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Time-Varying Simultaneous Stabilization,
Part I. Countable Families of LTI Systems

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Time-varying simultaneous stabilization, part I. Countable families of LTI systems

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Abstract

In this paper, we introduce a new method that enables us to prove that given any finite family of LTI (linear time-invariant) systems, there exists a continuous time-varying feedback law that simultaneously globally exponentially stabilizes this family. We then derive sufficient conditions for the simultaneous asymptotic stabilizability of countably infinite families of LTI systems. In both cases we provide simple design procedures as well as explicit controls.

Keywords: Continuous time-varying feedback, simultaneous stabilization, countable families of LTI systems.

1 Introduction

The simultaneous stabilization problem was first introduced in [12] and consists in finding a controller that stabilizes each one of the systems of a finite collection of systems. In [12, 13], tractable necessary and sufficient conditions for the simultaneous stabilizability of two linear systems by means of linear feedback, are given. Necessary and sufficient conditions for the existence of a linear feedback law that simultaneously stabilizes three linear systems are proposed in [2, 5, 13], but none of them is tractable.

To overcome the limitation of LTI controller, the design of time-varying feedback laws that simultaneously stabilize or simultaneously steer to the origin each one of the systems of a finite family of systems has been investigated in several papers. In [9], Khargonekar, Poolla and Tannenbaum consider finite collections of discrete time linear systems that individually admits a dead-beat controller. They show that given any such family, there exists a periodic linear time-varying (LTV) controller that simultaneously steers the systems of the family to the origin. In fact, the constructed feedback

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law periodically switches between the dead-beat controller of each system in order to steer each one of them to the origin. Along the same lines, Olbrot [10] proves that given any finite collection of continuous time or discrete time systems which are controllable in finite time to the origin, there exists a periodically time-varying feedback law that simultaneously steers each one of the systems of the family to the origin. Besides, discrete time LTV systems described by autoregressive moving average are considered in [6] and a procedure for the design of a simultaneous asymptotic stabilizer is proposed.

In the context of simultaneous stabilization of finite families of LTI systems by means of time-varying feedback, there exist three main results in the literature: In [7], Kabamba and Yang establish the simultaneous asymptotic stabilizability of such families by means of open loop periodically time-varying feedback. The controller that they find involves both the sampled output of the system and a periodic function of time. The resulting closed loop systems are therefore periodic linear systems whose asymptotic stability can be studied by means of Floquet Theory. On the other hand, Zhang and Blondel [14] propose a sufficient condition for the simultaneous asymptotic stabilizability of finite families of LTI systems by controllers based on LTI feedback laws together with zero-th order hold functions and samplers. While both of the aforementioned design procedures comprise a sampling scheme, Khargonekar et al. [8] adopt a method that does not involve any discretization strategy and prove that any finite family of stabilizable LTI systems can be simultaneously asymptotically stabilized by a periodic LTV controller which is piecewise continuous with respect to the time. Although a design procedure can be deduced from [8], neither the simultaneous stabilizer nor the rates of convergence of the closed-loop systems are explicit.

In Section 3, we consider finite families of LTI systems that can be asymptotically stabilized by means of LTI feedback. Given any such family, we establish the existence of a continuous time-varying feedback law that simultaneously globally exponentially stabilizes the family. Because our approach does not comprise a discretizing scheme, it should be compared to that used in Khargonekar et al. [8]. There are actually two main differences between the results derived in [8] by Khargonekar et al. and ours: First their controller is discontinuous and linear while ours is continuous and nonlinear. Secondly, we obtain explicit controllers as well as a lower bound on the exponential rate of convergence, while neither the controller nor the rates of convergence are explicit in [8].

In Section 4, we derive sufficient conditions for the simultaneous asymptotic stabilizability of infinite families of LTI systems. We do not know of any work that directly addresses the simultaneous stabilization of countably infinite families of LTI systems.

This paper is organized as follows: We review in section 2 a few definitions. The case of finite families is discussed in Section 3 while that of countably infinite families is presented in Section 4. In Section 5 we illustrate these results with some examples. Finally, Section 6 contains some technical results.
2 Definitions

In this section, we review a few definitions that will be needed in the sequel.

For $x$ in $\mathbb{R}^n$, we let $\|x\|$ denote its Euclidean norm, and for $r > 0$ and $x_0$ in $\mathbb{R}^n$ we let $B_r(x_0)$ denote the set $B_r(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$. Further, given an autonomous (resp. time-varying) dynamical system, we let $x(\cdot, x_0)$ (resp. $x(\cdot, x_0, t_0)$) denote its trajectory that starts at $x_0$ at time $t = 0$ (resp. $t = t_0$), for each $x_0$ in $\mathbb{R}^n$ (resp. for each $x_0$ in $\mathbb{R}^n$ and each $t_0 \geq 0$).

**Definition 2.1 (Stability of autonomous systems)** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping with $f(0) = 0$. The system $(S) : \dot{x} = f(x)$ is

i) stable if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $t \geq 0$ and each solution $x(\cdot, x_0)$ of $(S)$, we have $\|x(t, x_0)\| < \varepsilon$ whenever $\|x_0\| < \delta$.

ii) is locally asymptotically stable if it is stable and if there exists $\delta_0 > 0$ such that each solution $x(\cdot, x_0)$ of $(S)$ satisfies $x(t, x_0) \to 0$ as $t \to \infty$ whenever $\|x_0\| < \delta_0$.

iii) is globally asymptotically stable if (ii) holds for all $\delta_0 > 0$.

**Definition 2.2 (Stability of time-varying systems)** Let $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping such that $f(t, 0) = 0$ for each $t \geq 0$, and let $x(\cdot, x_0, t_0)$ denote the trajectory of the system $(S) : \dot{x} = f(t, x)$ for any given $x_0$ in $\mathbb{R}^n$ and $t_0 \geq 0$. The system $(S)$ is

i) stable, if for each $\varepsilon > 0$ and each $t_0 \geq 0$, there exists $\delta(\varepsilon, t_0) > 0$ such that

$$\|x(t, x_0, t_0)\| < \varepsilon, \quad t \geq t_0, \quad x_0 \in B_{\delta(\varepsilon, t_0)}(0).$$

ii) uniformly stable if (i) holds with $\delta$ independent of $t_0$.

iii) globally asymptotically stable, if it is stable according to (i) and for each $t_0 \geq 0$, we have

$$\lim_{t \to +\infty} x(t, x_0, t_0) = 0, \quad x_0 \in \mathbb{R}^n.$$

iv) globally uniformly asymptotically stable if it is uniformly stable according to (ii), and such that $x(t, x_0, t_0)$ converges uniformly in $x_0$ and $t_0$ to the origin as $t$ tends to $+\infty$, for each $x_0$ in $\mathbb{R}^n$ and each $t_0 \geq 0$.

v) globally exponentially stable if there exist some positive reals $\gamma$ and $L$ such that

$$\|x(t, x_0, t_0)\| \leq L\|x_0\|e^{-\gamma(t-t_0)}, \quad t \geq t_0,$$

for each $t_0 \geq 0$ and each $x_0$ in $\mathbb{R}^n$.

We now let $\mathcal{I}$ be a countable set (finite or infinite) and we let $\{S_i, i \in \mathcal{I}\}$ be a collection of control systems

$$S_i : \quad \dot{x} = f_i(x, u),$$

where the state $x$ is in $\mathbb{R}^n$, the input $u$ is in $\mathbb{R}^m$ and $f_i(0, 0) = 0$. 

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Definition 2.3 (Stabilization) Let \( i \) be in \( \mathcal{I} \).

i) A feedback law \( u : \mathbb{R}^n \to \mathbb{R}^m \) stabilizes (resp. asymptotically stabilizes) the system \( S_i \) if the closed-loop system \( \dot{x} = f_i(x, u(x)) \) is stable according to Definition 2.1 (i) (resp. locally asymptotically stable according to Definition 2.1 (ii)).

ii) A feedback law \( u : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m \) respectively globally asymptotically, globally uniformly asymptotically, and globally exponentially stabilizes \( S_i \) if the closed-loop system \( \dot{x} = f_i(x, u(t, x)) \) is respectively globally asymptotically, globally uniformly asymptotically, and globally exponentially stable according to Definition 2.2 (iii), (iv), and (v), respectively.

In the sequel, we will often omit the term “locally” and unless otherwise stated “asymptotically stable” will mean “locally asymptotically stable”.

Definition 2.4 (Simultaneous stabilization) A feedback law \( u \) simultaneously stabilizes (resp. asymptotically stabilizes) the family of control systems \( \{S_i, \ i \in \mathcal{I}\} \), if \( u \) stabilizes (resp. asymptotically stabilizes) each one of the system of the family, according to the appropriate definition.

Given a positive definite matrix \( M \), we let \( \lambda_{\min}(M) \) and \( \lambda_{\max}(M) \), denote respectively its smallest and largest eigenvalues. As usual, the infimum of a mapping over the empty set is taken to be equal to \( +\infty \). Further, we let \( \mathbb{R} \) and \( \mathbb{Z} \) denote the set of reals and integers respectively.

For a given integer \( k \geq 1 \) and two Banach spaces \( X \) and \( Y \), a mapping \( f : X \to Y \), is said to be \( C^k \) if it is \( k \) times continuously differentiable on \( X \). Further \( f \) is said to be \( C^0 \) if it is continuous on \( X \), while it is said to be “smooth” or equivalently \( C^\infty \) if it is \( C^k \) for all \( k \geq 0 \).

Assume that \( U \) is a neighborhood of the origin in \( \mathbb{R}^n \), and let \( k \geq 0 \) be an integer. A mapping \( f : U \to Y \) is said to almost \( C^k \) if it is \( C^k \) on \( U\setminus\{0\} \).

Because, we extensively use the concept of Lyapunov functions, we now define them.

Definition 2.5 (Lyapunov function) Let \( U \) be a neighborhood of the origin in \( \mathbb{R}^n \). A mapping \( V : U \to [0, \infty) \) is said to be a Lyapunov function if it is \( C^1 \) and if the following holds:

- \( V(x) = 0 \iff x = 0 \).
- There exists a continuous mapping \( f : U \to \mathbb{R}^n \) such that \( f(0) = 0 \) and \( \nabla V(x)f(x) < 0 \) for each \( x \) in \( U\setminus\{0\} \).

If in addition \( U = \mathbb{R}^n \) and \( V(x) \) converges to \( +\infty \) as \( ||x|| \) tends to \( +\infty \), the Lyapunov function \( V \) is said to be radially unbounded.
\(v(t,0) = 0\) for each \(t \geq 0\). This is a usual and natural requirement imposed on feedback laws that are used for stabilization purpose. Indeed, if the feedback law \(v\) stabilizes or asymptotically stabilizes a control system \(\dot{x} = f(x,u)\) satisfying \(f(0,0) = 0\), then the previous assumptions on \(v\) means that no control energy is necessary in order to maintain the corresponding closed-loop system at the origin once it has reached this state.

We finally introduce a notation that is extensively used throughout this paper.

**Definition 2.6** Let \(\{x_m, m = \ldots, -1, 0, 1, \ldots\}\) be a sequence of positive integers. Further, for each \(i = 1, \ldots, x_n\) and each \(n\) in \(\mathbb{Z}\), let \(Q_i^n\) belong to a given class of mathematical objects. Then, \(\{Q_i^n, i = 1, \ldots, x_n\}_{n=1}^{\infty}\), \(\{Q_i^n, i = 1, \ldots, x_n\}_{n=-1}^{-\infty}\), and \(\{Q_i^n, i = 1, \ldots, x_n\}_{n \in \mathbb{Z}}\) denote respectively the sequences

\[
Q_1^1, \ldots, Q_x^1, Q_2^0, \ldots, Q_{x+1}^2, Q_3^1, \ldots, Q_{x-3}^-2, Q_{x-2}^2, Q_{x-4}^-2, Q_{x-1}^-1, \ldots, Q_{x-1}^-1
\]

and

\[
\ldots, Q_1^-1, \ldots, Q_{x-1}^1, Q_0^0, Q_2^0, Q_1^1, \ldots, Q_{x-1}^1, \ldots
\]

We now look at the simultaneous asymptotic stabilization of finite families of LTI systems.

## 3 Finite families

Throughout this section, we consider a finite family \(\{S_i, i = 1, \ldots, I\}\) of linear systems

\[
S_i: \quad \dot{x} = A_i x + B_i u, \quad i = 1, \ldots, I,
\]

where \(I \geq 2\) is a positive integer, the state \(x\) lies in \(\mathbb{R}^n\), the input \(u\) is in \(\mathbb{R}^m\) and for each \(i = 1, \ldots, I\), the matrices \(A_i\) and \(B_i\) belong to \(\mathbb{R}^{n \times n}\) and \(\mathbb{R}^{n \times m}\) respectively. Finally, for each \(i = 1, \ldots, I\), we assume that there exists \(K_i\) in \(\mathbb{R}^{n \times m}\) such that the linear feedback law \(u_i: \mathbb{R}^n \to \mathbb{R}^m\) given by \(u_i(x) = K_i x\) asymptotically stabilizes \(S_i\).

Our goal is to prove the following theorem.

**Theorem 3.1** Assume that for each \(i = 1, \ldots, I\), there exists a linear feedback law \(u_i\) that asymptotically stabilizes the linear system \(S_i\). Then, there exists a time-varying feedback law \(v: [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n\), continuous on \([0, \infty) \times \mathbb{R}^n\), \(C^\infty\) on \([0, \infty) \times (\mathbb{R}^n \setminus \{0\})\), and which simultaneously globally exponentially stabilizes the family \(\{S_i, i = 1, \ldots, I\}\).

The general lines of the proof of this theorem are as follows: For each \(i = 1, \ldots, I\), we let \(V_i\) denote a Lyapunov function for the system \(\dot{x} = (A_i + B_i K_i)x\). We introduce a sequence \(\{b_i^*(t), i = 1, \ldots, I\}_{n \in \mathbb{Z}}\) of mappings defined from \([0, \infty)\) into \((0, +\infty)\), decreasing to 0 as \(t\) tends to +\(\infty\), and such that for each \(t \geq 0\), the sequence of neighborhoods \(\{V_i^{-1}([0, b_i^*(t))])\}_{n \in \mathbb{Z}}\) is a base at the origin. We then design a time-varying feedback law \(v(t,x)\) such that for each \(t \geq 0\) and each \(i = 1, \ldots, I\), we have
\( v(t, x) = u_i(x) \) for all \( x \) in \( V_i^{-1}(b^p_i(t)) \). Finally, we show that for each \( i = 1, \ldots, I \) and each \( t_0 \geq 0 \), each trajectory of the system \( \dot{x} = f_i(x, v(t, x)) \), that starts in the set \( V_i^{-1}([0, b^p_i(t_0)]) \) at time \( t = t_0 \), remains in the set \( V_i^{-1}([0, b^p_i(t)]) \) for all \( t \geq t_0 \). For each \( i = 1, \ldots, I \), we conclude that \( v \) asymptotically stabilizes the system \( S_i \), upon noting that the mapping \( b^p_i \) converges to 0 as \( t \) tends to \(+\infty\) for each \( n \) in \( \mathbb{Z} \).

We now present a technical lemma which is used to prove that the trajectory \( x(\cdot, x_0, t_0) \) of the system \( S_i \) lies in \( V_i^{-1}([0, b^p_i(t)]) \), for each \( t \geq t_0 \), whenever \( x_0 \) belongs to \( V_i^{-1}([0, b^p_i(t_0)]) \).

### 3.1 Invariance criteria

The following lemma is the key to prove the invariance of the set \( V_i([0, b^p_i(t)]) \) for each \( t \geq 0 \). It is also used in Section 4.

**Lemma 3.1** Let \( D \) be a bounded neighborhood of the origin in \( \mathbb{R}^n \) (resp. \( D = \mathbb{R}^n \)) and let \( V : \overline{D} \to (0, \infty) \) be a Lyapunov function (resp. a radially unbounded Lyapunov function). Further, let the mapping \( f : [0, \infty) \times \overline{D} \to \mathbb{R}^n \) be continuous, and let the mapping \( b : [0, \infty) \to (0, \inf_{x \in \partial D} V(x)) \) be \( C^1 \). Finally, for each \( \beta > 0 \) set

\[
W^\beta \triangleq \{ x \in D : V(x) < \beta \},
\]

and assume that

\[
\nabla V(x) f(t, x) < \dot{b}(t), \quad x \in \partial W^b(t), \quad t \geq 0. \tag{1}
\]

Then, for each \( t_0 \geq 0 \) and each \( x_0 \) in \( \overline{W}^b(t_0) \), the trajectory \( x(\cdot, x_0, t_0) \) of \( \dot{x} = f(t, x) \) starting from \( x_0 \) at time \( t_0 \) satisfies

\[
x(t, x_0, t_0) \in \overline{W}^b(t), \quad t \geq t_0.
\]

**Proof:** Fix \( t_0 \geq 0 \) and \( x_0 \) in \( \overline{W}^b(t_0) \), and let \( x(\cdot, x_0, t_0) \) denote the trajectory of \( \dot{x} = f(t, x) \) that starts from \( x_0 \) at time \( t_0 \).

For the sake of clarity, we split the proof into that of two claims.

**Claim 1:** Let \( t_3 > t_0 \) be such that \( x(t, x_0, t_0) \) lies in \( D \) for each \( t \) in \( [t_0, t_3] \). Then, \( V(x(t, x_0, t_0)) \leq b(t), \ t \in [t_0, t_3] \)

Assume that Claim 1 does not hold. Then, there exists \( t_2 \) in \( [t_0, t_3] \) such that

\[
V(x(t_2, x_0, t_0)) > b(t_2)
\]

Because \( V(x(t_0, x_0, t_0)) - b(t_0) \leq 0 \), continuity of the mapping \( V(x(\cdot, x_0, t_0)) - b(\cdot) \) yields the existence of \( t_1 \) in \( [t_0, t_2] \) and \( h_1 \) in \( (0, t_3 - t_1) \) such that

\[
V(x(t_1, x_0, t_0)) - b(t_1) = 0, \tag{2}
\]

and

\[
V(x(t_1 + h, x_0, t_0)) - b(t_1 + h) > 0, \quad h \in (0, h_1). \tag{3}
\]
By assumption $x(t_1, x_0, t_0)$ belongs to $D$, and we obtain from (2) that

$$x(t_1, x_0, t_0) \in \partial W^{b(t_1)}.$$  

(4)

This, together with Assumption (1) yields

$$\frac{d}{dt}_{t=t_1} V(x(t, x_0, t_0)) = \nabla V(x(t_1, x_0, t_0)) f(t_1, x(t_1, x_0, t_0)) < \dot{b}(t_1),$$

and continuity of the mappings $\nabla V(\cdot), f(\cdot, \cdot)$ and $\dot{b}(\cdot)$, combined with (2) yields

$$V(x(t_1 + h, x_0, t_0)) < b(t_1 + h), \quad \text{for} \ h > 0 \ \text{small enough},$$

a contradiction with (3). The proof of Claim 1 is thus complete.

**Claim 2:** $x(t, x_0, t_0)$ lies in $D$, for each $t \geq t_0$.

Because Claim 2 clearly holds if $D = \mathbb{R}^n$, we assume that $D$ is bounded.

Suppose that the claim does not hold. Because $x_0$ is in $D$, Lemma 6.5 (iii) yields the existence of $t_1 > t_0$ such that

$$x(t_1, x_0, t_0) \in \partial D,$$

(5)

with

$$x(t, x_0, t_0) \in D, \quad t \in [t_0, t_1).$$

(6)

This last relation combined with Claim 1, implies that

$$V(x(t, x_0, t_0)) \leq b(t), \quad t \in [t_0, t_1).$$

(7)

On the other hand, from (5) and the definition of $b$, we get

$$V(x(t_1, x_0, t_0)) \geq \inf_{x \in \partial D} V(x) > b(t_1),$$

and continuity of the mapping $V(x(\cdot, x_0, t_0)) - b(\cdot)$ at $t_1$ yields the existence of $h_1$ in $(0, t_1 - t_0)$ satisfying

$$V(x(t_1 - h, x_0, t_0)) > b(t_1 - h), \quad h \in (0, h_1),$$

which contradicts (7). Hence Claim 2.

Let $t_3 > t_0$. By Claim 2, the point $x(t, x_0, t_0)$ lies in $D$ for each $t \geq t_0$, and it follows from Claim 1 applied with $t_3$ that

$$V(x(t, x_0, t_0)) \leq b(t), \quad t \in [t_0, t_3).$$

The proof of the lemma is completed upon noting that this last argument holds for all $t_3 > t_0$, all $x_0$ in $\overline{W}^{b(t_0)}$ and all $t_0 \geq 0$.

We are now able to prove Theorem 3.1.

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3.2 A proof of Theorem 3.1

Proof of Theorem 3.1:

For each $i = 1, \ldots, I$, because the system $\dot{x} = A_i x + B_i u_i(x)$ is linear and asymptotically stable, it admits a Lyapunov function $V_i : \mathbb{R}^n \to (0, \infty)$ such that $V_i(x) = x^t P_i x$ where $P_i$ is a positive definite matrix. For each $i = 1, \ldots, I$, let $Q_i$ be the positive definite matrix defined by

$$\nabla V_i(x) (A_i x + B_i u_i(x)) = -x^t Q_i x, \quad x \in \mathbb{R}^n.$$ 

For each $i = 1, \ldots, I$ and each $n$ in $\mathbb{Z}$, we let $W_i^\beta$ denote the set

$$W_i^\beta \triangleq V_i^{-1}([0, \beta]).$$

For each $i = 1, \ldots, I$, let $\theta_i$ and $\pi_i$ satisfy the assumptions of Lemma 6.1. Further let \(\{\alpha_i^n, i = 1, \ldots, I\}_{n \in \mathbb{Z}}, \{\beta_i^n, i = 1, \ldots, I\}_{n \in \mathbb{Z}}\) and \(\{\gamma_i^n, i = 1, \ldots, I\}_{n \in \mathbb{Z}}\) be defined by the formulas (29)-(32) given in the statement of Lemma 6.1. Thus, by Lemma 6.1, for each $n$ in $\mathbb{Z}$ we have

\begin{align}
V_i^{-1}([0, \alpha_i^{n-1}]) & \supset V_i^{-1}([0, \gamma_i^n]), \quad i = 2, \ldots, I, \\
V_i^{-1}([0, \alpha_i^n]) & \supset V_i^{-1}([0, \gamma_i^{n+1}]),
\end{align}

and for each $i = 1, \ldots, I$ we have

$$\beta_i^n \to 0 \text{ as } n \to \infty \quad \text{with} \quad \beta_i^n \to \infty \text{ as } n \to -\infty.$$ 

Construction of the simultaneous stabilizer:

We now seek a $C^1$ mapping $h : [0, \infty) \to (0, \infty)$ such that, the mapping $b_i^n : [0, \infty) \to (0, \infty)$ given by

$$b_i^n(t) = \beta_i^n h(t), \quad t \geq 0,$$

satisfy

$$\nabla V_i(x) (A_i x + B_i u_i(x)) < \dot{b}_i^n(t), \quad x \in V_i^{-1}(b_i^n(t)), \quad t \geq 0,$$

or equivalently

$$-x^t Q_i x < \dot{b}_i^n(t), \quad x \in V_i^{-1}(b_i^n(t)), \quad t \geq 0,$$

for each $i = 1, \ldots, I$ and each $n$ in $\mathbb{Z}$. In what follows, we fix $i = 1, \ldots, I, n$ in $\mathbb{Z}$ and $t \geq 0$. Let $x$ be such that $x^t P_i x = b_i^n(t)$. Then, by elementary linear algebra, we get

$$-x^t x \leq -\frac{b_i^n(t)}{\lambda_{\max}(P_i)}$$

and because we also have $-x^t Q_i x \leq -\lambda_{\min}(Q_i) x^t x$, inequality (11) will be satisfied if

$$-\frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)} b_i^n(t) < \dot{b}_i^n(t).$$

(12)
Because we require that \( h(t) > 0 \) and \( b^n_i(t) = \beta^n_i h(t) \), inequality (12) will hold if

\[
\frac{\dot{h}(t)}{h(t)} > -\frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)} \cdot (13)
\]

We now set \( \rho \triangleq \min_{i=1,...,I} \left( \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)} \right) \) and we deduce from (13) that the desired assertion (10) will be satisfied for each \( i = 1, \ldots, I \) and each \( n \) in \( \mathbb{Z} \), if

\[
\frac{\dot{h}(t)}{h(t)} > -\rho, \quad t \geq 0.
\]

Let the mapping \( h : [0, \infty) \to (0, \infty) \) be given by

\[
h(t) = e^{-\frac{\eta}{\beta} t}, \quad t \geq 0,
\]

where \( \eta \) is a fixed constant in \((1, \infty)\). It is plain that \( h \) satisfies (14) so that for each \( i = 1, \ldots, I \) and each \( n \) in \( \mathbb{Z} \), the mapping \( b^n_i : [0, \infty) \to (0, \infty) \) defined by

\[
b^n_i(t) = \beta^n_i h(t) = \beta^n_i e^{-\frac{\eta}{\beta} t}, \quad t \geq 0,
\]

satisfies the desired assertion (10).

Next, for each \( i = 1, \ldots, I \) and each \( n \) in \( \mathbb{Z} \), we define the mappings \( a^n_i, c^n_i : [0, \infty) \to (0, \infty) \) by setting

\[
c^n_i(t) = \gamma^n_i h(t) \quad \text{and} \quad a^n_i(t) = \alpha^n_i h(t), \quad t \geq 0.
\]

For each \( t \geq 0 \), because we have

\[
b^n_i(t) = \theta_i c^n_i(t) \quad \text{with} \quad a^n_i(t) = \theta_i b^n_i(t), \quad i = 1, \ldots, I, \quad n \in \mathbb{Z},
\]

and

\[
c^{n+1}_i(t) = \pi_1 a^n_i(t) \quad \text{with} \quad c^n_i(t) = \pi_i a^n_{i-1}(t), \quad i = 2, \ldots, I, \quad n \in \mathbb{Z},
\]

it is easily checked that the sequences \( \{c^n_i(t), i = 1, \ldots, I\}_{n \in \mathbb{Z}}, \{b^n_i(t), i = 1, \ldots, I\}_{n \in \mathbb{Z}} \) and \( \{a^n_i(t), i = 1, \ldots, I\}_{n \in \mathbb{Z}} \) satisfy the assertions of Lemma 6.1. This together with the fact that \( h(\cdot) \) decreases to 0 as \( t \) tends to \( \infty \), yield for each \( t_0 \geq 0 \)

\[
\sup_{t \geq t_0} b^n_i(t) \to 0 \quad \text{as} \quad n \to +\infty, \quad i = 1, \ldots, I, \quad (15)
\]

\[
b^n_i(t_0) \to +\infty \quad \text{as} \quad n \to -\infty, \quad i = 1, \ldots, I. \quad (16)
\]

It also follows from Lemma 6.1 that for each \( t \geq 0 \), we have a double-sided sequence of neighborhoods

\[
W^n_i(t) \supset W^n_i(t) \supset \cdots \supset W^n_i(t) \supset W^n_i(t) \supset W^n_i(t) \supset \cdots \supset W^n_i(t) \supset \cdots \supset W^n_i(t) \supset \cdots.
\]

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such that each neighborhood contains the closure of the neighborhood that follows. In view of this comment, for each \( i = 1, \ldots, I \) and each \( n \) in \( \mathbb{Z} \), we define the mapping \( q_i^n : [0, \infty) \times \mathbb{R}^n \to [0, 1] \) by setting

\[
q_i^n(t, x) = \begin{cases} 
\frac{(V_i(x)-b_i^n(t))^2}{(V_i(x)-b_i^n(t))^2-(a_i^n(t)-b_i^n(t))^2} & \text{if } V_i(x) \in (a_i^n(t), b_i^n(t)] \\
\frac{(V_i(x)-b_i^n(t))^2}{(V_i(x)-b_i^n(t))^2-(c_i^n(t)-b_i^n(t))^2} & \text{if } V_i(x) \in (b_i^n(t), c_i^n(t)) \\
0 & \text{otherwise}
\end{cases}
\]

and we let the mapping \( v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m \) be given by

\[
v(t, x) = \sum_{i=1}^I \sum_{n \in \mathbb{Z}} u_i(x) q_i^n(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^n.
\]

We note that \( v(t, 0) = 0 \) for each \( t \geq 0 \) and we show that \( v \) is continuous on \([0, \infty) \times \mathbb{R}^n\) and \( C^\infty \) on \([0, \infty) \times (\mathbb{R}^n \setminus \{0\})\).

The feedback law \( v \) is continuous on \([0, \infty) \times \mathbb{R}^n\) and \( C^\infty \) on \([0, \infty) \times (\mathbb{R}^n \setminus \{0\})\):

Let \((t, x)\) be in \([0, \infty) \times \mathbb{R}^n \setminus \{0\}\). It is easily checked from (17) that there exists a unique pair of integer \((i, n)\) in \([1, \ldots, I] \times \mathbb{Z}\) such that either one of the following two assertions holds:

- We have \( V_i(x) \in [a_i^n(t), c_i^n(t)] \). In that case (17) together with the continuity of the mappings \( V_j, a_j^n \) and \( c_j^n \) for each \( j = 1, \ldots, I \) and each \( m \) in \( \mathbb{Z} \), yield the existence of a neighborhood \( U \) of \((t, x)\) in \([0, \infty) \times (\mathbb{R}^n \setminus \{0\})\) such that

\[
v(\tau, y) = u_i(y) q_i^n(\tau, y), \quad (\tau, y) \in U.
\]

- We have \( V_i(x) \in (a_i^n(t), c(t)) \) where the mapping \( c \) denotes either \( c_i^{n+1} \) if \( i = I \) or otherwise \( c_i^{n+1} \). In that case the continuity of the mappings \( V_i, a_j^n, c_j^n \) for each \( j = 1, \ldots, I \) and each \( m = 1, 2, \ldots \), yields the existence a neighborhood \( U \) of \((t, x)\) in \([0, \infty) \times (\mathbb{R}^n \setminus \{0\})\) such that \( V_i(y) \in (a_i^n(\tau), c(\tau)) \), \((\tau, y) \in U\) and it follows that

\[
v(\tau, y) = 0, \quad (\tau, y) \in U.
\]

Because \( q_i^n \) is \( C^\infty \) on \([0, \infty) \times \mathbb{R}^n\) [follows from Lemma 6.8] and \( u_i \) is \( C^\infty \) on \( \mathbb{R}^n \) for each \( i = 1, \ldots, I \) and each \( n \) in \( \mathbb{Z} \), (18) and (19) imply that \( v \) is \( C^\infty \) on \([0, \infty) \times (\mathbb{R}^n \setminus \{0\})\).

Moreover, because the mappings \( q_i^n \) take values in \([0, 1]\), the equalities (18) and (19) together with the fact that \( v(t, 0) = 0 \), \( t \geq 0 \) yield

\[
\|v(t, x)\| \leq \max \{\|u_1(x)\|, \ldots, \|u_I(x)\|\}, \quad (t, x) \in [0, \infty) \times \mathbb{R}^n,
\]

and continuity of \( u_i \) for each \( i = 1, \ldots, I \), implies that \( v \) is continuous at each point \((t, 0), \ t \geq 0\). Therefore, \( v \) is continuous on \([0, \infty) \times \mathbb{R}^n\).

Global exponential stability:
Throughout the rest of the proof, we fix $i = 1, \ldots, I$. From the definition of $v$, it is not hard to see that
\[
v(t, x) = u_i(x), \quad t \geq 0, \quad x \in V_{i}^{-1}(b^n_i(t)), \quad n \in \mathbb{Z}.
\]
Therefore, from (10) we get
\[
\nabla v_i(x) \left(A_i x + B_i v(t, x) \right) < \dot{b}_i^n(t), \quad x \in V_{i}^{-1}(b^n_i(t)), \quad t \geq 0, \quad n \in \mathbb{Z},
\]
and upon recalling that $\partial \mathcal{W}_i = V_{i}^{-1}(\beta)$ and $\overline{\mathcal{W}}_i = V_{i}^{-1}([0, \beta])$ for each $\beta > 0$ [follows from Lemma 6.6 (i)], Lemma 3.1 implies that for each $t_0 \geq 0$ and each $n$ in $\mathbb{Z}$, the trajectory $x(\cdot, x_0, t_0)$ of $\dot{x} = A_i x + B_i v(t, x)$ starting from $x_0$ at time $t = t_0$, satisfies
\[
V_i(x(t, x_0, t_0)) \leq b^n_i(t), \quad t \geq t_0, \quad x_0 \in \overline{V}_i b^n_i(t_0).
\]
(20)
Recall that from the definition of the sequence $\{\beta_i^n\}_{n \in \mathbb{Z}}$, we have
\[
\beta_i^{n+k} = (\pi_1 \cdots \pi_I \theta_i^2 \cdots \theta_i^2)^k \beta_i^n, \quad n, k \in \mathbb{Z},
\]
Upon setting $T = \frac{-\log(\pi_1 \cdots \pi_I \theta_i^2 \cdots \theta_i^2)}{\rho}$, the previous equality translates to
\[
\beta_i^n e^{-\frac{\rho}{\pi} k T} = \beta_i^{n+k}, \quad n, k \in \mathbb{Z},
\]
(21)
which in turn easily yields
\[
b_i^n(t+kT) = b_i^{n+k}(t), \quad t \geq 0, \quad n \in \mathbb{Z}, \quad k \in \mathbb{Z},
\]
(22)
Let $x_0$ be in $\mathbb{R}^n$. Because the sequence $\{b_i^n(0) = \beta_i^n\}_{n \in \mathbb{Z}}$ is strictly decreasing and converges to 0 and $+\infty$ as $n$ tends to $+\infty$ and $-\infty$ respectively, there exists an integer $\bar{n}$ such that
\[
b_i^{n+1}(0) < V_i(x_0) \leq b_i^n(0).
\]
(23)
Let $t_0$ be in $[0, \infty)$. Then, there exist an integer $k$ and $t'_0$ in $[0, T)$ such that $t_0 = kT + t'_0$. By combining (22) with the fact that the mapping $b_i^n$ is decreasing for each $n$ in $\mathbb{Z}$, we get
\[
b_i^n(0) = b_i^{n-1}((k+1)T) \leq b_i^{n-1}(t_0),
\]
so that (23) yields $x_0 \in V_i^{-1}([0, b_i^{n-1}(t_0)])$. Thus, assertion (20), implies that
\[
V_i(x(t, x_0, t_0)) \leq b_i^{n-1}(t), \quad t \geq t_0,
\]
and from the expression of $b_i^{n-1}(t)$ we obtain that
\[
V_i(x(t, x_0, t_0)) \leq \beta_i^{n-k-1} e^{-\frac{\rho}{\pi} (t-t_0+t'_0+kT)}, \quad t \geq t_0.
\]
Next, using (21), we can rewrite this last inequality as
\[
V_i(x(t, x_0, t_0)) \leq \beta_i^{n-1} e^{-\frac{\rho}{\pi} (t-t_0+t'_0)}, \quad t \geq t_0,
\]
\[
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\]
and from the non-negativeness of $t_0$, we get

$$V_i(x(t, x_0, t_0)) \leq \beta_i^{n-1} e^{-\frac{\epsilon}{n}(t-t_0)} , \quad t \geq t_0. \quad (24)$$

The identity (21) yields $\beta_i^{n-1} = e^{\frac{\epsilon}{n}T_i} \beta_i^{n+1}$, so that the inequality $\beta_i^{n-1} < e^{\frac{\epsilon}{n}T} V_i(x_0)$ follows from (23). Thus, (24) implies that

$$V_i(x(t, x_0, t_0)) \leq \left( e^{\frac{\epsilon}{n}T} V_i(x_0) \right) e^{-\frac{\epsilon}{n}(t-t_0)} , \quad t \geq t_0,$$

or equivalently

$$\sqrt{V_i(x(t, x_0, t_0))} \leq e^{\frac{\epsilon}{n}T} \sqrt{V_i(x_0)} e^{-\frac{\epsilon}{n}(t-t_0)} , \quad t \geq t_0.$$

Because $e^{\frac{\epsilon}{n}T}$ is a constant and the mapping $x \mapsto \sqrt{V_i(x)}$ is a norm on $\mathbb{R}^n$, we obtain from the equivalence of all norms on $\mathbb{R}^n$ that $v$ globally exponentially stabilizes $S_i$ [according to Definition 2.2 (v)]. The proof of the theorem is complete upon noting that the previous argument holds for each $i = 1, \ldots, I$.

We note the rates of convergence of the closed-loop systems corresponding to $S_i$ is greater than $\frac{\rho}{2\eta^n}$ for each $i = 1, \ldots, I$.

We now extend this result and we establish the simultaneous asymptotic stabilizability of a class of countably infinite family of stabilizable LTI systems.

4 Infinite families

Throughout this section, we consider a countably infinite family $\{S_i, i = 1, 2, \ldots\}$ of linear systems

$$S_i : \quad \dot{x} = A_i x + B_i u , \quad i = 1, 2, \ldots ,$$

where the state $x$ lies in $\mathbb{R}^n$, the input $u$ is in $\mathbb{R}^m$ and for each $i = 1, 2, \ldots$, the matrices $A_i$ and $B_i$ belong to $\mathbb{R}^{n \times n}$ and $\mathbb{R}^{n \times m}$ respectively. We assume that for each $i = 1, 2, \ldots$, there exists $K_i$ in $\mathbb{R}^{m \times n}$ such that the linear feedback $u_i : \mathbb{R}^n \to \mathbb{R}^m$ given by $u_i(x) = K_i x$ for each $x$ in $\mathbb{R}^n$, asymptotically stabilizes $S_i$. For each $i = 1, 2, \ldots$, we let $P_i$ be a positive definite matrix in $\mathbb{R}^{n \times n}$ and we let $V_i : \mathbb{R}^n \to [0, \infty)$ be a Lyapunov function for the system $\dot{x} = (A_i + B_i K_i) x$ given by $V_i(x) = x^T P_i x$ for each $x$ in $\mathbb{R}^n$. Further, we let $Q_i$ be the positive definite matrix defined by

$$\nabla V_i(x) ((A_i + B_i K_i) x) = -x^T Q_i x , \quad x \in \mathbb{R}^n .$$

The purpose of this section is to prove the following theorem. The proof is based on the same ideas as those of the proof of Theorem 3.1. The main difference lies in the structure of the sequence of mappings $\{b_i^n\}$ that must be considered here.
Theorem 4.1 Assume that, in addition to the enforced assumptions, the following holds:

i) There exists a positive real \( M \) such that \( \| K_i \| \leq M, \ i = 1, 2, \ldots \)

ii) The real \( \rho \triangleq \inf_{i=1, 2, \ldots} \left( \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)} \right) \) is strictly positive.

Then, there exists a time-varying feedback law \( v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m, \) continuous on \([0, \infty) \times \mathbb{R}^n, \ C^\infty \) on \([0, \infty) \times (\mathbb{R}^n \setminus \{0\}), \) and which simultaneously globally asymptotically stabilizes the family \( \{ S_i, i = 1, 2, \ldots \} \).

Proof:
Throughout, we let \( W_i^\beta \) denote the set

\[ W_i^\beta \triangleq V_i^{-1}([0, \beta)), \quad \beta > 0, \quad i = 1, 2, \ldots \]

Let \( \tilde{\gamma}_i^1 \) be a given positive real. By applying Lemma 6.2 with \( \tilde{\gamma}_i^1 \) and the family \( \{ V_i, i = 1, 2, \ldots \} \), we obtain three sequence of positive reals \( \{ \gamma_i^n, i = 1, \ldots, n \}_{n=1}^\infty, \quad \{ \beta_i^n, i = 1, \ldots, n \}_{n=1}^\infty \) and \( \{ \alpha_i^n, i = 1, \ldots, n \}_{n=1}^\infty \) converging to 0 as \( n \) tends to \( \infty \) and satisfying the assertions of the lemma.

Now by applying Lemma 6.3 with \( \gamma_i^n \) as defined in the sequence \( \{ \gamma_i^n, i = 1, \ldots, n \}_{n=1}^\infty, \) we obtain three sequences \( \{ \gamma_i^n, i = 1, \ldots, |n| \}_{n=1}^{-\infty}, \quad \{ \beta_i^n, i = 1, \ldots, |n| \}_{n=1}^{-\infty} \) and \( \{ \alpha_i^n, i = 1, \ldots, |n| \}_{n=1}^{-\infty} \) converging to \( +\infty \) as \( n \) tends to \( -\infty \) and satisfying the assertions of the lemma.

Let \( \rho \triangleq \inf_{i=1, 2, \ldots} \left( \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)} \right) \) and let \( \zeta > 1 \). We note that by Assumption (i) of Theorem 4.1, the real \( \rho \) is positive and we let the mapping \( h : [0, \infty) \to (0, \infty) \) be given by

\[ h(t) = \frac{\rho}{\zeta}, \quad t \geq 0. \]

For each \( i = 1, 2, \ldots \) and each \( n \) in \( \mathbb{Z} \setminus \{0\} \), we now define the mappings \( a_i^n, b_i^n, c_i^n : [0, \infty) \to (0, \infty) \) by setting

\[ a_i^n(t) \triangleq \alpha_i^n h(t), \quad b_i^n(t) \triangleq \beta_i^n h(t) \quad \text{and} \quad c_i^n(t) \triangleq \gamma_i^n h(t), \quad t \geq 0. \]

It is not hard to see from Lemmas 6.2 and 6.3 that for each \( t \geq 0 \) we have

\[ a_i^{-1}(t) > c_i^1(t) \quad \text{and} \quad a_i^{-n-1}(t) > \frac{M_{n-1}}{m_{n-1}} c_i^n(t), \quad n = \ldots, -2, -1, 2, 3, \ldots \]

together with

\[ a_i^n(t) > \frac{M_{n+1}}{m_i} c_{i+1}^n(t), \quad i = 1, \ldots, |n| - 1, \quad n = \ldots, -3, -2, 2, 3, \ldots \]

and

\[ c_i^s(t) > b_i^s(t) > a_i^n(t), \quad i = 1, \ldots, |n|, \quad n \in \mathbb{Z} \setminus \{0\}. \]
Thus, for each $t \geq 0$, we have a sequence of nested neighborhoods
\[
\begin{align*}
W_{1}^{c_{1}^{-2}(t)} & : \ \\
W_{1}^{c_{1}^{-2}(t)} & \supset W_{1}^{k_{1}^{-2}(t)} \supset W_{1}^{g_{1}^{-2}(t)} \supset W_{2}^{c_{2}^{-2}(t)} \supset W_{2}^{b_{2}^{-2}(t)} \supset W_{2}^{a_{2}^{-2}(t)} \supset \ \\
W_{1}^{c_{1}^{-1}(t)} & \supset W_{1}^{k_{1}^{-1}(t)} \supset W_{1}^{g_{1}^{-1}(t)} \supset W_{2}^{c_{2}^{-1}(t)} \supset W_{2}^{b_{2}^{-1}(t)} \supset W_{2}^{a_{2}^{-1}(t)} \supset \ \\
W_{1}^{c_{1}(t)} & \supset W_{1}^{k_{1}(t)} \supset W_{1}^{g_{1}(t)} \supset W_{2}^{c_{2}(t)} \supset W_{2}^{b_{2}(t)} \supset W_{2}^{a_{2}(t)} \supset \ \\
W_{1}^{c_{2}(t)} & \supset \ \\
W_{1}^{c_{2}(t)} & : \ \\
\end{align*}
\]

such that each neighborhood contains the closure of the neighborhood that follows. Moreover, for each $i = 1, 2, \ldots$, because $\alpha_{i}^{n}$, $\beta_{i}^{n}$ and $\gamma_{i}^{n}$ tend to 0 and $+\infty$ as $n$ tends respectively to $+\infty$ and $-\infty$, we obtain
\[
\begin{align*}
a_{i}^{n}(t), b_{i}^{n}(t), c_{i}^{n}(t) & \to 0 \ as \ n \to +\infty, \ t \geq 0, \\
a_{i}^{n}(t), b_{i}^{n}(t), c_{i}^{n}(t) & \to +\infty \ as \ n \to -\infty, \ t \geq 0.
\end{align*}
\]

In view of (25), we define the mapping $q_{i}^{n} : [0, \infty) \times \mathbb{R}^{n} \to [0, 1]$ by setting
\[
q_{i}^{n}(t, x) = \begin{cases} 
\frac{(V_{i}(x) - b_{i}^{n}(t))^{2}}{e^{(V_{i}(x) - b_{i}^{n}(t))^{2} - (b_{i}^{n}(t) - c_{i}^{n}(t))^{2}}} & \text{if } V_{i}(x) \in (a_{i}^{n}(t), b_{i}^{n}(t)) \ \\
\frac{(V_{i}(x) - b_{i}^{n}(t))^{2}}{e^{(V_{i}(x) - b_{i}^{n}(t))^{2} - (b_{i}^{n}(t) - c_{i}^{n}(t))^{2}}} & \text{if } V_{i}(x) \in (b_{i}^{n}(t), c_{i}^{n}(t)) \\
0, & \text{otherwise}
\end{cases}
\]

for each $i = 1, 2, \ldots$ and each $n$ in $\mathbb{Z} \setminus \{0\}$. Finally, we let the mapping $v : [0, \infty) \times \mathbb{R}^{n} \to \mathbb{R}^{m}$ be given by
\[
v(t, x) = \sum_{n=1}^{+\infty} \sum_{i=1}^{n} u_{i}(x) q_{i}^{n}(t, x) + \sum_{n=-1}^{-\infty} \sum_{i=1}^{n} u_{i}(x) q_{i}^{n}(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^{n}.
\]

Because the supports of the mappings of the collection $\{q_{i}^{n}, i = 1, \ldots, n\}_{n \in \mathbb{Z} \setminus \{0\}}$ are disjoint (follows from (25)), we have
\[
\|v(t, x)\| \leq \inf_{i=1,2,\ldots} \|u_{i}(x)\| \leq \left( \inf_{i=1,2,\ldots} \|K_{i}\| \right) \|x\|, \quad (t, x) \in [0, \infty) \times \mathbb{R}^{n},
\]
so that Assumption (ii) of Theorem 4.1 yields the continuity of $v$ at any point $(t, 0), \ t \geq 0$.

Further, in view of (25) and by using exactly the same argument as that used in the proof of Theorem 3.1 to show that $v$ is $C^{\infty}$ on $[0, \infty) \times (\mathbb{R}^{n} \setminus \{0\})$, we obtain that $v$ (as defined here) is $C^{\infty}$ on $[0, \infty) \times (\mathbb{R}^{n} \setminus \{0\})$.

**Stability :**

We now fix $i = 1, 2, \ldots$ and $n$ in $\mathbb{Z} \setminus \{0\}$. Recall that by definition, we have
\[
b_{i}^{n}(t) = \beta_{i}^{n} e^{-\xi_{i}t}, \quad t \geq 0,
\]

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with \( \zeta > 1 \) and \( \rho = \inf_{i=1,2,...} \left( \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)} \right) \). Using the inequalities

\[
x^tQ_ix \geq \lambda_{\min}(Q_i)x^tx \quad \text{and} \quad x^tP_ix \leq \lambda_{\max}(P_i)x^tx,
\]

together with an argument similar to that used in the proof of Theorem 3.1 to construct the mapping \( h \), we obtain

\[
\nabla V_i(x) (A_i x + B_i u_i(x)) < b^n_i(t), \quad x \in V_i^{-1}(b^n_i(t)), \quad t \geq 0,
\]

and because \( v(t, x) = u_i(x) \) whenever \( x \in V_i^{-1}(b^n_i(t)) \), we get

\[
\nabla V_i(x) (A_i x + B_i v(t, x)) < b^n_i(t), \quad x \in V_i^{-1}(b^n_i(t)), \quad t \geq 0. \tag{27}
\]

Next, we fix \( i = 1, 2, \ldots \) and we show that \( v \) stabilizes \( S_i \): Let \( \varepsilon > 0 \) and \( t_0 \geq 0 \) be given. Because \( b^n_i(t_0) \) converges to 0 as \( n \) tends to \( +\infty \), there exists an integer \( n \) such that \( \overline{W}_i^{b^n_i(t_0)} \subset B_\varepsilon(0) \). Let \( \delta > 0 \) be such that \( B_\delta(0) \subset \overline{W}_i^{b^n_i(t_0)} \) and let \( x_0 \) be in \( B_\delta(0) \). In view of (27) and Lemma 3.1, for each \( t \geq t_0 \), the trajectory \( x(t, x_0, t_0) \) of \( \dot{x} = A_i x + B_i v(t, x) \) lies in the set \( \overline{W}_i^{b^n_i(t)} \). Thus, because \( b^n_i(\cdot) \) is a decreasing function of time, it follows that this trajectory remains in \( B_\varepsilon(0) \). In short \( v \) stabilizes \( S_i \).

**Convergence to the origin:**

First, we fix \( i = 1, 2, \ldots \). Let \( x_0 \) be in \( \mathbb{R}^n \setminus \{0\} \) and let \( t_0 \geq 0 \). As \( b^n_i(t_0) \) converges to \( +\infty \) as \( n \) tends to \( -\infty \), there exists an integer \( n \) such that \( x_0 \in \overline{W}_i^{b^n_i(t_0)} \), so that (27) yields

\[
x(t, x_0, t_0) \in \overline{W}_i^{b^n_i(t)} \quad \text{i.e.} \quad V_i(x(t, x_0, t_0)) \leq b^n_i(t), \quad t \geq 0.
\]

Thus, it follows from the convergence to 0 of the mapping \( b^n_i(t) \) as \( t \) tends to \( +\infty \) together with the positive definiteness of \( V_i \), that \( x(t, x_0, t_0) \) converges to 0 as \( t \) tends to \( +\infty \). The proof of Theorem 4.1 is complete upon noting that the previous argument holds for each \( i = 1, 2, \ldots . \)

We stress that in order to extend to countably infinite families of LTI systems, the method introduced in Theorem 3.1 for finite families of LTI systems, two additional assumptions are needed: Assumption (i) of Theorem 4.1 ensures that the simultaneous stabilizer is continuous at any point \( (t, 0) \), \( t \geq 0 \), while Assumption (ii) is necessary for the construction of the sequence of mappings \( \{b^n_i\} \).

To the contrary of the finite case, our construction does not yield uniform stability and uniform asymptotic stability in general. On the other hand, the stabilizing feedback law \( v \) that we obtain through the previous construction depends on the sequences \( \{\pi_i\}_{i=1}^{+\infty}, \{k_i\}_{i=1}^{+\infty}, \{\eta_i\}_{i=1}^{+\infty} \) and \( \{\tau_i\}_{i=1}^{+\infty} \). Because these sequences are not uniquely defined we may consider that they are design parameters. It would be interesting to find what conditions should be imposed on these design parameters and on the systems \( S_i, i = 1, 2, \ldots \), in order that the system \( \dot{x} = f_i(x, v(t, x)) \) be uniformly asymptotically stable for each \( i = 1, 2, \ldots \). This is an issue that we do not investigate in the context of this paper and that we leave for further research.
5 Examples

We now present a few examples that illustrate Theorem 3.1 and 4.1.

In the following example, we consider a linear system in the plane $S$ that is asymptotically stabilizable by a linear feedback law. Given a Lyapunov function for the corresponding closed-loop system, we follow the construction of the proof of Theorem 3.1 and we produce a mapping $b : [0, \infty) \to (0, \infty)$ and a feedback law $v : [0, \infty) \times \mathbb{R}^2$ such the following holds: For each $t_0 \geq 0$ and each $x_0$ such that $V(x_0) \leq b(t_0)$, the trajectory of $S$ satisfies $V(x(t, x_0, t_0)) \leq b(t)$ for each $t \geq t_0$.

Example 1:

We consider the system

$$S : \begin{cases} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= x_1 + \frac{u}{2} \end{cases},$$

where $u$ is a scalar input and we let $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ denote the vector-field of $S$. Further, we let $P$ and $Q$ denote the matrices

$$P = \begin{pmatrix} 1.5 \\ -0.5 \\ 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$ 

It is easily checked that the feedback law $u : \mathbb{R}^2 \to \mathbb{R}$ given by $u(x) = -2x_2$, asymptotically stabilizes the system $S$. Moreover, the mapping $V : \mathbb{R}^2 \to [0, \infty)$ defined by setting $V(x) = x^t P x$ for each $x$ in $\mathbb{R}^n$ is a Lyapunov function for the corresponding closed-loop system $S$ and we have $\nabla V(x) f(x, u(x)) = -x^t Q x$ for each $x$ in $\mathbb{R}^2$. We note that the largest eigenvalue of $P$ is equal to 1.8090 and we set $\rho = \frac{1}{1.8090}$. Next, we define the mappings $a, b, c : [0, \infty) \to (0, \infty)$ by setting

$$a(t) = 9 e^{\frac{5t}{2}}, \quad b(t) = 10 e^{\frac{5t}{2}}, \quad \text{and} \quad c(t) = 11 e^{\frac{5t}{2}},$$

for each $t \geq 0$ and we let $x_0 = (1, 1)$. Finally, we let the mapping $v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ be given by

$$v(t, x) = -2x_2 q(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^n, $$

where $q(t, x)$ is obtained by replacing $V_i, a_i^n, b_i^n, c_i^n$ by $a, b, c$ respectively, in the formula (26). From the proof of Theorem 3.1, it should be clear that for each $t \geq 0$, the trajectory $x(\cdot, x_0, 0)$ of $\dot{x} = f(x, v(t, x))$ satisfies

$$V(x(t, x_0, t_0)) \leq b(t), \quad t \geq 0,$$

since $V(x_0) \leq b(0)$. Because $b(t)$ converges to 0 as $t$ tends to $+\infty$ it follows that the state $(x_1, x_2)$ converges to the origin as $t$ tends to $+\infty$. The simulation results in Fig. 1 confirm these facts.

We now give an example where we consider two stabilizable linear systems $S_1$ and $S_2$ with Lyapunov functions $V_1$ and $V_2$ respectively. Using the construction of Theorem 3.1, we then construct two mappings $b_1, b_2 : [0, \infty) \to (0, \infty)$ and a feedback law
Figure 1: Linear system in the plane
\[ v : [0, \infty) \times \mathbb{R} \to \mathbb{R} \] such that for each \( t_0 \geq 0 \) and each \( x_0 \) in \( V_1^{-1}([0, b_1(t_0)]) \) (resp. \( V_2^{-1}([0, b_2(t_0)]) \)) the trajectory \( x(\cdot, x_0, t_0) \) of \( S_1 \) (resp. \( S_2 \)) satisfies
\[
V_1(x(t, x_0, t_0)) \leq b_1(t), \quad \text{(resp. } V_2(x(t, x_0, t_0)) \leq b_2(t) \text{)}, \quad t \geq t_0.
\]

The feedback law \( v \) will actually represent two terms of the infinite sum that appears in the expression of the stabilizing feedback law \( v \) constructed in the proof of Theorem 3.1.

**Example 2:**

We consider the pair of scalar systems
\[
S_1: \quad \dot{x} = x - u \quad \text{and} \quad S_2: \quad \dot{x} = x + u,
\]
where \( u \) is a scalar input, and we let \( f_1, f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) denote the dynamics of \( S_1 \) and \( S_2 \) respectively. It is plain that the two linear feedback laws \( u_1(x) = 2x \) and \( u_2(x) = -2x \) stabilizes \( S_1 \) and \( S_2 \) respectively, and that there exists no continuous feedback law that simultaneously asymptotically stabilizes \( S_1 \) and \( S_2 \). However, by Theorem 3.1, there exists a time-varying feedback law that simultaneously stabilizes \( S_1 \) and \( S_2 \).

We let the Lyapunov function \( V_1 \) and \( V_2 \) be both equal to the mapping \( V \) given by \( V(x) = \frac{1}{2}x^2 \), and we note that
\[
\nabla V(x)f_1(x, u_1(x)) = \nabla V(x)f_2(x, u_2(x)) = -x^2, \quad x \in \mathbb{R}.
\]

Further, we let the mappings \( a_1, b_1, c_1, a_2, b_2, c_2 \) be given by
\[
a_1(t) = 3e^{-t}, \quad b_1(t) = (3 + \frac{1}{3})e^{-t}, \quad c_1(t) = (3 + \frac{2}{3})e^{-t}, \quad t \geq 0,
\]
\[
a_2(t) = 2e^{-t}, \quad b_2(t) = (2 + \frac{1}{3})e^{-t}, \quad c_2(t) = (2 + \frac{2}{3})e^{-t}, \quad t \geq 0,
\]
and we define the mapping \( v : [0, \infty) \times \mathbb{R} \to \mathbb{R} \) by setting
\[
v(t, x) = 2x q_1(t, x) - 2x q_2(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R},
\]
where \( q_1(t, x) \) and \( q_2(t, x) \) are obtained by replacing \( V_i, a^i, b^i, c^i \) by respectively \( a_1, b_1, c_1 \) and \( a_2, b_2, c_2 \) in the formula (26). Set \( x_0 \Delta 2 \). For each \( i = 1, 2 \), because \( V(x_0) \) is less than \( b_i(0) \), the trajectory \( x(t, x_0, 0) \) of the closed-loop system obtained once \( v \) is fed-back into \( S_i \) satisfies
\[
V_i(x(t, x_0, 0)) \leq b_i(t), \quad t \geq 0.
\]

The curves in Fig. 2 show the evolution of the Lyapunov function \( V_1 \), the state \( x \) and the control \( v(t, x) \). Although the curve \( \{(V_1(x(t, x_0, 0), t), \quad t \geq 0\} \) crosses the curve \( \{(b_2(t), t), \quad t \geq 0\} \), it remains below the curve \( \{(b_1(t), t), \quad t \geq 0\} \) as desired. In Fig. 3 are presented the simulation results for the system \( S_2 \).

We present in the next section, a technical lemma that was used in the proof of Theorem 4.1.
Figure 2: Scalar system $S_1$
Figure 3: Scalar system $S_2$
6 Technical lemmas

Lemma 6.1 Let \( I \geq 2 \) be an integer. For each \( i = 1, \ldots, I \), let \( P_i \) be a positive definite matrix, let \( \frac{1}{M_i} \) and \( \frac{1}{m_i} \) denote respectively its smallest and largest eigenvalue, let \( V_i : \mathbb{R}^n \to [0, \infty) \) denote the mapping given by \( V_i(x) = x^t P_i x \), and let \( \theta_i \) be in \((0,1)\). Furthermore, let \( \pi_i \) be in \((0, \frac{m_i}{M_i})\) and let \( \pi_i \) be in \((0, \frac{m_i-1}{M_i})\) for each \( i = 2, \ldots, I \). Assume that
\[
(\pi_1 \cdots \pi_I)(\theta_1^2 \cdots \theta_I^2) < 1
\]
and let \( \hat{\gamma}_1 \) be an arbitrary positive real. Finally, let the sequences of positive reals \( \{\gamma_i^n, i = 1, \ldots, I\}_{n \in \mathbb{Z}}, \{\beta_i^n, i = 1, \ldots, I\}_{n \in \mathbb{Z}} \) and \( \{\alpha_i^n, i = 1, \ldots, I\}_{n \in \mathbb{Z}} \) be defined by setting on one hand
\[
\gamma_1^0 = \hat{\gamma}_1, \quad \beta_i^0 = \theta_i \gamma_i^n, \quad \alpha_i^n = \theta_i \beta_i^n, \quad i = 1, \ldots, I, \quad n = 0, 1, \ldots
\]
with
\[
\gamma_1^{n+1} = \pi_1 \alpha_1^n \quad \text{and} \quad \gamma_i^n = \pi_i \alpha_i^{n-1}, \quad i = 2, \ldots, I, \quad n = 0, 1, \ldots
\]
and on the other hand
\[
\alpha_i^n = \frac{\gamma_i^{n+1}}{\pi_1} \quad \text{and} \quad \alpha_i^n = \frac{\gamma_i^{n+1}}{\pi_i+1}, \quad i = I-1, -1, 1, \quad n = -1, -2, \ldots
\]
with
\[
\beta_i^n = \frac{\alpha_i^n}{\theta_i} \quad \text{and} \quad \gamma_i^n = \frac{\beta_i^n}{\theta_i}, \quad i = I, \ldots, 1, \quad n = -1, -2, \ldots
\]
Then, for each \( n \) in \( \mathbb{Z} \) we have
\[
V_i^{-1}([0, \alpha_i^n]) \supset V_i^{-1}([0, \gamma_i^n]), \quad i = 2, \ldots, I, \quad n = 1, \ldots, I
\]
and for each \( i = 1, \ldots, I \) we have
\[
\beta_i^n \to 0 \quad \text{as} \quad n \to \infty,
\]
\[
\beta_i^n \to \infty \quad \text{as} \quad n \to -\infty.
\]

Proof:
In what follows we fix \( n = 1, 2, \ldots \). Let \( \delta > 0 \). It is well known that for each \( i = 1, \ldots, I \), the set \( V_i^{-1}([0, \delta]) \) is the volume bounded by an ellipsoid centered at the origin with smallest axis \( \sqrt{m_i} \delta \) and largest axis \( \sqrt{M_i} \delta \). Thus, (33) and (34) will hold if we have
\[
\gamma_i^n < \frac{m_i-1}{M_i} \alpha_i^{n-1} \quad \text{and} \quad \gamma_1^{n+1} < \frac{m_i}{M_i} \alpha_i^n
\]
Because \( \pi_i \) is in \((0, \frac{m_i}{M_i})\) and the real \( \pi_i \) is in \((0, \frac{m_i-1}{M_i})\) for each \( i = 2, \ldots, I \), we obtain (33) and (34).
Further, it is easily checked from the assumptions that for each \( i = 1, \ldots, I \) we have
\[
\beta_i^{n+1} = (\pi_1 \cdots \pi_I)(\theta_1^2 \cdots \theta_I^2) \beta_i^n, \quad n \in \mathbb{Z},
\]
and it follows from (28) combined with the definition of \( \alpha_i^n \) and \( \gamma_i^n \) that for each \( i = 1, \ldots, I \), the sequences \( \{\gamma_i^n\}_{n \in \mathbb{Z}} \), \( \{\beta_i^n\}_{n \in \mathbb{Z}} \) and \( \{\alpha_i^n\}_{n \in \mathbb{Z}} \) converge to 0 and \( +\infty \) as \( n \) tends to \( +\infty \) and \( -\infty \) respectively.

The following two technical lemmas were used in the proof of Theorem 4.1.

**Lemma 6.2** For each \( i = 1, 2, \ldots \), let \( P_i \) be a positive definite matrix, let \( \frac{1}{M_i} \) and \( \frac{1}{m_i} \) denote respectively its smallest and largest eigenvalue, let \( V_i : \mathbb{R}^n \to [0, \infty) \) denote the mapping given by \( V_i(x) = x^t P_i x \), and let \( \theta_i \) be in \((0,1)\). Further, let \( \gamma_i^0 \) be a given positive real and let the sequences of positive reals \( \{\pi_i\}_{i=2}^{\infty} \) and \( \{k_i\}_{i=1}^{\infty} \) be such that

\[
0 < \pi_i < \min\left(\frac{m_i-1}{M_i}, \frac{1}{\theta_i^2}\right), \quad i = 2, 3, \ldots, \tag{35}
\]

with

\[
k_i > 1 \quad \text{and} \quad \frac{m_i}{k_i M_i} \theta_i^2 < 1, \quad i = 1, 2, \ldots. \tag{36}
\]

Finally, let the sequences of positive reals \( \{\gamma_i^n, i = 1, \ldots, n\}_{n=1}^{\infty}, \{\beta_i^n, i = 1, \ldots, n\}_{n=1}^{\infty} \) and \( \{\alpha_i^n, i = 1, \ldots, n\}_{n=1}^{\infty} \) be defined by setting

\[
\gamma_1^0 = \gamma_1^0, \quad \beta_i^n = \theta_i \gamma_i^n, \quad \alpha_i^n = \theta_i \beta_i^n, \quad i = 1, \ldots, n, \quad n = 1, 2, \ldots,
\]

with

\[
\gamma_1^n = \frac{m_{n-1}}{k_{n-1} M_1} \alpha_{n-1}^{n-1}, \quad \gamma_i^n = \pi_i \alpha_i^{n-1}, \quad i = 2, \ldots, n, \quad n = 2, 3, \ldots.
\]

Then, for each \( n = 2, 3, \ldots \) we have

\[
V_{n-1}^{-1}([0, \alpha_{n-1}^{n-1}]) \supset V_i^{-1}([0, \gamma_i^n]), \quad i = 1, \ldots, n, \tag{37}
\]

\[
V_{n-1}^{-1}([0, \alpha_{n-1}^{n-1}]) \supset V_i^{-1}([0, \gamma_i^n]), \quad i = 2, \ldots, n, \tag{38}
\]

**Proof:**

In what follows we fix \( n = 1, 2, \ldots \) and we let \( \delta > 0 \). It is well known [1, p. 44] that for each \( i = 1, 2, \ldots \), the set \( V_i^{-1}([0, \delta]) \) is the volume bounded by an ellipsoid centered at the origin with smallest axis \( \sqrt{m_i} \delta \) and largest axis \( \sqrt{M_i} \delta \). Thus, (37) and (38) will hold if we have

\[
\gamma_i^n < \frac{m_{n-1}}{M_1} \alpha_{n-1}^{n-1} \quad \text{and} \quad \gamma_i^n < \frac{m_{i-1}}{M_i} \alpha_{i-1}^n, \quad i = 2, \ldots, n, \quad n = 2, 3, \ldots.
\]

Because we have \( \frac{m_i}{k_i M_i} < \frac{m_i}{M_i} \) for each \( n = 1, 2, \ldots \), and the real \( \pi_i \) lies in \((0, \frac{m_i-1}{M_i})\) for each \( i = 2, \ldots, n \) and each \( n = 2, 3, \ldots \), we already obtain the inclusions (37) and (38).
Next, we set
\[ y_1 \triangleq \ln\left( \frac{m_1}{k_1M_1} \theta_1^2 \right) \]
\[ y_n \triangleq \ln\left( \frac{m_n}{k_nM_1} \theta_1^2 \right) + \ln(\pi_2 \theta_2^2) + \ldots + \ln(\pi_n \theta_n^2), \quad n = 2, 3, \ldots. \]

It follows from (35) together with (36) that the reals \( \ln(\pi_i \theta_i^2) \) and \( \ln(\frac{m_i}{k_iM_1} \theta_1^2) \) are negative for each \( i = 2, 3, \ldots \) and each \( i = 1, 2, \ldots \) respectively. Thus, we have
\[ y_n \leq \ln(\pi_2 \theta_2^2), \quad n = 2, 3, \ldots. \]  \hspace{1cm} (39)

We now fix \( i = 1, 2, \ldots \). It is not hard to check from the definition of \( \gamma_i^n \), \( n = i, i+1, \ldots \), that
\[ \ln(\gamma_i^{i+1}) = y_i + y_{i+1} + \ldots + y_{i+l-1} + \ln(\gamma_i^l), \quad l = 1, 2, \ldots. \]

Therefore, (39) combined with the fact that \( \ln(\pi_2 \theta_2^2) < 0 \) yield the convergence to 0 of \( \gamma_i^{i+1} \) as \( l \) tends to \( \infty \), which completes the proof of the lemma. \( \blacksquare \)

**Lemma 6.3** For each \( i = 1, 2, \ldots \), let \( P_i \) be a positive definite matrix, let \( \frac{1}{M_i} \) and \( \frac{1}{m_i} \) denote respectively its smallest and largest eigenvalue, let \( V_i : \mathbb{R}^n \to [0, \infty) \) denote the mapping given by \( V_i(x) = x^T P_i x \), and let \( \theta_i \) be in \((0, 1)\). Further, let \( \gamma_i^1 \) be a given positive real and let \( \{\eta_i\}_{i=1}^{\infty} \) and \( \{r_i\}_{i=1}^{\infty} \) be sequences of positive reals such that
\[ r_i > \max(1, \frac{m_i}{M_i} \theta_i^2) \quad \text{with} \quad \eta_i > \max(\frac{M_{i+1}}{m_i} \theta_i^2), \quad i = 1, 2, \ldots \]  \hspace{1cm} (40)

Finally, let the sequences of positive reals \( \{\gamma_i^n, i = 1, \ldots, |n|\}_{n=-\infty}^{\infty}, \{\beta_i^n, i = 1, \ldots, |n|\}_{n=-\infty}^{\infty} \) and \( \{\alpha_i^n, i = 1, \ldots, |n|\}_{n=-\infty}^{\infty} \) be defined by setting
\[ \alpha_1^{-1} = 2\gamma_1^1, \quad \beta_i^n = \frac{\alpha_i^n}{\theta_i}, \quad \gamma_i^n = \frac{\beta_i^n}{\theta_i}, \quad i = 1, \ldots, |n|, \quad n = -1, -2, \ldots, \]

with
\[ \alpha_i^n = \eta_i \gamma_{i+1}^n, \quad i = 1, \ldots, |n| + 1, \quad n = -2, -3, \ldots, \]

Then, for each \( n = -2, -3, \ldots \) we have
\[ V_i^{-1}([0, \alpha_i^n]) \supset V_{i-1}^{-1}([0, \gamma_{i+1}^n]), \quad i = 1, \ldots, |n + 1|, \]  \hspace{1cm} (41)

together with
\[ V_{i-1}^{-1}([0, \alpha_{i-1}^{n-1}]) \supset V_i^{-1}([0, \gamma_i^n]), \quad n = -1, -2, \ldots, \]  \hspace{1cm} (42)

and
\[ \gamma_i^n \to +\infty \quad \text{as} \quad n \to -\infty, \quad i = 1, 2, \ldots. \]  \hspace{1cm} (43)
Proof: By using the fact that for each \( i = 1, 2, \ldots \) and each \( \delta > 0 \), the set \( V^{-1}([0, \delta]) \) is the volume of an ellipsoid, it follows that the inclusions (41) and (42) will hold if
\[
\alpha_{[n-1]}^{n-1} > \frac{M_1}{m_{[n-1]}} \gamma_1^n \quad \text{with} \quad \alpha_i^n > \frac{M_{i+1}}{m_i} \gamma_{i+1}^n.
\]
Thus, because \( r_i > 1 \) and \( \eta_i > \frac{M_{i+1}}{m_i} \) for each \( i = 1, 2, \ldots \), we obtain that (41) and (42) hold.

Next, we set
\[
y_l = \ln\left( \frac{M_1 r_{i+1}}{\theta_{i+1}^2 m_{l+1}} \right) + \ln\left( \frac{\eta_i}{\theta_i^2} \right) + \ldots + \ln\left( \frac{\eta_l}{\theta_l^2} \right), \quad l = 1, 2, \ldots.
\]
It is not hard to check that for each \( i = 1, 2, \ldots \), we have
\[
\ln(\gamma_i^{-i-l}) = y_i + y_{i+1} + \cdots + y_{i+l-1} + \ln(\gamma_i^{-i}), \quad l = 1, 2, \ldots,
\]
and because the definition of \( y_l, \ l = 1, 2, \ldots \) together with (40) yield
\[
y_l \geq \ln\left( \frac{\eta_l}{\theta_l^2} \right) > 0, \quad l = 1, 2, \ldots,
\]
we obtain that \( \ln(\gamma_i^{-i-l}) \) converges to \( +\infty \) as \( l \) tends to \( +\infty \). Hence, (43) and the lemma.

We now present more general facts, that are more or less easy to prove.

**Lemma 6.4** Let \( D \) be a bounded neighborhood of the origin in \( \mathbb{R}^n \) (resp. \( D = \mathbb{R}^n \)) and let \( V : D \to [0, \infty) \) be a Lyapunov function (resp. a radially unbounded Lyapunov function), let \( W^\beta \) denote the set \( W^\beta \triangleq \{ x \in D : V(x) < \beta \} \). Then, the family \( \{ W^\beta \}_{\beta > 0} \) is a base at the origin such that \( W^\alpha \subset W^\beta \) whenever \( \alpha \leq \beta \).

**Proof:** By definition, \( \{ W^\beta \}_{\beta > 0} \) is a neighborhood base at the origin if and only if for each \( \varepsilon > 0 \), there exists \( \beta > 0 \) such that \( W^\beta \subset B_\varepsilon(0) \). Suppose now that the assertion of the Lemma does not hold. Then, there exists \( \varepsilon > 0 \) and a sequence \( \{ x_n \}_{n=1}^\infty \) such that
\[
x_n \in W^n \quad \text{and} \quad x_n \not\in B_\varepsilon(0), \quad n = 1, 2, \ldots.
\]
By definition of \( W^n \), the sequence \( \{ x_n \}_{n=1}^\infty \) is included in the set \( \overline{D} \) and in the set \( \overline{W}^1 \).

By assumption, either \( D \) is bounded and in this case \( \overline{D} \) is bounded or \( D = \mathbb{R}^n \) and in this case the set \( \overline{W}^1 \) is bounded because \( V \) is radially unbounded. It follows that in both situations \( \{ x_n \}_{n=1}^\infty \) is included in a compact set so that there exists a subsequence \( \{ x_{n_k} \}_{k=1}^\infty \) which converges to some point \( x_0 \) in \( \overline{D} \). In view of (44), we have
\[
V(x_{n_k}) < \frac{1}{n_k}, \quad k = 1, 2, \ldots,
\]
and the continuity of \( V \) yields \( V(x_0) = 0 \). We conclude from the positive definiteness of \( V \) that \( x_0 \) is the origin, a contradiction with (44) and the fact that \( x_{n_k} \to 0 \) as \( k \to \infty \).

Furthermore, it is plain that \( W^\alpha \subset W^\beta \) for \( \alpha \leq \beta \), which completes the proof. \( \blacksquare \)
**Lemma 6.5** Let $D$ be an open subset of $\mathbb{R}^n$ and let $U$ and $F$ be respectively open and closed subsets of $\mathbb{R}^n$ included in $D$. Let $f : [0, \infty) \times D \to \mathbb{R}^n$ be continuous and let $(S)$ denote the system $\dot{x} = f(t,x)$. Throughout, let $x(\cdot, x_0, t_0) : [t_0, \infty) \to \mathbb{R}^n$ denote the trajectory of $(S)$ starting from a given point $x_0$ in $D$ at a given time $t_0 \geq 0$.

i) Let $t_0 \geq 0$, let $x_0$ be in $D$ and let $\tilde{t}$ be in $(t_0, \infty)$. If $x(\tilde{t}, x_0, t_0)$ lies in $U$ then there exists $\tilde{h} > 0$ such that

$$x(\tilde{t} + h, x_0, t_0) \in U$$

for each $h$ in $(-\tilde{h}, \tilde{h})$ if $\tilde{t} > t_0$ or for each $h$ in $[0, \tilde{h})$ if $\tilde{t} = t_0$.

ii) Let $t_0 \geq 0$, let $x_0$ be in $F$ and assume that the trajectory $x(\cdot, x_0, t_0)$ of $(S)$, does not remain in $F$ for ever. Then, there exists $\tilde{t} \geq t_0$ and $\tilde{h} > 0$ such that

$$x(\tilde{t}, x_0, t_0) \in \partial F \quad \text{and} \quad x(\tilde{t} + h, x_0) \notin F, \quad h \in (0, \tilde{h}).$$

iii) Let $t_0 \geq 0$, let $x_0$ be in $U$ and assume that the trajectory $x(\cdot, x_0, t_0)$ of the system $(S)$, does not remain in $U$ for ever. Then, there exists $t_1 \geq t_0$ such that

$$x(t, x_0, t_0) \in \partial U, \quad \text{with} \quad x(t, x_0, t_0) \in U, \quad t \in [t_0, t_1).$$

**Proof:** (i) Let $t_0 \geq 0$, let $x_0$ be in $D$ and let $\tilde{t}$ be in $(t_0, \infty)$. Further, assume that $x(\tilde{t}, x_0, t_0)$ lies in $U$. Because $x(\cdot, x_0, t_0)$ is continuous we obtain that $x^{-1}(U, x_0, t_0)$ is open in $[t_0, \infty)$ and it follows that $x^{-1}(U, x_0, t_0)$ is an union of open disjoint intervals in $[t_0, \infty)$ i.e.,

$$x^{-1}(U, x_0, t_0) = [t_0, b_0) \cup \bigcup_{\lambda \in \Lambda} (a_\lambda, b_\lambda).$$

Because $x(\tilde{t}, x_0, t_0) \in U$, we either have $\tilde{t} \in (t_0, b_0)$ or there exists $\tilde{\lambda}$ in $\Lambda$ such that $\tilde{t} \in (a_\lambda, b_\lambda)$. This yields the existence of $\tilde{h} > 0$ such that $\tilde{t} + h \in x^{-1}(U, x_0, t_0)$ for each $h$ in $(-\tilde{h}, \tilde{h})$. In other words, $x(\tilde{t} + h, x_0)$ lies in $U$ for each $h$ in $(-\tilde{h}, \tilde{h})$.

If $\tilde{t} = t_0$, the argument above can be easily modified in order to prove the claim.

(ii) Let $t_0 \geq 0$ and let $x_0$ be in $F$. If we assume that the trajectory $x(\cdot, x_0, t_0)$ of $(S)$ does not remain in $F$, then there exists $\tilde{t} > t_0$ such that $x(\tilde{t}, x_0, t_0) \in F^c$. Because the set $x^{-1}(F^c, x_0, t_0)$ is the union of open disjoint intervals in $[t_0, \infty)$ i.e.,

$$x^{-1}(F^c, x_0, t_0) = \bigcup_{\lambda \in \Lambda} (a_\lambda, b_\lambda),$$

there exists $\tilde{\lambda}$ in $\Lambda$ such that $\tilde{t} \in (a_{\lambda}, b_{\lambda})$. From (45), $x(a_{\lambda}, x_0, t_0)$ lies in $F$, and there exists $\tilde{h} > 0$ such that

$$x(a_{\lambda} + h, x_0, t_0) \in F^c, \quad h \in (0, \tilde{h}).$$

By (i), this implies that $x(a_{\lambda}, x_0, t_0) \notin \text{int}(F)$ and it follows that $x(a_{\lambda}, x_0, t_0) \in \partial F$. Therefore (ii) holds with $\tilde{t} = a_{\lambda}$. 

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(iii) Let \( t_0 \geq 0 \) and let \( x_0 \) be in \( U \). Then, \( x^{-1}(U, x_0, t_0) \) is an union of open disjoint intervals in \([t_0, \infty)\) i.e.,

\[
x^{-1}(U, x_0, t_0) = [t_0, b_0) \cup \bigcup_{\lambda \in A} (a_\lambda, b_\lambda),
\]

and it follows that \( x(b_0, x_0, t_0) \) lies in \( U^c \) with \( x(b_0 - h, x_0, t_0) \) in \( U \) for \( h > 0 \) small enough. By (i), this implies that \( x(b_0, x_0, t_0) \) lies in \( U^c \setminus \text{int}(U^c) \) or equivalently in \( \partial U \) (follows from elementary topology [11, Proposition 2, pp. 172]), which completes the proof. 

**Lemma 6.6** Let \( D \) be a bounded neighborhood of the origin in \( \mathbb{R}^n \) (resp. \( D = \mathbb{R}^n \)) and let \( V : \overline{D} \to [0, \infty) \) be a Lyapunov function (resp. a radially unbounded Lyapunov function) for some arbitrary system \((S) : \dot{x} = g(x)\), where the mapping \( g : \overline{D} \to \mathbb{R}^n \) is continuous on \( \overline{D} \). Let \( \beta \) be in the interval \((0, \inf_{x \in \partial D} V(x))\) and define the set \( U \) by setting \( U \triangleq D \cap V^{-1}([0, \beta]) \). Then, the following holds:

i) \( \overline{U} = D \cap V^{-1}([0, \beta]) \).

ii) Let \( x_0 \) be in \( U \) and let the mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) be continuous. If the trajectory \( x(\cdot, x_0) \) of the system \((S') : \dot{x} = f(x)\) does not remain in \( \overline{U} \) forever, then there exists \( \tilde{t} \geq 0 \) and \( \tilde{h} > 0 \) such that

\[
\begin{align*}
x(\tilde{t}, x_0) &\in \partial U \quad \text{with} \quad V(x(\tilde{t}, x_0)) = \beta, \\
x(\tilde{t} + h, x_0) &\not\in V^{-1}([0, \beta]), \quad h \in (0, \tilde{h}), \\
x(\tilde{t} + h, x_0) &\in D, \quad h \in (0, \tilde{h}).
\end{align*}
\]

**Proof:** (i) From the definition of \( U \) and elementary topology we find

\[
\overline{U} \subset \overline{D} \cap V^{-1}([0, \beta]). \quad (46)
\]

We now show that \( \overline{U} \subset D \). Because, this inclusion trivially holds if \( D = \mathbb{R}^n \), we assume that \( D \) is bounded and we prove the claim by contradiction. If the inclusion \( \overline{U} \subset D \) does not hold, then because \( \overline{U} \subset \overline{D} \), there exists \( y \) in \((\partial D) \cap \overline{U}\). Thus, we get

\[
V(y) \geq \inf_{x \in \partial D} V(x) > \beta,
\]

a contradiction with the fact that \( V(y) \leq \beta \) [follows from (46)]. Therefore, we have \( \overline{U} \subset D \) and it follows from [3, Proposition 5 pp. 24] combined with the definition of \( U \) that

\[
\overline{U} = D \cap V^{-1}([0, \beta]). \quad (47)
\]

On the other hand, it is easily checked from the continuity of \( V \) that

\[
D \cap V^{-1}([0, \beta]) \subset D \cap V^{-1}([0, \beta]). \quad (48)
\]

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We now show that these two sets are actually identical, by proving first that

\[ D \cap V^{-1}(\beta) \subset D \cap \overline{V^{-1}([0,\beta])}. \]

If \( D \cap V^{-1}(\beta) = \emptyset \), the inclusion trivially holds. Otherwise, let \( x_0 \) in \( D \cap V^{-1}(\beta) \) and consider the trajectory \( x(\cdot,x_0) \) of \( (S) \) starting from \( x_0 \) at time \( t = 0 \). Because \( V \) is a Lyapunov function for the system \( (S) \), we have

\[ \frac{d}{dt} V(x(t,x_0)) < 0, \quad t \geq 0, \]

so that

\[ V(x(t,x_0)) < V(x_0) = \beta, \quad t > 0. \]  \hspace{1cm} (49)

By continuity of the mapping \( x(\cdot,x_0) : [0,\infty) \to \mathbb{R}^n \), the sequence \( \{x(\frac{1}{n},x_0)\}_{n=1}^\infty \), converges to \( x_0 \) as \( n \) tends to \( \infty \). Since \( x_0 \) lies in the open set \( D \), there exists an integer \( N \) such that \( x(\frac{1}{n},x_0) \) belongs to \( D \) for each \( n = N,N+1,\ldots \). This together with (49) yields

\[ x(\frac{1}{n},x_0) \in D \cap V^{-1}([0,\beta]), \quad n = N,N+1,\ldots, \]

and it follows that \( x_0 \) belongs to \( \overline{D \cap V^{-1}([0,\beta])} \). In view of (47), this implies that \( x_0 \) lies in the set \( D \cap \overline{V^{-1}([0,\beta])} \) and we get

\[ D \cap V^{-1}(\beta) \subset D \cap \overline{V^{-1}([0,\beta])}. \]  \hspace{1cm} (50)

Because we clearly have \( D \cap V^{-1}([0,\beta]) \subset D \cap \overline{V^{-1}([0,\beta])} \), we conclude from (50) that

\[ D \cap V^{-1}([0,\beta]) \subset D \cap \overline{V^{-1}([0,\beta])}. \]

The assertion i) follows from this last inclusion, (47) and (48).

(ii) : Let \( x_0 \) be in \( \overline{U} \), and assume that the trajectory \( x(\cdot,x_0) \) of \( (S') \) that starts from \( x_0 \) at time \( t = 0 \), does not remain in \( \overline{U} \) forever. Then, by Lemma 6.5, there exists \( \hat{t} \geq 0 \) and \( \hat{h} > 0 \) such that

\[ x(\hat{t},x_0) \in \partial U \]  \hspace{1cm} (51)

and

\[ x(\hat{t} + h,x_0) \notin \overline{U}, \quad h \in (0,\hat{h}). \]  \hspace{1cm} (52)

From (i) and the definition of \( U \), we find

\[ \partial U = D \cap \left( V^{-1}([0,\beta]) \setminus V^{-1}([0,\beta]) \right), \]

so that (51) yields

\[ V(x(\hat{t},x_0)) = \beta. \]  \hspace{1cm} (54)

Moreover, in view of (51) and (53), \( x(\hat{t},x_0) \) lies in the open set \( D \). Thus, by Lemma 6.5 (ii), there exists \( \hat{h} \) in \( (0,\hat{h}) \) such that

\[ x(\hat{t} + h,x_0) \in D, \quad h \in (0,\hat{h}). \]  \hspace{1cm} (55)
Finally, upon noting that from (52) we have
\[ x(\hat{t} + h, x_0) \notin D \cap V^{-1}([0, \beta]), \quad h \in (0, \hat{h}), \]
relation (55) yields
\[ x(\hat{t} + h, x_0) \notin V^{-1}([0, \beta]), \quad h \in (0, \hat{h}). \label{56} \]
The assertion (ii) now follows from (51), (54), (55), and (56).

We note that the previous proof does not yield the assertions (i) and (ii) of Lemma 6.6 if the mapping \( V \) is only a positive definite mapping without being a Lyapunov function for some system.

**Lemma 6.7** The mapping \( q : \mathbb{R} \times \mathbb{R} \times (0, \infty) \times (0, \infty) \rightarrow [0, 1] \) defined by
\[
q(x, a, \alpha, \beta) = \begin{cases} 
\frac{(x - a - \alpha)^2}{e^{(x - a - \alpha)^2 - \alpha^2}} & \text{if } x \in (a, a + \alpha) \\
\frac{(x - a - \beta)^2}{e^{(x - a - \beta)^2 - \beta^2}} & \text{if } x \in [a + \alpha, a + \alpha + \beta) \\
0 & \text{otherwise}
\end{cases}
\]
is \( C^\infty \) on its domain of definition.

**Proof:** Let the mappings \( h, g : \mathbb{R} \times \mathbb{R} \times (0, \infty) \) be given by
\[
h(y, a, \alpha) = \begin{cases} 
0, & \text{if } y \leq a \\
\frac{(y - a - \alpha)^2}{e^{(y - a - \alpha)^2 - \alpha^2}}, & \text{if } y \in (a, a + \alpha) \\
1, & \text{if } y \geq a + \alpha
\end{cases}
\]
and
\[
g(y, b, \beta) = \begin{cases} 
1, & \text{if } y \leq b \\
\frac{(y - b - \beta)^2}{e^{(y - b - \beta)^2 - \beta^2}}, & \text{if } y \in (b, b + \beta) \\
0, & \text{if } y \geq b + \beta
\end{cases}
\]
respectively. Because we have
\[
q(y, a, \alpha, \beta) = h(y, a, \alpha) + g(y, a + \alpha, \beta) - 1,
\]
for each \((y, a, \alpha, \beta)\) in \( \mathbb{R} \times \mathbb{R} \times (0, \infty) \times (0, \infty) \), it suffices to show that \( h \) and \( g \) are \( C^\infty \) in order to prove that \( q \) is \( C^\infty \).

**The mapping \( h \) is \( C^\infty \):**

We fix \((x', a', \alpha')\) in \( \mathbb{R} \times \mathbb{R} \times (0, \infty) \) and we study the local behavior of the mapping \( h \) around \((x', a', \alpha')\). For each positive reals \( \delta x, \delta a \) and \( \delta \alpha \), with \( \delta \alpha \) in \((0, \alpha)\), we set
\[
I_{\delta x} \triangleq (x' - \delta x, x' + \delta x),
I_{\delta a} \triangleq (a' - \delta a, a' + \delta a),
I_{\delta \alpha} \triangleq (\alpha' - \delta \alpha, \alpha' + \delta \alpha),
\]
respectively. Because we have
and
\[ U_{\delta x, \delta a, \delta \alpha} \triangleq (x' - \delta x, x' + \delta x) \times (a' - \delta a, a' + \delta a) \times (\alpha' - \delta \alpha, \alpha' + \delta \alpha). \]

We now distinguish several cases:

i) If \( x' < a' \), then the positiveness of \( a' - x' \) yields the existence of \( \delta x \) and \( \delta a \) such that
\[ x' + \delta x < a' - \delta a, \]
and it follows that for all \((x, a)\) in \( I_{\delta x} \times I_{\delta a} \) we have \( x < a \). By picking \( \delta \alpha \) in \((0, \alpha')\), the previous comment yields
\[ h(x, a, \alpha) = 0, \quad (x, a, \alpha) \in I_{\delta x} \times I_{\delta a} \times I_{\delta \alpha}, \]
so that \( h \) is \( C^\infty \) on the neighborhood \( U_{\delta x, \delta a, \delta \alpha} \) of \((x', a', \alpha')\).

ii) If \( x' > a' + \alpha' \), by a similar argument, we find \( \delta x \), \( \delta a \) and \( \delta \alpha \) satisfying
\[ x \geq a + \alpha, \quad (x, a, \alpha) \in U_{\delta x, \delta a, \delta \alpha}. \]
Thus, we get
\[ h(x, a, \alpha) = 1, \quad (x, a, \alpha) \in U_{\delta x, \delta a, \delta \alpha}, \]
and it is plain that \( h \) is \( C^\infty \) on \( U_{\delta x, \delta a, \delta \alpha} \).

iii) Similarly, if \( x' \) is in \((a', a' + \alpha')\), there exists a neighborhood \( U_{\delta x, \delta a, \delta \alpha} \) such that
\[ x \in (a, a + \alpha), \quad (x, a, \alpha) \in U_{\delta x, \delta a, \delta \alpha}, \]
and we obtain from the definition of \( h \) that it is \( C^\infty \) on \( U_{\delta x, \delta a, \delta \alpha} \).

iv) If \( x' = a' \), we pick \( \delta \alpha \) in \((0, \alpha')\). Then, we select \( \delta x \) and \( \delta a \) satisfying
\[ x' + \delta x < a' - \delta a \quad \text{i.e.} \quad \delta x + \delta a < a' - \delta \alpha. \]
This implies that for each \((x, a, \alpha)\) in \( U_{\delta x, \delta a, \delta \alpha} \), we have \( x < a + \alpha \) and we get
\[
h(x, a, \alpha) = \begin{cases} 
0, & \text{if } x \leq a \\
\frac{(x-a)^2}{e^{(x-a)^2/a^2}}, & \text{if } x > a.
\end{cases}
\]

We now let the mapping \( \tilde{h} : \mathbb{R} \times (0, \infty) \to \mathbb{R} \) be given by
\[
\tilde{h}(x, \alpha) = \begin{cases} 
0, & \text{if } x \leq 0 \\
\frac{(x-\alpha)^2}{e^{(x-\alpha)^2/\alpha^2}}, & \text{if } x > 0.
\end{cases}
\]
It is readily seen from this definition that \( \tilde{h} \) is \( C^\infty \) on both sets \((-\infty, 0) \times (0, +\infty)\) and \((0, +\infty) \times (0, +\infty)\). Next, we fix \( n = 1, 2, \ldots \) and \( \bar{\alpha} > 0 \). Because, \( \frac{1}{x^n} e^{-\frac{1}{x}} \) converges to 0 as \( x \) tends to \( 0^+ \), for each \( m \) in \( \mathbb{Z} \), we easily obtain that for each \( \bar{\alpha} > 0 \), each
n-th order partial derivative of the mapping \((x, \alpha) \mapsto e^{\frac{(x-a)^2}{(x-a-\alpha)^2}}\), converges to 0 as \((x, \alpha)\) tends to \((0^+, \alpha)\). Therefore, we can extend by continuity each n-th order partial derivatives of \(\tilde{h}\) at the point \((0, \alpha)\), and it follows from [4, Lemma (8.12.8) pp. 185], that \(\tilde{h}\) is \(C^\infty\) on \(\mathbb{R} \times (0, \infty)\). Because \(h(x, a, \alpha) = \tilde{h}(x - a, \alpha)\), the previous result implies that \(h\) is \(C^\infty\) on \(U_{\delta x, \delta a, \delta \alpha}\).

(v) If \(x' = a' + \alpha'\), we pick \(\delta \alpha\) in \((0, \alpha')\) and we select \(\delta a\) and \(\delta x\) such that

\[a' + \delta a < x' - \delta x \quad \text{i.e.} \quad \delta a + \delta x < \alpha'.\]

This implies that \(a < x\) for each \((x, a, \alpha)\) in \(U_{\delta x, \delta a, \delta \alpha}\) and it follows that

\[h(x, a, \alpha) = \begin{cases} 
\frac{(x-a-\alpha)^2}{(x-a-\alpha-\delta a)^2}, & \text{if } x \in (a, a + \alpha) \\
1, & \text{if } x \geq a + \alpha
\end{cases}.

Let the mapping \(\bar{h} : \mathbb{R}^2 \to [0, \infty)\) be given by

\[\bar{h}(y, z) = \begin{cases} 
e^{-\frac{z^2}{(y-z)^2}}, & \text{if } z < 0 \\
1, & \text{if } z \geq 0
\end{cases}.

It is then easily seen that for each \(n = 1, 2, \ldots\) and each \(\bar{y}\) in \(\mathbb{R}\), every n-th order partial derivative of \(\bar{h}\) converges to 0 as \((y, z)\) tends to \((\bar{y}, 0^-)\). We now extend by continuity each n-th order partial derivatives of \(\bar{h}\) and we obtain that \(\bar{h}\) is \(C^\infty\) on \(\mathbb{R}^2\). Because \(h(x, a, \alpha) = \bar{h}(x - a, x - a - \alpha)\) for each \((x, a, \alpha)\) in \(U_{\delta x, \delta a, \delta \alpha}\), it follows that \(h\) is \(C^\infty\) on this neighborhood.

We conclude from (i), (ii), (iii), (iv) and (v), that \(h\) is \(C^\infty\) on \(\mathbb{R} \times \mathbb{R} \times (0, \infty)\).

The mappings \(g\) and \(q\) are \(C^\infty\):

It is not hard to check that we have

\[g(x, a, \alpha) = h(-x, -a - \alpha, \alpha), \quad (x, a, \alpha) \in \mathbb{R} \times \mathbb{R} \times (0, \infty),\]

which implies that \(g\) is \(C^\infty\) on \(\mathbb{R} \times \mathbb{R} \times (0, \infty)\).

Therefore, the mapping \(q\) is \(C^\infty\) on \(\mathbb{R} \times \mathbb{R} \times (0, \infty) \times (0, \infty)\).

The following result is a direct consequence of the previous lemma.

**Lemma 6.8** Let the mappings \(a, b, c : [0, \infty) \to \mathbb{R}\) be \(C^k\) on \([0, \infty)\) with

\[a(t) < b(t) < c(t), \quad t \in \mathbb{R},\]

and let \(V : \mathbb{R}^n \to \mathbb{R}\) be a \(C^k\) mapping. Then, the mapping \(q : [0, \infty) \times \mathbb{R}^n \to [0, 1]\) given by

\[q(t, x) = \begin{cases} 
\frac{e^{(V(x) - b(t))^2}}{(V(x) - b(t))^2 - (V(x) - a(t))^2} & \text{if } V(x) \in (a(t), b(t)) \\
\frac{e^{(V(x) - b(t))^2}}{(V(x) - b(t))^2 - (c(t) - b(t))^2} & \text{if } V(x) \in [b(t), c(t)] \\
0, & \text{otherwise}
\end{cases}, \quad (57)

for each \((t, x)\) in \([0, \infty) \times \mathbb{R}^n\), is \(C^k\) on \([0, \infty) \times \mathbb{R}^n\).
References


