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On the Stiffness of a Novel Six-DOF
Parallel Minimanipulator

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On the Stiffness of a Novel Six-DOF Parallel Minimanipulator

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Abstract

The dimensionally-uniform Jacobian matrix of a novel three-limbed, six degree-of-freedom (DOF) minimanipulator is used to derive its dimensionally-uniform stiffness matrix.

The minimanipulator limbs are inextensible and its actuators are base-mounted. The lower ends of the limbs are connected to bidirectional linear stepper motors which are constrained to move on a base plane. The minimanipulator is capable of providing high positional resolution and high stiffness.

It is shown that, at a central configuration, the stiffness matrix of the minimanipulator can be decoupled (diagonalized), if proper design parameters are chosen. It is also shown that the stiffness of the minimanipulator is higher than that of the Stewart platform. Guidelines for obtaining large minimanipulator stiffness values are established.

1 Introduction

In recent years, many researchers have studied parallel manipulators. Such mechanisms are most suitable for applications in which the requirements for accuracy, rigidity, load-to-weight ratio, and load distribution are more important than the need for a large workspace.

The stiffness matrix of a manipulator is generally defined as the transformation which relates the generalized force (force and torque) applied to the end-effector and its resulting displacement [1]. In a parallel manipulator, the platform plays the role of the end-effector. Stiffness matrices of parallel manipulators, which are closely related to their Jacobian matrices, have been studied by Kerr [3] and Gosselin [2].

The authors have designed and analyzed a new six-DOF parallel minimanipulator, which has three inextensible limbs ([6] - [9]). In this article, the Jacobian matrix of the minimanipulator is used to obtain its dimensionally-uniform stiffness matrix. It is shown that, if proper design parameters are chosen, the stiffness matrix of the minimanipulator can be decoupled at a central configuration. It is also shown that the stiffness of the minimanipulator is higher than that of the Stewart platform [4]. Finally, guidelines for increasing the minimanipulator stiffness are established.

2 Description of the Minimanipulator

Let subscript i in this section and the rest of this work represent numbers 1, 2, and 3 in a cyclic manner. The minimanipulator contains three inextensible limbs, $P_i R_i$, as shown in Figure 1.

The lower end of each limb is connected to a bidirectional linear stepper motor driver [10] and can be moved freely on the base plate. Note that bidirectional linear stepper motors act as X-Y positioning tables, but their stators are base-mounted. The desired minimanipulator motion is obtained by moving the lower ends of its three limbs on its base plate. Two-DOF universal joints connect the limbs to the moving platform. The lower ends of the limbs are connected to the drivers through three more universal joints. Note that one of the axes of the upper universal joint is collinear with the limb, while the other axis of the upper universal joint as well as one of the axes of the lower universal joint are always perpendicular to the limb. This arrangement is kinematically equivalent to a limb with a spherical joint at its lower end and a revolute joint at its upper end.
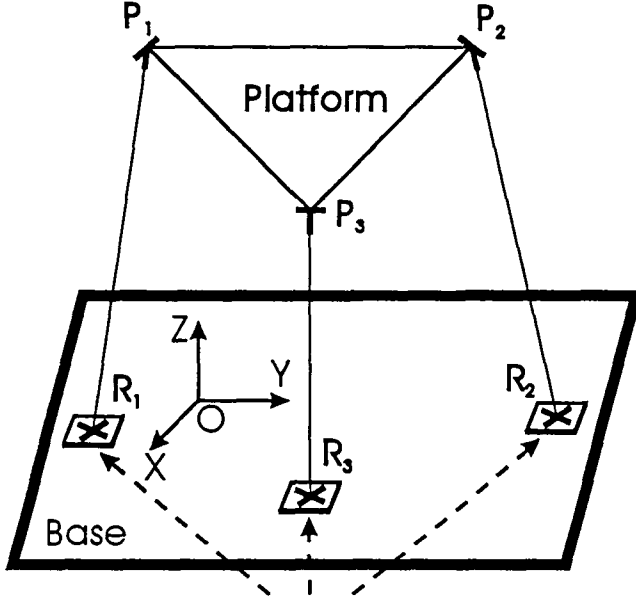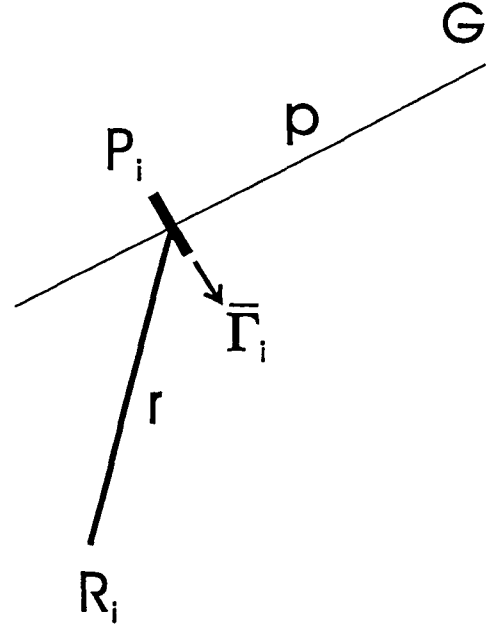
Figure 1: Representation of a minimanipulator.

Figure 2: Depiction of $\overline{\Gamma}_i$.

Other two-DOF mechanisms, such as five-bar linkages and pantographs, can also be used as the drivers for the minimanipulator ([6] - [9]).

Inextensible limbs are used to improve positional resolution and stiffness of the minimanipulator. Since the minimanipulator actuators are base-mounted; higher payload capacity, smaller actuator sizes, and lower power dissipation can be obtained. In addition, to maintain symmetry, the limbs are made equal in length, triangle $P_1P_2P_3$ is made equilateral, and the axes of the topmost joint at point P_i is made parallel to line $P_{i+2}P_{i+1}$.

3   The Jacobian Matrix

Let us define a fixed (base) reference frame, XYZ, as shown in Figure 1. The origin, point O, is a fixed point on the plane passing through R_1, R_2, and R_3. The X and Y axes lie on the same plane. The Z axis is defined by the right-hand-rule.

As shown in Figure 2, let $\overline{\Gamma}_i$ be a unit vector which is collinear with the axis of the topmost revolute joint at point P_i and points in the direction of vector $\overline{P_{i+2}P_{i+1}}$. Note that, in Figure 2, point G is the centroid of the platform, parameter r represents the limb length, and parameter p represents the circumradius of the platform.

Let us define vector $\overline{\mu}_i$ by

$$\overline{\mu}_i = \overline{\Gamma}_i \times \overline{P_iR_i} \tag{1}$$

In addition, let us define the generalized velocity vector of the platform, $\dot{\overline{x}}$, as

$$\dot{\overline{x}} = \begin{bmatrix} {}^B\overline{V}_x^G, & {}^B\overline{V}_y^G, & {}^B\overline{V}_z^G, & {}^B\overline{\omega}_x^P, & {}^B\overline{\omega}_y^P, & {}^B\overline{\omega}_z^P \end{bmatrix} \tag{2}$$

where ${}^B\overline{V}_x^G$, ${}^B\overline{V}_y^G$, ${}^B\overline{V}_z^G$ are the components of the velocity of point G in the base reference frame and ${}^B\overline{\omega}_x^P$, ${}^B\overline{\omega}_y^P$, ${}^B\overline{\omega}_z^P$ are the components of the angular velocity of the platform with respect to the base reference frame. Also, let

$$\dot{\overline{q}} = \begin{bmatrix} {}^B\overline{V}_x^{R_1}, & {}^B\overline{V}_y^{R_1}, & {}^B\overline{V}_x^{R_2}, & {}^B\overline{V}_y^{R_2}, & {}^B\overline{V}_x^{R_3}, & {}^B\overline{V}_y^{R_3} \end{bmatrix}^T \tag{3}$$

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where ${}^{B}\overline{V}_{x}^{R_i}$ and ${}^{B}\overline{V}_{y}^{R_i}$ are the X and Y components of the velocity of point R_i in the base reference frame, and superscript T denotes transpose. We can define the Jacobian matrix, $\mathbf{J}$, by

$$\dot{\bar{q}} = \mathbf{J}\dot{\bar{x}} \tag{4}$$

Note that due to dualities of parallel and serial manipulators, we have defined the Jacobian matrix as the transformation which maps the generalized velocity of the platform to the input rates. This is a common practice among most researchers who have studied parallel manipulators.

It has been shown [7] that the Jacobian matrix is given by

$$\mathbf{J} = \begin{bmatrix} 1 & 0 & -\mu'_{1,x} & -\mu'_{1,x}\,(Y_{R,1}-Y_G) & -Z_G - \mu'_{1,x}\,(X_G - X_{R,1}) & Y_G - Y_{R,1} \\ 0 & 1 & -\mu'_{1,y} & Z_G - \mu'_{1,y}\,(Y_{R,1}-Y_G) & -\mu'_{1,y}\,(X_G - X_{R,1}) & X_{R,1} - X_G \\ 1 & 0 & -\mu'_{2,x} & -\mu'_{2,x}\,(Y_{R,2}-Y_G) & -Z_G - \mu'_{2,x}\,(X_G - X_{R,2}) & Y_G - Y_{R,2} \\ 0 & 1 & -\mu'_{2,y} & Z_G - \mu'_{2,y}\,(Y_{R,2}-Y_G) & -\mu'_{2,y}\,(X_G - X_{R,2}) & X_{R,2} - X_G \\ 1 & 0 & -\mu'_{3,x} & -\mu'_{3,x}\,(Y_{R,3}-Y_G) & -Z_G - \mu'_{3,x}\,(X_G - X_{R,3}) & Y_G - Y_{R,3} \\ 0 & 1 & -\mu'_{3,y} & Z_G - \mu'_{3,y}\,(Y_{R,3}-Y_G) & -\mu'_{3,y}\,(X_G - X_{R,3}) & X_{R,3} - X_G \end{bmatrix} \tag{5}$$

The third, fourth, and fifth columns of $\mathbf{J}$ involve the terms $\mu'_{i,x}$ and $\mu'_{i,y}$ which are defined by

$$\mu'_{i,x} = \mu_{i,x}/\mu_{i,z} \quad , \quad \mu'_{i,y} = \mu_{i,y}/\mu_{i,z} \tag{6}$$

where $\mu_{i,x}$, $\mu_{i,y}$, and $\mu_{i,z}$ are the X, Y, and Z components of vector $\bar{\mu}_i$, respectively. Note that, in equation (5), X_G and Y_G represent the X and Y coordinates of point G. Also, $X_{R,i}$, and $Y_{R,i}$ denote the X and Y coordinates of point R_i.

3.1 Dimensionally-Uniform Jacobian Matrix

As shown in equation (2), the first three elements of $\dot{\bar{x}}$ have the dimension of length/time, whereas the last three elements of $\dot{\bar{x}}$ have the dimension of radian/time. As a result, as shown in equation (5), elements of the first three columns of $\mathbf{J}$ are dimensionless, whereas elements of the last three columns of $\mathbf{J}$ have the dimension of length.

In this section, we define a dimensionally-uniform generalized velocity vector for the platform in order to obtain a dimensionally-uniform Jacobian matrix. The result will be used in section 4.1 to obtain a dimensionally-uniform stiffness matrix, which can be diagonalized by means of a principal axis transformation.

Let $\dot{\tilde{x}}$ be a dimensionally-uniform generalized velocity vector for the platform, whose elements have the dimension of length/s. We define vector $\dot{\tilde{x}}$ as

$$\dot{\tilde{x}} = \mathbf{W_V}\dot{\bar{x}} \tag{7}$$

where $\mathbf{W_V}$ is a 6 × 6 diagonal, positive definite, weighting matrix given by

$$\mathbf{W_V} = diag(1, \ 1, \ 1, \ L, \ L, \ L) \tag{8}$$

In equation (8), L is a parameter which has the dimension of length (e.g., p). Solving equation (7) for $\dot{\bar{x}}$ and substituting the result into equation (4), we obtain

$$\dot{\bar{q}} = \tilde{\mathbf{J}}\dot{\tilde{x}} \tag{9}$$

where $\tilde{\mathbf{J}}$ is a dimensionally-uniform Jacobian matrix, which is given by

$$\tilde{\mathbf{J}} = \mathbf{J}\mathbf{W_V}^{-1} \tag{10}$$

The elements of $\tilde{\mathbf{J}}$ are all dimensionless.

4 The Stiffness Matrix

From equation (4), we can conclude that

$$\overline{\delta q} = \mathbf{J}\overline{\delta x} \tag{11}$$

where $\overline{\delta q}$ and $\overline{\delta x}$ represent infinitesimal displacements at the lower ends of the limbs and at the center of the platform, respectively. Equation (4) and the principle of virtual work can be used to derive the following equation[1].

$$\overline{\mathcal{F}} = \mathbf{J}^T \overline{f} \tag{12}$$

where

$$\overline{\mathcal{F}} = \left[\begin{array}{c} \overline{F}_P \\ \hline \overline{M}_P \end{array} \right] \tag{13}$$

and

$$\overline{f} = [f_{1,x}, \quad f_{1,y}, \quad f_{2,x}, \quad f_{2,y}, \quad f_{3,x}, \quad f_{3,y}]^T \tag{14}$$

Vectors $\overline{F}_P$ and $\overline{M}_P$ in equation (13) represent the force and moment applied to the platform. Also, $f_{i,x}$ and $f_{i,y}$ in equation (14) are the X and Y components of the actuator force applied at point R_i. The actuator forces and displacements at the lower ends of the limbs can be related by the following equation.

$$\overline{f} = \mathbf{k}\overline{\delta q} \tag{15}$$

where $\mathbf{k}$ is a 6×6 diagonal matrix whose elements have the dimension of force/length. Substituting equation (11) into equation (15) and the resulting equation into equation (12), yields

$$\overline{\mathcal{F}} = \mathbf{J}^T \mathbf{k} \mathbf{J}\overline{\delta x} \tag{16}$$

If κ represents the stiffness of each bidirectional linear stepper motor in the X and Y directions, then the diagonal elements of $\mathbf{k}$ are all equal to κ. Therefore, the stiffness matrix for the platform ($\mathbf{K}$) can be expressed as

$$\mathbf{K} = \kappa \mathbf{J}^T \mathbf{J} \tag{17}$$

It can be shown that $\mathbf{K}$ is a symmetric, positive semidefinite matrix.

4.1 Dimensionally-Uniform Stiffness Matrix and Its Principal Axis Transformation

As mentioned in section 3.1, matrix $\mathbf{J}$ is not dimensionally uniform. As a result, the stiffness matrix $\mathbf{K}$, which is defined by equation (17), is not dimensionally uniform either.

The upper left 3×3 portion of $\mathbf{K}$ represents the direct or translatory stiffness, and consists of elements which have the dimension of force/length. The lower right 3×3 portion of $\mathbf{K}$ represents the torsional stiffness, and consists of elements which have the dimension of force-length. The other 3×3 portions of $\mathbf{K}$ consist of cross-coupling elements which have the dimension of force.

We can ask ourselves if $\mathbf{K}$ can be diagonalized by means of a coordinate transformation. To diagonalize the matrix $\mathbf{K}$, its eigenvalue problem should be solved. Let us examine if such an eigenvalue problem makes sense physically.

Referring to equations (16) and (17), we note that the eigenvalue problem for $\mathbf{K}$ involves solving the following equation.

$$\overline{\mathcal{F}} = \mathbf{K}\overline{\delta x} = \lambda\overline{\delta x} \tag{18}$$

where λ is the eigenvalue of $\mathbf{K}$. In other words, we are looking for certain directions of $\overline{\delta x}$ which are identical to the corresponding directions of $\overline{\mathcal{F}}$. The first three elements of $\overline{\mathcal{F}}$ have the dimension of force and the first three elements of $\overline{\delta x}$ have the dimension of length. Therefore, the dimension of λ has to be force/length. However, the last three elements of $\overline{\mathcal{F}}$ have the dimension of force-length and the last three elements of $\overline{\delta x}$ have the dimension of radians. As a result, the dimension of λ should also be equal to force-length. Since the dimension of λ can not be both force/length and force-length, the eigenvalue problem for $\mathbf{K}$ is dimensionally inconsistent and does not make sense physically. This is due to the fact that $\mathbf{K}$, $\overline{\mathcal{F}}$, and $\overline{\delta x}$ are not dimensionally uniform.

Let us define a dimensionally-uniform generalized force applied to the platform ($\widetilde{\mathcal{F}}$) and use it in connection with the definitions of $\widetilde{x}$ and $\widetilde{\mathbf{J}}$ (see section 3.1) to obtain a dimensionally-uniform stiffness matrix ($\widetilde{\mathbf{K}}$).

Let the elements of $\widetilde{\mathcal{F}}$ have the dimension of force, and let $\widetilde{\mathcal{F}}$ be defined as

$$\widetilde{\mathcal{F}} = \mathbf{W_f}\mathcal{F} \tag{19}$$

where $\mathbf{W_f}$ is a 6×6 diagonal, positive definite, weighting matrix given by

$$\mathbf{W_f} = diag(1,\ 1,\ 1,\ L^{-1},\ L^{-1},\ L^{-1}) \tag{20}$$

In equation (20), L is a parameter with the dimension of length (e.g., p). Comparing equations (20) and (8), we note that

$$\mathbf{W_f}\mathbf{W_v} = \mathbf{I} \tag{21}$$

where $\mathbf{I}$ is the 6×6 identity matrix. Substituting equation (12) into equation (19) yields

$$\widetilde{\mathcal{F}} = \mathbf{W_f}\mathbf{J}^T f \tag{22}$$

Following the same procedure used for the derivation of equations (16) and (17) and using equation (21), we obtain

$$\widetilde{\mathcal{F}} = \widetilde{\mathbf{K}}\widetilde{\delta x} \tag{23}$$

where

$$\widetilde{\mathbf{K}} = \kappa\mathbf{W_f}\mathbf{J}^T\mathbf{J}\mathbf{W_f} \tag{24}$$

Note that $\widetilde{\mathbf{K}}$ is the dimensionally-uniform stiffness matrix and $\widetilde{\delta x}$ represents the dimensionally-uniform infinitesimal displacement of the platform. The elements of $\widetilde{\delta x}$ have the dimension of length and the elements of $\widetilde{\mathbf{K}}$ have the dimension of force/length. It can be shown that $\widetilde{\mathbf{K}}$ is symmetric and positive semidefinite. Comparison of equations (24) and (17) shows that

$$\widetilde{\mathbf{K}} = \mathbf{W_f}\mathbf{K}\mathbf{W_f} \tag{25}$$

Matrix $\widetilde{\mathbf{K}}$ can be diagonalized by means of a principal axis transformation. Since $\widetilde{\mathbf{K}}$ is symmetric, it has a complete set of orthonormal eigenvectors [5]. Let $\mathbf{E}$ be an orthogonal matrix whose columns represent orthonormal eigenvalues of $\widetilde{\mathbf{K}}$. Also, let $\mathbf{\Lambda}$ be a diagonal matrix, with the eigenvalues of $\widetilde{\mathbf{K}}$ along its diagonal. Then,

$$\widetilde{\mathbf{K}} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^T \tag{26}$$

Substituting equation (26) into equation (23), and modifying the resulting equation, we get

$$\mathbf{E}^T\widetilde{\mathcal{F}} = \mathbf{\Lambda}\mathbf{E}^T\widetilde{\delta x} \tag{27}$$

Matrix $\mathbf{E}^T$ represents the principal axis transformation and the elements of $\mathbf{E}^T\widetilde{\mathcal{F}}$ point in the same directions as the corresponding elements of $\mathbf{E}^T\widetilde{\delta x}$. These directions, which are determined by the orthonormal eigenvectors, are the principal axes of $\widetilde{\mathbf{K}}$.

To gain more insight into the physical meaning of the principal axes of $\widetilde{\mathbf{K}}$, let us examine $\widetilde{\delta x}$ when a dimensionally-uniform generalized force of unit magnitude is applied to the platform, i.e.

$$\widetilde{\mathcal{F}}^T\widetilde{\mathcal{F}} = 1 \tag{28}$$

Substituting equation (23) into equation (28), we get

$$\widetilde{\delta x}^T\widetilde{\mathbf{K}}^T\widetilde{\mathbf{K}}\widetilde{\delta x} = 1 \tag{29}$$

Since $\widetilde{\mathbf{K}}$ is symmetric, equation (29) reduces to

$$\widetilde{\delta x}^T\widetilde{\mathbf{K}}^2\widetilde{\delta x} = 1 \tag{30}$$

Equation (30) represents a six-dimensional ellipsoid whose axes point along the eigenvectors of $\widetilde{\mathbf{K}}$ (principal directions). The axes lengths are $1/\lambda_1,\ldots,1/\lambda_6$, where $\lambda_1,\ldots,\lambda_6$ are the eigenvalues of $\widetilde{\mathbf{K}}$ [5]. [1]

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[1] Note that eigenvectors of $\widetilde{\mathbf{K}}^2$ are the same as those of $\widetilde{\mathbf{K}}$ and eigenvalues of $\widetilde{\mathbf{K}}^2$ are $\lambda_1^2,\ldots,\lambda_6^2$.

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4.2 Central Stiffness Matrix

Since $\tilde{K}$ is symmetric, its norm is equal to its largest eigenvalue. Theoretically, the norm of $\tilde{K}$ can be used to measure and/or to compare the stiffness levels of different minimanipulators. However, to find the norm of $\tilde{K}$, its eigenvalue problem, which is a sixth order polynomial, should be solved. In general, there are no analytical expressions for the solutions of such a polynomial. Also, $\tilde{K}$ is configuration-dependent. Therefore, finding the eigenvalues and the norm of $\tilde{K}$ at all of the minimanipulator configurations becomes impractical. Even if we try to use the norm of $\tilde{K}$ at a specific configuration as a measure of stiffness, no analytical expression for such a measure can be obtained. As a result, establishing design guidelines for obtaining high stiffness becomes difficult.

In what follows, we define a central configuration for the minimanipulator and study the stiffness of the minimanipulator at the central configuration. Since the minimanipulator is intended for fine manipulation around such a designated posture, the stiffness will not deviate significantly from that corresponding to the central configuration. Hence, it is important that we understand fully the minimanipulator stiffness at the central configuration.

Let us define the "central configuration" of the minimanipulator workspace as the configuration where

1. The platform is not rotated with respect to the base.

2. The centroid of triangle $P_1 P_2 P_3$ (platform) is directly on top of the origin of the base reference frame.

3. The platform is positioned at a designated elevation, i.e., $Z_G = \zeta$ (a design value).

Next, the stiffness matrix at the central configuration of the minimanipulator (central stiffness matrix) will be derived. It will be shown that, if proper minimanipulator dimensions are used, the central stiffness matrix can be diagonalized (decoupled) without any coordinate transformations. The diagonalized central stiffness matrix will be used to obtain a simple measure of stiffness and to establish design guidelines for the minimanipulators. As mentioned before, a minimanipulator will be operated at or near the central configuration during most of its operations. Therefore, establishing design guidelines based on the central stiffness matrix is justified. The results of this section will also be used to compare the minimanipulator stiffness with that of the Stewart platform (see section 4.3).

Recall that $\left| \overline{GP_i} \right| = p$, and $Z_G = \zeta$ at the central configuration. Also, as shown in Figure 3, let $\left| \overline{OR_i} \right| = \nu$ at the central configuration. Using equations (5) and (17), the central stiffness matrix (K^+) is found to be

$$K^+ = 3\kappa \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{(2p-\nu)\zeta}{2(\nu-p)} & 0 \\ 0 & 1 & 0 & \frac{(\nu-2p)\zeta}{2(\nu-p)} & 0 & 0 \\ 0 & 0 & \frac{\zeta^2}{(\nu-p)^2} & 0 & 0 & 0 \\ 0 & \frac{(\nu-2p)\zeta}{2(\nu-p)} & 0 & \frac{(\nu^2-2p\nu+2p^2)\zeta^2}{2(\nu-p)^2} & 0 & 0 \\ \frac{(2p-\nu)\zeta}{2(\nu-p)} & 0 & 0 & 0 & \frac{(\nu^2-2p\nu+2p^2)\zeta^2}{2(\nu-p)^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \nu^2 \end{bmatrix} \tag{31}$$

It is desirable to eliminate the off-diagonal terms which couple the forces (moments) applied along (about) the X and Y axes to the rotations (translations) about (along) the Y and X axes, respectively. Fortunately, this can be easily accomplished by choosing

$$p = \nu/2 \tag{32}$$

as a design condition. In other words, the platform (triangle $P_1 P_2 P_3$) should be equal to one-half of the triangle $R_1 R_2 R_3$ at the central configuration. The above result is similar to that obtained by Kerr [3] in designing a Stewart-platform-based force/torque transducer. If the condition expressed in equation (32) is satisfied, then
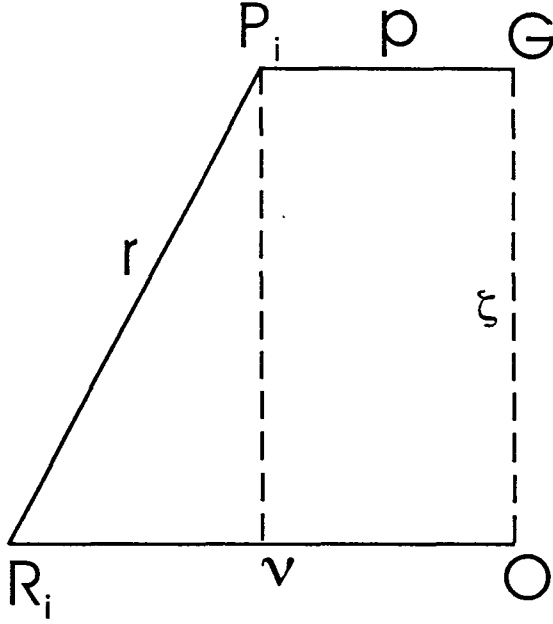
$$\zeta^2 = r^2 - p^2 \tag{33}$$
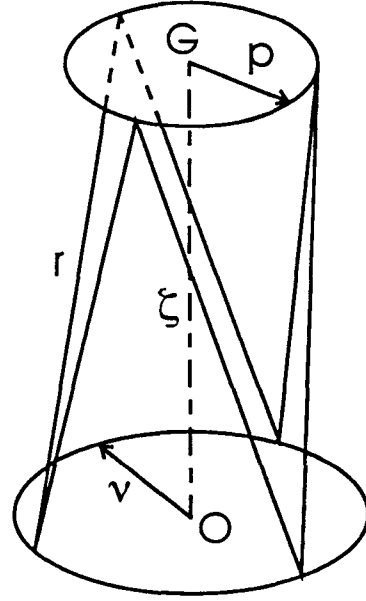
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Figure 3: Geometric parameters used in K*.

Figure 4: A Stewart platform configuration in which the limb lengths are equal.

If equations (32) and (33) are used to substitute for ν and ζ in equation (31), matrix $\mathbf{K}^+$ reduces to

$$\mathbf{K}^* = 3\kappa \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & (r^2-p^2)/p^2 & & & \\ & & & r^2-p^2 & & \\ & & & & r^2-p^2 & \\ 0 & & & & & 4p^2 \end{bmatrix} \tag{34}$$

We shall refer to matrix $\mathbf{K}^*$ as the decoupled central stiffness matrix. The first three diagonal elements of $\mathbf{K}^*$ have the dimension of force/length, whereas its last three diagonal elements have the dimension of force-length.

Using equation (25), and setting $L = p$ in $\mathbf{W_f}$, we obtain the following expression for the dimensionally-uniform decoupled central stiffness matrix

$$\tilde{\mathbf{K}}^* = 3\kappa \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & (r/p)^2-1 & & & \\ & & & (r/p)^2-1 & & \\ & & & & (r/p)^2-1 & \\ 0 & & & & & 4 \end{bmatrix} \tag{35}$$

Equation (35) can be used to determine the relative dimensions of the minimanipulator so that desirable characteristics are obtained.

Note that parameter ν in equation (32) can be determined from other design considerations such as the upper bound on the base plate size.

To move the platform in the X or Y direction, the lower ends of all three limbs should also move in the X or Y-direction. As a result, elements $\tilde{K}^*_{1,1}$ and $\tilde{K}^*_{2,2}$ are constants. From the statics point of view, external forces in the X and Y directions are shared equally among the three actuators. Element $\tilde{K}^*_{6,6}$ is also a constant. This is related to the fact that in order to rotate the platform about the Z-axis, the lower ends of the limbs should move on a circle, which passes through them, in the same direction and by an equal

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amount. Note that we can not increase the values of $\widetilde{K}_{1,1}^*$, $\widetilde{K}_{2,2}^*$, and $\widetilde{K}_{6,6}^*$ by changing the minimanipulator parameters r and p.

The other three non-zero elements $\widetilde{K}_{3,3}^*$, $\widetilde{K}_{4,4}^*$, and $\widetilde{K}_{5,5}^*$ are functions of minimanipulator dimensions and are equal to each other. Therefore, we can use any of these terms as the stiffness measure (S.M.) of the minimanipulator. Namely, we can define

$$S.M. = 3\kappa \left[(r/p)^2 - 1 \right] \tag{36}$$

The higher the value of r/p, the higher the stiffness of the minimanipulator. Another way to interpret the S.M. is to note that S.M. is proportional to tangent-squared of the angle between any of the limbs and the base plane at the central configuration. The closer this angle to 90 degrees, the higher the stiffness of the minimanipulator.

The first three diagonal elements of $\widetilde{K}^*$ are the direct stiffness terms. Equation (35) shows that if $r = \sqrt{2}p$, we obtain equal direct stiffness values in the X, Y, and Z directions. At this configuration, the angle between any of the limbs and the base plane becomes equal to 45 degrees.

The last three diagonal elements of $\widetilde{K}^*$ are the torsional stiffness terms. Referring to equation (35), we notice that if $r = \sqrt{5}p$, we obtain equal torsional stiffness values in the X, Y, and Z directions. At this configuration, the angle between any of the limbs and the base plane becomes equal to 63.43 degrees.

4.3 Comparison to the Stewart Platform

In this section, the results of Kerr [3] are used to compare the minimanipulator stiffness with that of the Stewart platform.

Duplicating Kerr's model, let us consider a Stewart platform with its linkage parameters identical to those of the minimanipulator. As shown in Figure 4, the six limbs connect three points on the base to three points on the platform. The three points on the platform form an equilateral triangle whose circumradius is equal to p. Similarly, The three points on the base form an equilateral triangle whose circumradius is equal to ν. Kerr [3] showed that the stiffness matrix of the Stewart platform is decoupled, when all of the limb lengths are equal, and $p = \nu/2$. When all of the limb lengths are equal, the platform is not rotated with respect to the base, and the centroid of the platform (point G) is right on top of the centroid of the base (point O). Therefore, the conditions under which the Stewart platform has a decoupled stiffness matrix are identical to those which decouple the stiffness matrix of the minimanipulator (see section 4.2). When the limb lengths are equal, we refer to the common limb length as r. The axial (actuator) stiffness of each limb, κ, is taken to be the same as the stiffness of each bidirectional linear stepper motor in the X and Y directions (see section 4).

Kerr's expression for the decoupled stiffness matrix ($K^{\#}$) is not dimensionally uniform and its elements have the same dimensions as the corresponding elements of K^*. Using the same procedure used in section 4.2 for obtaining $\widetilde{K}^*$, we find the following dimensionally-uniform decoupled stiffness matrix for the Stewart platform.

$$\widetilde{K}^{\#} = 3\kappa \begin{bmatrix} 3\,(p/r)^2 & & & & & 0 \\ & 3\,(p/r)^2 & & & & \\ & & 2\left[1 - 3\,(p/r)^2\right] & & & \\ & & & 1 - 3\,(p/r)^2 & & \\ & & & & 1 - 3\,(p/r)^2 & \\ 0 & & & & & 6\,(p/r)^2 \end{bmatrix} \tag{37}$$

The coordinate system, configuration, and parameters used for derivation of $\widetilde{K}^*$ in section 4.2 are identical to those used for obtaining $\widetilde{K}^{\#}$. Therefore, the diagonal elements of these two matrices can be used to compare the stiffness of the minimanipulator to that of the Stewart platform.

Equation (37) shows that decreasing p/r results in higher values for $\widetilde{K}_{3,3}^{\#}$, $\widetilde{K}_{4,4}^{\#}$, and $\widetilde{K}_{5,5}^{\#}$, and lower values for $\widetilde{K}_{1,1}^{\#}$, $\widetilde{K}_{2,2}^{\#}$, and $\widetilde{K}_{6,6}^{\#}$. To avoid having a negative or a zero value for any of the diagonal elements of $\widetilde{K}^{\#}$, p/r should be constrained as shown below.

$$1/\sqrt{3} > p/r > 0 \tag{38}$$

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r/p	1.75	2.0	2.5
$\widetilde{K}^*_{1,1} = \widetilde{K}^*_{2,2}$	3κ	3κ	3κ
$\widetilde{K}^\#_{1,1} = \widetilde{K}^\#_{2,2}$	2.9388κ	2.25κ	1.44κ
$\widetilde{K}^*_{3,3}$	6.1875κ	9κ	15.75κ
$\widetilde{K}^\#_{3,3}$	0.1224κ	1.5κ	3.12κ
$\widetilde{K}^*_{4,4} = \widetilde{K}^*_{5,5}$	6.1875κ	9κ	15.75κ
$\widetilde{K}^\#_{4,4} = \widetilde{K}^\#_{5,5}$	0.0612κ	0.75κ	1.56κ
$\widetilde{K}^*_{6,6}$	12κ	12κ	12κ
$\widetilde{K}^\#_{6,6}$	5.8776κ	4.5κ	2.88κ

Table 1: Sample values for the diagonal elements of $\widetilde{\mathbf{K}}^*$ and $\widetilde{\mathbf{K}}^\#$.

Due to the constraints expressed in equation (38), the diagonal elements of $\widetilde{\mathbf{K}}^\#$ are bounded, as shown below.

$$\widetilde{K}^\#_{1,1} < 3\kappa \tag{39a}$$

$$\widetilde{K}^\#_{2,2} < 3\kappa \tag{39b}$$

$$\widetilde{K}^\#_{3,3} < 6\kappa \tag{39c}$$

$$\widetilde{K}^\#_{4,4} < 3\kappa \tag{39d}$$

$$\widetilde{K}^\#_{5,5} < 3\kappa \tag{39e}$$

$$\widetilde{K}^\#_{6,6} < 6\kappa \tag{39f}$$

Comparing equations (35) and (39), we note that

$$\widetilde{K}^*_{1,1} = 3\kappa > \widetilde{K}^\#_{1,1} \tag{40a}$$

$$\widetilde{K}^*_{2,2} = 3\kappa > \widetilde{K}^\#_{2,2} \tag{40b}$$

$$\widetilde{K}^*_{6,6} = 12\kappa > \widetilde{K}^\#_{6,6} \tag{40c}$$

The other three non-zero elements of $\widetilde{\mathbf{K}}^*$ ($\widetilde{K}^*_{3,3}$, $\widetilde{K}^*_{4,4}$, and $\widetilde{K}^*_{5,5}$) are all functions of r/p. Writing equation (38) in terms of r/p, we obtain

$$\infty > r/p > \sqrt{3} \tag{41}$$

For comparison purposes, let us apply the lower limit of r/p in equation (41) to $\widetilde{K}^*_{3,3}$, $\widetilde{K}^*_{4,4}$, and $\widetilde{K}^*_{5,5}$. We find out that

$$\widetilde{K}^*_{3,3} = \widetilde{K}^*_{4,4} = \widetilde{K}^*_{5,5} > 6\kappa \tag{42}$$

Comparing equations (39) and (42), we note the

$$\widetilde{K}^*_{3,3} > \widetilde{K}^\#_{3,3} \tag{43a}$$

$$\widetilde{K}^*_{4,4} > \widetilde{K}^\#_{4,4} \tag{43b}$$

$$\widetilde{K}^*_{5,5} > \widetilde{K}^\#_{5,5} \tag{43c}$$

Table 1 shows the values for the diagonal elements of $\widetilde{\mathbf{K}}^*$ and $\widetilde{\mathbf{K}}^\#$ corresponding to three typical values of r/p, which satisfy the constraints imposed by equation (41).

The above results confirm that, due to the use of inextensible limbs, the minimanipulator has higher stiffness than the Stewart platform. In particular, direct stiffness in the Z direction, and torsional stiffness in the X and Y directions can be increased drastically by making the ratio of r to p large.

9

5 Summary

In this article, the dimensionally-uniform stiffness matrix of a new three-limbed six-DOF parallel minimanipulator is derived.

Based on the stiffness matrix, the following design guidelines are established for the minimanipulator.

- The central stiffness matrix can be diagonalized (decoupled) by making the platform (triangle $P_1P_2P_3$) one-half the size of the triangle passing through the lower ends of the limbs, i.e. $p = \nu/2$.

- If the central stiffness matrix is decoupled, then

  - The larger the ratio of the limb length to the platform circumradius (r/p), the larger the direct stiffness in the Z-direction, and the larger the torsional stiffness values in the X and Y-directions.

In addition, it is shown that the stiffness of the minimanipulator is higher than that of the Stewart platform.

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