# TECHNICAL RESEARCH REPORT

## Finite Buffer Realization of Input-Output Discrete Event Systems

*by R. Kumar, V.K. Garg, S.I. Marcus*

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# Corrections to "Finite Buffer Realization of Input-Output Discrete Event Systems" *

Ratnesh Kumar
Department of Electrical Engineering
University of Kentucky
Lexington, KY 40506-0046
Email: kumar@engr.uky.edu

Vijay K. Garg
Department of Electrical and Computer Engineering
University of Texas at Austin
Austin, Texas 78712-1084
Email: vijay@pine.ece.utexas.edu

Steven I. Marcus
Department of Electrical Engineering and
Institute for Systems Research
University of Maryland
College Park, MD 20742
Email: marcus@src.umd.edu

**Abstract**

This note presents a correction to [1, Theorem 4] which provides a necessary and sufficient condition for dispatchability.

## 1 Notation

The following additional notation is introduced: The notation $s \equiv_L s'$ is used to denote that strings $s, s'$ are equivalent under the Nerode equivalence induced by the language $L$.

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Given a DA $G := (Q, \Sigma_I \times \Sigma_O^*, \delta, q^0)$, we use $\pi_I(G)$ to denote its "projection" onto the input events, which is the automaton obtained by replacing each transition label $(\sigma, s) \in \Sigma_I \times \Sigma_O^*$ of $G$ by the label $\sigma$. The projection $\pi_O(G)$ is similarly defined. Note that the projection automaton $\pi_I(G)$ or $\pi_O(G)$ may be nondeterministic even when $G$ is deterministic. Thus [1, Proposition 5] can be restated as: If $\pi_I(G_{BO})$ is deterministic, then $(I, O, \overrightarrow{|B|})$ is dispatchable if and only if it is conditionally dispatchable. Next we define the *input-composition* of $G_I$ and $G_{BO}$, denoted $G_{IBO} := (Q_I \times Q_B \times Q_O, \Sigma_B, \delta_{IBO}, (q_I^0, \vec{0}, q_O^0))$, [1] and defined as $(q_{IBO} := (q_I, \vec{b}, q_O) \in Q_I \times Q_B \times Q_O, s_{IO} := (\sigma, s) \in \Sigma_B)$:

$$\delta_{IBO}(q_{IBO}, s_{IO}) := \begin{cases} \left(\delta_I(q_I, \sigma), \delta_{BO}((\vec{b}, q_O), s_{IO})\right) & \text{if } \delta_I(q_I, \sigma), \delta_{BO}((\vec{b}, q_O), s_{IO}) \text{ are defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Clearly, $L(G_{IBO}) = \{s \in L(G_B) \mid \pi_I(s) \in I \text{ and } \pi_O(s) \in O\}$. Consequently, $L(G_{IBO}) \subseteq L(G_B) \subseteq L(G)$, $\pi_I(L(G_{IBO})) \subseteq I$ and $\pi_O(L(G_{IBO})) \subseteq O$.

It follows from the definition of $G_{IBO}$ that $I \subseteq \pi_I(L(G_{BO}))$ if and only if $I = \pi_I(L(G_{IBO}))$. Hence [1, Theorem 2] can be rephrased as: A dispatching policy $(I, O, \overrightarrow{|B|})$ is conditionally dispatchable if and only if $I = \pi_I(L(G_{IBO}))$. Similarly, [1, Proposition 5] can be stated as: If $\pi_I(G_{IBO})$ is deterministic, then $(I, O, \overrightarrow{|B|})$ is dispatchable if and only if it is conditionally dispatchable.

## 2 Dispatchable Units

Note that in general $\pi_I(G_{IBO})$ may not be deterministic. However, it is easy to construct a subautomaton $G'_{IBO} \leq G_{IBO}$ such that $\pi_I(G'_{IBO})$ is deterministic. If such a subautomaton $G'_{IBO}$ satisfying $I = \pi_I(L(G'_{IBO}))$ exists, then $(I, O, \overrightarrow{|B|})$ is dispatchable. We show below that the converse is also true. We first define the notion of a "canonical" stable and causal input-output map, which requires that whenever the departure sequence pair for a pair of Nerode equivalent arrival sequences $s, s' \in I$ is Nerode equivalent and yields an identical buffer state, then the "future" departure sequence for any "future" arrival sequence $t$ should be identical for $s$ and $s'$. Formally,

**Definition 1** Given a dispatching unit $(I, O, \overrightarrow{|B|})$ and a stable and causal input-output map $\mathcal{D}_I$, it is called *canonical*, if for each $s, s' \in I$ satisfying $s \equiv_I s'$, $\mathcal{D}(s) \equiv_O \mathcal{D}_I(s')$ and $[\vec{s} - \overrightarrow{\mathcal{D}_I(s)}] = [\vec{s'} - \overrightarrow{\mathcal{D}_I(s')}]$, we have $\mathcal{D}_I(st)_{(|\mathcal{D}_I(s)|)} = \mathcal{D}_I(s't)_{(|\mathcal{D}_I(s')|)}$ for each $t \in \Sigma_I^*$.

The next lemma states that a stable and causal input-output map can always be chosen to be a canonical one.

**Lemma 1** Given a dispatching unit $(I, O, \overrightarrow{|B|})$, there exists a stable and causal input-output map over $I$ if and only if there exists a canonical such map.

[1] $\Sigma_B$ was incorrectly defined as $\Sigma_I \times \Sigma^{\leq \|B\|}$ in [1]; its correct definition is $\Sigma_I \times \Sigma^{\leq (\|B\|+1)}$.

**Proof:** It suffices to show the necessity. Suppose a stable and causal input-output map $\mathcal{D}_I : I \to O$ is given. If it is not canonical, then there exist $s, s' \in I$ and $t \in \Sigma_I^*$ such that $s \equiv_I s'$, $\mathcal{D}_I(s) \equiv_O \mathcal{D}_I(s')$, $[\vec{s} - \overrightarrow{\mathcal{D}_I(s)}] = [\vec{s'} - \overrightarrow{\mathcal{D}_I(s')}]$, but $\mathcal{D}_I(st)_{(|\mathcal{D}_I(s)|)} \neq \mathcal{D}_I(s't)_{(|\mathcal{D}_I(s')|)}$. A canonical stable and causal input-output map $\mathcal{D}'_I$ can be obtained from $\mathcal{D}_I$ by defining it to be the same as $\mathcal{D}_I$ except that for every prefix $\hat{t} \leq t$, it maps the arrival sequence $s'\hat{t}$ to the departure sequence $\mathcal{D}_I(s')\mathcal{D}_I(s\hat{t})_{(|\mathcal{D}_I(s)|)}$. By definition, $\mathcal{D}_I(s\hat{t}) = \mathcal{D}_I(s)\mathcal{D}_I(s\hat{t})_{(|\mathcal{D}_I(s)|)} \in O$, so $\mathcal{D}_I(s) \equiv_O \mathcal{D}_I(s')$ implies $\mathcal{D}'_I(s'\hat{t}) = \mathcal{D}_I(s')\mathcal{D}_I(s\hat{t})_{(|\mathcal{D}_I(s)|)} \in O$. Also since $\mathcal{D}_I$ is causal, it follows that $\mathcal{D}'_I$ is causal. Finally since $[\vec{s} - \overrightarrow{\mathcal{D}_I(s)}] = [\vec{s'} - \overrightarrow{\mathcal{D}_I(s')}]$, it follows that

$$
\begin{aligned}
[\overrightarrow{s\hat{t}} - \overrightarrow{\mathcal{D}_I(s\hat{t})}] &= [\vec{s} - \overrightarrow{\mathcal{D}_I(s)}] + [\vec{\hat{t}} - \overrightarrow{\mathcal{D}_I(s\hat{t})_{(|\mathcal{D}_I(s)|)}}] \\
&= [\vec{s'} - \overrightarrow{\mathcal{D}_I(s')}] + [\vec{\hat{t}} - \overrightarrow{\mathcal{D}'_I(s'\hat{t})_{(|\mathcal{D}'_I(s')|)}}] \\
&= [\overrightarrow{s'\hat{t}} - \overrightarrow{\mathcal{D}'_I(s'\hat{t})}],
\end{aligned}
$$

i.e., the buffer capacity constraint is also satisfied, which implies $\mathcal{D}'_I$ is stable. ■

The following theorem corrects the error in [1, Theorem 4].

**Theorem 1** A dispatching unit $(I, O, \overrightarrow{|B|})$ is dispatchable if and only if there exists a subautomaton $G'_{IBO} \leq G_{IBO}$ such that $\pi_I(G'_{IBO})$ is deterministic and $I = \pi_I(L(G'_{IBO}))$.

**Proof:** ($\Rightarrow$) First assume that $(I, O, \overrightarrow{|B|})$ is dispatchable. We need to show that there exists a subautomaton $G'_{IBO} \leq G_{IBO}$ such that $\pi_I(G'_{IBO})$ is deterministic and $I = \pi_I(L(G'_{IBO}))$. From hypothesis there exists a stable and causal input-output map $\mathcal{D}_I : I \to O$. From Lemma 1 it can be chosen to be canonical. Using this input-output map construct a subautomaton $G'_{IBO} := (Q_I \times Q_B \times Q_O, \Sigma_B, \delta'_{IBO}, (q_I^0, \vec{0}, q_O^0)) \leq G_{IBO}$, where the transition function is defined as $(q = (q_I, \vec{b}, q_O) \in Q_I \times Q_B \times Q_O, s_{IO} = (\sigma_I, s_O) \in \Sigma_B = \Sigma_I \times \Sigma_O^{\leq(\|B\|+1)}$):

$$
\delta'_{IBO}(q, s_{IO}) := \begin{cases} \delta_{IBO}(q, s_{IO}) & \exists s_I \in I \text{ s.t. } \delta^*_{IBO}((q_I^0, \vec{0}, q_O^0), (s_I, \mathcal{D}_I(s_I)) = q, \\ & \text{and } \mathcal{D}_I(s_I\sigma_I)_{(|\mathcal{D}(s_I)|)} = s_O \\ \text{undefined} & \text{otherwise} \end{cases}
$$

Note if there exists another arrival sequence $s'_I \in I$ such that $\delta^*_{IBO}((q_I^0, \vec{0}, q_O^0), (s'_I, \mathcal{D}_I(s'_I)) = q$, then $s_I \equiv_I s'_I$, $\mathcal{D}_I(s_I) \equiv_O \mathcal{D}_I(s'_I)$, and $[\vec{s_I} - \overrightarrow{\mathcal{D}_I(s_I)}] = [\vec{s'_I} - \overrightarrow{\mathcal{D}_I(s'_I)}] = \vec{b}$. Since $\mathcal{D}_I$ is canonical, this implies $\mathcal{D}(s_I\sigma_I)_{(|\mathcal{D}_I(s_I)|)} = s_O = \mathcal{D}(s'_I\sigma_I)_{(|\mathcal{D}(s'_I)|)}$, i.e., there is at most one departure sequence for the arrival event $\sigma_I$. So $\pi_I(G'_{IBO})$ is deterministic.

Since $\pi_I(L(G'_{IBO})) \subseteq \pi_I(L(G_{IBO})) \subseteq I$, it remains to show that the reverse inequality also holds. We use induction on the length of strings in $I$ to prove that if $s_I \in I$, then $s_I \in \pi_I(L(G'_{IBO}))$. In fact we prove a stronger claim:

$$
s_I \in I \Rightarrow \delta'^*_{IBO}\left((q_I^0, \vec{0}, q_O^0), (s_I, \mathcal{D}_I(s_I)\right) = \left(\delta_I^*(q_I^0, s_I), [\vec{s_I} - \overrightarrow{\mathcal{D}_I(s_I)}], \delta_O^*(q_O^0, \mathcal{D}_I(s_I))\right). \quad (1)
$$

Note that the condition of (1) implies that $(s_I, \mathcal{D}_I(s_I)) \in L(G'_{IBO})$, which in turn implies that $s_I \in \pi_I(L(G'_{IBO}))$. The condition of (1) certainly holds for the zero length string $\epsilon \in I$ since $\mathcal{D}_I(\epsilon) = \epsilon$. Hence the base step holds. In order to prove the induction step, consider $s_I \in I$ and $\sigma_I \in \Sigma_I$ such that $s_I\sigma_I \in I$. Then from induction hypothesis, (1) holds. Let $q_I := \delta_I^*(q_I^0, s_I)$, $\vec{b} := [\vec{s_I} - \overrightarrow{\mathcal{D}_I(s_I)}]$ and $q_O := \delta_O^*(q_O^0, \mathcal{D}_I(s_I))$. Then it follows from the definition of the transition function of $G'_{IBO}$ that

$$[\delta'_{IBO}((q_I, \vec{b}, q_O), (\sigma_I, s_O)) = \delta_{IBO}((q_I, \vec{b}, q_O), (\sigma_I, s_O))] \Longleftrightarrow [s_O = \mathcal{D}_I(s_I\sigma_I)_{(|\mathcal{D}_I(s_I)|)}]. \quad (2)$$

Thus by combining (1) and (2) we obtain the desired result of induction step:

$$\delta'^*_{IBO}\left((q_I^0, \vec{0}, q_O^0), (s_I\sigma_I, \mathcal{D}_I(s_I\sigma_I))\right) = \left(\delta_I^*(q_I^0, s_I\sigma_I), [\overrightarrow{s_I\sigma_I} - \overrightarrow{\mathcal{D}_I(s_I\sigma_I)}], \delta_O^*(q_O^0, \mathcal{D}_I(s_I\sigma_I))\right).$$

($\Leftarrow$) Next assume that there exists a subautomaton $G'_{IBO} \leq G_{IBO}$ such that $\pi_I(G'_{IBO})$ is deterministic and $I = \pi_I(L(G'_{IBO}))$. We need to show that $(I, O, \overrightarrow{|B|})$ is dispatchable. Construct an equivalent DMA, $M'_{IBO}$ for the DA $G'_{IBO}$. This is possible since $\pi_I(G'_{IBO})$ is deterministic. Then $I = \pi_I(L(G'_{IBO})) = L_I(M'_{IBO}), \pi_O(L(G'_{IBO})) = L_O(M'_{IBO}) \subseteq O$ and $L(G'_{IBO}) = L(G_{M'_{IBO}}) \subseteq L(G_B)$. Hence it follows from [1, Proposition 4] that $(I, O, \overrightarrow{|B|})$ is dispatchable. ■

**Example 1** Consider the setting of [1, Example 7]. As mentioned above, the corresponding DFA $G_{BO}$ is shown in [1, Figure 4(a)]. Also, as noted in [1, Example 9] condition C1 does not hold in this case. Thus although the triple $(I, O, \overrightarrow{|B|})$ is conditionally dispatchable the test for sufficiency of dispatchability as given in [1, Proposition 5] is not applicable. So we construct the DFA $G_{IBO}$ as shown in Figure 1. Clearly, $\pi_I(G_{IBO})$ is nondeterministic. However, the sub-automaton $G'_{IBO} \leq G_{IBO}$ lying within the dashed rectangular area of Figure 1 is such that $\pi_I(G'_{IBO})$ is deterministic and $I = \pi_I(L(G'_{IBO}))$. Thus it follows from Theorem 1 that the triple $(I, O, \overrightarrow{B})$ is dispatchable. The required dispatching policy is obtained as discussed in [1, Example 8]. ■

# Acknowledgment

The error in [1, Theorem 4] was found when Shigemasa Takai of University of Osaka asked a clarification for a proof step of this theorem.

# References

[1] R. Kumar, V. K. Garg, and S. I. Marcus. Finite buffer realization of input-output discrete event systems. *IEEE Transactions on Automatic Control*, 40(6):1042–1053, June 1995.

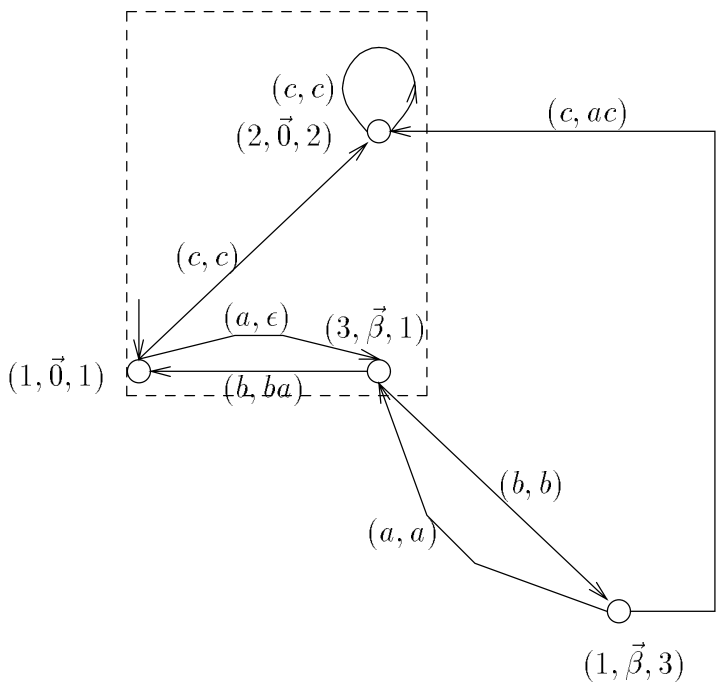


Figure 1: Diagram illustrating $G_{IBO}$ and $G'_{IBO}$