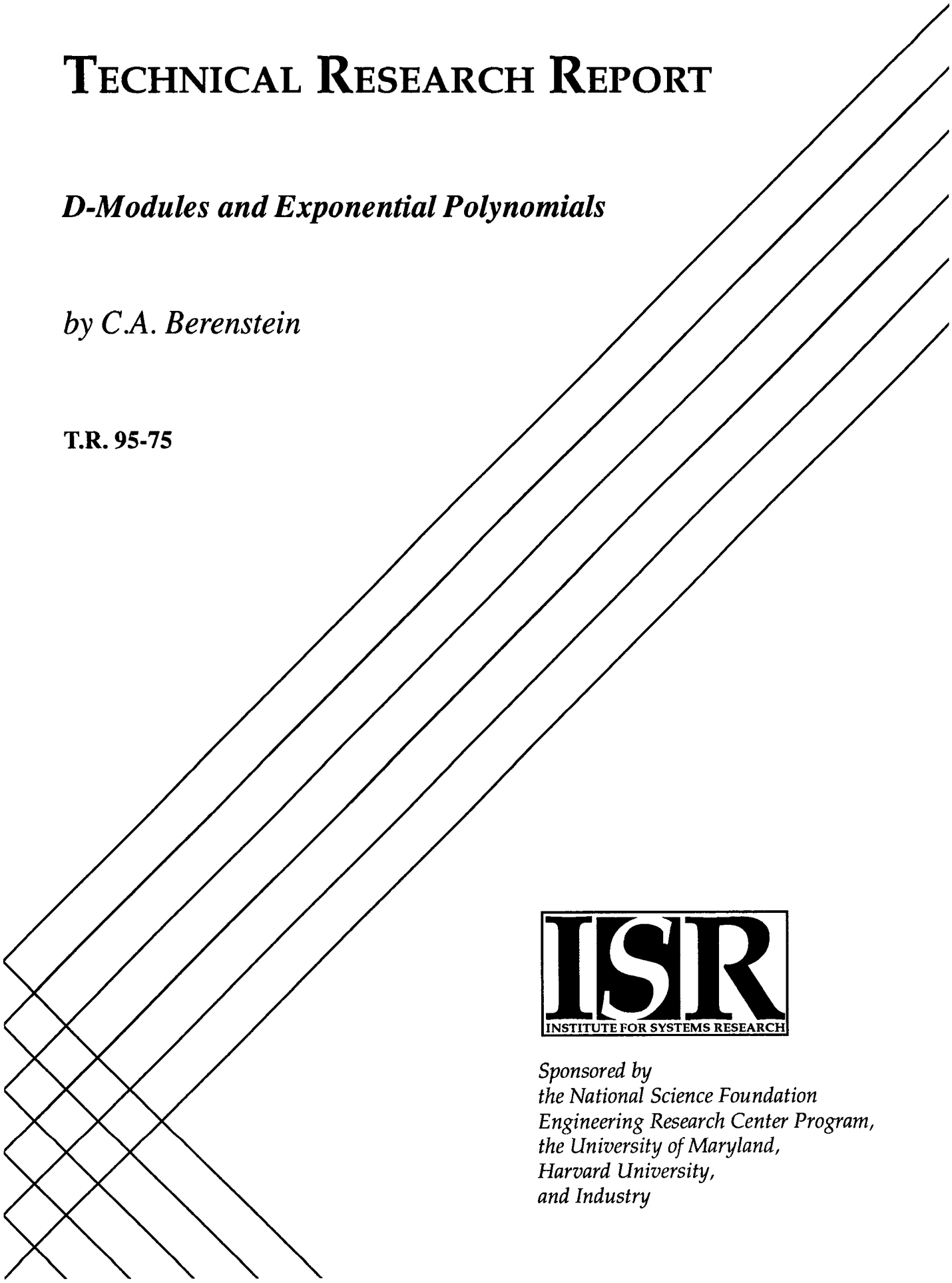
TECHNICAL RESEARCH REPORT

D-Modules and Exponential Polynomials

by C.A. Berenstein

T.R. 95-75



Sponsored by
the National Science Foundation
Engineering Research Center Program,
the University of Maryland,
Harvard University,
and Industry

$\mathcal{D}$-modules and exponential polynomials

Carlos A. Berenstein[1]

One of the fundamental problems of harmonic analysis in $C^\infty(\mathbf{R}^n)$ (or $\mathcal{D}'(\mathbf{R}^n)$) is to decide effectively whether a given homogeneous system of convolution equations

$$\mu_1 * f \;=\; \cdots \;=\; \mu_r * f \;=\; 0, \tag{1}$$

with $\mu_j \in \mathcal{E}'(\mathbf{R}^n)$, has solutions or not, and more generally finding all the possible solutions $f \in C^\infty(\mathbf{R}^n)(f \in \mathcal{D}'(\mathbf{R}^n))$. For $n = 1$ the procedure is "easy", just consider the analytic variety

$$V \;=\; V(\hat{\mu}_1, \ldots, \hat{\mu}_r) \;=\; \{\zeta \in \mathbf{C}^n : \; \hat{\mu}_j(\zeta) \;=\; 0, \; j \;=\; 1, \cdots, r\}, \tag{2}$$

where $\hat{\mu}_j(\zeta) \;=\; <\; \mu_j(x), exp(-ix \cdot \zeta) >$ are the Fourier transforms of the μ_j, and the Schwartz spectral synthesis theorem says that

(i) There is a non-trivial solution to (1) if and only if $V \neq \phi$

(ii) A polynomial exponential $f(x) = p(x)\, e^{i\alpha \cdot x}$, polynomial solves (1) if and only if $\alpha \in V$.

(iii) Every solution f of (1) can be represented as a possibly infinite linear combination of polynomial exponential solutions.

For $n > 1$, statement (ii) is easily seen to be correct but (i), and hence (iii), fail for $n \geq 2$ due to an example of Gurevich. On the other hand, for $r = 1$, the three statements are correct. For $r \geq 2$ the situation is very

[1]This is a report on joint work with A. Yger. This research was partly supported by NSF grant DMS 92-2504. Participation in this conference was made possible by the Taniguchi Foundation.

hard although there is a large class of systems, *slowly decreasing systems*, for which one can still prove (i), (ii) and (iii). (We also say that $(\hat{\mu}_1, \ldots, \hat{\mu}_r)$ form a s.d. system). We refer to the survey paper [BS] for references. We also recall that for systems of linear partial differential equations with constant coefficients, i.e., $\hat{\mu}_1, \ldots, \hat{\mu}_r$ are polynomials, the statements (i), (ii) and (iii) hold due to the Fundamental Principle of Ehrenpreis (see [BS]).

The difficulty in verifying a given system is slowly decreasing is that among other conditions it imposes that the number of equations r coincide with codim V. In particular, for $r = n$, it requires that either V be discrete or empty. In [BKS] we show how to weaken this condition slightly, for codim $V \geq n$ (we allow $V = \phi$), we can allow $r \geq n$. In any case, it seems reasonable to conjecture that when codim $V \geq n$ if the system (1) is reasonably simple then it should be slowly decreasing. The natural systems to consider are those where μ_j are difference-differential operators with integral steps:

$$\mu_j * f(x) = \sum_{k \in \mathbf{Z}^n} p_{jk}(D)\, f(x-k) \tag{3}$$

where the sum is finite and the $p_{jk}(D)$ are differential operators with constant coefficients. In fact, it was proved in [BTY] that for $r = n = 2$, the condition codim $V \geq 2$ implies the system is slowly decreasing. For $n \geq 3$ the last statement does not hold. The reason is a priori surprising. The arithmetic nature of the coefficients plays a role [BY1]. Recall that difference-differential operators μ_j have Fourier transforms that are *exponential polynomials*, that is, entire functions in $\mathbf{C}^n$ of the form

$$f(z) = \sum_k p_k(z)\, e^{ik \cdot z} \tag{4}$$

where to simplify we consider only frequencies $k \in \mathbf{Z}^n$ (the work in [BY1] is more general). One has thus the following natural conjecture. (Here $\bar{\mathbf{Q}}$ denotes the field of algebraic numbers.)

Conjecture *Assume $f_1, \ldots, f_n$ is a system of exponential polynomials in $\mathbf{C}^n$ with integral frequencies and polynomial coefficients $p_{jk} \in \bar{\mathbf{Q}}[z]$ is such that the dim$(V(f_1, \ldots, f_n)) \leq 0$. Then the system must be slowly decreasing.*

One of the properties of a slowly decreasing system is that the ideal generated by $f_1, \ldots, f_r$ localizes. Recall that this means the following. Let

$\widehat{\mathcal{E}}'(\mathbf{R}^n)$ be the space of Fourier transforms of distributions with compact support, then

$$\widehat{\mathcal{E}}'(\mathbf{R}^n) \;=\; \{\varphi \in \mathcal{H}(\mathbf{C}^n) \;:\; \exists A > 0 \text{ such that}$$

$$|\varphi(z)| \;\leq A(1+|z|)^A \; exp(A\;|Im\;z|)\;\forall\; z \in \mathbf{C}^n\}$$

We denote by $I = I(f_1, \ldots, f_m)$ the ideal generated by the functions $f_1, \ldots, f_m$ in $\widehat{\mathcal{E}}'(\mathbf{R}^n)$, $\bar{I}$ its closure, and I_{loc} the local ideal, i.e.,

$$I_{loc} \;=\; \{\varphi \in \widehat{\mathcal{E}}'(\mathbf{R}^n) : \forall\; z \in \mathbf{C}^n\; \exists U \text{ open}, z \in U, \varphi_j \in \mathcal{H}(U),$$

$$\text{so that } \varphi \;=\; \sum_j \; f_j\;\varphi_j \text{ in } U\}$$

An ideal $I \subseteq \widehat{\mathcal{E}}'(\mathbf{R}^n)$ localizes if $\bar{I} = I_{loc}$.

The evidence of [BY1] and [BY2] points out that the last conjecture is related to the following conjecture of Ehrenpreis:

Let φ be an exponential sum of a single variable with coefficients and frequencies that are algebraic, more precisely, let

$$\varphi(z) \;=\; \sum_j \; c_j\; e^{id_j z}, \quad c_j \in \bar{\mathbf{Q}}, \alpha_j \in \bar{\mathbf{Q}} \cap \mathbf{R}. \tag{5}$$

Then, its zeros are it well-separated.

The last statement means the following, let $\{\zeta_k\}$ denote the sequence of distinct zeros of φ, then they are *well-separated* if there are constants $\epsilon > 0, N > 0$ such that for every $j \neq k$

$$|\zeta_k \;-\; \zeta_j| \;\geq\; \frac{\epsilon}{(|\zeta_k| + |\zeta_j|)^N} \tag{6}$$

There are examples showing that the conditions on coefficients and frequencies are both necessary for the conjecture to hold [BY2].

In the case where $f_1, \ldots, f_m$ are polynomials, we have already mentioned that the Fundamental Principle implies the localization. In fact, we can now prove it in some cases, as well as "harder" problems as the effective Nullstellensatz via the study of the behavior of the distribution valued map

$$\lambda \longmapsto\; |f_1 \ldots f_m|^{2\lambda} \quad (Re\lambda >> 0) \tag{7}$$

(See [BGVY] and references, therein). One of the key points for polynomials is the Bernstein-Sato functional equation [Bj]

$$Q(x, \lambda, \partial_x) \; |f_1 \dots f_m|^{2(\lambda+1)} \; = \; b(\lambda) \; |f_1 \dots f_m|^{2\lambda} \tag{8}$$

where $Q(x, \lambda, \partial_x)$ is a linear partial differential operator in the x variables and polynomial coefficients in x and λ. Moreover, b is a monic polynomial. One of the first consequences of this identity is that $|f_1 \dots f_m|^{2\lambda}$ has an analytic continuation as a distribution-valued menomorphic function to the whole λ plane. The existence of (8) follows from the fact that the Weyl algebra $A_n(k)$ of differential operators with polynomial coefficients is holonomic in the sense of [Bj]. In the case of exponential polynomials the Weyl algebra needs to be replaced by another algebra $E_{n,\ell}(K)$ [BY3]. Here we write an exponential polynomial f in $\mathbf{C}^n$ as follows

$$f(z) \; = \; \sum_{\alpha} \; p_\alpha(z) \; e^{\alpha \cdot z}$$

where the sum is finite over the elements α in a lattice $\Gamma \subseteq \mathbf{C}^n$ and the coefficients $p_\alpha \in K[z]$. We let ℓ to be the rank of Γ. Then $E_{n,\ell}(K)$ is the algebra of operators on $K[x_1, \dots, x_n, y_1, \dots, y_\ell]$ generated by $X_i, D_i (1 \leq i \leq n)$ and $Y_j (1 \leq j \leq \ell)$ such that

$$\begin{aligned} X_i p(x,y) \; &= \; x_i p(x,y), \\ Y_i p(x,y) \; &= \; y_j p(x,y) \\ D_i X_k \; &= \; \delta_{ik} \\ D_i y_j \; &= \; \delta_{ij} y_j \end{aligned}$$

(to simplify we assume $\ell \leq n$). Note that a model is $y_j = e^{x_j}$. Note that $E_{n,0}(K) = A_n(K)$. In general $E_{n,\ell}(K)$ is not holonomic in the sense of [Bj]. Moreover, there are simple examples in $E_{n,1}(K)$ where the holonomicity depends on the *arithmetic* nature of K. So one does not expect the existence of a functional equation like (8). Nevertheless, one can find in some cases variations of the Bernstein-Sato functional equation of the following type in $E_{n,1}(K)$ [BY3]

$$\begin{cases} Q_1(\lambda, x, e^{x_1}, e^{-x_1}, \partial_x)\ f^{\lambda+1} &= b_1(x,\lambda)\ f^\lambda \\ Q_2(\lambda, x, e^{x_1}, e^{-x_1}, \partial_x)\ f^{\lambda+1} &= b_2(e^{x_1},\lambda)\ f^\lambda \end{cases} \tag{9}$$

where b_j are polynomials and Q_j differential polynomials. Note that this equation does not allow us to conclude that $\lambda \longmapsto |f|^{2\lambda}$ has an analytic continuation to the whole λ-plane, but this can be bypassed by using Hironaka's resolution of singularities. Nevertheless, there is enough information in (9) to prove a number of results about ideals generated by exponential polynomials in $E_{n,1}(\mathbf{C})$ [BY3].

From now on we only consider $f_1, \ldots, f_p \in E_{n,1}(\mathbf{C})$, i.e., finite sums of the form

$$f_j(z) = \sum_{k\in\mathbf{Z}} p_{j,k}(z)\ e^{kz_1} \tag{10}$$

They belong to the algebra $A_\Phi, \Phi(z) = log(2+|z|) + |Re\ z_1|$, of entire functions f such that

$$\exists\ C>0 \quad |f(z)| \leq exp\ (C\Phi(z)) \quad \forall z \in \mathbf{C}^n.$$

The definition of the local ideal I_{loc} is similar to the one given earlier, we also recall that the radical $\sqrt{I}$ and the local integral closure $\hat{I}$ are given by

$$\begin{aligned} \sqrt{I} &:= \{F \in A_\Phi\ :\ F^k \in I \text{ for some } k \in \mathbf{N}\} \\ \hat{I} &:= \{F \in A_\Phi\ :\ \forall z \in \mathbf{C}^n\ \exists U_z \text{ neighborhood of } z \text{ and } C_z > 0 \text{ such that} \\ &\quad |F(w)| \leq C_z(\textstyle\sum\ |f_j(w)|^2)^{1/2}\ \forall w \in \cup_z\} \end{aligned}$$

We always have

$$I \subseteq \bar{I} \subseteq I_{loc} \subseteq \hat{I} \subseteq I(V)$$

$$\sqrt{I} \subseteq I(V)$$

where $I(V) = \{F \in A_\Phi\ :\ F|V = 0\}, V = V(f_1, \ldots, f_p)$. In general, $I_{loc} \neq I(V)$ because of multiplicities.

In [BY3] we show that using (9) and the methods from [BGVY] one can prove the following:

Theorem 1 *If $f_1, \ldots, f_p \in E_{n,1}(\mathbf{C})$ define a complete intersection variety V, then I is localizable in A_Φ.*

Theorem 2 *Let $I = I(f_1, \ldots, f_p)$ (with no conditions on V), $f_j \in E_{n,1}(\mathbf{C})$, then $\sqrt{I} = I(V)$.*

There is also a variation of the theorem of Briançon-Skoda.

Theorem 3 *Let I be as in Theorem 2, $m = min\{p+1, n\}$, then $\hat{I}^{2m} \subseteq I$.*

When $p = n$ denote by J the Jacobian of $f_1, \ldots, f_n$. We refer to [BT2] for the fact that on discrete interpolation varieties we have good lower bounds on J.

Theorem 4 *Assume $p = n, dimV = 0$, and J is never zero on V. Then V is an interpolation variety for A_Φ.*

We can also use the arithmetic nature of the coefficients and frequencies as follows.

Theorem 5 *Let $\alpha \in \bar{\mathbf{Q}} \backslash \mathbf{Q}, f_1, \ldots, f_r$ polynomials in $e^{\alpha z_1}, e^{z_1}, z_2, \ldots, z_n$ with coefficients in $\bar{\mathbf{Q}}, dimV = n - p$. Then $I = I_{loc}$.*

Theorem 6 *Let $f_1 \ldots f_p$ be as in the last theorem (with no conditions on V) then $\sqrt{I} = I(V)$ and $\hat{I}^{2m} \subseteq I$.*

Theorem 7 *Let $f_1 \ldots f_p$ be the same as in Theorem 5, assume further V is discrete and all points are simple (or more generally, assume V is a manifold) then V is an interpolation variety.*

As a corollary of the above one can prove the following theorem about difference-differential systems. Let $P_j(D)$ represent a differential operator with time lag $T > 0$

$$P_j(D)\varphi(t,x) \;=\; \sum\, p_{j,k,\ell}\; D^\ell \varphi(t - kT, x) \tag{11}$$

where the sum is finite, $p_{j,k,\ell} \in \mathbf{C}, D = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \ldots, \frac{\partial}{\partial x_n})$. The corresponding symbols are the exponential polynomials

$$f_j(\zeta) \;=\; \sum\, p_{j,k,\ell}\; (-i\zeta)^\ell\; e^{ikT\tau}, \tag{12}$$

where $\zeta \;=\; (\tau, \xi), \tau \in \mathbf{C}, \xi \in \mathbf{C}^n$.

Theorem 8 *Let V be the variety in $\mathbf{C}^{n+1}$ defined by $f_1, \ldots, f_{n+1}$* as in (12). *Assume V is discrete and all its points are simple (i.e., $J \neq 0$ on V). Then,*

every solution $\varphi \in C^\infty(\mathbf{R}^{n+1})$ *(or* $\varphi \in \mathcal{D}'(\mathbf{R}^{n+1})$*) of the overdetermined system of differential equations with time lags*

$$P_1(D)\varphi \;=\; \cdots \;=\; P_{n+1}(D)\varphi \;=\; 0 \tag{13}$$

can be represented in a unique way in the form of a series of exponential solutions of the system (13), namely,

$$\varphi(t,x) \;=\; \sum_{\zeta \in V} c_\zeta \; e^{i(t\tau + x\cdot\xi)}$$

The series is convergent in the topology of $C^\infty(\mathbf{R}^{n+1})$ (resp. $\mathcal{D}'(\mathbf{R}^{n+1})$).

References

[BGVY] C. A. Berenstein, R. Gay, A. Vidras, and A. Yger, *Residue currents and Bezout identities*, Birkhäuser, 1993.

[BKS] C. A. Berenstein, T. Kawai, and D. Struppa, Interpolation varieties and the Fabry-Ehrenpreis-Kawai gap theorem, preprint, 1995.

[BS] C. A. Berenstein and D. Struppa, Complex analysis and convolution equations, Chapter 1, Vol. 54, *Several Complex Variables V*, G. Henkin (ed.), *Encyclopedia of Mathematical Sciences*, Springer- Verlag, 1993.

[BT1] C. A. Berenstein and B. A. Taylor, Interpolation problems in $\mathbf{C}^n$ with applications to harmonic analysis, J. Analyse Math. 37 (1980), 128-144.

[BT2] C. A. Berenstein and B. A. Taylor, On the geometry of interpolating varieties, Seminar Lelong-Skoda, 1980-81, Springer-Verlag Lecture Notes in Mathematics 919 (1983), 1-15.

[BTY] C. A. Berenstein, B. A. Taylor, and A. Yger, Sur les systemes d'equations difference-differentials, Ann. Inst. Fourier 33 (1983), 109-130.

[BY1] C. A. Berenstein and A. Yger, Ideals generated by exponential polynomials, Advances in Math. 60 (1986), 1-80.

[BY2] C. A. Berenstein and A. Yger, On Lojawiewicz type inequalities for exponential polynomials, J. Math. Anal. Appl. 129 (1988), 166-195.

[BY3] C. A. Berenstein and A. Yger, Exponential polynomials and $\mathcal{D}$-modules, Compositio Math 95 (1995), 131-181.

[Bj] J. E. Björk, *Rings of differential operators*, North-Holland Publ. Co., 1979.

[E] L. Ehrenpreis, *Fourier analysis in several complex variables*, John Wiley & Sons, 1970.

Carlos A. Berenstein
carlos@src.umd.edu
Institute for Systems Research
University of Maryland
College Park, MD 20742