TECHNICAL RESEARCH REPORT

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T.R. 95-97

INSTITUTE FOR SYSTEMS RESEARCH

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the National Science Foundation
Engineering Research Center Program,
the University of Maryland,
Harvard University,
and Industry

An Application of Graph Theory to the Detection of Fundamental Circuits in Epicyclic Gear Trains

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Abstract: In this paper, a systematic procedure is developed for the identification of fundamental circuits and transfer vertices in epicyclic gear trains. The method utilizes the well-established concepts of graph theory and related matrices. The procedure leads to automated formulation of the kinematic equations in a systematic manner.

Keywords: Graph Theory, Fundamental Circuits, Gear Trains

1 Introduction

A literature survey reveals that numerous methodologies have been proposed for the kinematic analysis of epicyclic spur gear trains (Allen, 1979, Gibson and Kramer, 1984, Levai, 1968, Olson et al., 1991, Smith, 1979). Using graph theory, Buchsbaum and Freudenstein (1970) investigated the structural characteristics associated with epicyclic gear trains (EGTs) and proposed a systematic methodology for the structure synthesis of such mechanisms. Subsequently, Freudenstein (1971) proposed a systematic methodology for the kinematic analysis of EGTs. Freudenstein's method utilizes the concept of fundamental circuits (f-circuits) which can be readily derived from the graph of an EGT. The method was elaborated in more detail by Freudenstein and Yang (1971).

The analysis of bevel-gear trains is more involved due to the nature of spatial motion of the gears and carriers. Freudenstein et al. (1984) suggested the tabular method using Rodrigues equation to keep track of the positions of the carriers. Gupta (1985) and, Ma and Gupta (1989) proposed the method of superposition. Tsai (1988) demonstrated that the f-circuit equations and coaxiality conditions can be applied to the kinematic analysis of complex robotic bevel-gear trains.

While the kinematic analysis for geared mechanisms has been nearly perfected, the challenge lies in automating the process of identifying fundamental circuits and subsequent generation of the kinematic equations. There are several matrices associated with the graph of an EGT. The adjacency matrix has been frequently used for structure synthesis of mechanisms. However, the incidence, edge set, circuit, and path matrices seem to be relatively unknown to most kinematicians. The adjacency, incidence, and edge set matrices determine the topology of a graph up to graph isomorphism, while the circuit matrix does not, because the presence or absence of an edge which lies on no circuit is not indicated in the matrix. In this paper, the definitions of these matrices and their relationships will be reviewed. Then these relationships will be applied to the detection of f-circuits and transfer vertices in EGTs.

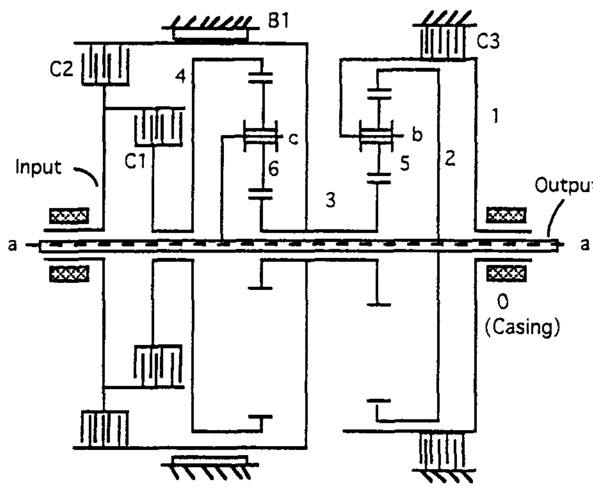

Figure 1: Schematic diagram of a typical EGM.

2 Graph Representations

Conventional Graph. Figure 1 shows the schematic diagram of a typical epicyclic gear mechanism (EGM). In a conventional graph representation, vertices denote links, edges denote joints, and the edges are labeled according to the type of pair connections. For EGTs, gear pairs are denoted by heavy edges, turning pairs by thin edges, and the thin edges are labeled according to their axis locations in space. In addition, the vertex denoting the fixed link is labeled by two concentric small circles. For example, the conventional graph representation for the mechanism shown in Fig. 1 is sketched in Fig. 2.

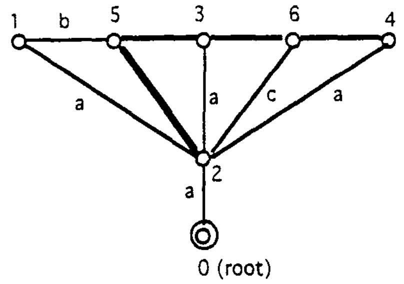

Figure 2: Conventional graph representation.

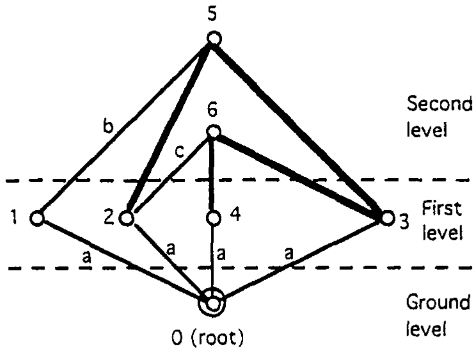

Figure 3: Canonical graph representation.

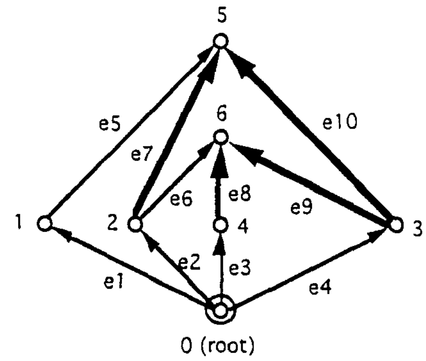

Figure 4: Directed canonical graph.

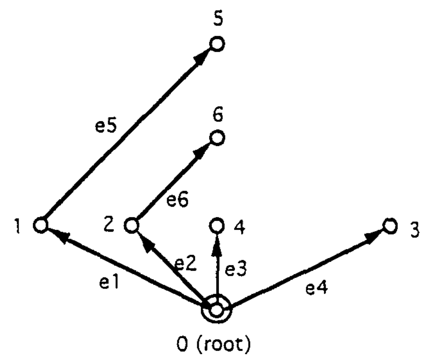

Figure 5: Spanning tree.

Canonical Graph. When there are several coaxial links in a mechanism, it is possible to reconfigure the revolute joints among these coaxial links without affecting the functionality of the mechanism. These kinematically equivalent mechanisms and their corresponding graphs are called pseudo isomorphic mechanisms and pseudo isomorphic graphs, respectively. In order to eliminate the problem of pseudo iosmorphism and to achieve a unique graph representation, a canonical graph was defined by Tsai (1988). The canonical graph is a rooted graph in which the root represents the fixed link. Furthermore, all edges lying on a thin edged path traced from the root to any other vertex in the graph have different edge labels.

Recently, Chatterjee and Tsai (1994) applied the concept of canonical graph for the enumeration of EGMs. The vertices in a canonical graph are divided into several levels. The ground-level vertex represents the root. The first-level vertices, which are one thin-edge away from the root, represent the coaxial links such as the suns, rings and carriers in an EGM. And the second-level vertices, which are two thin edges away from the root, represent the planet gears. For example, the canonical graph representation for the EGM shown in Fig. 1 is sketched in Fig. 3.

Directed Graph. In a directed graph, each edge is assigned a direction. For EGMs, we assign the direction to be pointing from its lower-level incident vertex to the higher-level incident vertex. Figure 4 shows a directed graph obtained by assigning a direction to each edge of the canonical graph shown in Fig. 3.

Spanning Tree. Buchsbaum and Freudenstein (1970) pointed out that removal all geared edges from the graph of an EGM results in a unique spanning tree. For this reason, we refer the thin edges as the arcs and the heavy edges as the chords. The arcs consists of all edges that belong to the spanning tree and the chords consists of all edges which are not in the tree. Figure 5 shows the spanning tree for the canonical graph shown in Fig. 4.

3 Matrix Representations

The information contained in the graph of an EGM can be conveniently stored in a matrix form. For a graph with v vertices and e edges, the vertices are labeled sequentially from 0 to $v-1$, the thin edges are numbered from 1 to $v-1$, and the heavy edges are numbered from v to e. The numbering of vertices and edges proceeds from the lowest level to the highest level, with no preferential order given to those edges (vertices) located at the same level, and the root is denoted as vertex 0. For example, Fig. 4 demonstrates a proper numbering of the vertices and edges.

Adjacency Matrix. The vertex-to-vertex adjacency matrix A is defined as

$$A(i,j) = \begin{cases} 1, & \text{if vertex } i \text{ is adjacent to vertex } j \\ 0, & \text{otherwise (including } i = j) \end{cases}$$

The adjacency matrix A is a $v \times v$ symmetric matrix. The row sums of A are the degrees of the corresponding vertices. The adjacency matrix identifies a graph up to graph isomorphism. The adjacency matrix is often used for structure synthesis of mechanisms and will not be elaborated in detail here.

Incidence Matrix. The edge-to-vertex incidence matrix B utilizes the concept of a directed graph. It elements are defined as

$$B(i,j) = \begin{cases} +1, & \text{if edge } j \text{ emanates from vertex } i \\ -1, & \text{if edge } j \text{ terminates at vertex } i \\ 0, & \text{otherwise} \end{cases}$$

Hence, the incidence matrix B is a $v \times e$ matrix. The i^{th} column represents the i^{th} edge and the j^{th} row represents the $(j-1)^{th}$ vertex. Since each edge has two end vertices, there are exactly two non-zero elements in each column and the column sum is always equal to zero. Similar to the adjacency matrix, the incidence matrix determines a graph up to isomorphism.

The **reduced incident matrix** $\tilde{B}$ is defined by striking out the first row of B.

Edge Set Matrix. It is not desirable to store the entire incidence matrix in a computer because it contains a large number of zero elements. For the purpose of computer programming, all the pertinent information contained in B can be conveniently stored in an edge set matrix. The edge set matrix E is defined as a $3 \times e$ matrix. Each column of E represents an edge: the element in the first row denotes the vertex from which the edge emanates, the element in the second row denotes the vertex to which the edge terminates, and the element in the third row denotes the edge label. The edge set matrix defines the structure topology of a mechanism up to isomorphism.

Circuit Matrix. The circuit matrix C is defined as

$$C(i,j) = \begin{cases} 1, & \text{if edge } i \text{ forms part of the boundary for circuit } j \\ 0, & \text{otherwise} \end{cases}$$

In a circuit matrix, each column represents a circuit and each row represents an edge. Obviously, those edges which do not lie on any circuit are not expressed in the matrix. Hence, the circuit matrix does not provide complete information of a graph. Unlike the adjacency and incidence matrices, the circuit matrix does not determine a graph up to isomorphism. Further, the column vectors of C are not necessarily independent.

It is well known that if the edges of a graph are partitioned into arcs and chords with respect to a spanning tree, then the addition of each chord to the spanning tree forms a circuit and the set of circuits determined from all the chords constitutes the basis for the circuit space. Any other circuits can be expressed as a linear combination of these base vectors.

Path Matrix. The path matrix T is defined for a rooted tree. The i^{th} row represents the i^{th} edge, the j^{th} column represent the j^{th} vertex, and the root is excluded from the matrix. The elements of a path matrix T are defined as

$$T(i,j) = \begin{cases} +1, & \text{if edge } i \text{ is on the path from the root to vertex } j \text{ and the edge is directed toward the root} \\ -1, & \text{if edge } i \text{ is on the path from the root to vertex } j \text{ and the edge is directed away from the root} \\ 0, & \text{otherwise} \end{cases}$$

Matrix Relationships. If we partition the edges of a graph into arcs and chords with respect to a spanning tree such that the arcs are labeled from 1 to $v-1$ and the chords from v to e, then the reduced incidence matrix can be partitioned as

$$\tilde{B} = \begin{bmatrix} \tilde{B}_a & \tilde{B}_c \end{bmatrix} \tag{1}$$

where $\tilde{B}_a$ denotes the portion of the reduced incidence matrix associated with the arcs and $\tilde{B}_c$ denotes the portion associated with the chords.

Similarly, the edge set matrix can also be partitioned as

$$E = [E_a \quad E_c] \tag{2}$$

where E_a denotes the portion of the edge set matrix associated with the arcs and E_c denotes the portion associated with the chords.

Using the partitioned matrix, it has been shown by Roberson and Schwertassek (1988) that

$$T\tilde{B}_a = I \tag{3}$$

where I is a unit matrix of order $v-1$, and

$$C = \begin{bmatrix} U \\ -I \end{bmatrix} \tag{4}$$

where $U = T\tilde{B}_c$ and I is a unit matrix of order $e-v+1$.

The nonzero elements in each column of C represent the edges of a circuit. The nonzero elements are negative for the chord itself and for those arcs that point in the same direction around the circuit as the chord, and positive for those arcs that point in the opposite direction. The number of circuits formed by Eq. (4) is equal to the number of chords. We call these circuits the fundamental circuits with respect to the spanning tree. Finally, it can also be shown that

$$\tilde{B}C = [0] \tag{5}$$

Hence, the columns of C span the null space of $\tilde{B}$.

4 Detection of Fundamental Circuits

An f-circuit as defined by Buchsbaum and Freudenstein (1970) consists of a heavy edge and several thin edges. Equation (4) can be applied for the detection of f-circuits. In what follows, we demonstrate the procedure by using the example mechanism shown in Fig. 1.

Neglecting the difference between the thin and heavy edges, the incidence matrix for the directed graph shown in Fig. 4 is given by

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \end{bmatrix} \tag{6}$$

The edge set matrix E is given by

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 2 & 4 & 3 & 3 \\ 1 & 2 & 4 & 3 & 5 & 6 & 5 & 6 & 6 & 5 \\ a & a & a & a & b & c & g & g & g & g \end{bmatrix} \tag{7}$$

The reduced incidence matrix associated with the arcs is given by

$$\tilde{B}_a = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{8}$$

The reduced incidence matrix associated with the chords is given by

$$\tilde{B}_c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{bmatrix} \tag{9}$$

The path matrix corresponding to the rooted tree shown in Fig. 5 is given by

$$T = \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{10}$$

Substituting Eqs. (8) and (10) into (3), yields

$$T\tilde{B}_a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

Substituting Eqs. (9) and (10) into (4), yields

$$C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{12}$$

Hence, there are four f-circuits: circuit 1 consists of edges e1, e2, e5 and e7; circuit 2 consists of edges e2, e3, e6 and e8; circuit 3 consists of edges e2, e4, e6 and e9; and circuit 4 consists of edges e1, e4, e5 and e10.

5 Detection of Transfer Vertices

In each f-circuit, there exists a vertex called the transfer vertex such that all thin edges lying on one side of the transfer vertex are at the same level and all the other thin edges lying on the opposite side are at a different level. The transfer vertex is also known as the carrier of a gear pair. Detection of the transfer vertex in an f-circuit is essential in order to develop the f-circuit equation.

The transfer vertex can be easily detected from the canonical graph. For each f-circuit, the edge labels are read from the edge set matrix, E_a, and the common vertex which belongs to two edges of different label is identified as the transfer vertex. For the example system, loop 1 has vertex 1 as the transfer vertex, loop 2 has vertex 2 as the transfer vertex, loop 3 has vertex 2 as the transfer vertex, and loop 4 has vertex 1 as the transfer vertex.

6 Kinematic Equations

Let links i and j be a gear pair and k be the carrier. Then links i, j and k form an f-circuit and the f-circuit equation can be written as (Freudenstein, 1971)

$$N_{ij}\omega_i - \omega_j + (1 - N_{ij})\omega_k = 0 \tag{13}$$

where ω_i denotes the angular velocity of link i, and N_{ij} denotes the gear ratio for the gear pair mounted on links i and j, positive or negative according as the gear mesh is internal or external.

For the example gear mechanism shown in Fig. 1, the f-circuit equations are

$$N_{25}\omega_2 - \omega_5 + (1 - N_{25})\omega_1 = 0 \quad (14)$$

$$N_{46}\omega_4 - \omega_6 + (1 - N_{46})\omega_2 = 0 \quad (15)$$

$$N_{36}\omega_3 - \omega_6 + (1 - N_{36})\omega_2 = 0 \quad (16)$$

and

$$N_{35}\omega_3 - \omega_5 + (1 - N_{35})\omega_1 = 0 \quad (17)$$

Equations (14) through (17) consist of four linear equations in six variables. However, under normal operating conditions, one of its coaxial links will be clutched to the casing and another to the input power source. Hence, two of the six angular velocities are known, and the remaining four can be solved in terms of these two.

7 Summary

A procedure for the detection of fundamental circuits in epicyclic gear mechanisms is outlined. The well established graph theory is used to represent the system and then to detect the fundamental circuits. The identification of fundamental circuits and the transfer vertices leads to automated formulation of the kinematic equations in a systematic and methodological manner. The method is also applicable to the kinematic analysis of bevel-gear robotic mechanisms.

8 Acknowledgment

This work was supported in part by the U.S. Department of Energy under Grant DEF05-88ER13977, and in part by the National Science Foundation under Grant No. NSF EEC 94-02384. Such support does not constitute an endorsement by the supporting agencies of the views expressed in the paper

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