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Note on a Generalized Sylvester Equation*

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ABSTRACT

In this note we show how to compute the minimum-norm, least squares solution of the generalized Sylvester equation

$$AX + YB = C,$$

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In a personal communication, Richard I. Shrager inquired about the the problem of solving the generalized Sylvester equation

$$AX + YB = C, \tag{1}$$

for X and Y. Here A is $m \times k$, B is $l \times n$, and C is $m \times n$ with $k < m$ and $l < n$. The equation arises in a generalization of a technique for measuring chemical transitions by spectra [1].

In general the system (1) is inconsistent, and we we will ask for a least squares solution:

$$\|C - AX - YB\|^2 = \min. \tag{2}$$

Here $\|\cdot\|$ denotes the Frobenius norm defined by

$$\|C\|^2 = \sum_{i,j} \gamma_{ij}^2.$$

Since the least squares problem (2) is in general underdetermined, we shall also require that

$$\|X\|^2 + \|Y\|^2 = \min; \tag{3}$$

i.e., that the combined solution solution be of minimal norm.

The solution can be expressed in terms of the singular value decomposition of A and B. Specifically, let

$$U_A^{\mathrm{T}} A V_A = \begin{pmatrix} \hat{A} & 0 \\ 0 & 0 \end{pmatrix},$$

where U_A and V_A are orthogonal and

$$\hat{A} = \mathrm{diag}(\hat{\alpha}_i)$$

has positive diagonal entries. Similarly let

$$U_B^{\mathrm{T}} B V_B = \begin{pmatrix} \hat{B} & 0 \\ 0 & 0 \end{pmatrix},$$

where U_B and V_B are orthogonal and

$$\hat{B} = \mathrm{diag}(\hat{\beta}_i)$$

has positive diagonal entries. If (with an obvious partitioning) we write

$$V_A^{\mathrm{T}} X V_B = \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix},$$

$$U_A^{\mathrm{T}} Y U_B = (\hat{Y}_1 \;\; \hat{Y}_2),$$

and

$$U_A^{\mathrm{T}} C V_B = \begin{pmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \end{pmatrix},$$

then we have

$$\begin{pmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \end{pmatrix} = \begin{pmatrix} \hat{A} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix} + (\hat{Y}_1 \;\; \hat{Y}_2) \begin{pmatrix} \hat{B} & 0 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

From (4) we see that the values of X_2 and Y_2 do not affect the norm (2). Consequently, by (3) they must be zero.

Now let us further partition

$$\hat{X}_1 = (\hat{X}_{11} \;\; \hat{X}_{12})$$

and

$$\hat{Y}_1 = \begin{pmatrix} \hat{Y}_{11} \\ \hat{Y}_{21} \end{pmatrix}.$$

Then our problem becomes one finding the minimum norm solution of

$$\|\hat{C}_{11} - \hat{A}\hat{X}_{11} - \hat{Y}_{11}\hat{B}\|^2 + \|\hat{C}_{12} - \hat{A}\hat{X}_{12}\|^2 + \|\hat{C}_{21} - \hat{Y}_{21}\hat{B}\|^2 = \min.$$

Clearly, we must have $\hat{X}_{21} = \hat{A}^{-1}\hat{C}_{21}$ and $\hat{Y}_{21} = \hat{C}_{21}\hat{B}^{-1}$. Thus the problem becomes one of determining minimal $\hat{X}_{11}$ and $\hat{Y}_{11}$ satisfying

$$\hat{C}_{11} = \hat{A}\hat{X}_{11} - \hat{Y}_{11}\hat{B}. \tag{5}$$

Because $\hat{A}$ and $\hat{B}$ are diagonal, the problem (5) uncouples into the independent problems

$$\begin{aligned} &\text{minimize} \quad \hat{\xi}_{ij}^2 + \hat{\eta}_{ij}^2 \\ &\text{subject to} \quad \hat{\alpha}_i \hat{\xi}_{ij} + \hat{\eta}_{ij}\hat{\beta}_j = \hat{\gamma}_{ij}. \end{aligned}$$

But this scalar problem is easily seen to have the solution

$$\hat{\xi}_{ij} = \frac{\hat{\gamma}_{ij}\hat{\alpha}_i}{\hat{\alpha}_i^2 + \hat{\beta}_j^2} \quad \text{and} \quad \hat{\eta}_{ij} = \frac{\hat{\gamma}_{ij}\hat{\beta}_j}{\hat{\alpha}_i^2 + \hat{\beta}_j^2}.$$

This completes the solution of the original problem.

References

[1] R. I. Shrager. Chemical transitions measured by spectra and resolved using singular value decomposition. *Chemometrics and Intelligent Laboratory System*, 1:59–70, 1986. Reference communicated by the author.