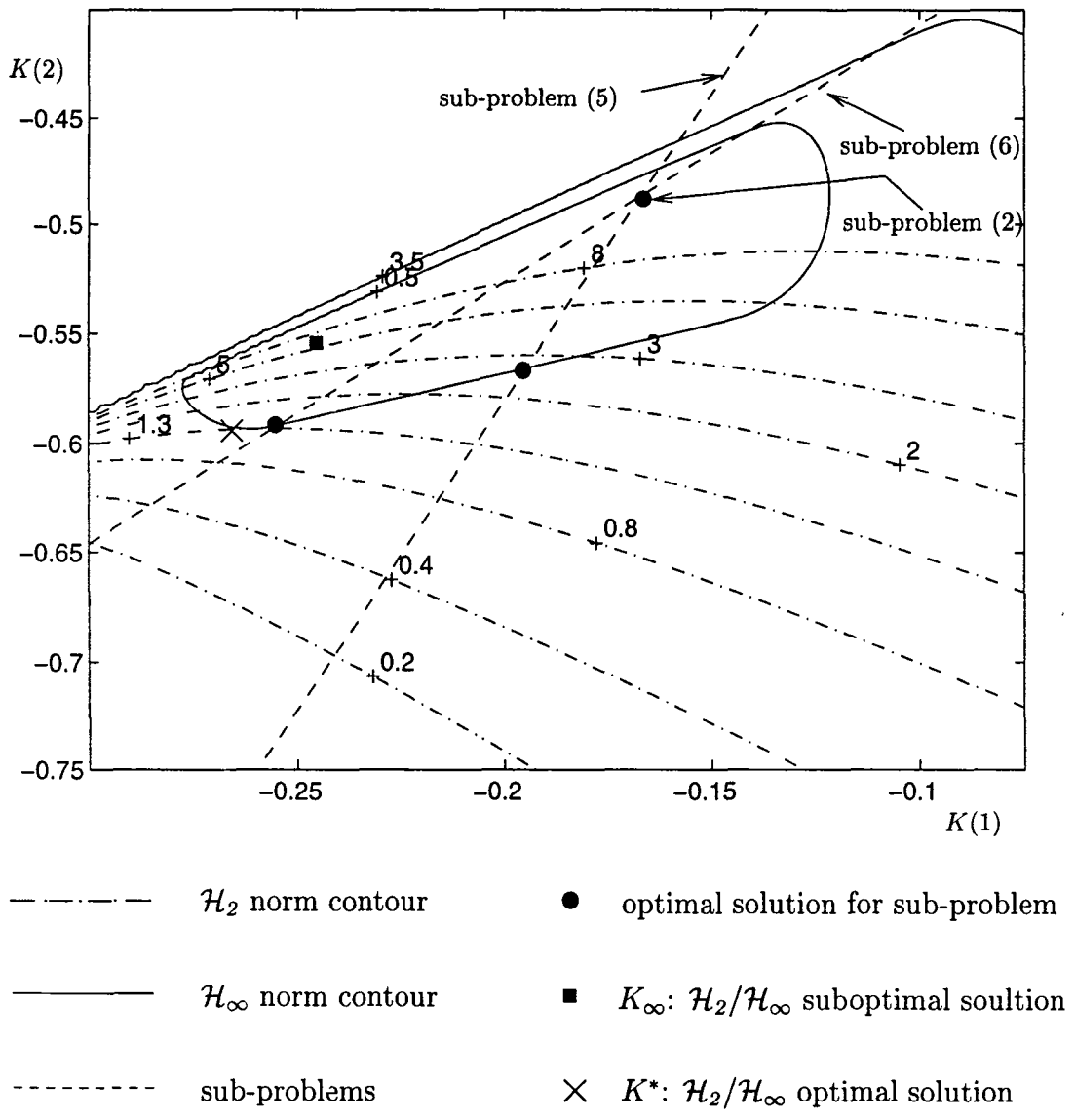


Figure 5.4: $\mathcal{H}_2/\mathcal{H}_\infty$ contour plot of Example 5.2

Figure 5.4 is the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm contour plot with respect to the state feedback K. The region within the solid line shows all feasible K, $\|T_{z_1 w}\|_\infty \leq 0.5$. The dashed-dotted lines are the constant $\mathcal{H}_2$ norm contours. It shows that K^* is at the position where the $\mathcal{H}_\infty$ norm constraint boundary contacts an $\mathcal{H}_2$ norm contour. It also shows even though K^* is not far away from the sub-optimal solution, it produces significantly smaller $\mathcal{H}_2$ norm than the sub-optimal solution. This is due the dense $\mathcal{H}_2$ contour lines in the neighborhood of K^*. In other words, the $\mathcal{H}_2$ norm is more sensitive to the variation of control than the $\mathcal{H}_\infty$ norm in the neighborhood of K^*.

The figure also shows the set of state feedbacks that will make V_1 unobservable from z_1, i.e. dash line (sub-problem (5)). And it shows the set of state feedbacks that will make V_2 unobservable from z_1, i.e. dash line (sub-problem (6)). The set of state feedbacks that will make U_1 unobservable from z_0 is out of the figure, therefore it is evident that they are not feasible and can be ignored. The way we perform the optimization starts from the sub-problem $\mathcal{P}([V_1, V_2], \emptyset)$. This sub-problem (2) contains the intersection of the above two lines. So no BMI solver is needed. After that, we perform perturbation from the intersection along each line separately in the direction that reduces the $\mathcal{H}_2$ norm. Then we perform the optimization along each line using the BMI solver to solve $\mathcal{P}(V_1, \emptyset)$ and $\mathcal{P}(V_2, \emptyset)$ separately. Finally, we pick the solution of $\mathcal{P}(V_2, \emptyset)$ because it has smaller optimal $\mathcal{H}_2$ norm, perturb it, and use it as an initial solution for solving $\mathcal{P}(\emptyset, \emptyset)$, i.e. the full problem.

5.5 The Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problem:Constant Output Feedback Case

One of the advantages of our BMI algorithm is that it can be extended to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem for systems with constant output feedback. Consider the

following system

$$\begin{cases} \dot{x} = Ax + +B_0w_0 + B_1w_0 + B_2u \\ z_0 = C_0x + D_0u \\ z_1 = C_1x + D_1u \\ y = C_3x \end{cases} \tag{5.64}$$

with the same assumptions as in the system (5.1) plus C_3 is a full row rank matrix of size $p_3 \times n$ and $p_3 < n$. Assuming that x is not accessible, one can apply feedback only through y. It is known that by using a dynamic filter of order $n-1$ or larger, one can obtain an estimator of x. However in our case, with the assumption of only constant feedback allowed and C_3 being a wide matrix, x can not be recovered completely. The problem is to find a control of the form $u = Gy$ such that $\|T_{z_0w_0}\|_2^2$ is minimal and $\|T_{z_1w_1}\|_\infty \leq \gamma$.

Compared to the constant state feedback case, the unobservability with respect to both z_1 and z_0 remains the same. The lattice of unobservable (A, B)-controlled invariants $\mathcal{L}_{VU}$ and their corresponding state feedback parameterizations $\mathcal{L}_{F_{UV}}$ are unchanged. The only difference lies in the fact that $u = Gy = GC_3x = Kx$. We need to extract a subset $\mathcal{G}(V, U) \subset \mathcal{K}(V, U)$ so that any $K \in \mathcal{G}(V, U)$ satisfies $K = GC_3$ for some G. In other words, given (U, V), we need to make the parameterization (5.10) equal to GC_3 , i.e. for some G, Z

$$(F_{V0} + J_VV^{\perp'}) + ZH_VV^{\perp'} = GC_3 \ or \tag{5.65}$$

$$(F_{U0} + J_UU^{\perp'}) + ZH_UU^{\perp'} = GC_3 \tag{5.66}$$

where $Z \in \mathcal{R}^{m_2 \times (n-q_v-q_u)}$ and $G \in \mathcal{R}^{m_2 \times p_3}$. The equality can be rearranged as

$$(F_{V0} + J_VV^{\perp'}) = [\,Z \quad G\,] \begin{bmatrix} -H_VV^{\perp'} \\ C_3 \end{bmatrix} \ or \tag{5.67}$$

$$(F_{U0} + J_UU^{\perp'}) = [\,Z \quad G\,] \begin{bmatrix} -H_UU^{\perp'} \\ C_3 \end{bmatrix} \tag{5.68}$$

A solution exists if the $(n - q_u - q_v + p_3) \times n$ matrices

$$\begin{bmatrix} -H_V V^{\perp'} \\ C_3 \end{bmatrix} \quad or \quad \begin{bmatrix} -H_U U^{\perp'} \\ C_3 \end{bmatrix}$$

are of full column rank. In general this means

$$p_3 \geq q_u + q_v \tag{5.69}$$

is sufficient and the solution can be parameterized as

$$Z = J_Z + W H_Z \tag{5.70}$$

where $W \in \mathcal{R}^{m_2 \times (p_3 - q_v - q_u)}$. We can thus define the set of constant output feedback $\mathcal{G}(U, V)$ that will make U and V unobservable as

$$\begin{aligned} & \mathcal{G}(V, U) \\ := \; & \left\{ K | K = (F_{V0} + (J_V + J_Z H_V) V^{\perp'}) + W H_Z H_V V^{\perp'} \;\; \forall W \in \mathcal{R}^{m_2 \times (p_3 - q_v - q_u)} \right\} \; or \\ := \; & \left\{ K | K = (F_{U0} + (J_U + J_Z H_U) U^{\perp'}) + W H_Z H_U U^{\perp'} \;\; \forall W \in \mathcal{R}^{m_2 \times (p_3 - q_v - q_u)} \right\} \end{aligned} \tag{5.71}$$

Applying this set of feedbacks to (5.64) with $u = Kx$ where $K \in \mathcal{G}(V, U)$ we obtain the closed-loop system for z_1 as

$$\begin{aligned} \dot{x} &= (A + B_2(F_{V0} + (J_V + J_Z H_V) V^{\perp'}))x + B_2 W H_Z H_V V^{\perp'} x + B_0 w_0 + B_1 w_1 \\ z_1 &= (C_1 + D_1(F_{V0} + (J_V + J_Z H_V) V^{\perp'}))x + D_1 W H_Z H_V V^{\perp'} x \end{aligned}$$

Under the coordinate transformation $T_V = [\, V \quad U \quad [V \quad U]^{\square} \,]$, we have

$$T_V^{-1}(A + B_2(F_{V0} + (J_V + J_Z H_V) V^{\perp'}))T_V = \begin{bmatrix} \Lambda_V & 0 & 0 \\ 0 & \Lambda_U & 0 \\ 0 & 0 & A_3 \end{bmatrix}$$

$$(C_1 + D_1(F_{V0} + (J_V + J_Z H_V) V^{\perp'}))T_V = [\, 0 \quad \hat{C}_{12} \quad \hat{C}_{13} \,]$$

using the same reasons as for the constant state feedback case. The difference is the term

$$T_V^{-1} B_2 W H_Z H_V V^{\perp'} T_V$$

$$
\begin{aligned}
&= T_V^{-1}B_2W\,[0 \quad H_ZH_VV^{\perp\prime}U \quad H_ZH_VV^{\perp\prime}[V\ U]^{\square}\,] \\
&= T_V^{-1}B_2W\,[0 \quad H_ZH_UU^{\perp\prime}U \quad H_ZH_VV^{\perp\prime}[V\ U]^{\square}\,] \\
&= [0 \quad 0 \quad T_V^{-1}B_2WH_ZH_VV^{\perp\prime}[V\ U]^{\square}\,] \\
&= [0 \quad 0 \quad \hat{B}_2W\hat{C}_3\,]
\end{aligned}
$$

Here we define $\hat{B}_2 = T_V^{-1}B_2$ and $\hat{C}_3 = H_ZH_VV^{\perp\prime}[V\ U]^{\square}$. $\hat{C}_3$ is of the size $(p_3 - q_v - q_u) \times (n - q_v - q_u)$ which is not square and invertible in the output feedback case. Therefore we can not incorporate $\hat{C}_3$ into W. Similarly we have

$$
D_1WH_ZH_VV^{\perp\prime}T_V = [0 \quad 0 \quad D_1W\hat{C}_3\,]
$$

After applying a similar procedure to z_0 we have the reduced sub-problem as

$$
\begin{cases}
\dot{\hat{x}}_1 = \Lambda_V\hat{x}_1 + \hat{B}_{01}w_0 + \hat{B}_{11}w + \hat{B}_{21}v \\
\dot{\hat{x}}_2 = \Lambda_U\hat{x}_2 + \hat{B}_{02}w_0 + \hat{B}_{12}w + \hat{B}_{22}v \\
\dot{\hat{x}}_3 = A_{33}\hat{x}_3 + \hat{B}_{03}w_0 + \hat{B}_{13}w + \hat{B}_{23}v \\
z_0 = \hat{C}_{21}\hat{x}_1 + \hat{C}_{23}\hat{x}_3 + D_0v \\
z_1 = \hat{C}_{12}\hat{x}_2 + \hat{C}_{13}\hat{x}_3 + D_1v \\
y = \hat{C}_3\dot{\hat{x}}_3 \\
v = Wy
\end{cases}
\tag{5.72}
$$

The only difference between the state feedback system and output feedback is the y equation in (5.17) and the y equation in (5.72).

Let

$$
A_U = \begin{bmatrix} \Lambda_V & \hat{B}_{21}W\hat{C}_3 \\ 0 & A_3 + \hat{B}_{23}W\hat{C}_3 \end{bmatrix} \quad A_V = \begin{bmatrix} \Lambda_U & \hat{B}_{22}W\hat{C}_3 \\ 0 & A_3 + \hat{B}_{23}W\hat{C}_3 \end{bmatrix}
$$

the BMI of this problem only needs slight modification as

$$
\min \lambda \tag{5.73}
$$

$$
\lambda - Tr(E) \geq 0 \tag{5.74}
$$

$$
\begin{bmatrix} E & [\hat{B}_{01}' \quad \hat{B}_{03}'] \\ \begin{bmatrix} \hat{B}_{01} \\ \hat{B}_{03} \end{bmatrix} & Q_0 \end{bmatrix} \geq 0 \tag{5.75}
$$

$$\begin{bmatrix} A_U Q_0 + Q_0 A_U' & Q_0 \begin{bmatrix} \hat{C}_{01}' \\ \hat{C}_{03}' + \hat{C}_3' W' D_0' \end{bmatrix} \\ [\hat{C}_{01} \quad \hat{C}_{03} + D_0 W \hat{C}_3] Q_0 & -I \end{bmatrix} \leq 0 \tag{5.76}$$

$$Q_0 \geq 0 \tag{5.77}$$

$$\begin{bmatrix} A_V Q_1 + Q_1 A_V' + \gamma^{-2} \begin{bmatrix} \hat{B}_{12} \\ \hat{B}_{13} \end{bmatrix} \begin{bmatrix} \hat{B}_{12} \\ \hat{B}_{13} \end{bmatrix}' & Q_1 \begin{bmatrix} \hat{C}_{12}' \\ \hat{C}_{13}' + C_3' W' D_1' \end{bmatrix} \\ [\hat{C}_{12} \quad \hat{C}_{13} + D_1 W \hat{C}_3] Q_1 & -I \end{bmatrix} \leq 0 \tag{5.78}$$

$$Q_1 \geq 0 \tag{5.79}$$

The procedure in the state feedback case can be applied to solve the output feedback BMI sub-problem without any problem, except that we need to take into account the requirement of (5.69) to ensure the state feedback parameterization exists. (5.69) means that when we solve for the full problem, we can ignore more sub-problems because $p_3 < n$.

Chapter 6

Application

In this chapter we demonstrate how our algorithm can be used to solve a more realistic problem. We chose the Grumman $F-14$ benchmark flight control problem [26, 52] for the following reasons.

The system is very sensitive to aircraft dynamic perturbation. In other words it can be made unstable easily. Therefore to find a control that can reduce its sensitivity is crucial. This can be interpreted as reducing the $\mathcal{H}_\infty$ norm of the closed-loop dynamics if we use the Δ block in Figure 6.1 to represent the aircraft dynamic variation due to modeling error or other reasons. At the same time, one needs to find a control that will suppress external inputs such as wind gusts or observation noise. Minimizing the $\mathcal{H}_2$ norm of the closed-loop dynamic achieves the goal of suppressing the interference induced by white noise. Our $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem is expected to be valuable in trading off the two competing requirements. A second reason is that the problem can be easily transformed into a standard $\mathcal{H}_2/\mathcal{H}_\infty$ constant output feedback problem of moderate size. Finally, there exist several solutions [26, 52] for this problem that can be used for comparison.

We first introduce the F-14 problem. Minor modifications of the problem will be made to meet the $\mathcal{H}_2/\mathcal{H}_\infty$ control requirement. Next we partition the problem into several sub-problems according to the lattice unobservable (A, B)-controlled invariant.

Each sub-problem is solved by our $\mathcal{H}_2/\mathcal{H}_\infty$ BMI optimization algorithm provided that a feasible solution exists. After that, we design a prefilter to meet the model-following requirement. Finally, we compare several results produced by different controllers such as a PID controller, optimal $\mathcal{H}_2$ controller , $\mathcal{H}_\infty$ suboptimal controller and our $\mathcal{H}_2/\mathcal{H}_\infty$ controller.

6.1 Problem Description and Design Requirements

The Grumman F-14 model is obtained under the condition that the aircraft is flying at 35,000 feet in straight and level flight at a speed of 690 ft/sec. The linearized time invariant system shown in Figure 6.1 consists of the following components

- short-period longitudinal aircraft dynamics; This is a second order system with stable poles at $-0.6478 \pm 2.0202i$. It has two outputs α(angle of attack) and q(pitch rate). The system parameters are
  - $U_0 = 689.4\ ft/sec$: longitudinal velocity.
  - $M_w = -0.00592\ (ft\ sec)^{-1}$: pitching moment due to the vertical velocity.
  - $M_q = -0.6571\ sec^{-1}$: pitching moment due to the pitch rate.
  - $Z_w = -0.6385\ sec^{-1}$: force in vertical direction due to vertical velocity.
- tail surface actuator: The output is surface deflection δ_c. It is used to control the aircraft dynamics. The parameters are
  - $\tau_a = 0.05\ sec$: actuator time constant
  - $Z_\delta = -63.9979\ ft/(rad\ sec^2)$: force in vertical direction due to displacement of tail surface
  - $M_\delta = -6.8847\ (rad\ sec^2)^{-1}$: pitching moment due to displacement of tail surface

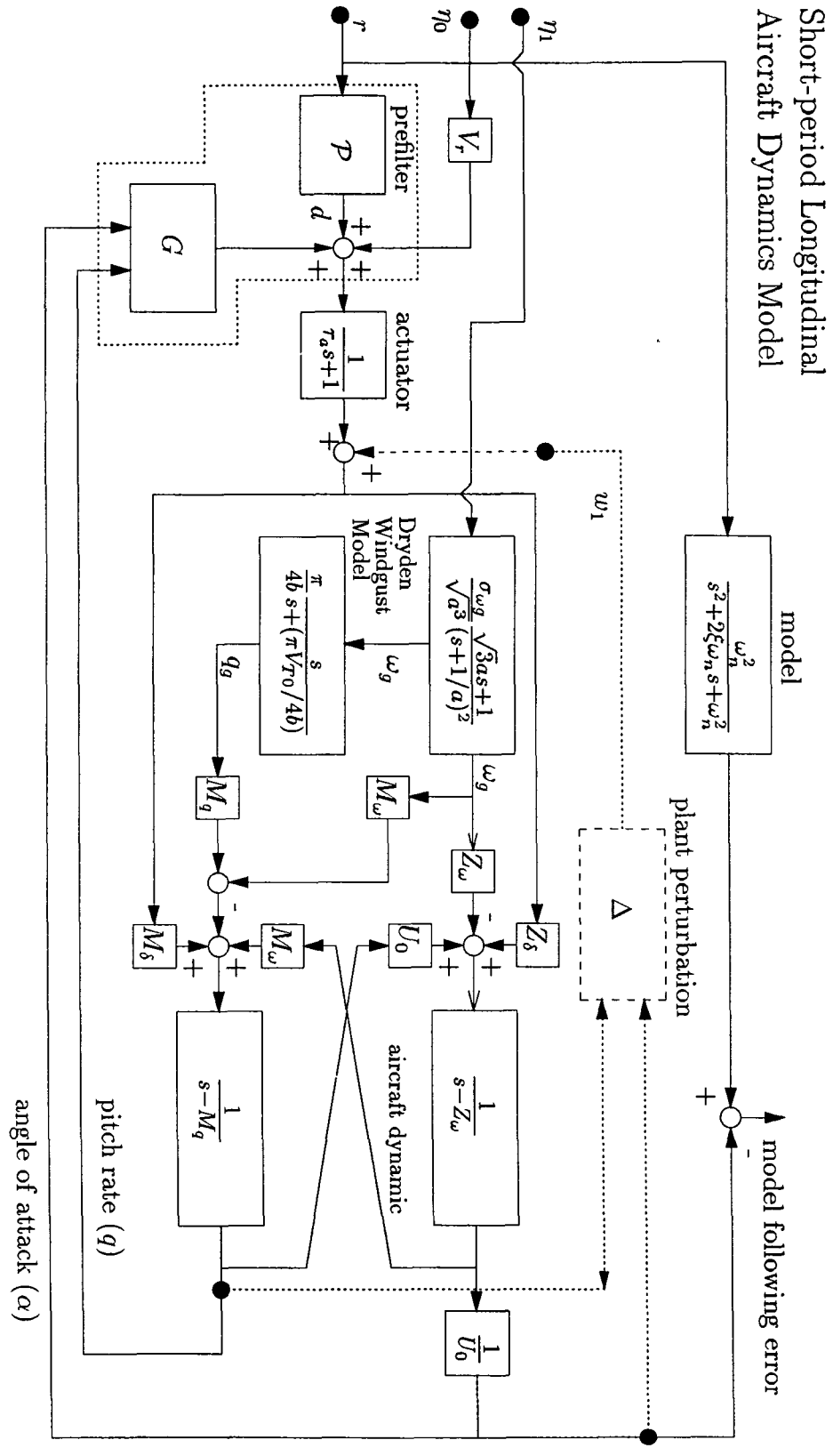


Figure 6.1: Block diagram of F14 control problem

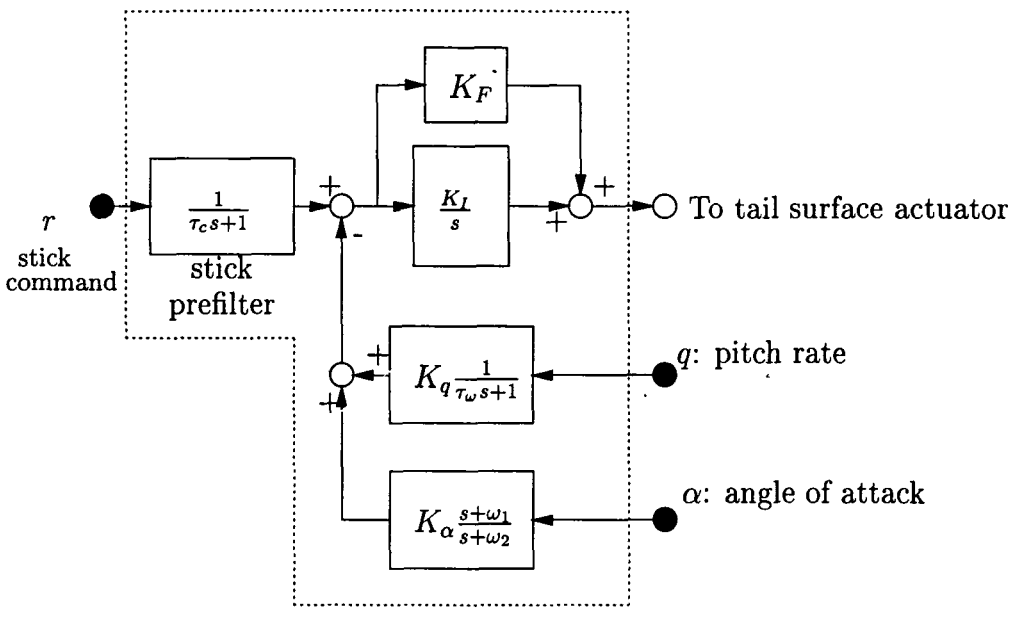


Figure 6.2: PID controller of F14 control problem

- Dryden wind-gust model: This models how vertical wind gusts affect the airplane dynamics. Two external disturbance signals ω_g(vertical velocity wind gust) and q_g (pitch rate wind gust) are generated by white noise η_1 with their spectrum modeled by the following parameters
  - $a = 2.5348$ sec: gust correlation time
  - $b = 64.13$ ft: reference span
  - $V_{To} = 690.4$ ft/sec: total air speed
  - $\sigma_{wg} = 3.0$ ft/sec: rms gust velocity
- desired α response T_M: This is a damped 2nd order model that we want the angle of attack α of the airplane to follow in response to a stick step input where
  - $\xi = 0.707$: damping ratio
  - $\omega_n = 2.49$ rad/sec: bandwidth

- controller: PID controller consists of stick prefilter, proportional/integral compensator, pitch rate lead filter and α filter command generator. The controller is shown in Figure 6.2. One of the typical controllers ($\mathcal{C}_p$)[26], using the following parameters $\omega_1 = 2.9710$, $\omega_2 = 4.1440$, $K_q = 0.8156$, $\tau_\omega = 0.3959$, $K_\alpha = 0.6770$, $\tau_c = 0.1$, $K_F = -1.7460$ and $K_I = -3.8640$, will be used to compare with the $\mathcal{H}_2/\mathcal{H}_\infty$ controller.

Using this particular controller $\mathcal{C}_p$, the response of angle of attack is able to follow the damped 2nd order model well under the prescribed level of wind gust noise. However this controller doesn't suppress the wind gust noise η_1 well. Especially, if the vertical wind gust rms speed increases from 3 ft/sec to 150 ft/sec, a considerable amount of noise will show up on the response. Moreover, if we use a feedback perturbation as the Δ block in Figure 6.1 to model the aircraft dynamic uncertainty, the closed-loop system with the controller $\mathcal{C}_p$ is very sensitive to Δ. In other words, for a small variation of Δ the closed-loop system will become unstable.

Our mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization algorithm offer a good alternative to find a controller that remedy these deficiencies.

6.1.1 System Modification

The original F-14 example is modified as follows to suit the requirements of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem. Figure 6.1 shows the modified system.

- Instead of using the PID controller $\mathcal{C}_p$, we choose to find a constant output feedback G that will satisfy the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ requirements on the system specification that will be defined shortly.

- A prefilter $\mathcal{P}$ is used to solve the model-matching problem after a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ solution G is found. The model to be followed is the step response of a second order damped system.

- The default rms vertical wind gust velocity $\sigma_{wg} = 3\ ft/sec$ is too small to have any effect even for the open-loop system. We deliberately exaggerate the wind gust by choosing $\sigma_{wg} = 150\ ft/sec$ to show the noise suppressing capability of various controllers.

- A white noise input η_0 is added to the actuator. It is used for the measurement of an additional performance requirement that the angle of attack α has a fast response to the actuator. We choose the weighting $V_r = 0.25$ so that noise level is in the same order as the wind gust noise.

- We draw a 2×1 feedback block Δ between (q:pitch rate, α:angle of attack) output and tail control surface to represent the variation of aircraft dynamic due to modeling error , shift of operating conditions or other reasons. The specification is to keep the aircraft model remain stable for all Δ such that noise$\|\Delta\|_\infty \leq 1/\gamma = \frac{1}{0.45}$.

6.1.2 Problem Specification

We need to find a controller that can achieve the following three objectives

1. The closed-loop transfer function between r and α, i.e. $T_{\alpha r}$ is to follow the damped second order transfer function T_M

$$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{6.1}$$

   as close as possible.

2. The 'effects' on α of the white noise imposed on η_0 and η_1 are to be suppressed. If one chooses the variance of α to measure the 'effects', this suppression can be achieved by minimizing the $\mathcal{H}_2$ norm of the transfer function between

$$\min_G \|T_{z_0 w_0}\|_2^2 \tag{6.2}$$

where w_0 and z_0 are defined as

$$w_0 := \begin{bmatrix} \eta_0 \\ \eta_1 \end{bmatrix} \quad and \quad z_0 := \alpha \tag{6.3}$$

3. Keep the airplane stable for any variation Δ such that $\|\Delta\|_\infty < \gamma^{-1}$. The sufficient condition for this requirement is

$$\|T_{z_1 w_1}\|_\infty \leq \gamma \tag{6.4}$$

where w_1 and z_1 are defined as

$$w_1 := \delta_c \quad and \quad z_1 := \begin{bmatrix} \alpha \\ q \end{bmatrix} \tag{6.5}$$

6.1.3 Design Procedure

The structure of the intended controller contains the prefilter $\mathcal{P}$ and the feedback controller G. We will assume $\mathcal{P}$ is a SISO stable proper transfer function and G is a constant output feedback gain of dimension 2×1. We will partition our design into 2 steps.

1. We will first use our $\mathcal{H}_2/\mathcal{H}_\infty$ optimization algorithm to find a G such that the closed-loop system will meet requirements 2 and 3, i.e.

$$\min_G \|T_{z_0 w_0}\|_2^2 \tag{6.6}$$

$$\textit{such that} \qquad \|T_{z_1 w_1}\|_\infty \leq \gamma \tag{6.7}$$

2. Let the closed-loop system between α and the elevator actuator be $T_{\alpha d}$ and T_M be the intended model. We will use the model matching algorithm to find a $\mathcal{P}$ such that the step response of $T_{\alpha a}\mathcal{P}$ and T_M are close, i.e.

$$\|\frac{T_{\alpha a}\mathcal{P}}{s} - \frac{T_M}{s}\| \tag{6.8}$$

is as small as possible to meet requirement 1. The measurement of (6.8) can be either the $\mathcal{H}_2$ or the $\mathcal{H}_\infty$ norm. In fact, in this example, we can find a stable

proper $\mathcal{P}$ that will match the model T_M exactly. Therefore it does not matter which norm is used.

6.2 $\mathcal{H}_2/\mathcal{H}_\infty$ Optimization Problem

First, we have to transform the system into our standard $\mathcal{H}_2/\mathcal{H}_\infty$ state space setup. For the modified F-14 control system we obtain

$$\begin{cases} \dot{x} = Ax + B_0 w_0 + B_1 w_1 + B_2 u \\ z_0 = C_0 x + D_0 u \\ z_1 = C_1 x + D_1 u \\ y = C_3 x \\ u = Gy \end{cases} \tag{6.9}$$

where

$$A = \begin{bmatrix} Z_w & U_0 & Z_\delta & -Z_w & 0 & 0 \\ M_w & M_q & M_\delta & -M_w - \frac{M_q \pi}{4b} & 0 & -M_q \\ 0 & 0 & \frac{-1}{\tau_a} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-2}{a} & 1 & 0 \\ 0 & 0 & 0 & \frac{-1}{a^2} & 0 & 0 \\ 0 & 0 & 0 & -(\frac{\pi}{4b})^2 & 0 & \frac{-\pi V_{T0}}{4b} \end{bmatrix}$$

$$B_0 = \begin{bmatrix} Z_\delta \\ M_\delta \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{V_r}{\tau_a} & 0 \\ 0 & \delta_{wg}\sqrt{\frac{3}{a}} \\ 0 & \delta_{wg}\sqrt{a^3} \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau_a} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_0 = [\,\tfrac{1}{U_0} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,], \quad D_0 = [\,1\,] \;\; or \;\; D_0 = [\,0.3\,]$$

$$C_1 = \begin{bmatrix} \frac{1}{U_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} \frac{1}{U_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

D_0 and D_1 are artificial parameters that we use to weight the control in the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm. In the F-14 example, the control is the tail elevator deflection. Without putting a weight on the control, $\mathcal{H}_2$ or $\mathcal{H}_2/\mathcal{H}_\infty$ controllers are likely to produce a large deflection on the tail surface which in reality can not be achieved. We will demonstrate a case with high penalty cost on control, i.e. $D_0 = 1$ as well as a case with low penalty cost on control, i.e. $D_0 = 0.3$. Because we choose nonzero D_0, the $\mathcal{H}_2$ norm of $T_{z_0 w_0}$ is not the $\mathcal{H}_2$ norm of the transfer function between α and noise w_0. Let $\tilde{z}_0$ be defined as

$$\tilde{z}_0 = C_0 x \tag{6.10}$$

The actual noise suppression capability between α and noise w_0 can be measured by the $\mathcal{H}_2$ norm of $T_{\tilde{z}_0 w_0}$. The $\|T_{z_0 w_0}\|_2$ is $\|T_{\tilde{z}_0 w_0}\|_2$ plus the control cost plus the cross coupling cost .

6.2.1 $\mathcal{H}_2/\mathcal{H}_\infty$ Solution for Case With $D_0 = 1$

We have $n = 6$, $m_0 = 1$, $m_1 = 1$, $m_2 = 1$, $p_0 = 1$, $p_1 = 2$ and $p_3 = 2$. Because $p_1 > m_2$ we know that z_1 will not have any unobservable (A, B)-controlled invariant. However $p_0 = m_2$, therefore there exist unobservable controlled invariants. D_0 is nonzero and of full rank, therefore the maximum unobservable controlled invariant is of dimension n.

This problem is a constant output feedback problem, therefore the condition that a state feedback parameterization exists for any unobservable (A, B)-controlled invariant is subject to (5.69).

$$2 \geq q_u + 0.$$

This means that all possible unobservable (A, B)-controlled invariants are at most two dimensional spanned by the following elementary unobservable controlled invariants.

Elementary Unobservable Controlled Invariants for $D_0 = 1.0$

$$U_1 = \begin{bmatrix} 0.994 \\ -0.0371 \\ -0.104 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; U_2 = \begin{bmatrix} 0.993 \\ -0.0115 \\ -0.00249 \\ 0 \\ 0 \\ 0.119 \end{bmatrix}, \; U_3 = \begin{bmatrix} -1 \\ 0.00292 \\ 0.00166 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$U_4 = \begin{bmatrix} 0.787 \\ 0.000702 \\ -0.00116 \\ -0.574 \\ -0.227 \\ 0.00738 \end{bmatrix} \; U_{4g} = \begin{bmatrix} 0.0799 \\ 0.000895 \\ -5.88e-05 \\ 0.291 \\ -0.46 \\ -0.00465 \end{bmatrix}, \; U_6 = \begin{bmatrix} -1 \\ -0.00221 \\ 0.00138 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

with the corresponding unobservable modes

$$\lambda_1 = -19.7219, \; \lambda_2 = -8.455, \; \lambda_3 = -2.5471,$$

$$\lambda_4 = -0.3945, \; \lambda_{4g} = -0.3945, \; \lambda_6 = 0.9734$$

Feasible Sub-Problem

Using the output feedback $\mathcal{H}_\infty$ algorithm in Chapter 2, we can determine that only the output feedback parameterization corresponding to the unobservable subspace U_3 contains a feasible solution for $\gamma = 0.45$. The reasons are as follows

- The output feedback parameterization for any unobservable controlled invariant of dimension equal to 2 is constant. More specifically the parameterization

$$K = [-0.0015 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]. \tag{6.11}$$

will make $\{U_1 \; U_2\}$, $\{U_1 \; U_2\}$, ..., $\{U_5 \; U_6\}$ unobservable. This particular feedback produce a closed-loop system with unstable hidden mode equal to λ_6. Therefore

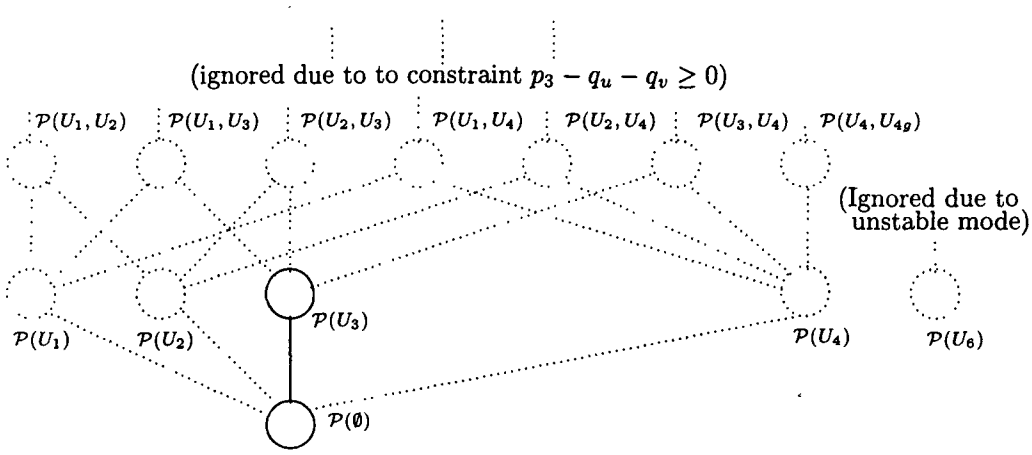


Figure 6.3: Lattice diagram of sub-problem for F-14 example with $D_0 = 1$

the subproblems associated with these controlled invariants can be ignored.

- $\{U_6\}$ corresponds to an unstable unobservable mode. It can also be ignored.
- $\{U_{4g}\}$ is a generalized space of $\{U_4\}$ that can not be an elementary unobservable controlled invariant by itself.
- The output feedback parameterizations $\mathcal{G}_{U_1}$, $\mathcal{G}_{U_2}$ and $\mathcal{G}_{U_4}$ of $\{U_1\}$, $\{U_2\}$ and $\{U_4\}$ are all not feasible, i.e. they will produce closed-loop systems that have the $\mathcal{H}_\infty$ norm of $T_{z_1 w_1}$ larger than 0.45.

The lattice diagram of the sub-problem corresponding to output feedback parameterizations for $D_0 = 1$ is shown in Figure 6.3. The two parameterizations need to be considered are

$$\begin{aligned} \mathcal{G}_{U_3} &= \{K | K = [-0.000479 \quad 0.332 \quad 0 \quad 0 \quad 0 \quad 0] \\ &\qquad + W[-0.00119 \quad -0.406 \quad 0 \quad 0 \quad 0 \quad 0] \ \forall W \in \mathcal{R}^{1\times 1}\} \quad (6.12) \\ \mathcal{G}_{\emptyset} &= \{K | K = WC_3 \ \forall W \in \mathcal{R}^{1\times 2}\}. \quad (6.13) \end{aligned}$$

The output feedback parameterizations are obtained by following the procedure in Section 5.5.

	$\lambda^{<k>}$	$\|T_{z_0w_0}(K^{<k>})\|_2^2$	$\|T_{z_1w_1}(K^{<k>})\|_\infty$	Newton's iterations
k=1	4.4407	1.8029	0.43383	33
k=2	2.1772	1.6624	0.43956	7
k=3	1.7379	1.5558	0.44461	11
k=4	1.5692	1.4988	0.44763	10
k=5	1.5006	1.4731	0.44907	10
k=6	1.4771	1.4642	0.44958	9
k=7	1.4609	1.4583	0.44992	10
k=8	1.4584	1.4574	0.44997	9
k=9	1.4575	1.4571	0.44999	16
k=10	1.4571	1.457	0.45	16
k=11	1.457	1.457	0.45	21

Table 6.1: BMI iterative steps for solving $\mathcal{P}(U_3)$

Solution for the Sub-Problems

Applying the output feedback parameterization to the system, we obtained two sub-problems $\mathcal{P}(U_3)$ and $\mathcal{P}(\emptyset)$. We first obtained an initial solution using the output feedback $\mathcal{H}_\infty$ algorithm in Chapter 2 for $\mathcal{P}(U_3)$. Next we used the BMI algorithm to solve for $\mathcal{P}(U_3)$ and obtained the optimal solution $K^*_{U_3}$

$$K^*_{U_3} = [\,0.0157 \quad 5.87 \quad 0 \quad 0 \quad 0 \quad 0\,] = [\,10.8 \quad 5.87\,]\,C_3.$$

The iterative steps of the BMI algorithm are shown in Table 6.1. This subproblem was solved using Matlab 4.2c on a SUN sparc-20 running sun-os 5.5 in multi-users mode. The execution time is roughly 50 seconds to achieve the precision indicated in Table 6.1.

After we solved $\mathcal{P}(U_3)$, we perturbed $K^*_{U_3}$ slightly to obtain an initial solution for

	$\lambda^{<k>}$	$\|T_{z_0w_0}(K^{<k>})\|_2^2$	$\|T_{z_1w_1}(K^{<k>})\|_\infty$	Newton's iterations
k=1	1.194	0.9924	0.4387	51
k=2	1.031	0.9269	0.4425	39
k=3	0.9377	0.8778	0.4449	37
k=4	0.8811	0.8487	0.447	24
k=5	0.8509	0.8336	0.4484	15
k=6	0.8328	0.8248	0.4493	12
k=7	0.8252	0.8213	0.4497	10
k=8	0.8212	0.8195	0.4499	9
k=9	0.8192	0.8186	0.45	10
k=10	0.8186	0.8184	0.45	13
k=11	0.8183	0.8183	0.45	16

Table 6.2: BMI iterative steps for solving $\mathcal{P}(\emptyset)$; $D_0 = 1$

$\mathcal{P}(\emptyset)$. We used the BMI algorithm to solve $\mathcal{P}(\emptyset)$ and obtained the optimal solution for the full problem.

$$K_\emptyset^* = [\,-0.000399 \quad 3.79 \quad 0 \quad 0 \quad 0 \quad 0\,] \; or \; [\,-0.275 \quad 3.79\,]\,C_3$$

where $K_\emptyset^* = G_\emptyset^* C_3$. We will relabel $G_\emptyset^*$ as $G_{2\infty H}$. Table 6.2 shows the iteration steps for $\mathcal{P}(\emptyset)$. The execution time is about 110 seconds under the same environment.

Result

The $\mathcal{H}_2/\mathcal{H}_\infty$ optimal output feedback gain $G_\emptyset^*$ for the case $D_0 = 1$ produced the following performance.

$$\|T_{z_1w_1}(G_\emptyset^*)\|_\infty = 0.45, \quad \|T_{z_0w_0}(G_\emptyset^*)\|_2^2 = 0.8183, \quad \|T_{\tilde{z}_0w_0}(G_\emptyset^*)\|_2^2 = 0.0328 \qquad (6.14)$$

The difference between $\|T_{z_0w_0}(G_\emptyset^*)\|_2^2$ and $\|T_{\tilde{z}_0w_0}(G_\emptyset^*)\|_2^2$ is due to the fact that we weight the control cost very high.

6.2.2 $\mathcal{H}_2/\mathcal{H}_\infty$ Solution for Case With $D_0 = 0.3$

This is the same problem except that we reduced the weighting of the control from $D_0 = 1$ to $D_0 = 0.3$. We have the following unobservable controlled invariants for this problem.

Elementary Unobservable Controlled Invariant for $D_0 = 0.3$

$$U_1 = \begin{bmatrix} 0.995 \\ -0.0353 \\ -0.0947 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; U_2 = \begin{bmatrix} 0.998 \\ -0.0121 \\ -0.00836 \\ 0 \\ 0 \\ 0.0649 \end{bmatrix}, \; U_3 = \begin{bmatrix} 1 \\ -0.00802 \\ -0.00678 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$, \; U_4 = \begin{bmatrix} 0.188 \\ 0.000827 \\ -0.000926 \\ -0.914 \\ -0.36 \\ 0.0117 \end{bmatrix}, \; U_{4g} = \begin{bmatrix} 0.0131 \\ -1.55e-05 \\ -1.76e-05 \\ 0.314 \\ -0.79 \\ -0.00549 \end{bmatrix}, \; U_6 = \begin{bmatrix} -1 \\ -0.00551 \\ 0.00413 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

with the corresponding unobservable modes

$$\lambda_1 = -18.985, \; \lambda_2 = -8.455, \; \lambda_3 = -5.736, \; \lambda_4 = \lambda_{4g} = -0.3945, \; \lambda_6 = 3.4254$$

λ_6 is unstable, λ_4 is a repeated unobservable mode from the Dryden wind gust model. U_{4g} is the generalized eigenvector of U_4.

Feasible Sub-Problem

Similar to the case that $D_0 = 1$, most of the sub-problems can be eliminated without actually solving them. For the case of $D_0 = 0.3$, even the sub-problem $\mathcal{P}(U_3)$ is not feasible. We only need to solve the problem $\mathcal{P}(\emptyset)$. The initial solution for $\mathcal{P}(\emptyset)$ can be obtained by the algorithm in Chapter 2. The optimal solution of this problem is

$$K_\emptyset^* = [0.00145 \quad 4.04 \quad 0 \quad 0 \quad 0 \quad 0] \; or \; G_{2\infty L} := G_\emptyset^* = [0.997 \quad 4.04]$$

where $K_\emptyset^* = G_{2\infty L} C_3$.

It can be obtained relatively fast as shown in Table 6.3. The execution time is 120 seconds. We will relabel $G_\emptyset^*$ as $G_{2\infty L}$

Result

The $\mathcal{H}_2/\mathcal{H}_\infty$ optimal output feedback gain $G_\emptyset^*$ for the the case $D_0 = 0.3$ produces the following performance.

$$\|T_{z_1 w_1}(G_\emptyset^*)\|_\infty = 0.45, \quad \|T_{z_0 w_0}(G_\emptyset)\|_2^2 = 0.1018, \quad \|T_{\tilde{z}_0 w_0}(G_\emptyset^*)\|_2^2 = 0.0182$$

Because of the reduced weighting on control, $\|T_{z_0 w_0}(G_\emptyset^*)\|_2^2$ and $\|T_{\tilde{z}_0 w_0}(G_\emptyset^*)\|_2^2$ improve to a lesser value. When comparing the performance for different values of D_0, only $\|T_{\tilde{z}_0 w_0}(G_\emptyset^*)\|_2^2$ is meaningful because it excludes the penalty for the control cost and it represents the true performance of the response to the inputs.

6.2.3 Output Feedback LQ Solution and Output Feedback Solution

We will compare the performance of our $\mathcal{H}_2/\mathcal{H}_\infty$ solution with the performance of one $\mathcal{H}_\infty$ feasible solution G_∞ and output feedback LQ solutions G_{2H} for $D_0 = 1$ and G_{2L} for $D_0 = 0.3$.

To obtain G_∞, we use the algorithm in Chapter 2 for $\gamma = 0.45$. We have one such controller gain

$$G_\infty = [-8.35 \quad 25.2]$$

	$\lambda^{<k>}$	$\|T_{z_0w_0}(K^{<k>})\|_2^2$	$\|T_{z_1w_1}(K^{<k>})\|_\infty$	Newton's iterations
k=1	0.5191	0.2517	0.3738	51
k=2	0.2668	0.1696	0.4359	51
k=3	0.1982	0.1566	0.4401	18
k=4	0.1628	0.1364	0.4409	24
k=5	0.1415	0.1238	0.4421	18
k=6	0.1275	0.1154	0.4434	14
k=7	0.1189	0.1104	0.4448	13
k=8	0.1124	0.1067	0.4462	13
k=9	0.1079	0.1044	0.4475	12
k=10	0.1048	0.1029	0.4487	13
k=11	0.1034	0.1024	0.4493	11
k=12	0.1025	0.102	0.4497	12
k=13	0.1021	0.1019	0.4499	11
k=14	0.1019	0.1019	0.45	11

Table 6.3: BMI iterative steps for solving $\mathcal{P}(\emptyset)$; $D_0 = 0.3$

*	$G_{2\infty L}$	G_{2L}	$G_{2\infty H}$	G_{2H}	G_∞
$\|T_{z_1w_1}\|_\infty$	0.45	0.9926	0.45	2.3470	0.3460
$\|T_{z_0w_0}\|_2^2$	0.1019	0.0588	0.8183	0.1862	0.5715
$\|T_{\tilde{z}_0w_0}\|_2^2$	0.0182	0.0129	0.0328	0.2033	0.0673

Table 6.4: $\mathcal{H}_2$ norm and $\mathcal{H}_\infty$ norm of F-14 example with various controllers

To obtain the output feedback LQ solution, we use the same algorithm as $\mathcal{H}_2/\mathcal{H}_\infty$ but with γ large enough. We have the optimal output feedback LQ solution for both $D_0 = 1$ and $D_0 = 0.3$ as

$$G_{2H} = [-0.325 \quad 0.335] \; and \; G_{2L} = [2.4 \quad 1.04]$$

The performance produced by different controllers is presented in Table 6.4. The $\mathcal{H}_\infty$ norms of the closed-loop $T_{z_1w_1}$ with $G_{2\infty L}$ and $G_{2\infty L}$ are 0.45, i.e. located on the boundary of the $\mathcal{H}_\infty$ constraint. The $\mathcal{H}_\infty$ norm corresponding to the sub-optimal solution G_∞ is less than 0.45. The LQ solutions G_{2L} and G_{2H} produce large $\mathcal{H}_\infty$ norm. It will be shown later that because of this, the corresponding closed-loop system can be made unstable more easily. The table also shows that even though the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ solutions produce much better $\mathcal{H}_\infty$ performance, they do not yield much $\mathcal{H}_2$ performance but $T_{\tilde{z}_0w_0}$ like the sub-optimal $\mathcal{H}_\infty$ controller does. One interesting observation from the table is that the $\|T_{\tilde{z}_0w_0}\|_2^2$ is actually larger than $\|T_{z_0w_0}\|_2^2$ for the controller G_{2H}. This is not normal because $\|T_{\tilde{z}_0w_0}\|_2^2$ is obtained without the control cost and cross-coupling cost. It tells that the cross-coupling between the state and control is negative.

6.3 Model Matching

One of the objectives of the F-14 example is to match the step response of the angle of attack α with a damped 2nd order model. Let $T_{\alpha d}$ be the closed-loop transfer function between d and α shown in Figure 6.1. Also let T_M be the damped second order model to follow. The purpose of model matching is to find a P such that

$$\|T_{\alpha d}P - T_M\|_i \tag{6.15}$$

is minimum. For the model matching problem of a step response we minimize

$$\|\frac{T_{\alpha d}P}{s} - \frac{T_M}{s}\|_i \tag{6.16}$$

$\|.\|_i$ can be the $\mathcal{H}_\infty$ or $\mathcal{H}_2$ norm. If one chooses the $\mathcal{H}_\infty$ norm to measure the difference, the problem can be transformed into a Nehari problem by inner-outer factorization on $T_{\alpha d}$. Because, for the F-14 example, the model matching is SISO, the problem is particularly easy to solve. Let $T_{\alpha di}$ and $T_{\alpha do}$ be the inner and outer factors of $T_{\alpha d}$. (6.16) can be transformed as

$$\|\frac{T_{\alpha di}T_{\alpha do}P}{s} - \frac{T_M}{s}\|_\infty \tag{6.17}$$

Use the fact that the $\mathcal{H}_\infty$ norm of (6.17) will not be changed by pre-multiplying by $T_{\alpha di}^{-1}$(the inverse of an inner function is also inner). (6.17) becomes

$$\|\frac{T_{\alpha do}P}{s} - T_{\alpha di}^{-1}\frac{T_M}{s}\|_\infty \tag{6.18}$$

Let M_+ be the unstable part of $T_{\alpha di}^{-1}\frac{T_M}{s}$ and M_{-0} be such that

$$T_{\alpha di}^{-1}\frac{T_M}{s} = M_+ + M_{-0}$$

and define

$$\tilde{P} = \frac{T_{\alpha do}P}{s}$$

We have the Nehari problem

$$\min_{\tilde{P}} \|\tilde{P} - M_+ - M_{-0}\|_\infty \tag{6.19}$$

The solution of the Nehari problem (6.19) is to let $\tilde{P} = M_{-0}$. The model matching difference is therefore $\|M_+\|_\infty$.

We can recover P by computing

$$P = sT_{\alpha do}^{-1}\tilde{P} \tag{6.20}$$

using the fact that the inverse of an outer function remains stable. The s in (6.20) will be canceled by the pole at 0 contained in $\tilde{P}$. P will be a proper stable transfer function.

If we apply $G_{2\infty L}$ to the system, we have

$$T_{\alpha d} = \frac{-1.8566(s+74.8207)}{(s+0.9722)(s+10.1617+21.5829i)(s+10.1617-21.5829i)}$$

$$\begin{aligned} T_{\alpha di} &= 1 \\ T_{\alpha do} &= \frac{-1.8566(s+74.8207)}{(s+0.9722)(s+10.1617+21.5829i)(s+10.1617-21.5829i)} \end{aligned}$$

$$\begin{aligned} M_+ &= 0 \\ M_{-0} &= \frac{T_M}{s} = \frac{5.5696}{s(s+1.7936+1.5338i)(s+1.7936-1.5338i)} \end{aligned}$$

The solution to the Nehari problem is

$$\tilde{P} = \frac{5.5696}{s(s+1.7936+1.5338i)(s+1.7936-1.5338i)}$$

The prefilter P can be recovered as

$$P = sT_{\alpha do}^{-1}\tilde{P} = \frac{-2.9999(s+0.9722)(s+10.1617+21.5829i)(s+10.1617-21.5829i)}{(s+74.8207)(s+1.7936+1.5338i)(s+1.7936-1.5338i)}$$

Because $M_+ = 0$, the model matching problem turns out to be an exact match. Similar prefilter design also applies to the cases with output feedback gain is $G_{2\infty H}$, G_{2L}, G_{2H} or G_∞.

In the next section we will denote the controller, the prefilter plus its corresponding output feedback gain, as $\mathcal{C}_{2\infty L}$, $\mathcal{C}_{2L}$, $\mathcal{C}_{2\infty H}$, $\mathcal{C}_{2H}$, and $\mathcal{C}_{\infty}$.

6.4 Result and Comparison

In this section we will compare the performance of each controller for the following situations

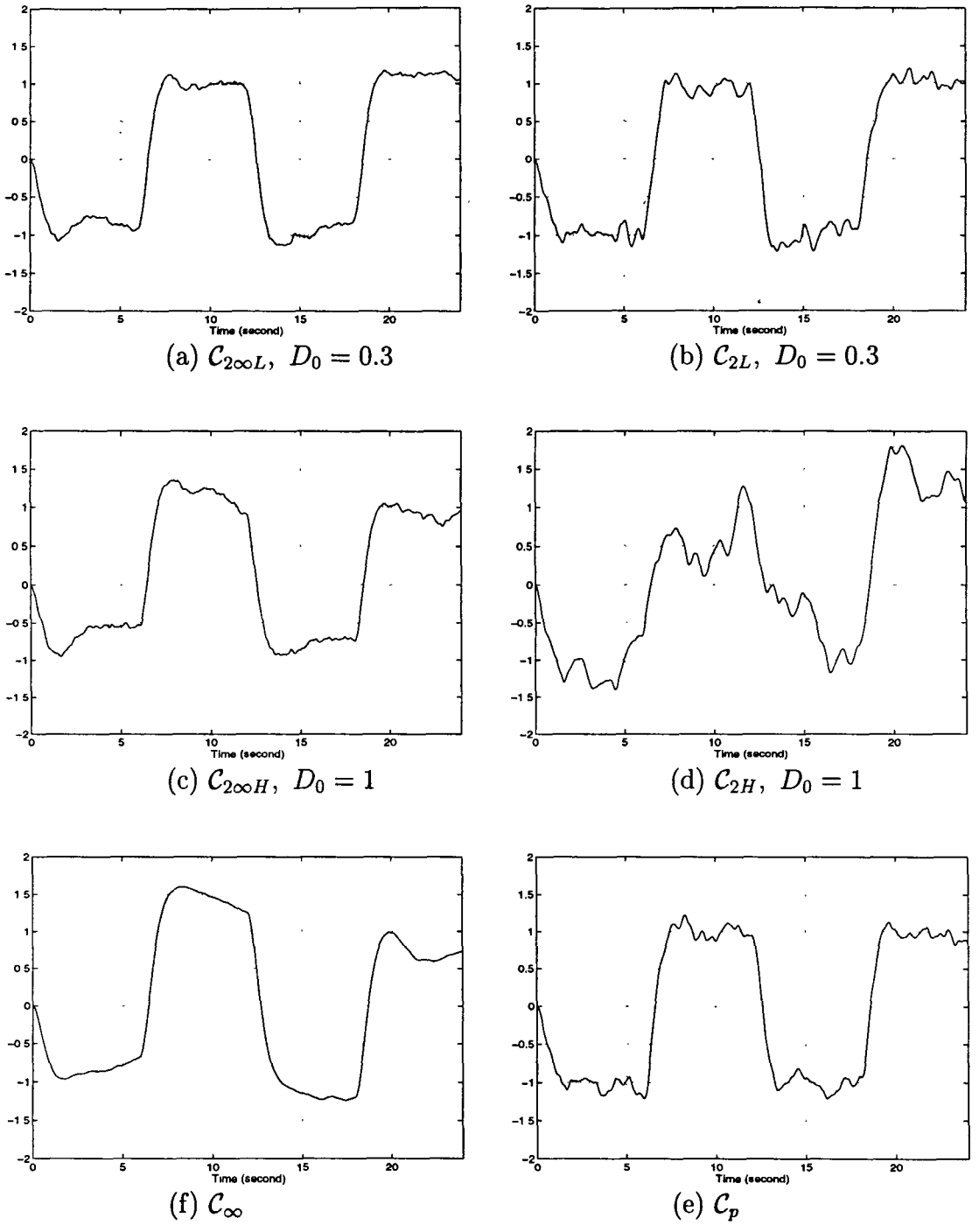
- The angle of attack α response to the step input command under heavy wind gust noise($\sigma_{wg} = 150$) and actuator noise($V_r = 0.25$).
- The response of α under only heavy wind gust noise ($\sigma_{wg} = 150$) without command input.
- The response of α under only actuator noise ($V_r = 0.25$) without command input.
- Closed-loop aircraft stability under feedback perturbation Δ where $\|\Delta\|_{\infty} = 1/\gamma$.

6.4.1 Step Response

First we show the response of α produced by a step input from the stick command. For the problem without noise each controller matches the model perfectly. We will illustrate the results of the step response subject to wind gust noise with vertical rms speed equal 150 ft/sec, i.e. $\sigma_{wg} = 150$ and actuator noise with variance $V_r = 0.25$.

Figure 6.4 (a),(b),(c),(d),(e),(f) shows the step response of the $\mathcal{H}_2/\mathcal{H}_{\infty}$ controller $\mathcal{C}_{2\infty L}$, $\mathcal{H}_2$ controller $\mathcal{C}_{2L}$, $\mathcal{H}_2/\mathcal{H}_{\infty}$ controller $\mathcal{C}_{2\infty H}$, $\mathcal{H}_2$ controller $\mathcal{C}_{2H}$, $\mathcal{H}_{\infty}$ controller $\mathcal{C}_{\infty}$ and dynamic controller $\mathcal{C}_p$ respectively.

It can be seen that only the $\mathcal{C}_{2\infty L}$ and $\mathcal{C}_{2L}$ and $\mathcal{C}_p$ produce satisfactory results. As expected $\mathcal{C}_{2L}$ produces a result slightly better than $\mathcal{C}_{2\infty L}$.



(a) $\mathcal{C}_{2\infty L}$, $D_0 = 0.3$

(b) $\mathcal{C}_{2L}$, $D_0 = 0.3$

(c) $\mathcal{C}_{2\infty H}$, $D_0 = 1$

(d) $\mathcal{C}_{2H}$, $D_0 = 1$

(f) $\mathcal{C}_{\infty}$

(e) $\mathcal{C}_{p}$

Figure 6.4: Response of the indicated closed-loop system to a square wave of peak to peak amplitude equal to one and period equal to 5 second with both wind gust noise and actuator noise

	$\mathcal{C}_{2\infty L}$	$\mathcal{C}_{2L}$	$\mathcal{C}_{2\infty H}$	$\mathcal{C}_{2H}$	$\mathcal{C}_{\infty}$	$\mathcal{C}_p$
σ_α^2	0.0054	0.0031	0.0077	0.0232	0.0375	0.0045

Table 6.5: The variance of α for all 6 controllers with only wind gust noise

	$\mathcal{C}_{2\infty L}$	$\mathcal{C}_{2L}$	$\mathcal{C}_{2\infty H}$	$\mathcal{C}_{2H}$	$\mathcal{C}_{\infty}$	$\mathcal{C}_p$
σ_α^2	0.0011	0.0066	0.0023	0.0647	0.0002	0.0066

Table 6.6: The variance of α for all 6 controllers with only actuator noise

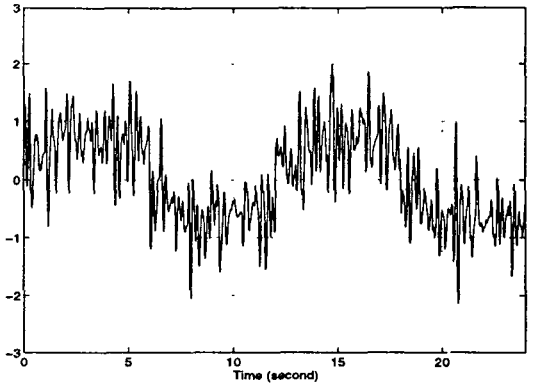
The tail elevator deflection for the different controllers is shown in Figure 6.5. The deflections for $\mathcal{C}_{2\infty H}$ and $\mathcal{C}_{2H}$ are slightly smaller than those for $\mathcal{C}_{2\infty L}$ and $\mathcal{C}_{2L}$. Because we penalize the control, therefore the elevator does not have excessive deflection.

6.4.2 Wind Gust Noise Suppression

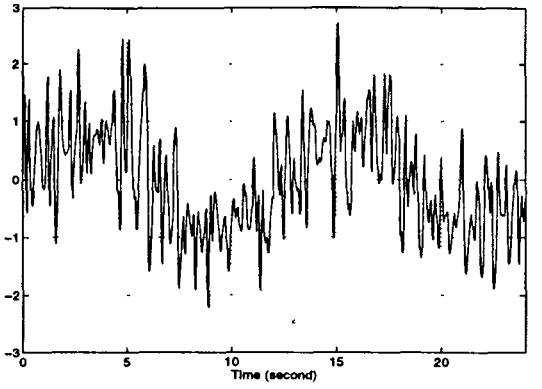
The wind gust noise($\sigma_{wg} = 150$) suppression capability of all 6 controllers is shown in Figure 6.6. Note that the stick command is inactive. The variance of α for all 6 controllers is displayed in Table 6.5. This shows that $\mathcal{C}_{2\infty L}$, $\mathcal{C}_{2L}$, and $\mathcal{C}_p$ are better at suppressing the wind gust noise. The variance is consistent with $\|T_{\tilde{z}_0 w_0}\|_2^2$ because (a)the real performance of the closed-loop system is $\alpha := \tilde{z}_0 = C_0 x$ instead of $z_0 = (C_0 + D_0 G C_3)x$. and (b)the square of the $\mathcal{H}_2$ norm can also be interpreted as the sum of the variance in response to the white noise on each input.

6.4.3 Actuator Noise suppression

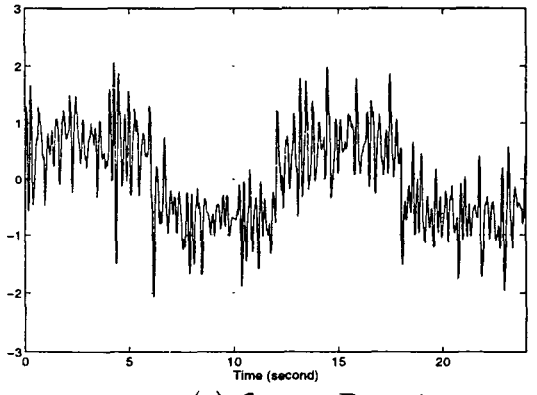
We turn the wind gust noise and stick command off in this simulation. We apply a white noise of variance 0.0625 into the closed-loop systems for all 6 controllers. The responses to the actuator noise are shown in Figure 6.7.



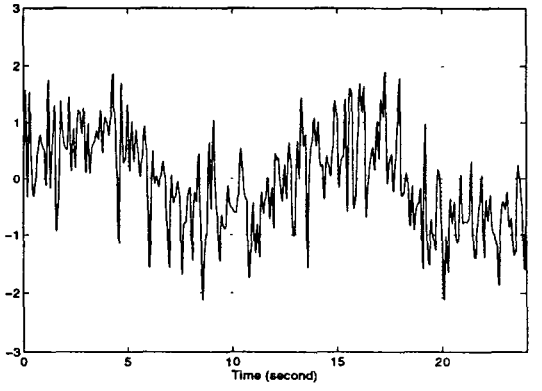
(a) $\mathcal{C}_{2\infty L},\ D_0 = 0.3$



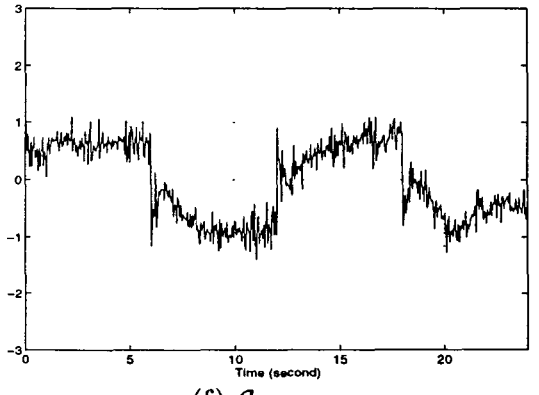
(b) $\mathcal{C}_{2L},\ D_0 = 0.3$



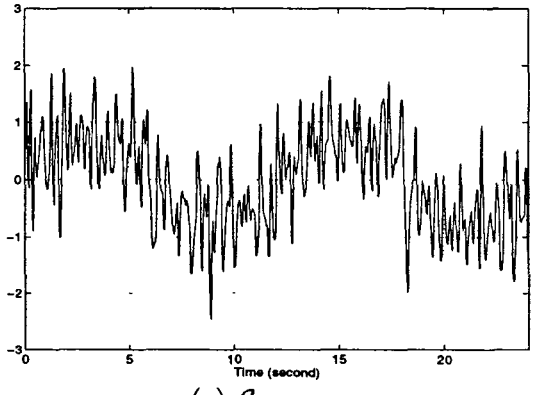
(c) $\mathcal{C}_{2\infty H},\ D_0 = 1$



(d) $\mathcal{C}_{2H},\ D_0 = 1$

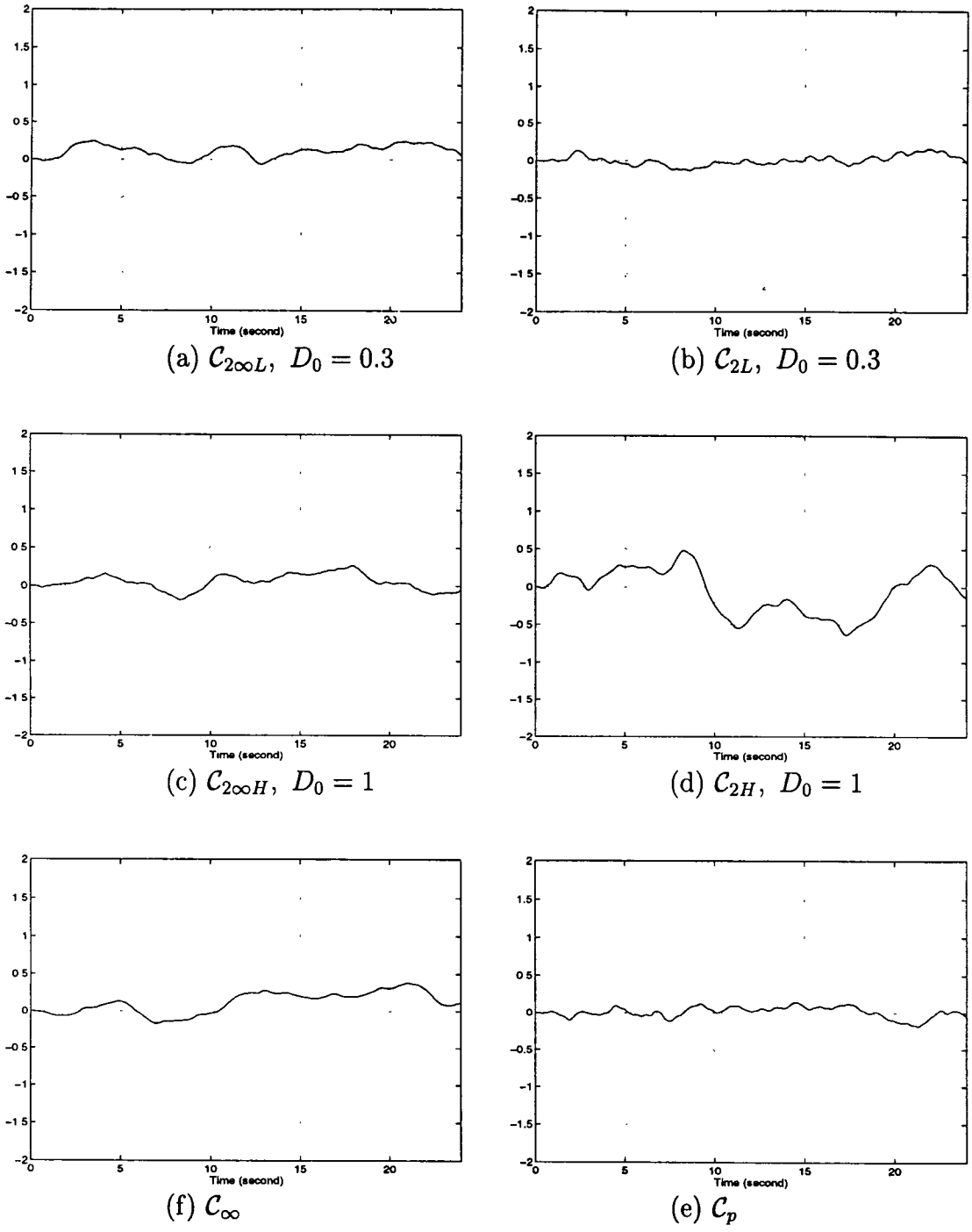


(f) $\mathcal{C}_{\infty}$



(e) $\mathcal{C}_p$

Figure 6.5: Tail elevator deflection of the indicated closed-loop system for same input as in Figure 6.4 and the same noise



(a) $\mathcal{C}_{2\infty L}$, $D_0 = 0.3$

(b) $\mathcal{C}_{2L}$, $D_0 = 0.3$

(c) $\mathcal{C}_{2\infty H}$, $D_0 = 1$

(d) $\mathcal{C}_{2H}$, $D_0 = 1$

(f) $\mathcal{C}_{\infty}$

(e) $\mathcal{C}_{p}$

Figure 6.6: Angle of attack α of the indicated closed-loop system in response to wind gust noise and without the command input

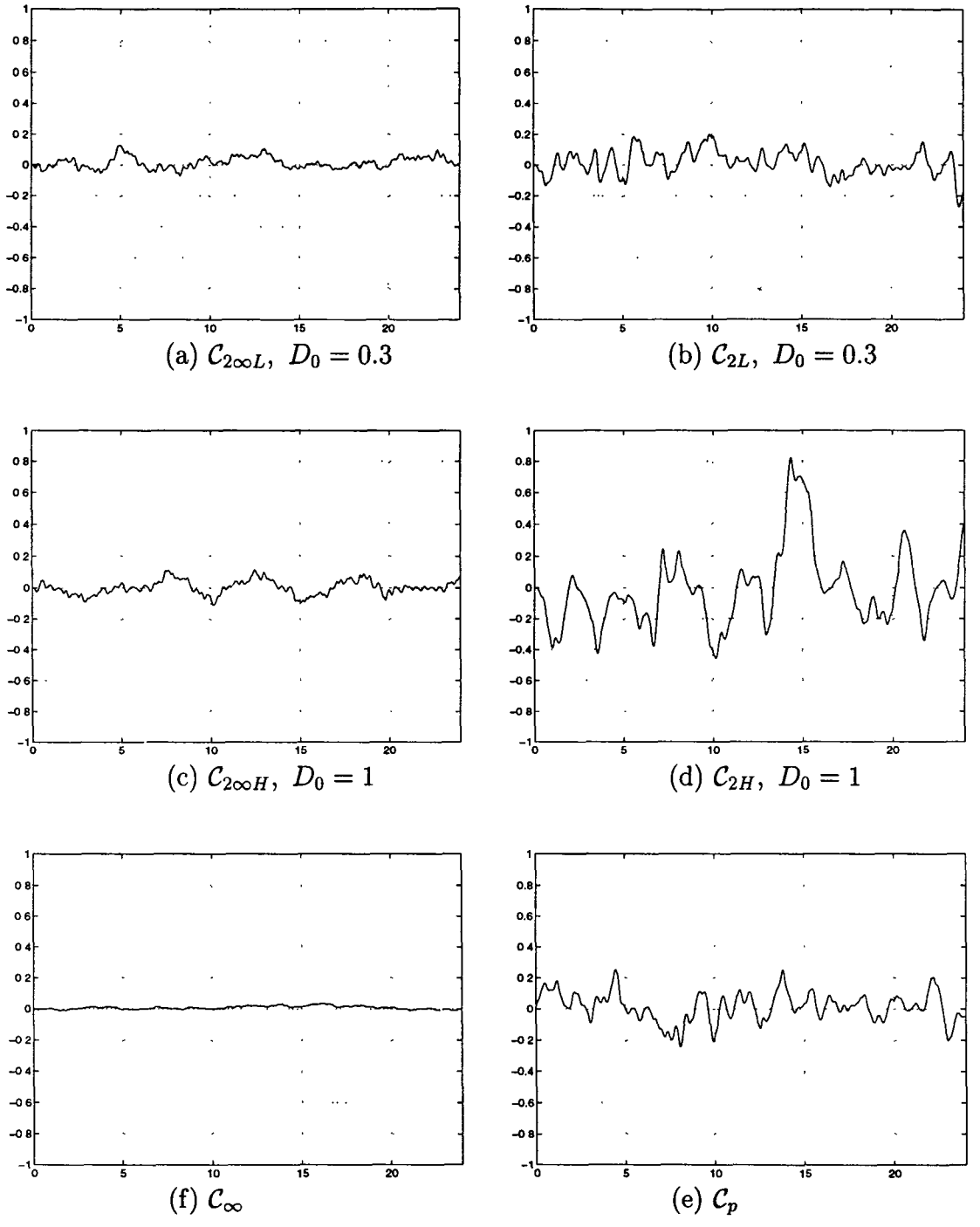


Figure 6.7: Angle of attack α of the indicated closed-loop system in response to actuator noise without the command input

6.4.4 Closed-Loop Aircraft Stability

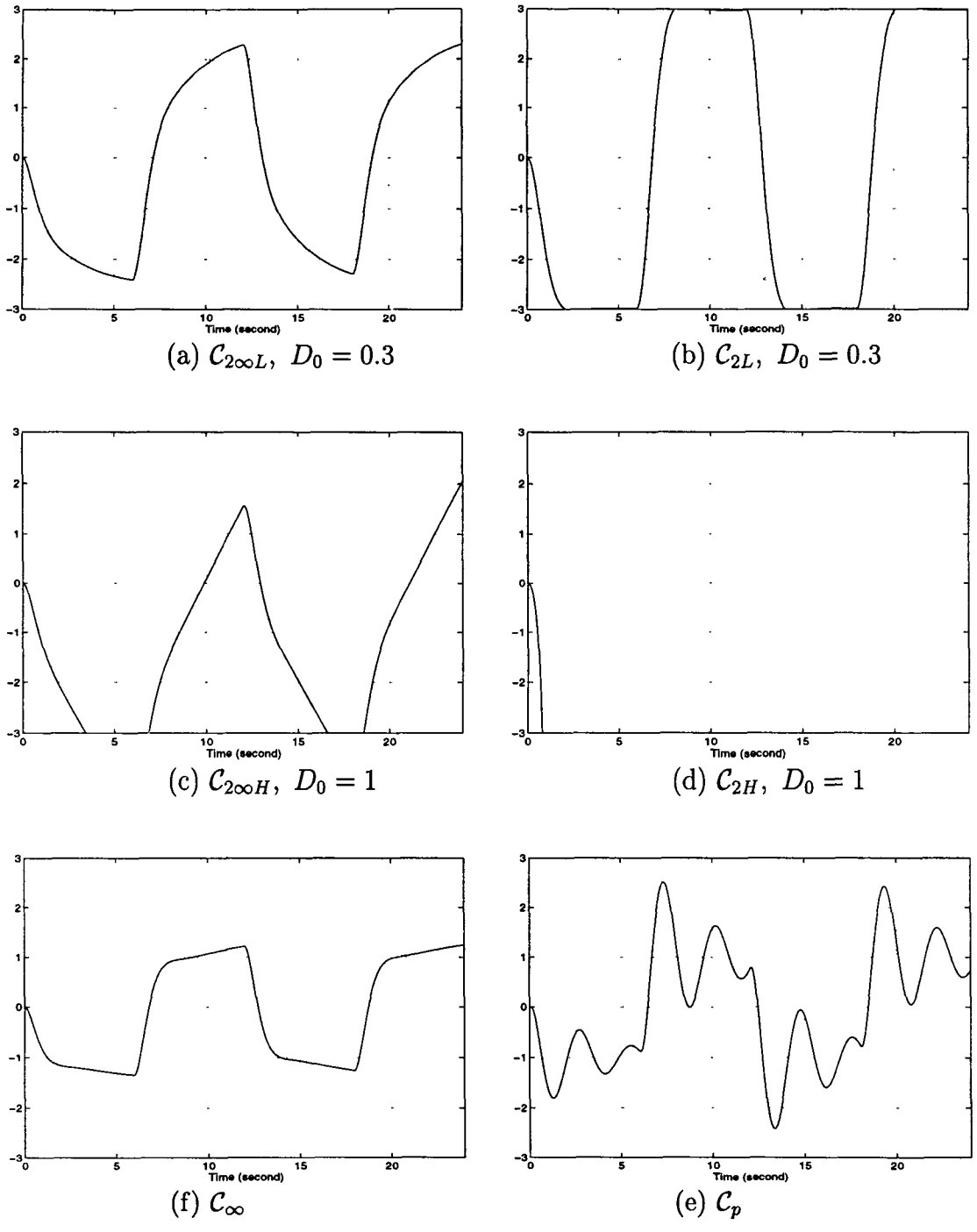
As mentioned earlier, the closed-loop transfer functions between w_0 and z_0 for $\mathcal{C}_{2\infty L}$, $\mathcal{C}_{2L}$, $\mathcal{C}_{2\infty H}$, $\mathcal{C}_{2H}$, $\mathcal{C}_{\infty}$ and $\mathcal{C}_p$ have $\mathcal{H}_\infty$ norm of 0.45, 0.9926, 0.45, 2.3470, 0.3460, and 0.9543 respectively. Therefore, for plant perturbation $\|\Delta\|_\infty$ less than 1/0.45, 1/0.9926, 1/0.45, 1/2.3470, 1/0.3460 and 1/0.9543 respectively, the closed-loop system is guaranteed to remain stable.

In the following simulation, we chose four different Δ's with the $\mathcal{H}_\infty$ norm(singular value for constant matrix) equal to 2.2222=1/0.45 as

$$\Delta = [-2.1828 \quad -0.4158], [2.1828 \quad 0.4158], [-0.4158 \quad 2.1828], [0.4158 \quad -2.1828]$$

These are perturbations in 4 opposite directions. The simulation results shown in Figure 6.8-Figure 6.11 are as expected in that the closed-loop systems using $\mathcal{C}_{2\infty L}$, $\mathcal{C}_{2\infty H}$ or $\mathcal{C}_\infty$ remain stable by the Small Gain Theorem for all Δ's. The closed-loop systems using $\mathcal{C}_{2L}$, $\mathcal{C}_{2H}$, $\mathcal{C}_p$ are destabilized or close to being destabilized by the perturbation $\Delta = [0.4158 \quad -2.1828]$ and $\Delta = [-2,1828 \quad -0.4158]$.

We have demonstrated the value of the $\mathcal{H}_2/\mathcal{H}_\infty$ controller because it provide a good compromise between two competing objectives. The BMI algorithm we developed is able to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem with a relatively low computing cost.



(a) $\mathcal{C}_{2\infty L}$, $D_0 = 0.3$

(b) $\mathcal{C}_{2L}$, $D_0 = 0.3$

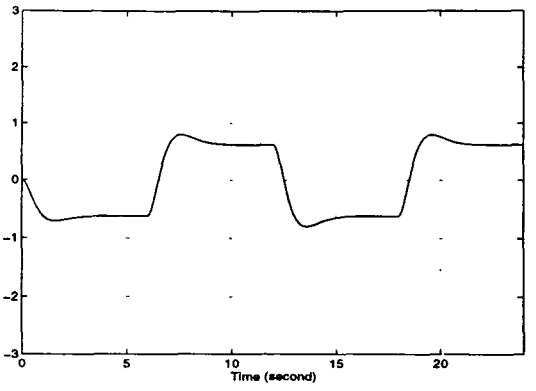
(c) $\mathcal{C}_{2\infty H}$, $D_0 = 1$

(d) $\mathcal{C}_{2H}$, $D_0 = 1$

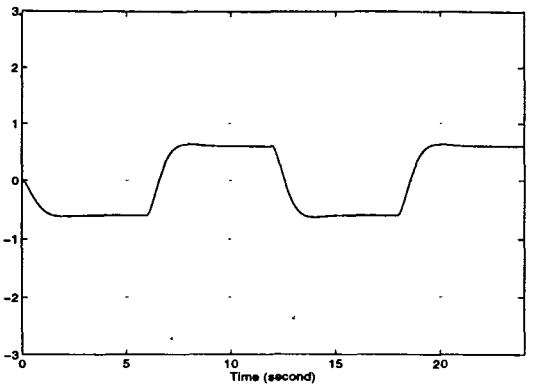
(f) $\mathcal{C}_{\infty}$

(e) $\mathcal{C}_{p}$

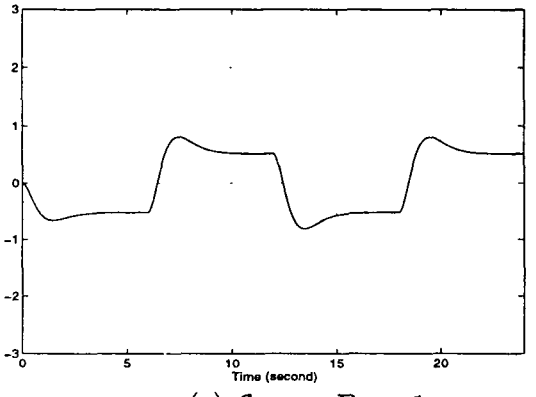
Figure 6.8: Response of the indicated closed-loop system to the same square wave with $\Delta = [-2.1828 \;\; -0.4158]$ and no noise



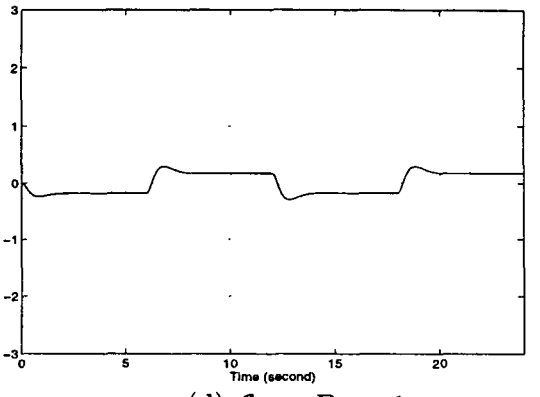
(a) $\mathcal{C}_{2\infty L}$, $D_0 = 0.3$



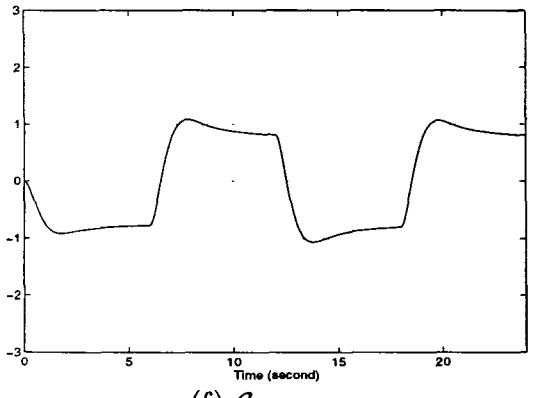
(b) $\mathcal{C}_{2L}$, $D_0 = 0.3$



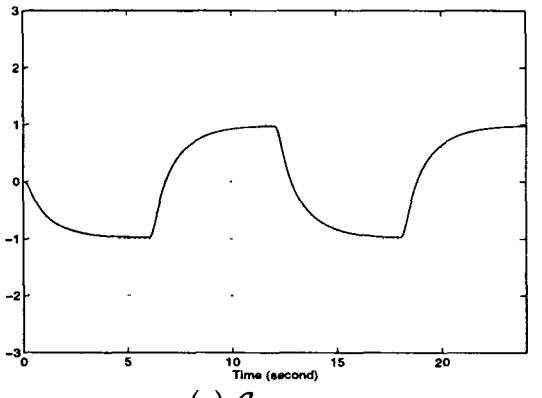
(c) $\mathcal{C}_{2\infty H}$, $D_0 = 1$



(d) $\mathcal{C}_{2H}$, $D_0 = 1$

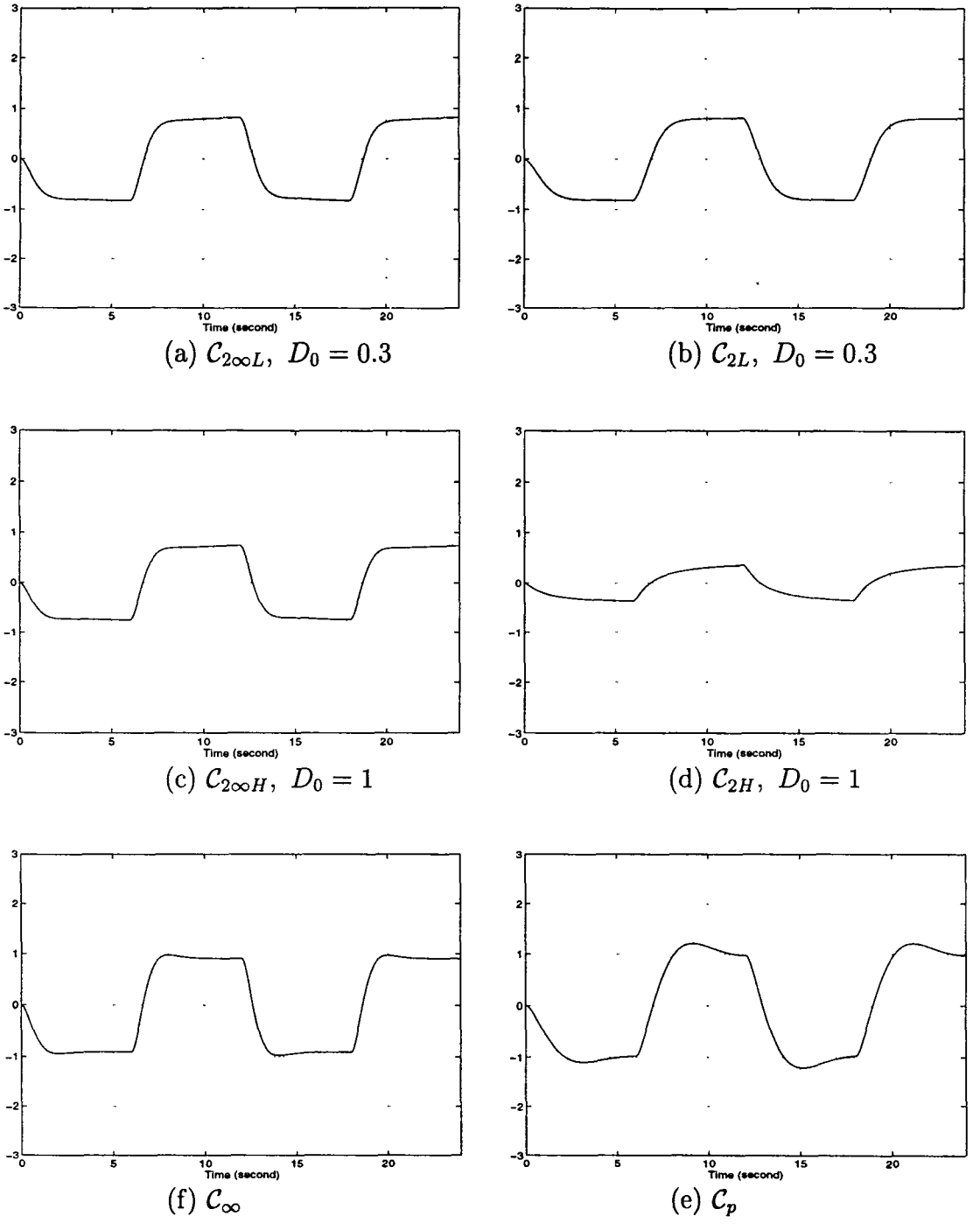


(f) $\mathcal{C}_{\infty}$



(e) $\mathcal{C}_p$

Figure 6.9: Response of the indicated closed-loop system to the same square wave with $\Delta = [2.1828\ 0.4158]$ and no noise



(a) $\mathcal{C}_{2\infty L}$, $D_0 = 0.3$

(b) $\mathcal{C}_{2L}$, $D_0 = 0.3$

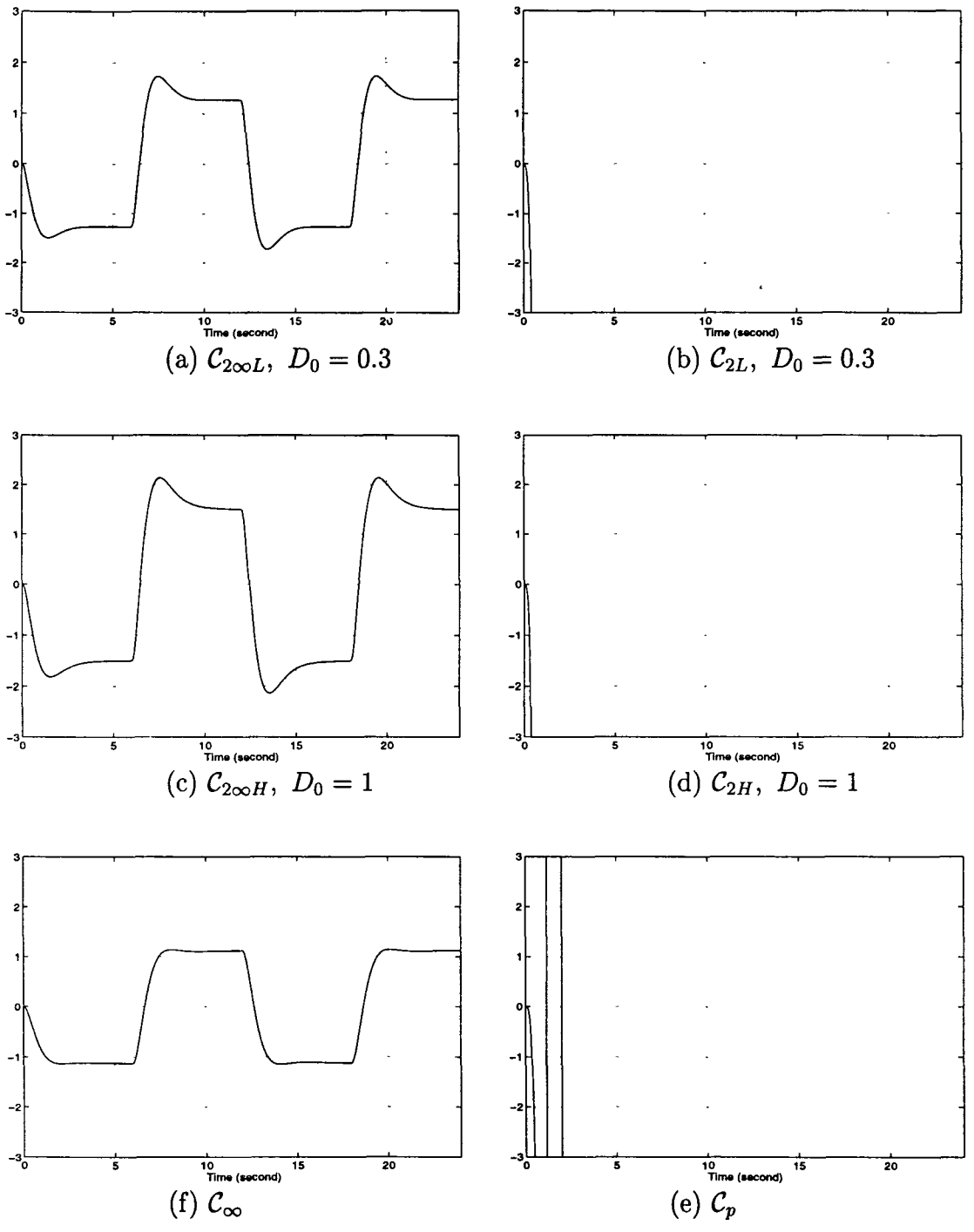
(c) $\mathcal{C}_{2\infty H}$, $D_0 = 1$

(d) $\mathcal{C}_{2H}$, $D_0 = 1$

(f) $\mathcal{C}_{\infty}$

(e) $\mathcal{C}_p$

Figure 6.10: Response of the indicated closed-loop system to the same square wave with $\Delta = [-0.4158\ 2.1830]$ and no noise



(a) $\mathcal{C}_{2\infty L}$, $D_0 = 0.3$

(b) $\mathcal{C}_{2L}$, $D_0 = 0.3$

(c) $\mathcal{C}_{2\infty H}$, $D_0 = 1$

(d) $\mathcal{C}_{2H}$, $D_0 = 1$

(f) $\mathcal{C}_{\infty}$

(e) $\mathcal{C}_{p}$

Figure 6.11: Response of the indicated closed-loop system to the same square wave with $\Delta = [0.4158 \quad -2.1830]$ and no noise

Chapter 7

Conclusions and Suggestions for Future Research

In this dissertation, we presented a numerical algorithm to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for both constant state feedback and constant output feedback cases. We showed the difficulties in solving the BMI optimization problem caused by the unobservability of the closed-loop system. We proposed to partition the $\mathcal{H}_2/\mathcal{H}_\infty$ problem into a lattice of sub-problems according to the lattice of unobservable (A, B)-controlled invariants. Each sub-problem is a BMI optimization problem itself and can be solved effectively using the interior point algorithm if started from an initial solution obtained by perturbing the optimal solution of its ancestral sub-problems. Eventually, we obtain the optimal solution of the full problem when we reach the bottom of the lattice. We can ignore many of the sub-problems even for a realistic example of moderate size such as the F-14 flight control problem.

Performance of the F-14 example using the $\mathcal{H}_2/\mathcal{H}_\infty$ output feedback gain shows that the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ solution offers a very good compromise between robustness and noise suppression. This has not been an accomplished by the LQ controller or $\mathcal{H}_\infty$ sub-optimal controller and other benchmark controllers.

As a side product, we also developed an algorithm for finding an $\mathcal{H}_\infty$ sub-optimal

controller using alternating LMIs. This algorithm is useful for obtaining a feasible solution for our mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem. It is also valuable in its own right if only the output feedback $\mathcal{H}_\infty$ controller is sought.

Related topics to be done or for further analysis are listed below.

- It is not clear whether the solution we obtained is global although none of the numerical experiments we've done contradicts the conjecture that the solution is global.

- The dynamic controller case of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem may be solved by transforming the problem into an output feedback problem.

- We use a Newton's method version of an interior point algorithm to solve the nonconvex BMI eigenvalue problem. Other efficient methods may be used to obtain the global solution for each problem.

- BMI optimization can be obtained easily for many control problems. Other multi-criteria control problems with Lyapunov or Riccati type inequality will likely have the same problem as our $\mathcal{H}_2/\mathcal{H}_\infty$ problem and can be treated in the same way.

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