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Adaptive Control of Nonlinear Systems with Applications to Flight Control Systems and Suspension Dynamics

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Adaptive Control of Nonlinear Systems with Applications to Flight Control Systems and Suspension Dynamics

by

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Abstract

Title of Dissertation: Adaptive Control of Nonlinear Systems with Applications to Flight Control Systems and Suspension Dynamics

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Dissertation directed by: Professor G. L. Blankenship
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In this dissertation, we employ recent theoretical advances in differential geometric formulation of nonlinear control theory and adaptive control to develop a practical adaptive nonlinear control strategy.

We first present a new scheme for tracking and decoupling of multi-input/multi-output nonlinear systems with parametric uncertainty in their dynamics. We obtain an adaptive right-inverse that can be used as a decoupling prefilter for the original system and to generate the input necessary such that the outputs track a desired path. The procedures are systematic and have been implemented into a computer code. An integrated symbolic-numerical software system, written in **Mathematica** and C, has been developed that includes capabilities for automatic generation of model equations, for design of nonlinear tracking, regulation, stabilization, and adaptive control laws, and for generation of simulation codes (in C) for performance evaluations. This system is then used to design a nonlinear adaptive control algorithm for active suspensions for vehicles with the objective to effectively isolate the sprung body dynamics from the road disturbances. We also consider the design of a magnetic levitation control system.

For systems that do not satisfy the restrictive regularity assumptions of the current adaptive nonlinear control methodologies, commonly based on exact

feedback linearization technique, we develop a technique of adaptive approximate tracking and regulation. This technique achieves reasonable stable tracking performance under parameter uncertainty in nonlinear dynamics for a large class of nonlinear systems with guaranteed bounds on the tracking error and parameter estimates. While the controller structure is designed using the approximate system, the adaptive loop is constructed around the true system in order to avoid any parameter drift typically caused by dynamic uncertainty in the system. Furthermore, for adaptive regulation, our scheme removes the linear parameter dependence assumption on the location of the unknown parameters. It also replaces the involutivity condition for exact feedback linearization with an order n involutivity assumption for approximate feedback linearization. For nonlinear systems that are linearly controllable, we give a simple systematic design procedure using a dynamic state feedback that achieves adaptive quadratic linearization.

We then investigate the use of this technique in the design of flight control systems using a simplified planar VTOL aircraft model. While due to the non-minimum phase property of the VTOL system the previous results in adaptive nonlinear control theory are not applicable, a comparison between the performance of our adaptive controller to the non-adaptive case reveals that the adaptive controller performs about 90% better in signal tracking.

Dedication

To My Wife, Tara Barkley, and

My Son, Julien Mehdi

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Part I

Preliminaries

Chapter 1

Introduction

Design of high-performance control systems is essentially a problem of obtaining a nominal model for the dynamics characteristics of the physical system considered for compensation and deriving a control law that achieves a desired objective using the nominal model. If the model is accurate, i.e. system dynamics are known with sufficient accuracy, then the specified controller will continue to meet the performance specifications as long as the system characteristics do not change. However, should the changes become large, the controller, as originally designed with fixed parameters, will fail to meet the design specifications, e.g. stability, output tracking, or state regulation.

In practice, the assumed nominal model most likely differs from the true system. This difference, generally called modeling errors or **system uncertainties**, can lead to a wide range of problems, from performance degradation to eventually loss of stability. Therefore, any effective control system should, up to some degree, be able to deal with the changes and uncertainties that occur in the system over various operating conditions. For sometime, there has been interest in the design of **robust** controllers that compensate for system uncertainties and

still achieve objectives such as output tracking and regulation. System uncertainties may arise in several different ways, and in the design of robust controllers for uncertain systems it is important to know the cause of the modeling error in a given situation.

In engineering practice two of the most important sources of modeling error are the presence of nonlinearities in the system and lack of exact knowledge of some of the system parameters. This is common in flight control systems, robotic manipulators with rigid or flexible joints, automotive vehicles, electric motors, chemical processes, etc. For instance, the moment of inertia in a simple industrial robot changes as a function of the load and geometry, with typical variations of the order 1:5 [8]. Or consider the control of high-performance airplanes where the dynamics depend on speed, altitude, mass, etc. These parameters continually change during airplane operation. Therefore, in many situations it is useful to update a model of the system on-line while the system is in operation. The update rules should then be based on observation signals up to the current time.

Nonlinearities are an intrinsic part of almost all physical systems. Over the past fifteen years there has been a considerable progress in geometric nonlinear control theory leading to powerful tools for nonlinear feedback design. Often, when the nonlinearities are considered in the nominal model, the common approach for output tracking is, if possible, to employ a nonlinear state feedback control that results in **exact** cancellation of the nonlinear terms appearing in the input-output map. In theory, this renders the input-output behavior of the resulting feedback system linear. For the problem of regulation, using coordinate changes, one employs a nonlinear state feedback to achieve exact state linearization [51, 81]. Major limitations to these approaches come from the fact

that they require certain regularity conditions such as involutivity, existence of a (vector) relative degree, and the minimum phase property. The main drawback in **implementing** exactly linearizing control laws from differential geometric control theory is that they rely on **exact** cancellation of nonlinear terms. As a result of the above limitations, the applications of nonlinear differential geometric design techniques to engineering practice have been limited. In most cases when the forms of nonlinearities are known from physical properties, some of the parameters appearing in the nonlinear terms, such as mass, moment of inertia, resistance, damping coefficients, etc., are unknown. This problem arises in many practical situations where system parameters are either not exactly known, or they change over time while the controller with constant parameters is originally designed based on the nominal values of these parameters. Note that this type of uncertainty is even more significant when nonlinearities are present in the true system.

The necessity to consider the above modeling errors in the feedback control design motivates the idea of **nonlinear adaptive control**; system nonlinearities are kept in the **model** as much as it is possible and practical, and uncertain parameters in the resulting nonlinear model are updated on-line based on the response of the **true** system. Such combination of tools in nonlinear analysis and adaptive design has a significant potential to outperform any nonadaptive nonlinear or adaptive linear control scheme by introducing a more robust nonlinear controller over a broader range of operating conditions [61].

Parameter adaptive control theory [78, 88, 3, 97, 57, 98, 12, 56] has offered a promising approach to compensate for the parameter mismatch problem in nonlinear feedback design. With a broad definition, the adaptive control problem

is to generate a feedback controller for an **unknown** plant subject to some clear control objectives such as setpoint regulation, asymptotic tracking, or stability. Applications of adaptive control to aircraft control, autopilots for tankers and ships, process control, robotics, and power systems among others have been investigated in the past [8].

Adaptive control of **nonlinear** systems has received a lot of attention in the last five years and many exciting new theoretical results [78, 88, 3, 97, 57, 98, 12, 56], and applications [13, 16, 14] have been obtained. This is mainly due to the recent advances in differential geometric techniques in control theory over the last decade. The work in this area involves the names of Akhrif, Arapostathis, Bastin, Blankenship, Campion, Isidori, Kanellakopoulos, Kokotovic, Nam, Marino, Morse, Pomet, Praly, Sastry, Slotine, Taylor, and Teel among others. The available geometric nonlinear adaptive control schemes are based on exact feedback linearization theory and they suffer from several limitations due to the stringent regularity conditions required for exact feedback linearization.

In this dissertation, we employ recent theoretical advances in differential geometric formulation of nonlinear control theory to develop practical adaptive nonlinear control design tools. Since the language of our formulation is differential geometry, which is not readily accessable to most design engineers, we have developed a prototype integrated symbolic-numerical software system, in **Mathematica**, for computer aided design and analysis of nonlinear models. This software system includes capabilities for automatic generation of model equations from the Lagrangian formulation of nonlinear multibody systems, for design of nonlinear tracking, regulation, stabilization, and adaptive control laws, partly developed in this dissertation, and for generation of simulation codes (in

C) for performance evaluation. The outline of this dissertation is as follows: Part One is the preliminary element to this dissertation. In Chapter 2 we review some of the basic notions of geometric nonlinear control theory that we shall use in the sequel.

In Chapter 3 we shall give an introduction to adaptive nonlinear control theory. We will review some of the important contributions in this rapidly developing field such as the works of Sastry and Isidori [88], Nam and Arapostathis [78], Akhrif and Blankenship [4, 5, 3], Kanellakopoulos, Kokotovic, Marino, Morse, [55, 56, 57], Teel [98], and Krstic and Kokotovic [68]. We shall describe the techniques of direct, indirect, and semi-indirect adaptive control and illustrate adaptive backstepping procedure of Kokotovic and co-workers [68].

In Part Two we present the theoretical contribution of this dissertation: For the adaptive tracking problem, we have removed several restrictive assumptions in the adaptive nonlinear control literature, namely:

- existence of a well-defined (vector) relative degree (necessary condition for feedback input-output linearization)
- exponentially minimum phase property (exponentially stable zero-dynamics)
- conic continuity assumptions on growth of system nonlinearities.

For the adaptive regulation problem, we have removed the following restrictive assumptions:

- involutivity condition (necessary for state feedback linearization).
- linear parameter dependence of the unknown parameters.

- conic continuity assumptions on growth of system nonlinearities.

In each case we substitute much milder assumptions such as existence of a **robust relative degree**, i.e. relative degree for the Jacobian linearization, **slightly** minimum phase property in the sense of [39], order ρ approximate involutivity assumption, and local Lipschitz condition. Note that, as we shall see in Chapter 6, any linearly controllable nonlinear system can be made order ρ involutive, for any ρ, using a **dynamic state feedback**. The problem of adaptive quadratic regulation is treated in detail.

In Chapter 4 we present two new schemes for the tracking and decoupling of multi-input/multi-output (MIMO) nonlinear systems with parametric uncertainty in their dynamics. The first approach is an adaptive version of the well-known inversion algorithm developed by Hirschorn [42] and Singh [91] for the inversion of the input-output map of a MIMO nonlinear system. After deriving a sequence of subsystems by remapping the outputs, an adaptive right-inverse is obtained. The inverse system then can be used as a decoupling prefilter that produces the input to the original system such that the outputs track a desired path. Therefore, using output feedback and precompensation, adaptive asymptotic functional reproducibility can be achieved. The second approach is based on the generalized normal form of nonlinear systems [51]. These schemes do not use any restrictive growth condition on the nonlinearities. They both assume that unknown parameters appear linearly in the model. We illustrate the features of this scheme using a nonlinear system arising in the outer-loop design of an aircraft [7, 90].

In Chapter 5 we describe a technique of **indirect** adaptive control for **approximate** linearization of nonlinear systems. The adaptive controller can

achieve tracking of reasonable trajectories with small error for **slightly non-minimum phase** systems. It can also be applied to nonlinear systems where the relative degree is not well defined. In this scheme an observer-based parameter identifier is employed that continuously adjusts the parameter estimates on-line based on observation error. We show that although the system can not be **exactly** input-output linearized to achieve exact output tracking of reference trajectories, we can still achieve good tracking of small trajectories. We then demonstrate the results on the familiar ball and beam laboratory experiment, and compare the performance of the adaptive controller with the non-adaptive control under parameter uncertainty in the mass of the ball.

In Chapter 6 we present a **direct** adaptive control scheme for nonlinear systems that fail to meet the restrictive regularity conditions of the current nonlinear adaptive control schemes in the literature necessary for feedback state or input-output linearization. We provide an adaptive **approximate** tracking scheme for systems that do not have a well-defined (vector) relative degree, nor can they achieve a vector relative degree through the dynamic inversion or extension of Chapter 4 and 2. The results are also applicable to **slightly** non-minimum phase nonlinear systems with unknown parameters in their dynamics. We prove that the design scheme results in an asymptotically stable closed loop system and show that the controller can achieve adaptive tracking of reasonable (small magnitude slowly varying) trajectories with bounds on the tracking error. We also present a state regulation scheme based on **state approximate linearization**. This scheme is applicable to a large class of nonlinear systems that are not necessarily feedback linearizable. In particular, we present a systematic adaptive regulation technique for **linearly** controllable uncertain nonlinear systems

where unknown parameters do not necessarily appear linearly in their dynamics. Simulations are provided for two "benchmark" examples of adaptive nonlinear control design.

In Part Three we consider several applications of adaptive nonlinear control theory and illustrate some of the design tools we developed in the previous part.

In Chapter 7 we consider a simplified model of the Harrier aircraft studied in [40] and compare the performance of the adaptive tracking controller, developed in Chapter 6, to the performance of the non-adaptive tracking controller under parametric uncertainties in the mass and the moment of inertia of the aircraft. We will show that this system is very sensitive to the parameter uncertainties in its dynamics, and when these parameters are not exactly known for feedback control, the performance is very poor and unacceptable in practice. Due to the non-minimum phase nature of the aircraft dynamics, none of the existing adaptive nonlinear schemes, in particular [78, 88, 3, 97, 57, 98, 12, 56], is applicable to this situation. This illustrates the significance of the approximation technique used to develop the theoretical framework of our new adaptive control design scheme of Chapter 6.

In Chapter 8 we shall describe some computational tools developed for the design and analysis of nonlinear control systems. We use results in nonlinear control theory and adaptive control scheme of Chapter 4 to derive adaptive nonlinear control laws for such systems using an integrated symbolic-numerical software system. This system is developed for computer aided system analysis and design of nonlinear multibody models for vehicle and robot subsystems and implementation of the design techniques for nonlinear tracking/regulation and adaptive control. We illustrate the use of this system by considering the design

of nonlinear adaptive control algorithms for active suspensions for vehicles, focusing on a model of the Army High Mobility Multipurpose Wheeled Vehicle (HMMWV). We will show that it is possible, using adaptive asymptotic tracking control laws, to effectively isolate the sprung body dynamics (center of mass velocity and pitch rate) from the road disturbances. This allows substantial enhancements in vehicle performance in several areas including increased travel speed and platform stability, and decreased passenger absorbed power for better ride quality, among others. In this chapter we will also consider applications to the design of a magnetic levitation control system under parameter uncertainty.

Finally in Chapter 9 we shall discuss topics for future research in this field.

Chapter 2

Introduction to Geometric Control Theory

In this chapter we give a brief review of some of the tools of geometric control theory [81, 51, 18, 107, 96, 41]. In particular, concepts of the linearization theory, minimum phase nonlinear systems, and dynamic extension algorithm are described.

The design and analysis of nonlinear control systems using tools from differential geometry had its starting point in the important papers of Brockett and Krener in the early 1970's [19, 20, 63]. Other important early contributions were made by Hunt, Su, Meyer, Jakubczyk, Respondek, Hirshorn, and Singh, among many others [42, 43, 49, 48, 47, 52, 90, 91, 92]. While the subject continues to evolve, a body of techniques has been developed and have proven useful in the treatment of a wide range of problems [13, 16, 31, 36, 77, 100, 95, 94, 101].

2.1 Mathematical Tools

We first introduce some very basic mathematical tools from differential geometry in the context of nonlinear dynamic systems. We shall use the term smooth function f to mean that each component of the function $f(x)$ is differentiable as many times as required. A set $M \in \mathcal{R}$ is a smooth manifold of dimension n if it is locally diffeomorphic to $\mathcal{R}^n$. A **diffeomorphism** is a smooth bijective (one to one, onto) function $\Phi : \mathcal{R}^n \to \mathcal{R}^n$, defined in a region $\Omega \subset \mathcal{R}^n$, which has a smooth inverse Φ^{-1}. If the region Ω is the whole space $\mathcal{R}^n$, then $\Phi(x)$ is called a **global diffeomorphism**, otherwise it is called a **local diffeomorphism**. In systems theory, global diffeomorphisms are very rare and one often looks for local diffeomorphism. In general, manifolds are locally diffeomorphic to $\mathcal{R}^n$.

The concept of diffeomorphism can be viewed as a generalization of the concept of coordinate transformation in linear systems. This concept can be used as a tool to transform a nonlinear system into another nonlinear system in terms of a new set of states. For example, consider a nonlinear system of the form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

and suppose that a new set of states is defined by a diffeomorphism: $z = \Phi(x)$. The new state space description in states z is then obtained by differentiating z to get:

$$\begin{aligned} \dot{z} &= \tilde{f}(z) + \tilde{g}(z)u \\ y &= \tilde{h}(z) \end{aligned}$$

where $x = \Phi^{-1}(z)$ has been used to obtain new smooth functions $\tilde{f}, \tilde{g}$ and $\tilde{h}$.

A vector field on $\mathcal{R}^n$ is a smooth function $f : \mathcal{R}^n \to \mathcal{R}^n$:

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} \tag{2.1}$$

Given a smooth scalar function $h(x)$, the gradient of h is denoted by $\mathcal{D}h$ represented by a $row - vector$:

$$\mathcal{D}h = \frac{\partial h}{\partial x}$$

Given a vector field $f(x)$, the Jacobian of f, denoted by $\mathcal{D}f$, is an $n \times n$ matrix:

$$\mathcal{D}f = \frac{\partial f}{\partial x} = \left[\frac{\partial f_i}{\partial x_j}\right]$$

The following lemma is a straightforward consequence of the implicit function theorem which can be used to check whether a function is a local diffeomorphism:

Lemma 2.1.1 Let $\Psi(x)$ be a smooth function defined in a region $\Omega \subset \mathcal{R}^n$. If the Jacobian matrix $\mathcal{D}\Phi$ is nonsingular at a point $x_0 \in \Omega$, then $\Phi(x)$ defines a local diffeomorphism in a neighborhood of x_0 in Ω.

Given two vector fields f and g, we define their Lie bracket to be the vector field:

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = \mathcal{D}g \cdot f - \mathcal{D}f \cdot g \tag{2.2}$$

where $\mathcal{D}f, \mathcal{D}g$ are the Jacobian matrices. Another way of representing (2.2) is:

$$[f, g] \cdot e_i = \sum_{j=1}^{n} \left\{ f_j \frac{\partial g_i}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j} \right\} \tag{2.3}$$

where e_i is the ith coordinate function. Since the Lie bracket of two vector fields is another vector field, we may repeat the operation recursively, defining the **ad** operator:

$$\begin{aligned} ad_f^0 g &\triangleq g \\ ad_f^k g &\triangleq [f, ad_f^{k-1} g] \qquad k > 0 \end{aligned} \tag{2.4}$$

Note that the Lie bracket of two constant vectors is a zero vector. Lie brackets satisfy the following properties:

(i) bilinearity:

$$\begin{aligned} [\alpha_1 f_1 + \alpha_2 f_2, g] &= \alpha_1 [f_1, g] + \alpha_2 [f_2, g] \\ [f, \alpha_1 g_1 + \alpha_2 g_2] &= \alpha_1 [f, g_1] + \alpha_2 [f, g_2] \end{aligned}$$

where α_1, α_2 are constant scalars.

(ii) skew-community:

$$[f, g] = -[g, f]$$

(iii) Jacobi identity:

$$[f, [g, h]] + [h, [f, g]] + [g, [h, f]] = 0$$

where $h(x)$ is a smooth scalar function of x.

The Lie derivative of a smooth scalar function $h(x)$ with respect to a vector field $f : \mathcal{R}^n \to \mathcal{R}^n$ is a new scalar function defined as:

$$\mathcal{L}_f h(x) = \mathcal{D}h \cdot f(x) \tag{2.5}$$

Similarly, repeated Lie derivatives are defined recursively by:

$$\begin{aligned} \mathcal{L}_f^0 h &= h \\ \mathcal{L}_f^i h(x) &= \mathcal{L}_f(\mathcal{L}_f^{i-1} h(x)) = \mathcal{D}(\mathcal{L}_f^{i-1} h(x)) \cdot f(x) \end{aligned}$$

Given a smooth mapping $f : U \to V$, where U and V are open subsets of $\mathcal{R}^p$ and $\mathcal{R}^q$ respectively with $p \geq q$, we say $x \in U$ is a regular point of f if the rank of $\mathcal{D}f$ is equal to q. Similarly, a point $y \in f(U) \subset V$ is called a regular value of f if the inverse image of y under f contains only regular points.

Lemma 2.1.2 If $f : M \to N$ is a smooth mapping between manifolds of dimension $m \geq n$, and if $y \in N$ is a regular value, then the set $f^{-1}(y) \subset M$ is a smooth manifold of dimension $m - n$.

A smooth k-dimensional distribution $\triangle$ on a manifold M is the span of a set of linearly independent vector fields $f_i,\ i = 1, \ldots, k$:

$$\triangle = \mathbf{span}\{f_1, f_2, \ldots f_n\} \tag{2.6}$$

A distribution $\triangle$ is called involutive if it is closed under Lie brackets, i.e. if $f_1, f_2 \in \triangle$, then we have $[f_1, f_2] \in \triangle$. In other words, a linearly independent set of vector fields $\{f_1, \ldots, f_n\}$ is said to be involutive if and only if:

$$[f_i, f_j](x) = \sum_{k=1}^{n} c_{ijk}(x) f_k(x) \quad \forall i, j \tag{2.7}$$

for some scalar functions $c_{ijk}(x)$.

Given a set of vector fields $\{f_1, \ldots, f_n\}$, it is straightforward to check whether it is involutive by checking the following involutivity condition:

$$rank\ [f_1(x), \ldots, f_k(x)] = \mathbf{rank}\ [f_1(x), \ldots, f_k(x), [f_i, f_j](x)] \quad \forall x, \forall i, j \tag{2.8}$$

Note that as an easy consequence of this test, constant vector fields are always involutive since their Lie brackets are zero. Sets composed of a single vector field are also always involutive since $[f, f] = 0$.

A linearly independent set of vector fields $\{f_1, \ldots, f_m\}$ is said to be **completely integrable** if and only if there exists $n-m$ scalar functions $h_1(x), \ldots,$ $h_{n-m}(x)$ satisfying the system of partial differential equations:

$$\mathcal{L}_{f_j} h_i(x) = 0 \qquad 1 \leq i \leq n-m, 1 \leq j \leq m, x \in U \subset \mathcal{R}^n$$

and the Jacobians $\mathcal{D}h_i$ are linearly independent.

The following theorem relates the two concepts of involutivity and integrability:

Theorem 2.1.3 (Frobenius) A set of linearly independent vector fields $\{f_1, \ldots,$ $f_m\}$ on $\mathcal{R}^n$ is completely integrable if and only if it is involutive.

2.2 Full and Partial State Linearization by Feedback

In this section we discuss state linearization of a nonlinear system of the form:

$$\dot{x} = f(x) + g(x)u \tag{2.9}$$

using input and state transformations. We also discuss the conditions for the existence of such transformations along with the design of controllers based on this technique.

Consider a nonlinear system of the form (2.9) with initial state x^0. This system is fully state linearizable if there exists a diffeomorphism $\Phi : U \rightarrow \mathcal{R}^n$, and a nonlinear feedback control law :

$$u_i = \alpha(x) + \beta(x)v \tag{2.10}$$

such that the system (2.9) becomes, in terms of the new state variables $z = \Phi(x)$ and the new input v, a linear time-invariant system of the form:

$$\dot{z} = Az + Bv \tag{2.11}$$

in some neighborhood of x^0. Often, (A, B) are in Brunovsky canonical form of dimensions $n \times n$ and $n \times 1$.

Since $z = \Phi(x)$ is a (local) diffeomorphism assures that one can pass back and forth between the state vectors x and z. If such a linearizing transformation exists, then the problem of designing a stabilizing state feedback controller for system (2.9) reduces to choosing a linear state feedback of the form $v = -Kz$ for the equivalent system (2.11), and then transforming this control law into the original coordinate system of (2.10).

The following theorem gives the necessary and sufficient conditions for the existence of such a linearizing transformation:

Theorem 2.2.1 The nonlinear system (2.9) can be fully state linearized by a state feedback (2.10) and coordinate transformation $z = \Phi(x)$, if and only if there exists a region Ω such that the following conditions hold:

(i). the vector fields $\{g, ad_f g, \ldots, ad_f^{n-1} g\}$ are linearly independent in Ω;

(ii). the set $\{g, ad_f g, \ldots, ad_f^{n-2} g\}$ is involutive in Ω

Remark 2.2.1 The first condition can be interpreted as the nonlinear analog of the well-known linear controllability condition. For linear systems, this condition requires the controllability matrix $[b, Ab, \ldots, A^{n-1}b]$ to have full rank. This condition which one defines a controllability distribution states that the nonlinear system is locally controllable if this condition holds in a region Ω. The second

condition is always satisfied in the linear case. Note that because of (ii) not all locally controllable nonlinear systems (through its Jacobian linearization) can be locally fully state linearized. As shown in [27], this is true in general under any static state feedback control law of the form (2.10) or any dynamic state feedback of the form:

$$\begin{aligned} \dot{\eta} &= a(\eta) + b(\eta)v \\ u &= c(\eta) + d(\eta)v \end{aligned} \tag{2.12}$$

However, as we will see in Chapter 6, any linearly controllable nonlinear system, i.e. its Jacobian linearization is a controllable linear system, can be locally fully state **approximately** linearized up to any order (of approximation) using a dynamic state feedback.

Theorem (2.2.1) suggests a systematic procedure to check whether state linearization is feasible, and a procedure to compute the necessary state transformation. These procedures are summarized below:

Step 1: Compute the vector fields $ad_f^{i-1}g(x), 1 \leq i \leq n$, and construct the controllability matrix:

$$\mathcal{C} = \left[g(x), ad_f g(x), \ldots, ad_f^{n-1}g(x)\right] \tag{2.13}$$

Step 2: Check items (i) and (ii) in Theorem (2.2.1) for controllability and involutivity conditions.

Step 3: If exact state space linearization is possible then solve the following

set of n linear equations for $\frac{\partial z_1}{\partial x_i}$:

$$\left[g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)\right] \cdot \begin{bmatrix} \frac{\partial z_1}{\partial x_1} \\ \vdots \\ \frac{\partial z_1}{\partial x_{n-1}} \\ \frac{\partial z_1}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2.14}$$

where $\mathcal{C}$ in (2.13) is invertible from the controllability condition. Therefore, the partial derivatives can be computed uniquely. By sequentially integrating these partial derivatives, we can find the transformation z_1, and further compute the rest of the transformation z_i from z_1:

$$z_i = \mathcal{L}_f^{i-1} z_1(x)$$

2.3 Input-Output Linearization

In this section we discuss the input-output linearization by state feedback for single-input single-output nonlinear systems of the form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{2.15}$$

where $x \in \mathcal{R}^n$, f and g are smooth vector fields, and $h(x)$ is a smooth nonlinear scalar function. Our goal is to derive a linear differential mapping between the specified output y above and a **new** input v. By a nonlinear change of coordinates $z = \Phi(x) \triangleq h(x)$ and a choice of nonlinear state feedback control law u, a large class of nonlinear systems can be made to have linear input-output behavior which are often subject to a nonlinear unobservable subsystem called the **internal dynamics**.

Let differentiate y to obtain

$$\begin{aligned} \dot{y} &= \tfrac{\partial h}{\partial x}(f(x) + g(x)u) \\ &= \mathcal{L}_f h(x) + \mathcal{L}_g h(x) u \end{aligned} \tag{2.16}$$

If the scalar coefficient of u, ($\frac{\partial h}{\partial x} g(x)$), is zero we differentiate repeatedly until a nonzero control coefficient appears.

Hence assuming $L_g(h) = 0$ in a neighborhood Ω, we differentiate again to obtain

$$\ddot{y} = L_f^2(h) + L_g(L_f(h))u. \tag{2.17}$$

If $L_g(L_f^{k-1}(h(x))) = 0 \ \forall x \in \Omega$ for $k = 1, \ldots, r-1$, but $L_g(L_f^{r-1}(h)) \neq 0$, then the process terminates at the rth step with:

$$\frac{d^r y}{dt^r} = L_f^r(h) + L_g(L_f^{r-1}(h))u. \tag{2.18}$$

The system (8.12) can be effectively linearized by introducing a feedback transformation of the form

$$u = \frac{1}{L_g L_f^{r-1}(h)}[v - L_f^r(h)] \tag{2.19}$$

which results in an input-output response from $v \to y$ given by

$$\frac{d^r y}{dt^r} = v, \tag{2.20}$$

a linear differential map.

The integer $r > 0$ can be viewed as a relative degree for the nonlinear system (2.15) (single input single output case), an extension of the usual definition of relative degree for linear systems. Note that given an output y in (2.15), the number of required differentiations is a fundamental system invariant of the system (2.15). This motivates the following definition:

Definition 2.3.1 (Relative Degree) The SISO system (2.15) is said to have relative degree r, in a region Ω if: $(\forall x \in \Omega)$

$$\begin{aligned} \mathcal{L}_g \mathcal{L}_f^i h(x) &= 0 \qquad 0 \leq i \leq r-1 \\ \mathcal{L}_g \mathcal{L}_f^{r-1} h(x) &\neq 0 \end{aligned} \tag{2.21}$$

Now if we define new state coordinates $z \in \Re^r$ as

$$z_k = L_f^{k-1}(h), \quad k = 1, \ldots, r \tag{2.22}$$

for the r-dimensional nonlinear system (2.18), then the system model can be written in state space form as,

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \mathcal{A}(x) + \mathcal{B}(x)u \end{bmatrix} v \tag{2.23}$$

where

$$\mathcal{A}(x) = L_f^r(h), \quad \mathcal{B}(x) = L_g(L_f^{r-1}(h)). \tag{2.24}$$

These coordinates express the original system in the normal form whose importance has been emphasized by Byrnes and Isidori [21, 51]. Note that if $r < n$, the original nonlinear system is not linearized completely. It is only (partially) input-output linearized resulting in an $(n-r)$-dimensional unobservable subsystem called **internal dynamics**.

The above procedure can be readily generalized to the multi-input/multi-

output (MIMO) case. Consider a MIMO nonlinear system of the form:

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) \cdot u_i \\ y &= h(x) \end{aligned} \tag{2.25}$$

where $x \in \mathcal{R}^n, u \in \mathcal{R}^m$, and $y \in \mathcal{R}^m$. We differentiate the outputs y_i until the inputs appear. Let's assume that r_i is the smallest integer such that at least one of the inputs appears in $y^{(r_i)}$. We have for the ith output:

$$y_i^{(r_i)} = \mathcal{L}_f^{r_i} h_i(x) + \sum_{j=1}^{m} \mathcal{L}_{g_j} \mathcal{L}_f^{r_i - 1} h_i(x) \cdot u_j$$

where $\mathcal{L}_{g_j} \mathcal{L}_f^{r_i - 1} h_i(x) \neq 0$ for some j. We can rewrite this as:

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_f^{r_1} h_1(x) \\ \vdots \\ \mathcal{L}_f^{r_m} h_m(x) \end{bmatrix} + A(x) \cdot u \tag{2.26}$$

where $A(x)$ is the $m \times m$ decoupling matrix:

$$A(x) = \begin{bmatrix} \mathcal{L}_{g_1} \mathcal{L}_f^{r_1 - 1} h_1(x) & \dots & \mathcal{L}_{g_m} \mathcal{L}_f^{r_m - 1} h_1(x) \\ \vdots & & \vdots \\ \mathcal{L}_{g_1} \mathcal{L}_f^{r_1 - 1} h_m(x) & \dots & \mathcal{L}_{g_m} \mathcal{L}_f^{r_m - 1} h_m(x) \end{bmatrix} \tag{2.27}$$

Definition 2.3.2 (Vector Relative Degree) The system (2.25) is said to have vector relative degree $(r_1, \dots, r_m)$ at a point x_0 if there exists a neighborhood U of x_0 such that $\forall x \in U$,

(i). $\mathcal{L}_{g_j} \mathcal{L}_f^k h_i(x) = 0 \qquad 0 \leq k \leq r_i - 1, 1 \leq i, j \leq m$

(ii). $A(x)$ is nonsingular in U.

Assuming that a vector relative degree exists for the system (2.25), $A(x)$ is nonsingular on U, and hence, the following control law is well-defined on U.

$$u = A^{-1}(x) \cdot \begin{bmatrix} v_1 - \mathcal{L}_f^{r_1} h_1(x) \\ \vdots \\ v_m - \mathcal{L}_f^{r_m} h_m(x) \end{bmatrix} \tag{2.28}$$

Application of this control law to system (2.15) yields:

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \tag{2.29}$$

a linear differential mapping where all the outputs are decoupled as well. Note that due to this decoupling, the same results as in SISO case can be used (at each separate channel) for tracking and regulation of the outputs. The case where the relative degree is undefined due to the rank condition (ii) of the matrix A, is more involved and there are algorithms to achieve a vector relative degree for systems that are **invertible**.

Remark 2.3.1 Often, we are interested in operating the system around a point, say x_0. If after r number of differentiations, the input u appears with nonzero coefficient **at** x_0, i.e. $\mathcal{L}_g \mathcal{L}_f^{r-1} h(x_0) \neq 0$, then by continuity, this coefficient stays nonzero over a neighborhood of x_0. Hence, from the definition (2.3.1), if there exists a region Ω such that (2.21) holds, we say that the system has a relative degree r **at the point** x_0. However, if when the input u appears and it happens that $\mathcal{L}_g \mathcal{L}_f^{r-1} h(x_0)$ is zero at x_0, but nonzero at some point x arbitrary close to x_0, then the relative degree of this system is not well-defined at x_0. In such cases, the **exact** input-output linearization scheme fails. As an alternative, we

may seek **approximate** linearization techniques to avoid this problem. This approach will be discussed in Chapters 5 and 6.

The linearized system (2.23) can be used to design a state feedback controller for asymptotic stabilization, regulation, and tracking for the original system (2.15). For stabilization, we may choose the new control v in (2.20) as a linear pole-placement controller to stabilize the (observable) linear subsystem (3.53). Of course, the stability of the overall system including the (unobservable) internal subsystem must be investigated under such a feedback. The overall control input u can be written as:

$$u(x) = \frac{1}{\mathcal{L}_g \mathcal{L}_f^{r-1} y}[-\mathcal{L}_f^r y - \alpha_{r-1} y^{(r-1)} - \ldots - \alpha_1 \dot{y} - \alpha_0 y] \tag{2.30}$$

where α_i are chosen so that $s^r + \alpha_{r-1} s^{r-1} + \ldots + \alpha_1 s + \alpha_0$ is a Hurwitz polynomial. The tracking controller can also be easily formulated once the nonlinear system (2.15) is transformed into its input-output linearized form of (2.23). The tracking control law is given by:

$$u(x) = \frac{1}{\mathcal{L}_g \mathcal{L}_f^{r-1} y}[-\mathcal{L}_f^r y - \alpha_{r-1}(y^{(r-1)} - y_d^{(r-1)}) - \ldots - \alpha_1(\dot{y} - \dot{y}_d) - \alpha_0(y - y_d)] \tag{2.31}$$

Remark 2.3.2 The above control laws do not guarantee the stability of the closed loop system except for the case $r = n$. The internal behavior of the system needs to be checked when $r < n$. The stability of the closed loop system will be investigated in the next two sections.

2.4 Normal Forms

Recall that for input-output linearization $z_k = \mathcal{L}_f^{k-1}(h), \quad k = 1, \ldots, r$, where $r \leq n$.

Theorem 2.4.1 ([51]) Suppose that system (2.15) has relative degree r at x_0. Then the gradients $\mathcal{D}z_i(x), 1 \leq i \leq r$, with z_i defined as above, are linearly independent on a neighborhood of x_0.

Note that as a result of the above theorem, we can use the first r derivatives of the output y, i.e. $z_k = \mathcal{L}_f^{k-1}(h), \quad k = 1, \ldots, r$, as a partial change of coordinates. The reminding $n-r$ coordinates are found as a direct application of the Frobenius Theorem. It guarantees the existence of $n - r$ independent functions $\lambda_i, i = 1, \ldots, n - r$ such that:

$$\mathcal{L}_g \lambda_i(x) = 0 \quad x \in U, i = 1, \ldots, n - r \tag{2.32}$$

This allows us to to choose $n - r$ more functions η_i that satisfy (2.32) such that the collection $z_i = y^{(i-1)}, 1 \leq i \leq r, \ \eta_i, \ 1 \leq i \leq n - r$ are independent at U. Consider the following transformation defined on U:

$$\Phi(x) = \begin{bmatrix} z_1 \\ \vdots \\ z_r \\ \eta_1 \\ \vdots \\ \eta_{n-r} \end{bmatrix} \tag{2.33}$$

From Theorem (2.4.1) and the fact that $\mathcal{L}_g\Phi_r(x) \neq 0, \forall x \in U$, the gradient

$$\mathcal{D}\Phi(x) = \begin{bmatrix} \mathcal{D}h(x) \\ \vdots \\ \mathcal{D}(\mathcal{L}_f^{r-1}h(x)) \\ \mathcal{D}\eta_1(x) \\ \vdots \\ \mathcal{D}\eta_{n-r}(x) \end{bmatrix} \tag{2.34}$$

is nonsingular in U, and hence, (2.33) defines a valid diffeomorphism on U. In a neighborhood U of a point x_0, the normal form of (2.15) is then given by [22, 51]:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= \alpha(z,\eta) + \beta(z,\eta)\cdot u \\ \dot{\eta} &= q(z,\eta) \\ y &= z_1 \end{aligned} \tag{2.35}$$

where $\alpha(z,\eta) = \mathcal{L}_f^r h(x), \beta(z,\eta) = \mathcal{L}_g\mathcal{L}_f^{r-1}h(x)$, and $q_i(z,\eta) = \mathcal{L}_f\eta_i(x)$.

2.5 Zero Dynamics

From (2.35) it is clear that by means of input/output linearization, the dynamics of (2.15) is decomposed into an input/output (external) linear part z-dynamics and an unobservable internal part η-dynamics. Since any input designed for the external part affects the internal subsystem as well, the behavior of the internal part, namely the states η_i, has to be analyzed carefully. Generally, the internal

dynamics depends on the output states z, and it is typically time-varying and hard to address. As an alternative, we can study the internal behavior of a nonlinear system subject to a control law such that the output is maintained at zero.

Definition 2.5.1 The zero-dynamics of a nonlinear system of the form (2.35) is given by:

$$\dot{\eta} = q(0, \eta) \tag{2.36}$$

i.e. dynamics of the system when its output is constrained to zero by the input.

Note that the subset:

$$Z_0 = \{x \in U | h(x) = \mathcal{L}_f h(x) = \ldots = \mathcal{L}_f^{r-1} h(x) = 0\} \tag{2.37}$$

can be made invariant by choosing the control law given in (2.19), i.e. given an initial condition belonging to Z_0, the entire solution trajectory of (2.15) can be made to stay in Z_0. The zero dynamics of (2.15) are then the dynamics of (2.15) subject to the control law (2.19) with $v = 0$ restricted on this subspace Z_0.

Recall that linear systems whose zero-dynamics are stable are called minimum phase. Extending this notion to nonlinear systems, we have:

Definition 2.5.2 System (2.15) is said to be locally (globally) asymptotically minimum phase if its zero-dynamics are locally (globally) asymptotically stable.

The system is called **exponentially minimum phase** when the zero dynamics are exponentially stable.

Remark 2.5.1 For stabilization and tracking problems in **nonlinear systems**, it can be shown that **local** asymptotic stability of the zero dynamics (locally

minimum phase) is enough to guarantee that the internal dynamics is locally asymptotically stable. However, unlike the linear case, even if a nonlinear system is **globally exponentially** minimum phase, no results on the global stability or large range stability can be drawn for the internal dynamics. Only local stability of the internal dynamics is guaranteed in this case.

Theorem 2.5.1 Assume that the system (2.15) is (locally) asymptotically stable, and has a relative degree r. Then the tracking control law (2.31) yields a locally asymptotically stable close loop system.

Remark 2.5.2 Computing a system's zero dynamics (or internal dynamics) does not necessarily require solution of the partial differential equations (2.32) to obtain the normal forms (2.35) defining η. Very often, specially in the MIMO case, it is easier to find $n - r$ vector fields to complete a state transformation (by checking that the Jacobian of the transformation is invertible). Then, since the control law u in (2.19) is known, we can simply replace u by its expression. Recall that the zero dynamics of a system is an intrinsic property of that system independent of the choice of state transformation.

2.6 Dynamic Extension

For multi-input multi-output (MIMO) nonlinear systems, the input/output linearization technique can be achieved only when the decoupling matrix $A(x)$ in (2.27) is invertible in a region U. The invertibility condition of the decoupling matrix is a necessary condition for the existence of a well-defined vector relative degree. There are two ways that a MIMO nonlinear system can fail to have a well-defined relative degree. As in the SISO case, a control coefficient $\mathcal{L}_{g_i}\mathcal{L}_f^k h_j(x)$

could be neither identically zero nor bounded away from zero on U. As a result the decoupling matrix $A(x)$ is not well-defined. However, the decoupling matrix $A(x)$ could be well-defined without having a full rank, i.e. it is singular. In such a case, the vector relative degree is not well-defined, but we may still be able to generate input/output linearization by remapping either the inputs, via a "dynamic extension algorithm", or the outputs, via a "dynamic inversion algorithm". The dynamic inversion algorithm will be discussed in detail in the next chapter. In this section, we will review the dynamic extension algorithm, where, by adding integrators to certain input channels, one obtains an **extended** system with a well-defined vector relative degree. The procedure is as follows:

Assume the decoupling matrix $A(x)$ has a constant rank $r < m$.

Step 0: Refer to the system (2.15) as system P_0, and set $k = 0$.

Step 1: Let r_i be the relative degree of the ith output of system P_k in U and define the decoupling matrix $A_k(x)$ in the usual way with entries: $a_{ij} = \mathcal{L}_{g_j}\mathcal{L}_f^{r_i-1}h_i(x)$. Let:

$$s_k = generic\ rank\ of\ A_k(x) \tag{2.38}$$

If $s_k = m$, **stop**.

Step 2: If $s_k < m$, without loss of generality, assume that the first s_k rows of $A_k(x)$ are linearly independent in U. Note that this is always possible with a simple reordering of the outputs. Apply the static state feedback:

$$u = \alpha_k(x) + \beta_k(x)v \tag{2.39}$$

so that the resulting decoupling matrix of the closed-loop system under

this control law is of the form:

$$\tilde{A}(x) = \begin{bmatrix} I_{s_k \times s_k} & 0 \\ M(x) & 0 \end{bmatrix}$$

Note that this is always possible with the help of an elementary column operation so that the last $m - r$ columns of the resulting matrix are zero.

Step 3: There exists q_k columns of $\tilde{A}(x)$ with two or more nonzero elements. Without loss of generality assume that these are the first q_k columns. Put an **integrator** in series with q_k corresponding input channels by defining:

$$\dot{\omega}_i = v_i \quad 1 \leq i \leq q_k \tag{2.40}$$

Step 4: Define the new (extended) system P_{k+1} by composing system P_k with (2.40) subject to u in (2.39). Note that the new state is $x_{k+1} = (x_k, \omega)$. Return to step 1, with $(k+1) \rightarrow k, v \rightarrow u$.

Remark 2.6.1 Descusse and Moog [28] showed that this algorithm converges in a finite number of steps to a system of dimension $n + p$ with a well-defined vector relative degree if the original system is right-invertible. We also note that $p \leq n$.

In Chapter 7 we provide an application of this algorithm to an aircraft control system problem.

Chapter 3

Review of Previous Results

This chapter gives an introduction to adaptive nonlinear control theory and briefly reviews some of the previous results in this field. By no means do we provide a complete survey of the earlier results. For a better understanding of this subject, the interested reader should consult one of the texts or recent surveys in this topic, for example [82, 79, 87, 8, 24, 72, 35, 70, 37]. We begin by motivation and introduction to some of the basic notions in adaptive control theory in section I. We then review several of the recent contributions in adaptive nonlinear control theory.

3.1 Introduction

As discussed in chapter one, any effective control system should, up to some degree, be able to deal with changes and uncertainties in the plant under compensation. It should also be robust against modeling errors in the design. Otherwise, the resulting control system will very likely fail to meet the design objectives, such as stability and tracking, over a period of time. We also noted that a major source of modeling error which could potentially cause instability in the closed

loop system is due to the lack of **exact** knowledge of some or all the parameters in the plant. Often, only nominal values of these parameters are known, and in many cases, the value of these parameters may change over time. For example, consider a nonlinear system:

$$\begin{aligned} \dot{x} &= f(x,\theta) + g(x,\theta)\cdot u \\ y &= h(x,\theta) \end{aligned} \tag{3.1}$$

where $x \in \Re^n$ is the state, f, g, h are smooth (C^∞) vector fields, and $\theta = [\theta_1, \ldots, \theta_p]^T$ is the vector of unknown or uncertain parameters in the system. We assume that only a nominal value θ_0 for the parameter vector θ is known, and suppose that the design objective is to stabilize the system and to make the output y asymptotically track a reference signal y_r. Therefore, to a control engineer, the model used for the design and stability analysis is an evaluation of system (3.1) at $\theta = \theta_0$, i.e.

$$\begin{aligned} \dot{x} &= f(x,\theta_0) + g(x,\theta_0)\cdot u \\ y &= h(x,\theta_0) \end{aligned} \tag{3.2}$$

For this **model**, under some regularity conditions discussed in previous chapter, a nonlinear state feedback control law exists that makes the resulting feedback **model** have **linear** input-output behavior. Stability and asymptotic tracking can then be achieved with application of a linear controller to the resulting **linear** system. This is achieved by exact cancellation of nonlinear terms appearing in the input-output map. However, since θ is not exactly known, this cancellation is not exact and the application of this control law, derived from model (3.2), to the true plant (3.1) will leave some of the nonlinear terms in (3.1) unaccounted for. In general, the main restriction in implementing **exactly** linearizing control laws from differential geometric control theory is that they are based on exact

cancellation of nonlinear terms. Therefore, our goal is to design a control law that is robust against parametric uncertainties in the system. In some cases, the control law may achieve asymptotic cancellation of the nonlinear terms appearing in the input-output map in (3.1). But as we shall see, this is not necessary and is not our goal as long as stability and asymptotic tracking is achieved. Typically, the control law is based on a vector $\hat{\theta}$ which is an on-line estimate of the true parameter vector θ. The update laws for these adjusted parameters are determined as part of the design and shall be such that the closed loop system stability is preserved. The convergence of these parameter estimates to their true value θ is a necessary condition in **indirect** schemes of adaptive control. This is because a **parameter identifier** is used in the outer loop design that continuously adjusts the parameter estimates based on observation error. The

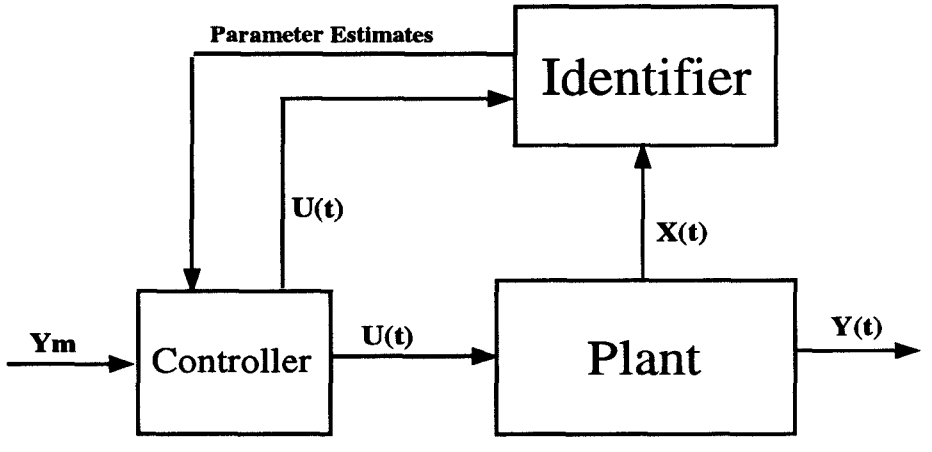

Figure 3.1: Schematic diagram for indirect adaptive control schemes

certainty equivalence principle suggests that these parameter estimates that are converging to their true values may be employed to asymptotically achieve the desired objective as the parameter estimates converge to their true value. Figure (3.1) show the schematic diagram for indirect adaptive control schemes.

The adaptive scheme developed in chapter five of this dissertation is of this form.

The other two adaptive schemes, in chapters four and six, are examples of **direct** adaptive control schemes, depicted in figure (3.2). In schemes of this form, parameters do not need to converge to their **true** value but they are required to stay bounded and converge to some constant. Typically, if the system is **persistently exciting**, as will be discussed later, then all the parameter estimates will converge to their true values.

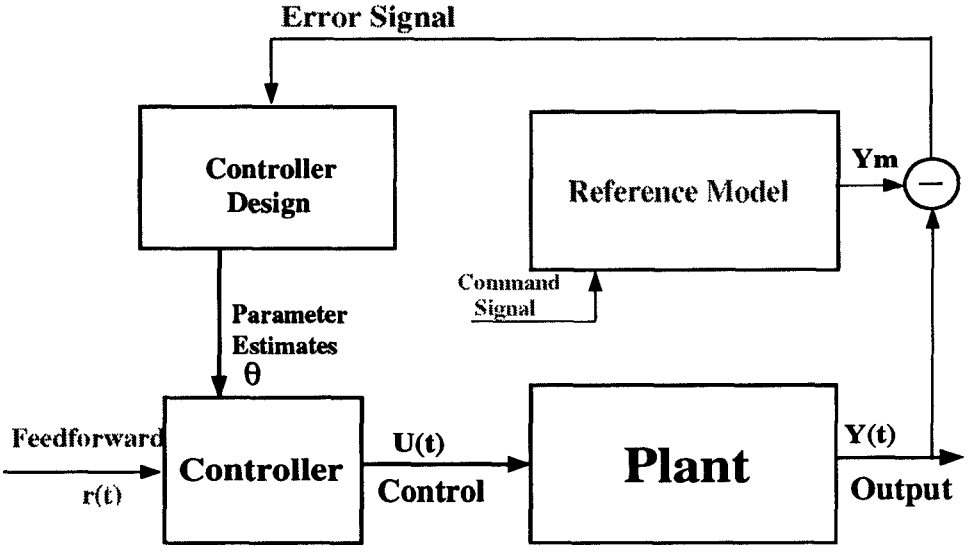

Figure 3.2: Schematic diagram for direct adaptive control schemes

In the next four sections we review some of the previous results in adaptive nonlinear control theory. The key differences are in the assumptions necessary to achieve adaptive asymptotic tracking/regulation with bounded state and bounded parameter estimates. They all require the system under consideration to be exponentially minimum phase and input-output feedback linearizable. Also, they require that the uncertain parameters appear linearly in the system dynamics. In addition, some of the earlier schemes suffer from overparametriza-

tion of the uncertain parameters to be estimated, and they assume that nonlinearities satisfy a global Lipschitz condition. Some other schemes that avoid the overparametrization, assume very restrictive conditions on system nonlinearities. In the last section, we review an adaptive tracking and regulation scheme that is based on **backstepping** design and is more practical than others in the literature. This scheme assumes that the nonlinear system is transformable into the so called **parametric-strict-feedback** form to be discussed later.

3.2 Adaptive Control of Linearizable Systems

Consider a SISO nonlinear system of the form (3.1) under parameter uncertainty in θ. The following assumption constrains the position of uncertain parameters appearing in the system dynamics.

Assumption 3.2.1 (Linear Parameter Dependence) The vector fields f and g in (3.1) may be parametrized linearly in unknown parameters θ:

$$\begin{aligned} f(x,\theta) &= \sum_{i=1}^{p} \theta_i \cdot f_i(x) \\ g(x,\theta) &= \sum_{i=1}^{p} \theta_i \cdot g_i(x) \end{aligned} \tag{3.3}$$

where vector fields f_i, and g_i are known functions of x.

The estimates of these functions are given by:

$$\begin{aligned} \hat{f}(x) &= f_0(x) + \sum_{i=1}^{p} \hat{\theta}_i \cdot f_i(x) \\ \hat{g}(x) &= g_0(x) + \sum_{i=1}^{p} \hat{\theta}_i \cdot g_i(x) \end{aligned} \tag{3.4}$$

where $\hat{\theta}_i$ are the estimates of the unknown parameters θ_i.

Remark 3.2.1 Most adaptive nonlinear control schemes are developed for systems that satisfy the above assumption. Under this assumption, system (3.1) can be rewritten as:

$$\begin{aligned} \dot{x} &= f_0(x) + \sum_{i=1}^{p} \theta_i f_i(x) + \left[g_0(x) + \sum_{i=1}^{p} \theta_i g_i(x) \right] \cdot u \\ y &= h(x) \end{aligned} \tag{3.5}$$

where f_i, g_i are smooth vector fields in a neighborhood of the origin $x = 0$, with $f_i(0) = 0,\ g(0) \neq 0$.

The next assumption states that (3.1) is input-output feedback linearizable with a constant relative degree:

Assumption 3.2.2 (Constant Relative Degree) For all $\hat{\theta}$ in a sphere around θ, and for all x in a neighborhood of x^0 we have:

$$\begin{aligned} &\mathcal{L}_{g(x)} \mathcal{L}^{i}_{f(x)} h(x) = 0 \qquad i = 0, \cdots, r-2 \\ &\mathcal{L}_{g(x)} \mathcal{L}^{r-1}_{f(x)} h(x) \neq 0 \end{aligned} \tag{3.6}$$

where r is the relative degree of (3.1).

Under the above assumption, (3.1) can be written as:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= b(x) + a(x) \cdot u \\ \dot{\eta} &= q(\xi, \eta) \end{aligned} \tag{3.7}$$

where $\xi_i \triangleq \mathcal{L}_f^{i-1} h(x), i = 1, \ldots, r,\ a(x) \triangleq \mathcal{L}_g \mathcal{L}_f^{r-1} h(x) \neq 0$, and $b(x) \triangleq \mathcal{L}_f^r h(x)$.

As shown in chapter two, the following control law can be used to achieve asymptotic tracking when there is no parameter uncertainty present:

$$u = \frac{1}{a(\xi, \eta)} [b(\xi, \eta) + v] \tag{3.8}$$

with:

$$v = y_d^{(r)} + \alpha_{r-1}(y_d^{(r-1)} - \xi_r) + \ldots + \alpha_0(y_m - \xi_1) \tag{3.9}$$

where α_i are chosen so that $s^r + \alpha_{r-1}s^{r-1} + \ldots + \alpha_0$ is a Hurwitz polynomial, and y_m is a bounded smooth reference trajectory for asymptotic output tracking satisfying the following:

Assumption 3.2.3 (Reference Signal) The reference trajectory $y_m(t)$ and its first r derivatives are bounded. i.e $|y_m^{(i)}| \leq b_m \quad i = 0, 1, \ldots, \gamma$ for some $b_m > 0$.

Thus the control law u in (3.8) linearizes the system (3.1) from input v to output y. Now let's replace the control law (3.8) by:

$$u_{ad} = \frac{1}{L_g \widehat{L_f^{r-1}h}}[-\widehat{L_f^r h}(\xi, \eta) + v_{ad}] \tag{3.10}$$

and:

$$v_{ad} = y_d^{(r)} + \alpha_{r-1}(y_d^{(r-1)} - \hat{\xi}_r) + \ldots + \alpha_0(y_m - \hat{\xi}_1) \tag{3.11}$$

where α_i are chosen as before and $\xi_{i-1} = L_f^i h$ are replaced by their estimates $\widehat{L_f^i h}$:

$$\begin{aligned} \hat{\xi}_i = \widehat{L_f^{i-1}h} &\triangleq L_{\hat{f}(x,\hat{\theta})}^{i-1} h(x) \\ L_g \widehat{L_f^{r-1}h} &\triangleq L_{\hat{g}(x,\hat{\theta})} L_{\hat{f}(x,\hat{\theta})}^{r-1} h(x) \end{aligned} \tag{3.12}$$

Since these estimates are not linear in the unknown parameters θ_i, we define each of the parameter products to be a new parameter. For example:

$$\widehat{L_f^2 h} = \sum_{i=1}^{p} \sum_{j=1}^{p} \theta_i \theta_j L_{f_j} L_{f_i} h(x) \tag{3.13}$$

and we let $\Theta \in \Re^P$ be the large P-dimensional vector of all multilinear parameter products: $\theta_i, \theta_j\theta_k, \theta_i\theta_j\theta_k, \ldots$. The vector containing all the estimates is denoted

by $\hat{\Theta} \in \Re^P$ with $\Phi \triangleq \Theta - \hat{\Theta}$ representing the parameter error. Using the control law (3.10) in (3.7) yields:

$$\begin{aligned} \dot{\xi}_r &= L_f^r h + \left[L_g L_f^{r-1} h - L_g \widehat{L_f^{r-1}} h\right] \cdot u_{ad} - \widehat{L_f^r h} + v_{ad} \\ &= \left[L_f^r h - \widehat{L_f^r h}\right] + \left[L_g L_f^{r-1} h - L_g \widehat{L_f^{r-1}} h\right] \cdot u_{ad} + v_{ad} \end{aligned} \tag{3.14}$$

Subtracting v in (3.9) from both sides gives:

$$\begin{aligned} e^{(r)} + \alpha_{r-1} e^{(r-1)} + \ldots \alpha_0 e &= \left[L_g L_f^{r-1} h - L_g \widehat{L_f^{r-1}} h\right] \cdot u_{ad} + \left[L_f^r h - \widehat{L_f^r h}\right] \\ &\quad + \alpha_{r-1}\left(L_f^{r-1} h - \widehat{L_f^{r-1}} h\right) + \ldots + \alpha_1 \left(L_f h - \widehat{L_f h}\right) \\ &= \Phi^T \cdot w(x, u_{ad}(x)) \end{aligned} \tag{3.15}$$

where: $w^T \triangleq \left[L_{g_i} L_{f_j}^{r-1} h_k u_{ad}(x) | \ldots | L_{f_j} h_k\right]$.

Therefore, in the closed loop, the resulting system, can be written in a compact form:

$$\begin{aligned} \dot{e} &= Ae + W^T(x, \hat{\theta}) \cdot \Phi \\ \dot{\eta} &= \tilde{q}(\xi, \eta) \end{aligned} \tag{3.16}$$

where A is a Hurwitz matrix.

Assumption 3.2.4 (Exponentially Stable Zero Dynamics) The zero dynamics of system (3.1), or equivalently (3.7), are exponentially stable, i.e. the dynamics:

$$\dot{\eta} = q(0, \eta) \tag{3.17}$$

is exponentially stable.

Remark 3.2.2 Assumptions (3.2.1), (3.2.2), and (3.2.4) are very common in adaptive schemes for nonlinear systems and in combination with other conditions specific to each scheme, they specify the class of systems that are considered by these schemes. Assumption (3.2.1) constrains the location of the uncertain

parameters. Assumption (3.2.3) is a generic assumption on the properties of the reference trajectory to be tracked. On the other hand, assumptions (3.2.2) and (3.2.4) are due to the deterministic (with no uncertainties) approach in the control scheme. In the next three chapters we shall relax some of these conditions and substitute them with some milder conditions, hence, enlarging the class of nonlinear system to be considered.

Sastry and Isidori [88] considered adaptive asymptotic tracking for a class of nonlinear systems that are exactly input-output linearizable by state feedback. The systems considered are assumed to be **minimum phase** and subject to parametric uncertainty. Adaptive scheme of [88] employs the following assumption in addition to the above four:

Assumption 3.2.5 (Globally Lipschitz Condition) $f, g, \mathcal{L}_f^k h$, and $\mathcal{L}_g \mathcal{L}_f^{k-1} h$, $k = 1, \ldots, r$ are globally Lipschitz continuous functions of x, and $W(x, \hat{\theta})$ in (3.16) has bounded derivatives in x and $\hat{\theta}$.

Theorem 3.2.1 ([88]) Consider the control law of (3.10) applied to a SISO nonlinear system of the form (3.1) satisfying assumptions (3.2.1)-(3.2.5). If $L_g \widehat{L_f^{r-1}} h$ is bounded away from zero, then the parameter update law:

$$\begin{aligned} \dot{\phi} &= \tfrac{-e_1 \zeta}{1+\zeta^T \zeta} \\ \zeta &= L^{-1}(s) W \end{aligned} \tag{3.18}$$

where $L(s) \triangleq s^r + \alpha_{r-1} s^{r-1} + \ldots + \alpha_0$, yields bounded tracking. i.e. $y \to y_d$ as $t \to \infty$, and x and $\hat{\theta}$ are bounded.

Remark 3.2.3 If W is **persistently exciting** (PE), i.e. $\exists \alpha_1, \alpha_2, \sigma > 0$ such that:

$$\alpha_1 I \leq \int_{t_0}^{t_0+\sigma} W W^T dt \leq \alpha_2 I \tag{3.19}$$

then both $y - y_d$ and $\phi = \theta - \hat{\theta}$ converge **exponentially** to zero.

Nam and Arapostathis [78] considered the case where the relative degree of (3.1) is n for the so called **pure-feedback** nonlinear systems:

$$\begin{aligned} \dot{x} &= f_{pf}(x) + u g_{pf}(x) \\ y &= h(x_1) \end{aligned} \tag{3.20}$$

where:

$$f_{pf}(x) = \begin{bmatrix} \theta_1 f_1(x_1, x_2) \\ \theta_2 f_2(x_1, x_2, x_3) \\ \vdots \\ \theta_n f_n(x_1, \ldots, x_n) \end{bmatrix}, g_{pf}(x) = \begin{bmatrix} 0 \\ 0 \\ \\ \vdots \\ \theta_{n+1} g_1(x_1, \ldots, x_n) \end{bmatrix} \tag{3.21}$$

This scheme uses a time-varying change of coordinates $\hat{\xi} = T_{\hat{\theta}}(x)$ by replacing θ in $\xi \triangleq \mathcal{L}_{f(x,\theta)} h(x)$ with its estimate $\hat{\theta}$ to transfer (3.1) into an equivalent system of the form:

$$\begin{aligned} \dot{\hat{\xi}} &= A \cdot \hat{\xi} + W(x, u, \hat{\theta}) \cdot \Phi + M(x, \hat{\theta}) \cdot \dot{\hat{\theta}} \\ \dot{\eta} &= q(\hat{\xi}, \eta) \end{aligned} \tag{3.22}$$

where:

$$A = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 1 \\ -k_1 & -k_2 & \ldots & -k_n \end{bmatrix}, M(x, \hat{\theta}) = \frac{\partial \hat{\xi}(x, \hat{\theta})}{\partial \hat{\theta}}$$

$$W(x, u, \hat{\theta}) = \begin{bmatrix} \mathcal{L}_{f(x)} h(x) \\ \vdots \\ \mathcal{L}_{f(x)} \mathcal{L}^{n-2}_{f(x,\hat{\theta})} h(x) \\ \mathcal{L}_{f(x)} \mathcal{L}^{n-1}_{f(x,\hat{\theta})} h(x) + u \cdot \mathcal{L}_g(x) \mathcal{L}^{n-1}_{f(x,\hat{\theta})} h(x) \end{bmatrix} \tag{3.23}$$

In this case, assumption (3.2.4) is not necessary since there is no zero-dynamics. This scheme replaces the globally Lipschitz condition of assumption (3.2.5) with the following less restrictive sector-type assumption on the nonlinearities:

Assumption 3.2.6 There exist positive constants c_1, c_2, d_1 and d_2 such that $\forall x \in \Re^n, \forall \hat{\theta} \in B(\theta, \rho)$:

$$\begin{aligned} ||W(x,u,\hat{\theta})|| &\leq c_1 + c_2||T_{\hat{\theta}}(x)|| \\ ||M(x,\hat{\theta})|| &\leq d_1 + d_2||T_{\hat{\theta}}(x)|| \end{aligned} \tag{3.24}$$

We now summarize the results of the adaptive algorithm given in [78]:

$$\begin{aligned} u_{ad} &= \frac{1}{\mathcal{L}_{g(x,\hat{\theta})}\mathcal{L}^{n-1}_{f(x,\hat{\theta})}h(x)}[-\mathcal{L}^{n}_{f(x,\hat{\theta})}h(x) + k^T\hat{\xi} + y_m] \\ \dot{\hat{\theta}} &= qH_t^T(W)K(t)\zeta(t) \\ \dot{K} &= -K(t)\cdot(\Phi_t(W) + \Phi_t^T(W))\cdot K(t) \ , \ K(0) = \tfrac{1}{\lambda}I \\ \Phi_t(W) &= [(A+bk^T)H_t(W) + W]\cdot H_t^T(W) \\ \hat{\xi}_i &= \mathcal{L}^{i-1}_{f(x,\hat{\theta})}h(x) \quad i = 1,\ldots,n \\ \zeta(t) &= \hat{\xi}(t) - H_t(by_m) - e^{(A+bk^T)t}\hat{\xi}(0) - \epsilon(t) - H_t(M(x,\hat{\theta})\dot{\hat{\theta}}) \\ \epsilon(t) &= H_t(W)\cdot\hat{\theta} - H_t(W\cdot\hat{\theta}) \end{aligned} \tag{3.25}$$

where $H_t : PC(\Re^+, \Re^n \times \Re^m) \rightarrow C(\Re^+, \Re^n \times \Re^m)$, PC being the space of piecewise continuous functions, is a linear operator defined as:

$$H_t(Q) = \int_0^t e^{(A+bk^T)(t-\tau)}Q(\tau)d\tau \tag{3.26}$$

for any function $Q \in PC(\Re^+, \Re^n \times \Re^m)$, and $W(x,u,\hat{\theta})$ and $M(x,\hat{\theta})$ are defined in (3.23).

Remark 3.2.4 The scheme of [78] is less restrictive on the type of nonlinearities in the system than that of [88], and it does not introduce overparameterization in

the adaptive loop. It, however, does increase the dynamic order of the parameter update rules since the matrix $K(t)$ needs to be updated on-line as the solution to an $n \times n$ differential equation. This scheme is only applicable to pure-feedback nonlinear systems defined in (3.20) with relative degree n and at most $n+1$ unknown parameters. These results can be extended to the multi-input case with any number of unknown parameters.

Consider now a multi-input multi-output (MIMO) nonlinear system of the form:

$$\begin{aligned} \dot{x}(t) &= f(x) + \sum_{i=1}^{m} u_i \cdot B_i(x) \qquad x \in M \\ y(t) &= h(x(t)) \end{aligned} \tag{3.27}$$

where the state space is a connected, n-dimensional, real analytic manifold; $f, g_i \in V(M)$ is the real vector space of the real analytic vector fields on M, $u_i \in U$ is the class of real analytic functions from $[0 \ \ \infty)$ into $\Re$, and $h(\cdot) : M \rightarrow \Re^l$ is a real analytic mapping. Theorem (3.2.1) can be extended to the MIMO case provided that system (3.27) can be **decoupled** by static state feedback. The following assumption is necessary (assuming a square system with equal number of outputs and inputs):

Assumption 3.2.7 (Nonsingular Decoupling Matrix) The $p \times p$ decoupling matrix $A(x)$:

$$A(x) = \begin{bmatrix} \mathcal{L}_{g_1}\mathcal{L}_f^{r_1-1}h_1(x) & \dots & \mathcal{L}_{g_p}\mathcal{L}_f^{r_p-1}h_1(x) \\ \vdots & & \vdots \\ \mathcal{L}_{g_1}\mathcal{L}_f^{r_p-1}h_p(x) & \dots & \mathcal{L}_{g_p}\mathcal{L}_f^{r_p-1}h_p(x) \end{bmatrix} \tag{3.28}$$

is invertible.

Theorem 3.2.2 Consider a square multivariable nonlinear system with a non-singular decoupling matrix under linear parametric uncertainty in assumption (3.2.1) and satisfying assumptions (3.2.4)-(3.2.7). This system can be decoupled (and linearized) with the following adaptive control law:

$$u = \hat{A}^{-1}(x)\left(-\begin{bmatrix} \mathcal{L}_f^{r_1}\widehat{h_1}(x) \\ \vdots \\ \mathcal{L}_f^{r_p}\widehat{h_p}(x) \end{bmatrix} + \begin{bmatrix} \hat{v_1} \\ \vdots \\ \hat{v_p} \end{bmatrix}\right) \tag{3.29}$$

Proof. Straightforward from results in [88]. □

Remark 3.2.5 If the multivariable nonlinear system can not be decoupled by static feedback, in particular if the decoupling matrix $A(x)$ is not invertible at some point, the above results do not hold. In this case, if the system is linearizable with **dynamic-state feedback**, then the above results on adaptive linearization/tracking can still be achieved under some more restrictive assumptions.

Now consider a MIMO nonlinear system of the form (3.27) Di Benedetto and Sastry [12] considered the case where this system has a singular decoupling matrix $A(x)$ but can be decoupled by dynamic state feedback resulting to an extended system described in the previous chapter.

Assumption 3.2.8 (Exponentially Attractive Zero-Dynamics) The zero-dynamics is exponentially attractive to a large ball in X, in the following sense:

$$\eta^T q(0,\eta) \leq -\alpha|\eta|^2 \quad \forall|\eta| \geq k_\eta \tag{3.30}$$

Assumption 3.2.9 (Conic Continuity) The zero-dynamics satisfies the following conic continuity assumption:

$$|q(\xi,\eta) - q(0,\eta)| \leq k|\xi| \tag{3.31}$$

Assumption 3.2.10 After possibly a new parameterization of the unknown parameters, $(A_e)^{-1}b_e, (A_e)^{-1}, \mathcal{L}_{g_j}\mathcal{L}_f^k h_i(x), \mathcal{L}_f^k h_i(x)$ depend linearly on the unknown parameters.

The following theorem summarizes the results in [12] on multivariable adaptive tracking using the dynamic extension algorithm reviewed in section 2.6.

Theorem 3.2.3 (MIMO Adaptive Tracking with Dynamic Extension) Consider a multivariable nonlinear system with singular decoupling matrix (3.28) under parameter uncertainty stated in assumption (3.2.1). Assume that the zero-dynamics are exponentially attractive with conic continuity condition of assumptions (3.2.8) and (3.2.9). Further assume that the extended system can be globally converted into normal form coordinates on X, the regressor $W(x,\hat{\theta})$ has bounded derivatives in both its arguments, and that assumption (3.2.10) holds. Then given a bounded trajectory y_m satisfying assumption (3.2.3), it follows that the control law:

$$u = \hat{A}_e^{-1}(x,\hat{\theta})\left(-\begin{bmatrix} \mathcal{L}_{f_e}{}^{r_1}\widehat{h_1}(x) \\ \vdots \\ \mathcal{L}_{f_e}{}^{r_p}\widehat{h_p}(x) \end{bmatrix} + \begin{bmatrix} \hat{v_1^e} \\ \vdots \\ \hat{v_p^e} \end{bmatrix}\right) \tag{3.32}$$

where subscript e is for the extended system under dynamic extension, with the parameter update law:

$$\dot{\phi} = \frac{-We_1}{1+W^TW} \tag{3.33}$$

yields bounded tracking provided that the state trajectory is confined to X, and the initial conditions of the states $x_e(0)$, the initial parameter error $\phi(0)$, the tracking reference y_m, and their appropriate derivatives are small enough.

Proof See [12] □

Remark 3.2.6 The above schemes assume very restrictive **growth conditions** on the nonlinearities in the system. As a result, the class of nonlinear systems that these schemes can be applied to is very small. They also suffer from over-parametrization and higher order update rules in the adaptation procedure. This will in turn result in a controller with a high dynamic order. Nonlinear systems with high dynamic order are generally difficult to work with.

3.3 Strict and Extended Matching Conditions

Slotine and Coetsee [93] introduced an adaptive sliding controller for nonlinear systems. The scheme can handle any **form** of smooth nonlinearity in the system with no growth constrain. This scheme, later extended in a more general setting by Taylor and co-workers [97], is based on the following assumption, known as **strict matching condition** (SMC), that restricts the location of the unknown parameters :

Assumption 3.3.1 (SMC) In (3.5), f_i, g_i are such that:

$$f_i, g_i \in sp\{g_0\} \quad i = 1, \ldots, p \tag{3.34}$$

Other schemes developed later by Kanellakopoulos, Kokotovic, and Marino [55, 56, 76] and Campion and Bastin [23], substitute the SMC condition with the

following less restrictive assumption known as **extended matching condition** (EMC):

Assumption 3.3.2 (EMC) In (3.5), f_i, g_i are such that:

$$\begin{aligned} f_i &\in sp\{g_0, ad_{f_0} g_0\} \\ g_i &\in sp\{g_0\} \quad i = 1, \ldots, p \end{aligned} \tag{3.35}$$

For feedback linearizable nonlinear systems under parametric uncertainty in the form (3.5), the EMC assumption is a necessary and sufficient condition for the existence of a **parameter independent** diffeomorphism which transforms (3.5) into the following form:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_i &= z_{i+1} \\ &\vdots \\ \dot{z}_{n-1} &= z_n + \sum_{k=1}^{p} \theta_i \psi_k(z) \\ \dot{z}_n &= \alpha_0 + \sum_{k=1}^{p} \theta_i \alpha_k(z) + \left[\beta_0(z) + \sum_{k=1}^{p} \theta_i \beta_k(z)\right] \cdot u \end{aligned} \tag{3.36}$$

This parameter independent transformation is very useful because the first $n-1$ coordinates $\hat{z}_i \triangleq z_i, i = 1, \ldots n-1$ that are parameter independent can then be used with the following parameter dependent coordinate:

$$\hat{z}_n = z_n + \sum_{k=1}^{p} \hat{\theta}_i \psi_k(z) \tag{3.37}$$

to obtain a closed-loop system of the form:

$$
\begin{aligned}
\dot{\hat{z}}_1 &= \hat{z}_2 \\
&\vdots \\
\dot{\hat{z}}_i &= \hat{z}_{i+1} \\
&\vdots \\
\dot{\hat{z}}_{n-1} &= \hat{z}_n + \phi^T \psi_1(z) \\
\dot{\hat{z}}_n &= [u + \theta^T \Psi_2(z)]\beta(z,\hat{\theta}) + \hat{\theta}^T \Psi_3(z) + \hat{\theta}^T \Psi_4(z)\theta + \dot{\hat{\theta}}^T \Psi_1(z)
\end{aligned}
\tag{3.38}
$$

and using the control law:

$$
\begin{aligned}
u(t) &= -\hat{\theta}^T \Psi_2(z) - \tfrac{1}{\beta(z,\hat{\theta})}\Big[k_1 z_1 + \ldots + k_{n-1} z_{n-1} + k_n(z_n + \hat{\theta}^T \Psi_1(z)) \\
&\quad + \hat{\theta}^T \Psi_3(z) + \hat{\theta}^T \Psi_4(z)\hat{\theta} + \dot{\hat{\theta}}^T \Psi_1(z)\Big]
\end{aligned}
\tag{3.39}
$$

in a more compact form:

$$
\begin{aligned}
\dot{\hat{z}} &= A\hat{z} + W(z)\cdot\phi + \Psi^T(z)\dot{\hat{\theta}} \\
W(\hat{z}) &= [0,\ldots,0,\psi^T(z), w_n(x,\hat{\theta})]^T
\end{aligned}
\tag{3.40}
$$

The parameter update law is given by:

$$
\dot{\phi} = \Omega W^T(\hat{z},\hat{\theta}) P\hat{(z)} \tag{3.41}
$$

where P is the solution to the Lyapunov equation $PA + A^T P = -I$. The following theorem summarizes the results on EMC-based adaptive control schemes [56, 55, 76]:

Theorem 3.3.1 ([56]) The equilibrium $x = x^e, \hat{\theta} = \theta$ of the adaptive system (3.39)-(3.41) is stable for every $\hat{\theta}(0) \in B_\theta$. Moreover, $\forall (x(0), \hat{\theta}(0)) \in B_x \times B_\theta$, a neighborhood around the equilibrium (x^e, θ), the state $x(t)$ converges to its equilibrium value x^e, that is:

$$
\lim_{t\to\infty} x(t) = x^e
$$

Remark 3.3.1 The main advantage of EMC-based adaptive control schemes is that there is no growth condition on the nonlinear terms, i.e. the global Lipschitz condition of assumption (3.2.5) and the sector-type assumption (3.2.6) are not required. On the other hand, the geometric condition of assumption (3.3.2) is very restrictive on the location of the unknown parameters in the system dynamics.

3.4 Semi-Indirect Adaptive Schemes

In the this section, we briefly review the adaptive schemes proposed in [98] and [3]. These schemes eliminate the overparametrization problem discussed in remark (3.2.6). Therefore, the dynamic order of the closed loop system is lower which enhances the stability of the overall closed loop system. The design procedure in both of these schemes are very similar. They both use an observer type auxiliary system and consider the stability properties of an augmented error. They both assume that the system is feedback linearizable with constant relative degree and is linearly parameterized with exponentially stable zero dynamics. The scheme of [98], however, assumes some restrictive assumptions on the nonlinearities which is avoided in [3].

Consider system (3.1) satisfying assumptions (3.2.1) and (3.2.2). Since the transformation ξ in (3.7) depends on the unknown parameters θ, consider its estimate by replacing all the unknown parameters appearing in ξ with their

estimates:

$$\begin{aligned}\hat{\xi} &= \phi_{\hat{\xi}}(x,\hat{\theta}) \triangleq \mathcal{L}^{i-1}_{\hat{f}(x,\hat{\theta})} h(x) \\ \eta &= \phi_{\eta}(x,\theta)\end{aligned} \tag{3.42}$$

where as before θ is a vector of unknown constant parameters and $\hat{\theta}$ is a time-varying, on-line estimate of θ. Also, η is such that as before:

$$\mathcal{L}_{g(x,\theta)}\eta = 0 \tag{3.43}$$

The time derivative of this transformation along the trajectories of (3.1) is:

$$\begin{aligned}\dot{\hat{\xi}}_1 &= \mathcal{L}_{f(x,\theta)}h(x) \\ \dot{\hat{\xi}}_2 &= \mathcal{L}_{f(x,\theta)}\mathcal{L}_{f(x,\hat{\theta})}h(x) + \frac{\partial\hat{\xi}_2(x,\hat{\theta})}{\partial\hat{\theta}}\cdot\dot{\hat{\theta}} \\ &\vdots \\ \dot{\hat{\xi}}_{r-1} &= \mathcal{L}_{f(x,\theta)}\mathcal{L}^{r-2}_{f(x,\hat{\theta})}h(x) + \frac{\partial\hat{\xi}_{r-1}(x,\hat{\theta})}{\partial\hat{\theta}}\cdot\dot{\hat{\theta}} \\ \dot{\hat{\xi}}_r &= \mathcal{L}_{f(x,\theta)}\mathcal{L}^{r-1}_{f(x,\hat{\theta})}h(x) + \mathcal{L}_{g(x,\theta)}\mathcal{L}^{r-1}_{f(x,\hat{\theta})}h(x)\cdot u + \frac{\partial\hat{\xi}_r(x,\hat{\theta})}{\partial\hat{\theta}}\cdot\dot{\hat{\theta}} \\ \dot{\eta} &= q(\hat{\xi},\eta)\end{aligned} \tag{3.44}$$

From assumption (3.2.1), we have:

$$\begin{aligned}\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + \sum_{i=1}^{p}(\theta_i - \hat{\theta}_i)\cdot\mathcal{L}_{f_i(x)}h(x) \\ \dot{\hat{\xi}}_2 &= \hat{\xi}_3 + \sum_{i=1}^{p}(\theta_i - \hat{\theta}_i)\cdot\mathcal{L}_{f_i(x)}\mathcal{L}_{f(x,\hat{\theta})}h(x) + \frac{\partial\hat{\xi}_2(x,\hat{\theta})}{\partial\hat{\theta}}\cdot\dot{\hat{\theta}} \\ &\vdots \\ \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + \sum_{i=1}^{p}(\theta_i - \hat{\theta}_i)\cdot\mathcal{L}_{f_i(x)}\mathcal{L}^{i-1}_{f(x,\hat{\theta})}h(x) + \frac{\partial\hat{\xi}_i(x,\hat{\theta})}{\partial\hat{\theta}}\cdot\dot{\hat{\theta}} \quad i = r,\ldots,r-1 \\ \dot{\hat{\xi}}_r &= \mathcal{L}_{f(x,\theta)}\mathcal{L}^{r-1}_{f(x,\hat{\theta})}h(x) + \mathcal{L}_{g(x,\theta)}\mathcal{L}^{r-1}_{f(x,\hat{\theta})}h(x)\cdot u + \frac{\partial\hat{\xi}_r(x,\hat{\theta})}{\partial\hat{\theta}}\cdot\dot{\hat{\theta}} \\ \dot{\eta} &= q(\hat{\xi},\eta)\end{aligned} \tag{3.45}$$

From assumption (3.2.2), and applying the control law:

$$u_{ad} = \frac{1}{\mathcal{L}_{g(x,\hat{\theta})}\mathcal{L}^{r-1}_{f(x,\hat{\theta})}h(x)}[-\mathcal{L}^{r}_{f(x,\hat{\theta})}h(x) + v_{ad}] \tag{3.46}$$

with:

$$v_{ad} = y_m^{(r)} + \alpha_{r-1}(y_m^{(r-1)} - \hat{\xi}_r) + \ldots + \alpha_0(y_m - \hat{\xi}_1) \tag{3.47}$$

and α_i chosen such that $s^r + \alpha_{r-1}s^{r-1} + \ldots + \alpha_0$ is a Hurwitz polynomial, we can rewrite (3.44) in a more compact form with $\phi = \theta - \hat{\theta}$:

$$\begin{aligned}
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + w_1(x,\hat{\theta}) \cdot \Phi \\
\dot{\hat{\xi}}_2 &= \hat{\xi}_2 + w_2(x,\hat{\theta}) \cdot \Phi + \frac{\partial \hat{\xi}_2(x,\hat{\theta})}{\partial \hat{\theta}} \cdot \dot{\hat{\theta}} \\
&\vdots \\
\dot{\hat{\xi}}_r &= v_{ad} + w_r(x,\hat{\theta},u_{ad}) \cdot \Phi + \frac{\partial \hat{\xi}_r(x,\hat{\theta})}{\partial \hat{\theta}} \cdot \dot{\hat{\theta}} \\
\dot{\eta} &= q(\hat{\xi},\eta)
\end{aligned} \tag{3.48}$$

where:

$$w_r(x,\hat{\theta},u) \cdot \Phi = \sum_{i=1}^{p} (\theta_i - \hat{\theta}_i) \cdot \left[\mathcal{L}_{f_i(x)}\mathcal{L}_{f(x,\hat{\theta})}^{r-1}h(x) + u \cdot \mathcal{L}_{g_i(x)}\mathcal{L}_{f(x,\hat{\theta})}^{r-1}h(x)\right] \tag{3.49}$$

Finally:

$$\begin{aligned}
\dot{\hat{\xi}} &= A \cdot \hat{\xi} + B \cdot v + W(x,u,\hat{\theta}) \cdot \Phi + M(x,\hat{\theta}) \cdot \dot{\hat{\theta}} \\
\dot{\eta} &= q(\hat{\xi},\eta)
\end{aligned} \tag{3.50}$$

where:

$$A = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 1 \\ 0 & 0 & \ldots & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, W = \begin{bmatrix} w_1 \\ \vdots \\ \\ w_r \end{bmatrix}, M = \frac{\partial \hat{\xi}(x,\hat{\theta})}{\partial \hat{\theta}} \tag{3.51}$$

The error signal e is defined as:

$$e_i = \hat{\xi}_i - y_m^{(i-1)} \qquad i = 1, \ldots, r \tag{3.52}$$

with $e_1 = \hat{\xi}_1 - y_m = y - y_m$. The objective is to force: $e_i(t) \to 0$ as $t \to \infty$. To do this, the following state observer form system, also used in schemes [83, 3],

is used to obtain a regressor form equation suitable to obtain parameter update laws:

$$\begin{aligned} \dot{\bar{\xi}} &= \Omega \cdot (\bar{\xi} - \hat{\xi}) + A \cdot \hat{\xi} + B \cdot v_{ad} + M(x, \hat{\theta}) \cdot \dot{\hat{\theta}} \\ \bar{\xi}(0) &= \bar{\xi}_0 = \hat{\xi}(0) \end{aligned} \tag{3.53}$$

where Ω is a Hurwitz matrix and is a design parameter, $\hat{\xi} \in \Re^r$ from (3.42), and $M(x, \hat{\theta})$ as in (3.51). Let's define the augmented error $s(t)$ as:

$$s = \bar{\xi} - \hat{\xi} \tag{3.54}$$

where $\hat{\xi}$ is defined by (3.42), and its dynamics are given by (3.50), subject to control $v = v_{ad}$ as in (3.47). Note that $s(t)$ satisfies:

$$\dot{s} = \Omega \cdot s + W.\Phi \tag{3.55}$$

We are now ready to state the main results on the **semi indirect** adaptive scheme of [98]. The following assumptions in addition to assumptions (3.2.1)-(3.2.4) identify the class of nonlinear systems under parameter uncertainty to which this scheme can be applied.

Assumption 3.4.1 (Globally Lipschitz Continuous) $q(\xi, \eta)$ is globally Lipschitz in ξ and η. Also, transformation $\phi_\xi(x, \theta)$ in (3.7),(3.42) is globally Lipschitz in θ and uniform in x, i.e.

$$|\phi_\xi(x, \theta) - \phi_{\hat{\xi}}(x, \hat{\theta})| \leq l_\phi |\phi|, \quad \forall x \in \Re^n \tag{3.56}$$

Assumption 3.4.2 (Conic Boundedness) $M(x, \hat{\theta}) \cdot W^T(x, u, \hat{\theta})$ is cone bounded in x and uniform in u and $\hat{\theta}$, where: $M(x, \hat{\theta}) \triangleq \frac{\partial \phi_{\hat{\xi}}(x, \hat{\theta})}{\partial \hat{\theta}}$ and $W(x, u, \hat{\theta})$ as in (3.50). i.e.

$$\left| \frac{\partial \phi_{\hat{\xi}}(x, \hat{\theta})}{\partial \hat{\theta}} \cdot W^T \right| \leq l_M |x|, \forall u \in \Re, \quad \forall \hat{\theta} \in \Re^p \tag{3.57}$$

Theorem 3.4.1 ([98]) Consider the plant (3.1) and the control objective of tracking the trajectory y_m. If in addition to assumptions (3.2.1)-(3.2.4), assumptions (3.4.1) and (3.4.2) are satisfied, then for sufficiently small $|\phi(0)|$ the control law u_{ad} given in (3.46) with the parameter update law:

$$\dot{\phi} = -W^T(x, u, \hat{\theta}) \cdot P_0 \cdot s(t) \tag{3.58}$$

where P_0 is the solution to the Lyapunov equation: $\Omega^T P_0 + P_0 \Omega = -I$, results in bounded tracking for the system (3.1).

Remark 3.4.1 Adaptive scheme of theorem (3.4.1) resolved the overparametrization problem of previous adaptive schemes in the last section (see the remark (3.2.6)), i.e. if p is the number of unknown parameters θ in (3.1), then (3.58) gives exactly p parameter update laws necessary in the adaptation loop. This results in a lower dynamic order for the overall closed loop system and significantly improves the stability of the resulting adaptive system. On the other hand, the assumptions (3.4.1) and (3.4.2) are highly restrictive and impractical.

Under some milder conditions, a model reference adaptive scheme was developed by Akhrif [3]. After a time-varying change of coordinates an observer system (3.53) is constructed. Using the continuity assumption of $M(x, \hat{\theta})$ and $W(x, u, \hat{\theta})$ in (3.50) and boundedness of $||M|||$ and $||W||$, it can be shown that the tracking error $e(t)$ approaches zero and that $e(t)$ and $\hat{\theta}$ remain bounded. In this scheme, however, the design objective is slightly different than the one in [98] in that the tracking objective is to force the **transformed** states $\hat{z}$ to track

the states z_m of a **reference model**. Consider a nonlinear system satisfying assumption (3.2.1) with no output specified:

$$\dot{x} = f_0(x) + \sum_{i=1}^{p} \theta_i f_i(x) + \left[g_0(x) + \sum_{i=1}^{p} \theta_i g_i(x) \right] \cdot u \tag{3.59}$$

with $x(0) = x_0$.

Assumption 3.4.3

$$g_i \in \mathcal{G} \triangleq span\{g\} \quad i = 1, \ldots, n, \quad \forall x \in U, \forall \theta \in B_\theta \tag{3.60}$$

Assumption 3.4.4 System (3.59) is feedback linearizable at a nominal value $\theta = \theta_0$, i.e:

I.

$$\textbf{dim. } sp\{g(0), ad_f g(0), \ldots, ad_f^{n-1} g(0)\} = n \tag{3.61}$$

II. Distribution:

$$\mathcal{M}_{n-2} = sp\ \{g, ad_f g, \ldots, ad_f^{n-2} g\} \tag{3.62}$$

is involutive in a neighborhood U of the origin.

Assumption 3.4.5

$$\begin{gathered} \textbf{dim. } \mathcal{M}_{n-1}^{\theta} = n \quad \forall \theta \in B_\theta \\ [f_i, \mathcal{M}_j] \subset \mathcal{M}_{j+1} \quad j = 0, \ldots, n-3 \end{gathered} \tag{3.63}$$

where:

$$\begin{gathered} \mathcal{M}_j \triangleq sp\ \{g, ad_f g, \ldots, ad_f^j g\} \\ \mathcal{M}_0 \triangleq sp\ \{g\} \end{gathered} \tag{3.64}$$

Remark 3.4.2 The above assumptions guarantee that system (3.1) is feedback linearizable $\forall\theta \in B_\theta$ [3], and that there exists a diffeomorphism $T(x,\theta)$ that linearizes this system and its first component is independent of θ.

Therefore, consider the change of coordinates $\hat{z}_i \triangleq T_i(x,\hat{\theta})$, where $\hat{z}_1 = z_1 = T_1(x)$. Assume that the reference model is an asymptotically stable **linear** system of the form:

$$\dot{z}_m = A_m z_m + b_m r \tag{3.65}$$

where $z_m(0) = z_{m_0}$, $r(t)$ is a uniformly bounded reference input, and (A_m, b_m) is in the controllable normal form. We wish the **transformed** state z_1 to track z_{m_1}. The error signal defined as:

$$e(t) = \hat{z} - z_m \tag{3.66}$$

can be shown to satisfy the following regressor form differential equation:

$$\dot{e} = A_m e + W(x,u,\hat{\theta})(\hat{\theta} - \theta) + M(x,\hat{\theta})\dot{\hat{\theta}} \tag{3.67}$$

where W and M are defined as in (3.46). Following a procedure very similar to the scheme of [98], and assuming $||W(x,u,\hat{\theta})||$ and $||M(x,\hat{\theta})||$ are bounded close to the origin $(0,\theta)$ such that:

$$\begin{aligned} ||W(x,u,\hat{\theta})|| &\leq c_1 \\ ||M(x,\hat{\theta})|| &\leq c_2 \end{aligned} \tag{3.68}$$

for some positive constants c_1, c_2, the following holds:

Theorem 3.4.2 ([3]) Consider the plant (3.59) satisfying assumptions (3.2.1), (3.4.3)-(3.4.5), with the following control law:

$$u = \frac{1}{\mathcal{L}_{g(x,\hat{\theta})}\mathcal{L}^{r-1}_{f(x,\hat{\theta})}h(x)}[-\mathcal{L}^r_{f(x,\hat{\theta})}h(x) + v] \tag{3.69}$$

with:

$$v = k\hat{z} + r(t) \tag{3.70}$$

k chosen such that $A + kb = A_m$, and (A, b) in Brunovsky form. There exists an open neighborhood $B(\theta, \mu)$ of θ, depending on the norm of the initial state $||\hat{z}(0)||$, such that if $\hat{\theta}(0) \in B(\theta, \mu)$, then $\hat{z}$ and $\hat{\theta}$ are bounded and $\lim_{t\to\infty} ||e(t)|| = \lim_{t\to\infty} (||\hat{z}(t)|| - ||z_m(t))||) = 0$.

3.5 Parametric-Strict-Feedback Systems

In this section we briefly review a recursive design procedure, based on backstepping desing technique of Kokotovic and co-workers, for adaptive control of nonlinear systems transformable into the so called **parametric-strict-feedback** form [57, 68]:

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \theta^T \psi_i(x_1, \ldots, x_i) \quad 1 \leq i \leq n-1 \\ \dot{x}_n &= \psi_0(x) + \theta^T \psi_n(x) + \beta_0(x) u \end{aligned} \tag{3.71}$$

where ψ_i, β_0 are all smooth nonlinear functions in $\Re^n$, and $\beta_0(x) \neq 0 \ \forall x \in \Re^n$. The early results [57] for adaptive control of systems in this special form suffered from overparameterization, np parameter estimates $\hat{\theta}$ for p unknown parameters θ in the system. Recent results obtained in [68] has solved this overparameterization problem resulting in exactly p parameter estimates for p unknown parameters. These schemes are based on **backstepping** design technique where a recursive procedure is used to, at each step, stabilize a subsystem containing the previous one. The idea is to start with a subsystem which is stabilizable with respect to a fictitious input (some state) with a **known** feedback law for a **known** Lyapunov function, and then to add an integrator to this input. In the

next step, a new subsystem with relative degree one with respect to the previous input is considered, and for the **augmented** system a new stabilizing feedback law is designed with an augmented Lyapunov function. After n steps, the true control law is explicitly formulated and shown to be globally stabilizing.

In this section we first review the regulation results of [68] followed by the tracking results of [57] The following two assumptions identify the class of nonlinear systems known as strict feedback systems that are considered in this scheme for **global adaptive regulation**:

Assumption 3.5.1 (Global Diffeomorphism) There exists a global diffeomorphism $z = \phi(x), \phi(0) = 0$, transforming the system (known part of the system (3.5)):

$$\dot{x} = f_0 + g_0(x)u \tag{3.72}$$

into the system:

$$\begin{aligned} \dot{z}_i &= z_{i+1} \quad 1 \leq i \leq n-1 \\ \dot{z}_n &= \gamma_0(z) + \beta_0 u \end{aligned} \tag{3.73}$$

where $\gamma_0(0) = 0, \beta_0(z) \neq 0 \; \forall z \in \Re^n$.

Assumption 3.5.2 (Strict Feedback Systems) In system (3.5), f_i and g_i satisfy the following conditions:

$$\begin{aligned} g_i &\equiv 0 \quad 1 \leq i \leq p \\ [X, f_i] &\in \mathcal{G}^j, \; \forall X \in \mathcal{G}^j, \; 0 \leq j \leq n-2 \end{aligned} \tag{3.74}$$

where $\mathcal{G}^j$ is defined as:

$$\mathcal{G}^j = span\{g_0, ad_{f_0} g_o, \ldots, ad^i_{f_0} g_0\} \tag{3.75}$$

At the kth step of the design, a k dimensional subsystem is stabilized using a stabilizing function α_k and a tuning function τ_k with respect to a Lyapunov function V_k. At the final step, $k = n$, the control law u and the parameter update law are designed. The control objective is to regulate x_1 to zero and to stabilize the corresponding equilibrium.

Step One: Introduce $z_1 = x_1$ and $z_2 = x_2 - \alpha_1$, where α_1 is a **stabilizing function** to be determined. Rewrite $\dot{x}_1 = x_2 + \theta^T\psi(x_1)$ as:

$$\dot{z}_1 = z_2 + \alpha_1 + \theta^T\psi_1(x_1) \tag{3.76}$$

and stabilize (3.76) with respect to the following Lyapunov candidate function:

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}(\hat{\theta} - \theta)^T\Omega^{-1}(\hat{\theta} - \theta) \tag{3.77}$$

using α_1 as control. Since:

$$\dot{V} = z_1(z_2 + \alpha_1 + \hat{\theta}^T\psi_1) + (\hat{\theta} - \theta)^T\Omega^{-1}(\dot{\hat{\theta}} - \Omega z_1\psi_1)$$

we choose the following stabilizing and tuning functions at this step:

$$\begin{aligned} \alpha_1(x_1, \hat{\theta}) &= -c_1 z_1 - \hat{\theta}^T\psi_1(x_1) \\ \tau_1(x_1) &= \Omega z_1\psi_1(x_1) \end{aligned} \tag{3.78}$$

which makes:

$$\dot{V}_1 = -c_1 z_1^2 + z_1 z_2 + (\hat{\theta} - \theta)^T\Omega^{-1}(\dot{\hat{\theta}} - \tau_1) \tag{3.79}$$

a negative definite function.

Step i: Introduce $z_{i+1} = x_{i+1} - \alpha_i$ and rewrite $\dot{x}_i = x_{i+1} + \theta^T\psi_i(x_1, \ldots, x_i)$ as:

$$\dot{z}_i = z_{i+1} + \alpha_i + \theta^T\psi_i - \sum_{k=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_k}(x_{k+1} + \hat{\theta}^T\psi_k) - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} \tag{3.80}$$

and use α_1 to stabilize the $(z_1, \ldots, z_i)$ subsystem using the Lyapunov candidate function $V_i = V_{i-1} + \frac{1}{2}z_i^2$:

$$\begin{aligned}\alpha_i(x_1, \ldots, x_i, \hat{\theta}) &= -z_{i-1} - c_i z_i + \sum_{k=1}^{i-1} \tfrac{\partial \alpha_{i-1}}{\partial x_k}(x_{k+1}) + \tfrac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\tau_i \\ &\quad + \left[\sum_{k=1}^{i-2} z_{k+1}\tfrac{\partial \alpha_k}{\partial \hat{\theta}}\Omega - \hat{\theta}^T\right] \cdot \left[\psi_i - \sum_{k=1}^{i-1} \tfrac{\partial \alpha_{i-1}}{\partial x_k}\psi_k\right]\end{aligned} \tag{3.81}$$

The tuning function at this step is defined as:

$$\tau_i(x_1, \ldots, x_i, \hat{\theta}) = \tau_{i-1} + \Omega z_i \left[\psi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\psi_k\right] \tag{3.82}$$

Step n: In the last step of the design the actual update law $\dot{\hat{\theta}} = \tau_n$ and the control law u are computed using the previously introduced stabilizing functions α_i and tuning functions τ_i with respect to the Lyapunov candidate function $V_n = V_{n-1} + \frac{1}{2}z_n^2$. The resulting parameter update law is given as:

$$\dot{\hat{\theta}} = \tau_n(x, \hat{\theta}) = \tau_{n-1} + \Omega z_n \left[\psi_n - \sum_{k=1}^{i-1} \frac{\partial \alpha_{n-1}}{\partial x_k}\psi_k\right] \tag{3.83}$$

The control law u is:

$$\begin{aligned}u &= \tfrac{1}{\beta_0}\Bigg[-z_{n-1} - -c_n z_n - \psi_0 + \sum_{k=1}^{n-1} \tfrac{\partial \alpha_{n-1}}{\partial x_k}(x_{k+1}) - \tfrac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \\ &\quad + \left[\sum_{k=1}^{n-2} z_{k+1}\tfrac{\partial \alpha_k}{\partial \hat{\theta}}\Omega - \hat{\theta}^T\right] \cdot \left[\psi_n - \sum_{k=1}^{n-1} \tfrac{\partial \alpha_{n-1}}{\partial x_k}\psi_k\right]\Bigg]\end{aligned} \tag{3.84}$$

It can be shown that:

$$\dot{V}_n = -\sum_{k=1}^{n} c_k z_k^2$$

The resulting closed loop adaptive system will be the form:

$$
\begin{aligned}
\dot{z}_1 &= -c_1 z_1 + z_2 + (\theta - \hat{\theta})^T \omega_1 \\
\dot{z}_2 &= -z_1 - c_2 z_2 + z_3 + (\theta - \hat{\theta})^T \omega_2 - \sum_{k=3}^{n} \tfrac{\partial \alpha_1}{\partial \hat{\theta}} \Omega z_k \omega_k \\
&\vdots \\
\dot{z}_i &= -z_{i-1} - c_i z_i + z_{i+1} + (\theta - \hat{\theta})^T \omega_i - \sum_{k=i+1}^{n} \tfrac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Omega z_k \omega_k + \sum_{k=1}^{i-2} z_{k+1} \tfrac{\partial \alpha_k}{\partial \hat{\theta}} \Omega \omega_i \\
\dot{z}_n &= -z_{n-1} - c_n z_n + (\theta - \hat{\theta})^T \omega_n + \sum_{k=1}^{n-2} z_{k+1} \tfrac{\partial \alpha_k}{\partial \hat{\theta}} \Omega \omega_n \\
\dot{\hat{\theta}} &= \Omega \sum_{i=1}^{n-1} z_i \omega_i
\end{aligned}
\tag{3.85}
$$

The following theorem summarizes the results on adaptive regulation for parametric-strict-feedback nonlinear systems:

Theorem 3.5.1 ([68]-Adaptive Regulation) Assume that assumptions (3.5.1) and (3.5.2) are satisfied and that (3.5) is transformed into the parametric-strict-feedback system (3.71). Suppose that the design procedure above is applied to this system. Then, the equilibrium $x = x^e, \hat{\theta} = \theta$ of the resulting adaptive system is globally stable. Furthermore, its state $(x, \hat{\theta})$ converges to a $(p - r)$-dimensional manifold:

$$
M = \left\{ (x, \hat{\theta}) \in \Re^{n+p} : x = x^e, (\theta - \hat{\theta})^T \psi_i^e = 0, i = 1, \ldots, n \right\} \tag{3.86}
$$

where p is the number of unknown parameters θ_i in (3.71) and:

$$
r \triangleq rank[\psi_1^e, \ldots, \psi_n^e]
$$

The equilibrium $x = x^e, \hat{\theta} = \theta$, is globally asymptotically stable if and only if $r = p$.

The adaptive tracking counterpart of this scheme is summarized below with procedures similar to the above scheme. The following three assumptions replace assumptions (3.5.1) and (3.5.2) and are necessary for input-output formulation of this scheme. Suppose that the relative degree assumption (3.2.2) is globally satisfied with r being the relative degree:

Assumption 3.5.3 There exists $n-r$ smooth functions $\phi(x), r+1 \leq i \leq n$, such that the change of coordinates:

$$\begin{aligned} z_i &= \mathcal{L}_{f_0}^{i-1} h(x) \quad 1 \leq i \leq r \\ z_i &= \phi_i(x) \quad r+1 \leq i \leq n \end{aligned} \tag{3.87}$$

is a global diffeomorphism $z = \phi(x)$ transforming the system:

$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x)u \\ y &= h(x) \end{aligned} \tag{3.88}$$

into the following normal form:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= b_0(z) + a_0(z)u \\ \dot{z}^r &= \Phi_0(y, z^r) \\ y &= z_1 \end{aligned} \tag{3.89}$$

where $a_0(z) \triangleq \mathcal{L}_{g_0} \mathcal{L}_{f_0}{}^{r-1} h(z) \neq 0, \forall z \in \Re^n$.

Assumption 3.5.4 (Strict Feedback Systems with Relative Degree r) In system (3.5) with relative degree r, f_i, g_i globally satisfy the following:

$$\begin{aligned} g_i &\equiv 0 \quad 1 \leq i \leq p \\ [X, f_i] &\in \mathcal{G}^j, \ \forall X \in \mathcal{G}^j, \ 0 \leq j \leq r-2 \end{aligned} \tag{3.90}$$

where $\mathcal{G}^j, 1 \leq j \leq r-1$ is defined as:

$$\mathcal{G}^j = span\{g_0, ad_{f_0}g_o, \ldots, ad^i_{f_0}g_0\} \tag{3.91}$$

Assumption 3.5.5 (Bounded-Input Bounded-Output z^r Subsystem) The z^r subsystem of (3.89) has the bounded-input bounded-output (BIBS) property with respect to y as its input.

The design procedure is very similar to the adaptive regulation discussed above. We briefly review these procedures from [57]. Under the assumptions (3.5.3) and (3.5.4), consider the parametric-strict-feedback system in (3.89) and the following change of coordinates:

$$\begin{aligned}
\xi_1 &= z_1 - y_m \\
\xi_{i+1} &= c_i\xi_i + z_{i+1} + \psi_i(\xi_1, \ldots, \xi_i, z^i, \vartheta_1, \ldots, \vartheta_{i-1}, y_m, \ldots, y_m^{(i)}) \\
&\quad + \hat{\theta}^T\gamma_i(\xi_1, \ldots, \xi_i, z^i, \vartheta_1, \ldots, \vartheta_{i-1}, y_m, \ldots, y_m^{(i-1)}) \quad i = 2, \ldots, r-1
\end{aligned} \tag{3.92}$$

where $\psi(\cdot)$ is a nonlinear term obtained from previous iteration:

$$\begin{aligned}
\dot{\xi}_i &= z_{i+1} + \psi(\xi_1, \ldots, \xi_i, z^i, \vartheta_1, \ldots, \vartheta_{i-1}, y_m, \ldots, y_m^{(i)}) \\
&\quad + \hat{\theta}^T\gamma_i(\xi_1, \ldots, \xi_i, z^i, \vartheta_1, \ldots, \vartheta_{i-1}, y_m, \ldots, y_m^{(i-1)})
\end{aligned}$$

and:

$$\dot{\vartheta}_i = \xi_i\gamma_i(\xi_1, \ldots, \xi_i, z^i, \vartheta_1, \ldots, \vartheta_{i-1}, y_m, \ldots, y_m^{(i-1)}) \quad i = 1, \ldots, r \tag{3.93}$$

Finally, using the control law:

$$u = \frac{1}{a_0(z)}[-c_r\xi_r - \psi_r - \vartheta_r^T\gamma_r] \tag{3.94}$$

the resulting closed-loop adaptive system is given by:

$$
\begin{aligned}
\dot{\xi}_1 &= -c_1\xi_1 + \xi_2 + (\theta - \vartheta_1)^T\gamma_1(\xi_1, z^r, y_m) \\
&\vdots \\
\dot{\xi}_r &= -c_r\xi_r + (\theta - \vartheta_r)^T\gamma_i(\xi_1, \ldots, \xi_r, z^r, \vartheta_1, \ldots, \vartheta_{i-1}, y_m, \ldots, y_m^{(r-1)}) \\
\dot{z}^r &= \Phi_0(y, z^r) + \sum_{i=1}^{p} \theta_i\Phi_i(y, z^r) \\
\dot{\vartheta}_i &= \xi_i\gamma_i(\xi_1, \ldots, \xi_i, z^i, \vartheta_1, \ldots, \vartheta_{i-1}, y_m, \ldots, y_m^{(i-1)}) \quad i = 1, \ldots, r \\
y &= \xi_1 + y_r
\end{aligned}
\tag{3.95}
$$

The following theorem summarizes the results on global adaptive tracking for parametric-strict-feedback nonlinear system:

Theorem 3.5.2 ([57]) Suppose that the system (3.1) satisfies the linear parameter assumption (3.2.1) and relative degree assumption (3.2.2) globally, and that under assumptions (3.5.3)-(3.5.5), it can be globally transformed into the parametric-strict-feedback form in (3.89). Furthermore, suppose that the reference signal y_m satisfies the assumption (3.2.3). Then, all the signals in the resulting closed-loop adaptive system in (3.95) are bounded and asymptotic tracking is achieved for all initial conditions in $\Re^{n+rp}$.

Part II

Adaptive Control of Nonlinear Systems

Chapter 4

Adaptive Control of Invertible MIMO Nonlinear Systems

4.1 Introduction

In this chapter we present two new schemes for the tracking and decoupling of multi-input multi-output (MIMO) nonlinear systems with parametric uncertainty in their dynamics. The first approach is an adaptive version of the well-known invertion algorithm developed by Hirschorn [42] and Singh [91] for the invertion of the input-output map of a MIMO nonlinear system. The second approach is based on the generalized normal form of nonlinear systems [51].

Most of the current techniques for adaptive control of nonlinear systems are based on feedback linearization by means of coordinate changes and assume the existence of a (vector) relative degree at a point of interest. The nonlinear systems to which these schemes can be applied are characterized by very restrictive coordinate-free conditions on the geometric properties of the system and constraints on the form of the nonlinearities present in the system [56, 57, 98, 23, 3, 55, 76, 97, 51, 78, 93]. The primary focus of these schemes

has been on single-input single-output (SISO) nonlinear systems. In the MIMO case, it is known that the possibility of using state feedback for input-output linearization is not restricted to systems with a certain relative degree, but holds for a broader class of nonlinear systems [51]. In particular, one can utilize the well-known structure algorithm developed by Hirschorn and Singh for the inversion of multi-input multi-output nonlinear systems and construct a right-inverse system. The inverse system then can be used as a decoupling prefilter that produces the input to the original system such that the outputs track a desired path. Therefore, using output feedback and precompensation, asymptotic functional reproducibility can be achieved. The structure of the resulting tracking controller using inverse dynamics is shown in figure (4.1).

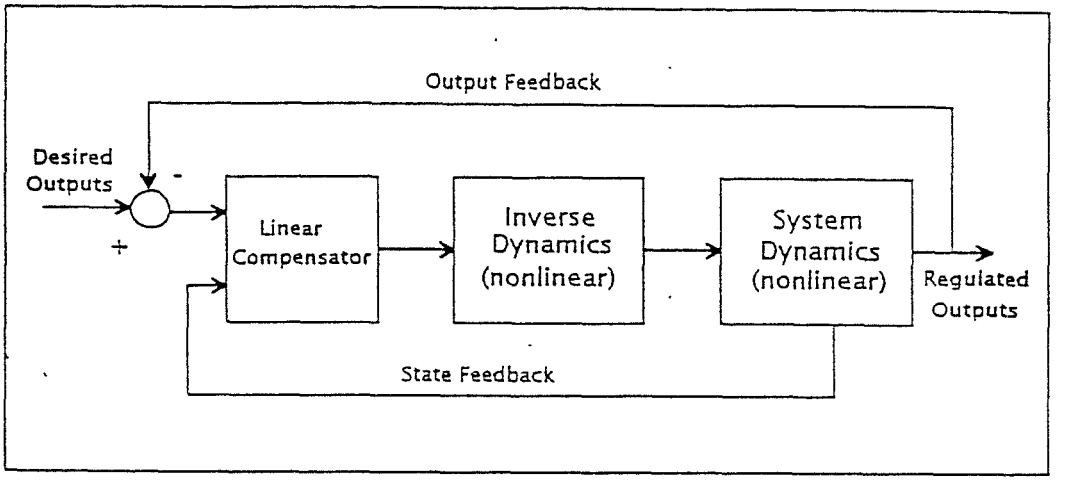

Figure 4.1: Control structure for asymptotic output tracking using dynamic inversion.

We first review the inversion algorithm of [42, 91]. In section III, we present

the adaptive version of this algorithm for nonlinear MIMO systems under parameter uncertainty. We illustrate the features of this scheme in section IV using a nonlinear system arising in the outer-loop design of an aircraft [7, 90]. This scheme has also been successfully applied to active vehicle suspension control [16, 15] and magnetic levitation systems [15] which will be presented in the second part of this thesis. In section V, we develop an alternative design approach which is based on the generalized normal form for MIMO nonlinear systems [51].

4.2 Structure Algorithm

Consider the following nonlinear system with m inputs and l outputs with $l \leq m$:

$$\begin{aligned} \dot{x}(t) &= A(x) + \sum_{i=1}^{m} u_i \cdot B_i(x) \qquad x \in M \\ y(t) &= C(x(t)) \end{aligned} \tag{4.1}$$

where the state space is a connected, n-dimensional, real analytic manifold; $A, B_i \in V(M)$ is the real vector space of the real analytic vector fields on M, $u_i \in U$ is the class of real analytic functions from $[0\ \infty)$ into $\Re$, and $C(\cdot) : M \rightarrow \Re^l$ is a real analytic mapping. The output of this system can be made to track various signals depending on controls u_i and the choice of initial states. We are interested in deriving a control law $u = (u_1, u_2, \ldots, u_m)$ for asymptotic tracking of a given signal $y_m = f(\cdot)$ under parametric uncertainty in (4.1) so that the output $y(t, u, x_0)$ of (4.1) converges to y_m as $t \rightarrow \infty$.

The inversion algorithm, developed by Hirschorn [42] and Singh [91], gives a systematic scheme to obtain a sequence of systems associated with (4.1). These systems are derived by performing a series of simple operations such as differentiation, row ordering, and row reduction on the output $y(\cdot)$ of system

(4.1). In what follows, we skip the details of the algorithm and summarize the end results. Associated with (4.1), we construct a sequence of systems in the form:

$$\begin{aligned} \dot{x}(t) &= A(x) + B(x).u \qquad x \in M_k \\ z_k(t) &= C_k(x) + D_k(x).u \end{aligned} \tag{4.2}$$

where $B(x) = (B_1, B_2, \ldots, B_m)$, M_k is an open dense submanifold of M, $D_k(x)$ has all but the first r_k rows zero and has rank r_k for all $x \in M_k$. The tracking order β of the system (4.1) is defined as the least positive integer k such that $r_k = l$ or $\beta = \infty$ if $r_k < l$ for all $k > 0$. Hence, $D_\beta(x)$ is an $l \times m$ matrix with rank l ($l \leq m$) for all $x \in M_\beta$. Therefore, if $\beta < \infty$, any given analytic function $f(\cdot)$ is functionally reproducible in the sense of [92] by system (4.2) for $k = \beta$. $z_k(t)$ is partitioned in the form:

$$z_k(t) = \begin{bmatrix} \bar{z}_k(t) \\ \hat{z}_k(t) \end{bmatrix} = \begin{bmatrix} \bar{C}_k(x) \\ \hat{C}_k(x) \end{bmatrix} + \begin{bmatrix} D_{k1}(x) \\ 0 \end{bmatrix} u(t) \tag{4.3}$$

where rank $D_{k1}(x) = r_k$, for all $x \in M_k$, and $\bar{z}_k(t)$ and $\bar{C}_k(x)$ consist of the first r_k elements of $z_k(t)$ and C_k. $z_k(t)$ can also be written in terms of the derivatives of the output $y(t, u, x_0)$ up to kth order:

$$z_k(t) = \begin{bmatrix} \bar{z}_k(t) \\ \hat{z}_k(t) \end{bmatrix} = \begin{bmatrix} H_k(x) \\ J_k(x) \end{bmatrix} Y_k(t) \tag{4.4}$$

where:

$$Y_k(t) = [(y^{(1)})^T, (y^{(2)})^T, \ldots, y^{(k)})^T]^T$$

The system β is:

$$\begin{aligned} \dot{x}(t) &= A(x) + B(x).u \qquad x \in M_\beta \\ z_\beta(t) &= C_\beta(x) + D_\beta(x).u \end{aligned} \tag{4.5}$$

with

$$z_\beta(t) = H_\beta(x) \cdot Y_\beta(t) \tag{4.6}$$

and $J_\beta(x) = 0$. We can rewrite this as:

$$z_\beta(t) = N(x) \begin{bmatrix} y_1^{(n_1)} \\ \vdots \\ y_l^{(n_l)} \end{bmatrix} + M(x)\tilde{y} \tag{4.7}$$

where n_i and N_i are the lowest and highest order derivatives of $y^{(i)}$ appearing in (4.6),

$$\tilde{y} = \left[y_1^{(n_1+1)}, \cdots, y_1^{(N_1)}, y_2^{(n_2+1)}, \cdots, y_l^{(N_l)}\right]^T \tag{4.8}$$

and $N(x)$ is an $l \times l$ nonsingular matrix with determinant of $N(x) = \pm 1$ for all $x \in M_\beta$.

With perfect knowledge of A, B and C in (4.1), (4.5) and (4.7) can be utilized to achieve asymptotic tracking by applying the following control law introduced in [92]:

$$u(t) = D_\beta^\dagger(x) \cdot \{-C_\beta(x) + M(x)\tilde{y} + N(x) \cdot K\} \tag{4.9}$$

where:

$$K \triangleq \begin{bmatrix} y_m{}^{(n_1)} + \sum_{j=0}^{n_1-1} p_{1j}(y_{m1}^{(j)} - y_1^{(j)}) + v_1(t) \\ \vdots \\ y_m{}^{(n_l)} + \sum_{j=0}^{n_l-1} p_{lj}(y_{ml}^{(j)} - y_l^{(j)}) + v_l(t) \end{bmatrix}$$

and $D_\beta^\dagger(x)$ is the pseudoinverse of $D_\beta(x)$, p_{ij} are some constant coefficients, and $v_i(t)$ is a servocompensator of the form:

$$\dot{v}_i = \gamma_{i0}(y_{m_i}(t) - y_i(t))$$

for robustness against disturbances in the system. The coefficients p_{ij} and γ_{i0} are chosen such that all the roots of the corresponding characteristic polynomial have negative real parts:

$$\sum_{j=0}^{n_i+1} \gamma_{ij} \cdot s^j = 0 \qquad i = 1, \ldots, l$$

with $\gamma_{ij} = p_{i,j-1}, j = 1, \ldots, n_i + 1$.

The control law (4.9) does not guarantee the internal stability of system (4.1) under this feedback. For the states to remain bounded, all the unobservable modes of the system under such feedback must remain stable, and of course, all the states must remain in M_β. These conditions must also hold for adaptive version of the above control law discussed in the next section.

4.3 Adaptive Control of Invertible MIMO Nonlinear Systems

We are interested in solving the decoupling/tracking problem under parametric uncertainty in the original system, **i.e.** in $A(x), B(x)$, or $C(x)$. Consider system (4.1) under parametric uncertainty:

$$\begin{aligned} \dot{x}(t) &= A(x,\theta) + \sum_{i=1}^{m} u_i \cdot B_i(x,\theta) \qquad x \in M \\ y(t) &= C(x(t),\theta) \end{aligned} \tag{4.10}$$

where θ represents the vector containing unknown parameters. Recall that following the above inversion algorithm, system (k) in (4.2) was obtained by a sequence of linear operations (row ordering and reduction) on the original sys-

tem. Hence, the β system under parametric uncertainties will be:

$$\begin{aligned} \dot{x}(t) &= A(x,\theta) + B(x,\theta)\cdot u \qquad x \in M_\beta \\ z_\beta(t) &= C_\beta(x,\bar{\theta}) + D_\beta(x,\bar{\theta})u \end{aligned} \tag{4.11}$$

where $\bar{\theta}$ is now possibly a new vector of unknown constants that is related to the original vector θ, but it may be of higher dimension, and

$$z_\beta(t) = H_\beta(x(t),\bar{\theta})\cdot Y_\beta(t) \tag{4.12}$$

which may be rewritten in the form:

$$z_\beta(t) = N(x(t),\bar{\theta})\cdot\hat{y} + M(x(t),\bar{\theta})\cdot\tilde{y} \tag{4.13}$$

where $\tilde{y}$ was defined in (4.8) and:

$$\hat{y} = \left[y_1^{(n_1)},\cdots,y_l^{(n_l)}\right]^T \tag{4.14}$$

and, as before, n_i and N_i are the lowest and highest order derivatives of $y^{(i)}$ appearing in (4.12). Recall also that $N(x(t),\bar{\theta})$ is an $l\times l$ nonsingular matrix with determinant of $N(x(t),\bar{\theta}) = \pm 1$ for all $x \in M_\beta$. We now make the following assumption, which together with the algorithmic assumptions made above ($l \leq m$ and $\beta < \infty$) represents the class of smooth nonlinear MIMO systems that our adaptive tracking scheme is applicable to.

Assumption 4.3.1 (Linear Parameter Dependence) The vector fields $C_\beta(x(t),\bar{\theta})$, $D_\beta(x(t),\bar{\theta})$ and $H_\beta(x(t),\bar{\theta})$ in (4.11) and (4.12) depend linearly on the unknown parameters $\bar{\theta}$.

Next, we introduce the following control law which is the same as the control law introduced in [92] for asymptotic reproducibility of nonlinear systems, except

that the uncertain parameters θ in system (4.10) have been replaced by the adjustable parameters $\hat{\theta}$ that are on line estimates of the true parameters $\bar{\theta}$ with some updating rules yet to be determined:

$$u(t) = D_\beta^\dagger(x,\hat{\theta}) \cdot \left[-C_\beta(x,\hat{\theta}) + \hat{M}\tilde{y} + N(x,\hat{\theta}) \cdot K\right] \tag{4.15}$$

where:

$$K = \begin{bmatrix} y_{m1}^{(n_1)} + \sum_{j=0}^{n_1-1} p_{1j}(y_{m1}^{(j)} - y_1^{(j)}) \\ \vdots \\ y_{ml}^{(n_l)} + \sum_{j=0}^{n_l-1} p_{lj}(y_{ml}^{(j)} - y_l^{(j)}) \end{bmatrix} \tag{4.16}$$

where $D_\beta^\dagger(x)$ is the pseudoinverse of $D_\beta(x)$, and p_{ij} are some constant coefficients such that the roots of the corresponding characteristic polynomials have negative real parts. Using Lyapunov stability theory, we will now derive a suitable updating rule for the adjustable parameter vector $\bar{\theta}$ such that output tracking is achieved.

From assumption (4.3.1), we have:

$$\begin{aligned} C_\beta(x(t),\bar{\theta}) &= \sum_{i=1}^{p} C_{\beta i}(x) \cdot \bar{\theta}_i + C_{\beta 0}(x) = \tilde{C}_\beta(x) \cdot \bar{\theta} + C_{\beta 0}(x) \\ D_\beta(x(t),\bar{\theta}) \cdot u &= \sum_{i=1}^{p} D_{\beta i}(x,u) \cdot \bar{\theta}_i + D_{\beta 0}(x,u) = \tilde{D}_\beta(x,u) \cdot \bar{\theta} + D_{\beta 0}(x,u) \\ M(x(t),\bar{\theta}) \cdot \tilde{y} &= \sum_{i=1}^{p} M_i(x,\tilde{y}) \cdot \bar{\theta}_i + M_0(x,\tilde{y}) = \tilde{M}(x,\tilde{y}) \cdot \bar{\theta} + M_0(x,\tilde{y}) \\ N(x(t),\bar{\theta}) \cdot \hat{y} &= \sum_{i=1}^{p} N_i(x,\hat{y}) \cdot \bar{\theta}_i + N_0(x,\hat{y}) = \tilde{N}(x,\hat{y}) \cdot \bar{\theta} + N_0(x,\hat{y}) \end{aligned} \tag{4.17}$$

Substituting (4.15) into (4.11) and using (4.13) for $z_\beta(t)$ gives:

$$N(x(t),\bar{\theta})\hat{y} + M(x(t),\bar{\theta})\tilde{y} = C_\beta(x(t),\bar{\theta}) + D_\beta(x(t),\bar{\theta})u$$

$$N(x,\hat{\theta})[K-\hat{y}] = \tilde{C}_\beta(x)\cdot\phi + \tilde{D}_\beta(x,u)\cdot\phi - \tilde{M}(x,\tilde{y})\cdot\phi - \tilde{N}(x,\hat{y})\cdot\phi$$

$$N(x,\hat{\theta})[K-\hat{y}] = W^1(x,u,\tilde{y},\hat{y})\cdot\phi \quad (4.18)$$

where $\phi = \hat{\theta} - \bar{\theta}$. Since $N(x,\hat{\theta})$ is nonsingular by construction with determinant ± 1, we have:

$$[K-\hat{y}] = N^{-1}(x,\hat{\theta})\cdot W^1(x,u,\tilde{y},\hat{y})\cdot\phi = W^2(x,u,y^{(i)},\hat{\theta})\cdot\phi \qquad (4.19)$$

Now let $\epsilon = [e_1, \dot{e}_1, \ldots, e^{(1)}n_1 - 1, e_2, \ldots, e^{(l)}n_l - 1]^T$ and choose p_{ij} such that the corresponding characteristic polynomials $\sum_{j=0}^{n_i-1} p_{ij}\cdot s^j = 0$ are asymptotically stable. Consider the following Lyapunov candidate function:

$$V = \epsilon^T R\epsilon + \phi^T\Omega\phi$$

where $\Omega^T = \Omega = diag(1/g_i) > 0$ and $R = R^T > 0$ is the solution of the Lyapunov equation:

$$R\cdot P + P^T\cdot R = -Q$$

for some $Q = Q^T > 0$ and:

$$P = diag(P_i)$$

$$P_i = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -p_{i0} & -p_{i1} & & \ldots & -p_{i,n_i-1} \end{pmatrix}$$

Note that:

$$\dot{\epsilon} = P \cdot \begin{bmatrix} e_1 \\ \vdots \\ e^{(1)}n_1 - 1 \\ e_2 \\ \vdots \\ e^{(l)}n_l - 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ W_1^2 \\ 0 \\ \vdots \\ W_l^2 \end{bmatrix} \cdot \phi = P\epsilon + W\phi \tag{4.20}$$

Then, the derivative of V evaluated along the solution trajectories of the error equation (4.20) is:

$$\dot{V} = -\epsilon^T Q\epsilon + 2\epsilon^T RW(x, u, y^{(i)}, \hat{\theta})\phi + 2\phi\Omega\dot{\phi}$$

Taking parameter update laws as:

$$\dot{\phi} = -\Omega^{-1} \cdot W^T \cdot R \cdot \epsilon \tag{4.21}$$

gives:

$$\dot{V} = -\epsilon^T Q\epsilon \leq -\gamma\|\epsilon\|^2 \leq 0$$

This proves that V is bounded. Hence e_i and $\hat{\theta}_i$ are bounded, and $\dot{V}$ is bounded and integrable. If, moreover, the system is internally stable, then $\|e_i\| \to 0$ as $t \to \infty$. Of course, for the system be internally stable under such feedback, all the unobservable modes must remain stable. In fact, if the zero dynamics of the system is not asymptotically stable, then it is possible that for some reference signals, the tracking control law producing a linear input-output response may result in unbounded unobservable states. To achieve asymptotic tracking for **all** reference signals, sufficient conditions are typically too restrictive and are often hard to satisfy. Given a specific class of reference signals

one might search for **bounded-input bounded-state** (BIBS) property of the unobservable subsystem under the above decoupling feedback control, treating the output y as input. This is a generalization of BIBS assumption in [57]. This subsystem is obtained using the generalized normal form transformation of [51]. The following theorem summarizes the main result of this section:

Theorem 4.3.1 (Adaptive Tracking) Suppose that the system described by (4.10) has a finite tracking order ($\beta < \infty$), has at least as many inputs as there are outputs ($l \leq m$), and that assumption (4.3.1) holds (linear parameter dependence). Then given any smooth bounded signal $y_m = [y_{m1}, \ldots, y_{ml}]$ with bounded derivatives up to order $n_i - 1$, with the control law (4.15) and (4.21), and p_{ij} chosen such that the corresponding characteristic polynomials are asymptotically stable, if the resulting unobservable subsystem is BIBS with respect to output as its input, the output $y(\cdot)$ of (4.10) approaches y_m.

Theorem (4.3.1) is the adaptive version of tracking theorems of [92, 44, 90] which are based on constructing a right inverse system. Figure (4.2) shows the design structure for adaptive output tracking developed in this section. To illustrate the proposed design technique, in the next section we will consider its potential application to the control of nonlinear system arising in the outer-loop design of an aircraft.

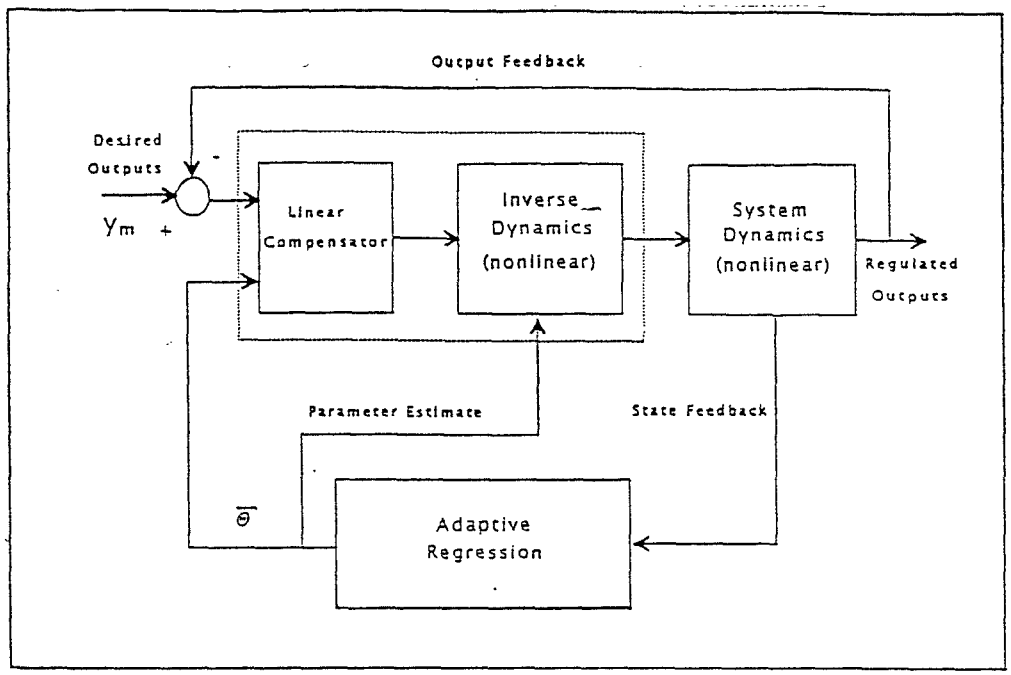

Figure 4.2: Control structure for adaptive output tracking using a dynamic right-inverse.

4.4 Applications to Aircraft Control Problem

Consider the nonlinear system arising in the outer-loop design of an aircraft [7, 90]:

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} x_2 \\ 0 \\ 0 \\ (\frac{-g}{v_0}) \cdot \sin^2(x_1) + x_3 \cos(x_1) \\ (\frac{g}{2v_0}) \cdot \sin(2x_1) + x_3 \sin(x_1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
y &= \begin{pmatrix} x_4 \\ x_5 \end{pmatrix}
\end{aligned}
\tag{4.22}
$$

with:

$$x = (\phi, p, q_w, \gamma, \psi)^T$$

where ϕ is the roll angle, p is the roll rate, q_w is the wind referenced pitch rate, γ is the vertical path flight angle, and ψ is the horizontal path flight angle of the airplane. g is the gravitational constant, and v_0 is the air speed. The objective of the outer-loop design is to decouple γ and ψ. It is desired to design a robust control law $u(t) = [u_1, u_2]^T$ such that under parametric uncertainty and slow variations in v_0, γ and ψ will remain decoupled and follow pilot command inputs.

Let's define $\theta_1 = (g/2v_0)$ and using the structure algorithm, one gets:

$$\begin{aligned} z_3(t) &= \begin{bmatrix} -x_2 \cdot (2\theta_1 \sin(2x_1) + x_3 \sin(x_1)) \\ x_3 {x_2}^2 \ \tan(x_1) sec(x_1) \end{bmatrix} \\ &\quad + \begin{bmatrix} \cos(x_1) & 0 \\ x_2 sec(x_1) & 2\theta_1 + x_3 sec(x_1) \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ z_3(t) &\triangleq C_3(x, \bar{\theta}) + D_3(x, \bar{\theta}) u \end{aligned} \tag{4.23}$$

Also:

$$\begin{aligned} z_3(t) &= \begin{bmatrix} 1 & 0 \\ -x_2 sec^2(x_1) & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1^{(2)} \\ y_2^{(3)} \end{bmatrix} + \begin{bmatrix} 0 \\ -\tan(x_1) \end{bmatrix} y_1^{(3)} \\ z_3(t) &\triangleq N(x)\hat{y} + M(x)\tilde{y} \end{aligned} \tag{4.24}$$

In view of (4.15), let's choose the following control law:

$$u(t) = D_3^{-1}(x, \hat{\theta}) \cdot \left[-C_3(x, \hat{\theta}) + M(x) \cdot \tilde{y} + N(x) \cdot K \right] \tag{4.25}$$

where:

$$D_3(x, \hat{\theta}) \triangleq \begin{bmatrix} \cos(x_1) & 0 \\ x_2 sec(x_1) & 2\hat{\theta}_1 + x_3 sec(x_1) \end{bmatrix}$$

$$C_3(x,\hat{\theta}) \triangleq \begin{bmatrix} -2\hat{\theta}_1 x_2 \sin(2x_1) + x_2 x_3 \sin(x_1) \\ x_3 {x_2}^2 \tan(x_1) sec(x_1) \end{bmatrix} \tag{4.26}$$

and:

$$K = \begin{bmatrix} {y_m}_1^{(2)} + \alpha_{11}\dot{e}_1 + \alpha_{10} e_1 \\ {y_m}_2^{(3)} + \alpha_{22}\ddot{e}_2 + \alpha_{21}\dot{e}_2 + \alpha_{20} e_2 \end{bmatrix} \tag{4.27}$$

where $e \triangleq y_m - y$. Applying (4.25) to (4.23), using (4.24) for $z_3(t)$ and regrouping terms gives:

$$\begin{bmatrix} \ddot{e}_1 + \alpha_{11}\dot{e}_1 + \alpha_{10} e_1 \\ e^{(2)}3 + \alpha_{22}\ddot{e}_2 + \alpha_{21}\dot{e}_2 + \alpha_{20} e_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \sin(2x_1) \\ -2{x_2}^2 \sin(2x_1) sec^2(x_1) + 2u_2 \end{bmatrix} \cdot \phi \tag{4.28}$$

where $\phi \triangleq \hat{\theta}_1 - \theta_1$. Now let $\epsilon = [e_1, \dot{e}_1, e_2, \dot{e}_2, \ddot{e}_2]^T$ and choose α_{ij} such that the corresponding characteristic polynomials are asymptotically stable, for example:

$$\alpha_{11} = 20, \alpha_{10} = 100, \alpha_{22} = 30, \alpha_{21} = 300, \alpha_{20} = 1000$$

which results in five poles at $s = -10$. Consider the following Lyapunov candidate function:

$$V = \epsilon^T R \epsilon + 1/g \cdot \phi^T \cdot \phi$$

where $g > 0$ is the adaptation gain for parameter θ, and $R = R^T > 0$ is the solution of the Lyapunov equation:

$$R \cdot P + P^T \cdot R = -Q$$

with $Q = Q^T > 0$ chosen for our simulation to be $Q \triangleq 1000 \cdot I_5$ where I_5 is the

5×5 identity matrix, and:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -100 & -20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1000 & -300 & -30 \end{pmatrix} \tag{4.29}$$

Using (4.28), we have the following relationship:

$$\dot{\epsilon} = P \cdot \epsilon + \begin{pmatrix} 0 \\ -2x_2 \sin(2x_1) \\ 0 \\ 0 \\ -2{x_2}^2 \sin(2x_1) sec^2(x_1) + 2u_2 \end{pmatrix} \cdot \phi \triangleq P \cdot \epsilon + W(x,u) \cdot \phi \tag{4.30}$$

The derivative of V along the solution trajectories of (4.22) is:

$$\dot{V} = -\epsilon^T Q \epsilon + 2\epsilon^T \cdot R \cdot W(x,u) \cdot \phi + 2g\phi \cdot \dot{\phi}$$

The design procedure of last section applied to this system results in the updating law:

$$\dot{\phi} = -1/g \cdot W^T \cdot R \cdot \epsilon \tag{4.31}$$

The generalized normal form for this system is obtained with new coordinates as ($\xi = (\xi_1, \cdots, \xi_5)^T$):

$$\xi = (h_4, f_4, h_5, f_5, \psi)^T$$

with $\psi \triangleq -\tan(X_1) \cdot \mathcal{L}f_4 + \mathcal{L}f_5$ where the transformation Φ given by $x \to \xi$ is a local diffeomorphism. This system does not have any zero dynamics and the BIBS condition of theorem (4.3.1) is automatically satisfied. Hence, since $\dot{e}_i$

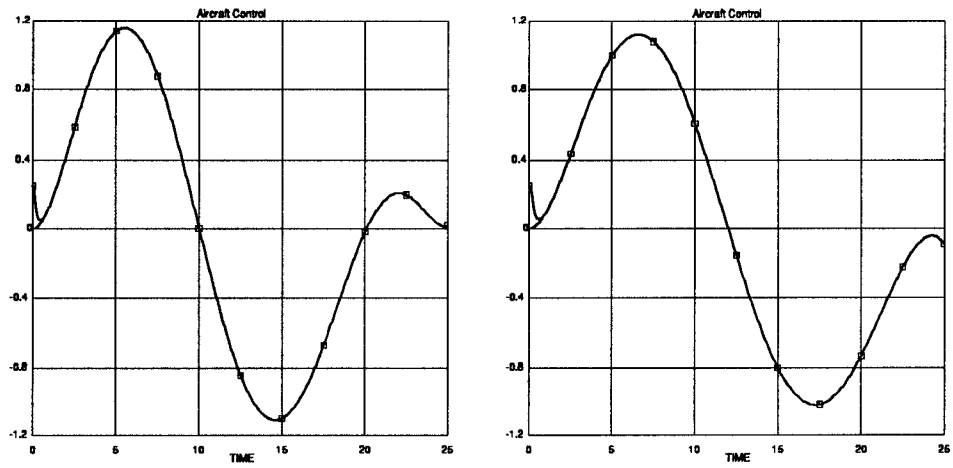

Figure 4.3: Reference trajectories and controlled outputs for 25% uncertainty in the air speed v_0 and 20% error in initial conditions.

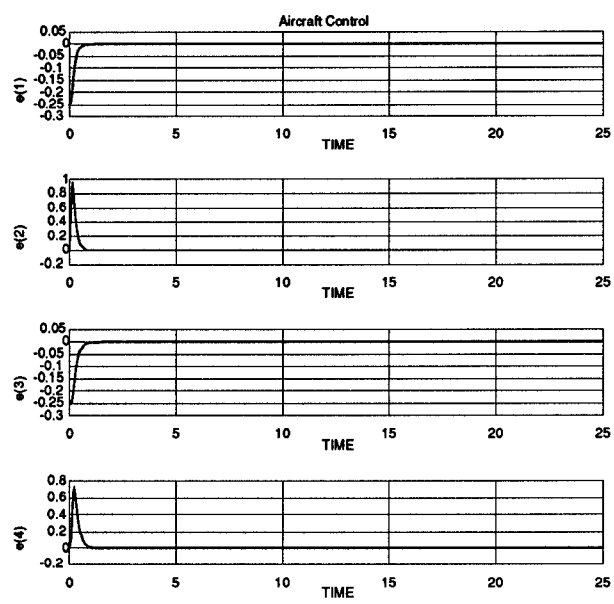

Figure 4.4: Error trajectories for 25% uncertainty in the air speed v_0 and 20% error in initial conditions.

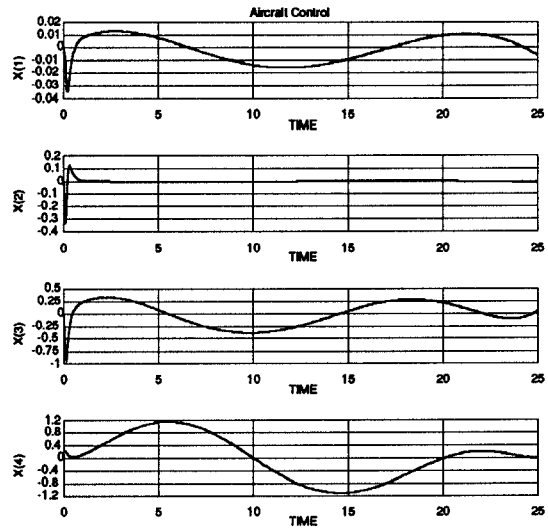

Figure 4.5: State trajectories for 25% uncertainty in the air speed v_0 and 20% error in initial conditions.

and e_i are bounded and $\|e_i\| \in \mathcal{L}_\infty$, we conclude that $\|e_i\| \to 0$ as $t \to \infty$, and consequently: $y_i \to y_{m_i}$ as $t \to \infty$.

Figure (4.3) shows the vertical and horizontal flight path angles tracking the command inputs. Figure (4.4) indicates that the errors (ϵ) converge to zero, and shows the response (all the states) of the closed-loop aircraft to pilot command inputs. It is clear that the responses are stable and decoupled, and adaptive output tracking is achieved.

4.5 Dynamic High Gain Feedback

We now extend our results to MIMO nonlinear systems that do not necessarily have a finite tracking order β, hence are not invertible in the sense defined in [42, 91]. Consider the nonlinear system given in (4.10) with m inputs and l outputs and assume $l \leq m$. We obtain the following system by differentiating y:

$$
\begin{aligned}
\frac{dy}{dt} &= \dot{y}(t) = dc_{x(t)}(\dot{x}(t)) \\
&= dc_{x(t)}\left(A(x,\theta) + \sum_{i=1}^{m} u_i \cdot B_i(x,\theta)\right) \\
&= (AC)(x,\bar{\theta}) + \sum_{i=1}^{m} u_i (B_i C)(x,\bar{\theta})
\end{aligned}
\tag{4.32}
$$

Define $D(x,\bar{\theta}) \triangleq [B_1C(\cdot), B_2C(\cdot), \ldots, B_mC(\cdot)]$, an $l \times m$ matrix for each $x \in M$, and with this notation we write:

$$
\dot{y} = AC(x,\bar{\theta}) + D(x,\bar{\theta})u \qquad x \in M \tag{4.33}
$$

where θ represents the vector containing unknown parameters in the system, $\bar{\theta}$ is a new vector of unknown constants that is related to the original vector θ in (4.10), possibly of higher dimension, and it is assumed that $\bar{\theta}_i's$ appear linearly in (4.33). Note that this was the first step in the structure algorithm used in section (4.2) with $r_1 \triangleq max_{x\in M}\{rankD(x,\bar{\theta})\}$. If $r_1 = l$, then we have the case where $\beta = 1$, and one can apply the design scheme developed in section (4.2) to this system since the $l \times m$ matrix $D(x,\bar{\theta})$ is of full rank on $M_1 \triangleq \{x : rankD(x,\bar{\theta}) = r_1\}$ and $D \cdot D^\dagger = I$ on M_1, where $D^\dagger$ is the pseudoinverse of D. In the case where $r_1 < l$, the structure algorithm would continue to the next step as explained before since $D^\dagger$ no longer exist. However, consider a matrix $\hat{D}(x,\bar{\theta},\hat{\vartheta})$ where ϑ is a vector of some "fictitious" parameters appearing linearly in $\hat{D}(x,\bar{\theta},\hat{\vartheta})$ and are injected in $D(x,\bar{\theta})$ such that $rank\hat{D}(x,\bar{\theta},\hat{\vartheta}) = l$ for all $x \in M$ with $\hat{\vartheta}$ an estimate of ϑ. So: $\hat{D}(x,\bar{\theta},0) = \hat{D}(x,\bar{\theta})$. Define $\hat{\theta}$ to be the estimates of the vector $\eta = [\bar{\theta}^T, \vartheta^T]^T$ and $\phi \triangleq \hat{\theta} - \eta$. The idea now is to find updating laws for $\hat{\theta}$ such that $y_i \to y_{m_i}$ while $rank\ \hat{D}(x,\hat{\theta})$ remains constant. To do this we will assign small values to the gains corresponding to $\hat{\vartheta}$, the estimates of the fictitious parameters, so that they change very slowly. Moreover, one can use a suitable

projection algorithm in order to prevent the convergence of these parameters to their true values (usually zero) and *rank* $\hat{D}(x,\hat{\theta})$ remains constant. The control will then be based on estimates of the fictitious parameters with updating rules determined such that the stability is preserved and the tracking is achieved. This procedure can also be viewed as a dynamic state feedback control.

With this in mind consider (4.33) and in view of (4.15)

$$u(t) = \hat{D}^{\dagger}(x,\hat{\theta}) \cdot \left[-C(x,\hat{\theta}) + \ddot{y}_m + \alpha_1 \dot{e} + \alpha_0 e\right] \tag{4.34}$$

where $\hat{\theta}$ are estimated parameters with updating laws to be determined, and $e \triangleq y_m - y$. Applying (4.34) to (4.33) gives:

$$\begin{aligned} \ddot{e} + \alpha_1 \dot{e} + \alpha_0 e &= \left[\hat{D}(x,\hat{\theta}) - D(x,\theta)\right] \cdot u + C(x,\hat{\theta}) - C(x,\theta) \\ \ddot{e} &= -\alpha_1 \dot{e} - \alpha_0 e + W^1(x,u,\hat{\theta}) \cdot \phi \end{aligned} \tag{4.35}$$

We have:

$$\dot{\epsilon} = P \cdot \begin{bmatrix} e_1 \\ \dot{e}_1 \\ e_2 \\ \vdots \\ \dot{e}_l \end{bmatrix} + \begin{bmatrix} 0 \\ W_1^1 \\ 0 \\ \vdots \\ W_l^1 \end{bmatrix} \cdot \phi = P\epsilon + W\phi \tag{4.36}$$

where $\epsilon \triangleq [e_1, \dot{e}_1, e_2, \ldots, \dot{e}_l]^T$ and:

$$P = diag(P_i)$$

$$P_i = \begin{pmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{pmatrix}$$

Consider the following Lyapunov candidate function:

$$V = \epsilon^T R \epsilon + \phi^T \Omega \phi$$

where $\Omega^T = \Omega = diag(1/g_i) > 0$ and $R = R^T > 0$ is the solution of the Lyapunov equation:

$$R \cdot P + P^T \cdot R = -Q$$

for some $Q = Q^T > 0$. Taking the derivative of V evaluated along the solution trajectories of (4.36) and the updating laws:

$$\dot{\phi} = -\Omega^{-1} \cdot W^T \cdot R \cdot \epsilon \tag{4.37}$$

gives:

$$\dot{V} = -\epsilon^T Q \epsilon \leq -\gamma \|\epsilon\|^2 \leq 0$$

Hence, since $\dot{e}_i$ and e_i are bounded and $\|e_i\| \in \mathcal{L}_\infty$, we conclude that $\|e_i\| \to 0$ as $t \to \infty$. Consequently: $y_i \to y_{m_i}$ as $t \to \infty$.

Of course, as in any adaptive control strategy, the matrix $\hat{D}(x, \hat{\theta})$ has to be monitored, on line, to remain nonsingular as long as there are nonzero errors in the system. Assuming these adjustable parameters for the fictitious parameters do not converge to their true values (zero), we can state the following theorem:

Theorem 4.5.1 Suppose that (4.33) is linear in $\bar{\theta}_i$. Then given any smooth bounded signal $y_m = [y_{m1}, \ldots, y_{ml}]$ with bounded derivatives such that the estimates of ϑ_i converge to a nonzero value (nonpersistently exciting signal), with the control law (4.37) and (4.34), $n_i \geq 2$, and p_{ij} chosen such that the corresponding characteristic polynomials are asymptotically stable, the output $y(\cdot)$ of (4.10) tracks y_m, for all $x_0 \in M_\beta$.

Note that the update law (4.37) guarantees the convergence of the estimates of ϑ_i to some values. However, theorem (4.5.1) assumes that these values are bounded away from zero during the adaptation process in order to have a nonsingular

decoupling matrix. Note that if one chooses $n_i = 2$ in (4.34), then the control law (4.34) with (4.37) is of a PID type controller coupled with state feedback. In order to have asymptotic tracking with internal stability, x_0 and y_m need to be such that all the states remain in a compact subset of M_β in the operating region of interest. Sufficient conditions to globally achieve this are typically very restrictive in general. The application of this theorem is, however, more useful for a given nonlinear system where the boundedness of states can be shown explicitly under this feedback, or where the operating region of interest is known to be contained in the domain of internal stability of our system. An example, where this is the case is treated in the next section. It is also possible to use other Lyapunov type functions to guarantee that the estimates ϑ_i converge to a nonzero value in order to guarantee a nonsingular decoupling matrix.

4.6 Example

Although the proposed schemes here are intended more for MIMO nonlinear systems, for the sake of comparison and to illustrate the design procedure, we apply the scheme developed in the last section to the problem considered in [57] in which the output y of the system:

$$\begin{aligned} \dot{z}_1 &= z_2 + \theta z_1^2 \\ \dot{z}_2 &= z_3 + u \\ \dot{z}_3 &= -z_3 + y \\ y &= z_1 \end{aligned} \tag{4.38}$$

is required to track the reference signal $y_r = 0.1\sin(t)$. Differentiating y gives (from (4.32)):

$$\dot{y} = z_2 + \theta z_1^2 \tag{4.39}$$

where $D \equiv 0$, and we introduce a fictitious parameter in (4.39) with true value zero and estimate $\hat{\theta}_2$, and apply in view of (4.34):

$$u(t) = \frac{1}{\hat{\theta}_2}\left\{-z_2 - \hat{\theta}_1 z_1^2 + \dot{y}_m + \alpha(y_m - y)\right\} \tag{4.40}$$

to (4.38) with $\alpha > 0$. We have from (4.35) and (4.39):

$$\begin{aligned}
\dot{y} &= z_2 + \theta z_1^2 - \hat{\theta}_2 u + \left\{-z_2 - \hat{\theta}_1 z_1^2 + \dot{y}_m + \alpha(y_m - y)\right\} \\
\dot{e} &= -\alpha e + z_1^2 \phi_1 + u\phi_2 \\
\dot{e} &= -\alpha e + \left[z_1^2, u\right] \cdot \phi \triangleq -\alpha \cdot e + W(z_1, u) \cdot \phi
\end{aligned} \tag{4.41}$$

where $\phi_1 \triangleq (\hat{\theta}_1 - \theta), \phi_2 \triangleq \hat{\theta}_2$, and $\phi \triangleq [\phi_1, \phi_2]$. From (4.37), we have:

$$\begin{aligned}
\dot{\phi} &= -\Omega^{-1} \cdot W^T \cdot e \\
\dot{\phi}_1 &= -g_1 z_1^2 e = -g_1 y^2 (y_m - y) \\
\dot{\phi}_2 &= -g_2 u e
\end{aligned} \tag{4.42}$$

Clearly boundedness of $y = z_1$ is guaranteed. In fact, in systems of this type, where all the observable states are chained as in [57], it suffices to go up to the first derivative of the output in the control law. In practice, this can be easily and effectively implemented using PID controllers. Note that since the unobservable subsystem z_3 is BIBS with y regarded as input, boundedness of z_3 follows. Moreover, the boundedness of z_2 is clear from (4.38) with the control law (4.40) in place. Hence, $\dot{e}$ is bounded and $y \to y_m$ as $t \to \infty$ for any initial condition $y(0)$.

For simulation, we chose $\alpha = 100,\ g_1 = 1, g_2 = 10^{-4},\ \theta = 1, \hat{\theta}_1(0) = 0.6$, and $\hat{\theta}_2(0) = 0.03$. The results of the simulation, shown in figures (4.6) and (4.7), indicate that the tracking error converges to zero as predicted. Compare to the

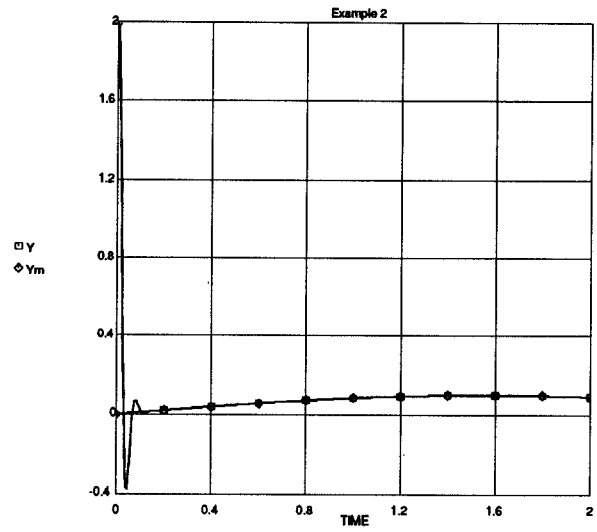

Figure 4.6: Reference trajectory and controlled output trajectory in example 2 for 40% parameter uncertainty and with $e(0) = 2$

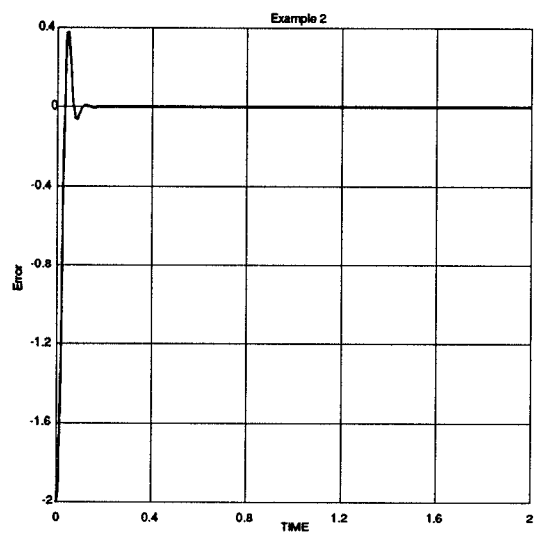

Figure 4.7: Error trajectories in example 2 for 40% parameter uncertainty and with $e(0) = 2$

results claimed in [57], the rate of convergence is fast, and the results hold for any initial conditions as shown in [57]. In [57], this problem was also solved, for comparison, using another adaptive scheme developed in [88], and it was shown that the later scheme works only locally when $e(0) < 0.45$.

4.7 Conclusion

In this chapter, we have described an adaptive control scheme for MIMO nonlinear systems where the vector relative degree is not (necessarily) well-defined. For right-invertible systems, we utilized the structure algorithm of Hirschorn and Singh for the inversion of the nonlinear input-output map under parametric uncertainty such that adaptive tracking was achieved. For non-invertible systems, we presented an algorithm based on introducing some fictitious parameters with associated update laws such that tracking was achieved. The resulting control law could also be considered as a dynamic high gain feedback control for the original system.

Chapter 5

Adaptive Approximate Linearization: Indirect Scheme

5.1 Introduction

In this chapter we present a technique of **indirect** adaptive control for **approximate** linearization of nonlinear systems. The adaptive controller can achieve tracking of reasonable trajectories with small error for **slightly non-minimum phase** systems. It can also be applied to nonlinear systems where the relative degree is not well defined.

There has been considerable research on the application of nonlinear adaptive control theory for improving the feedback linearization in the input-output response of nonlinear systems under parametric uncertainty. Most of the current research, [57, 98, 88, 97] among others, is based on feedback linearization [51, 81] and assumes some restrictive conditions such as existence of relative degree, bounded input bounded state property of the unobservable subsystem regarding the output as the input, or minimum phase property of the nonlinear system. In this chapter, using the results of [39], we provide adaptive **approxi-**

mate tracking of a wide class of reference signals for nonlinear systems that fail to meet some of the above conditions **slightly**. Our indirect adaptive controller scheme is motivated by the fact that, with the knowledge of the system parameters, **approximate** input-output linearization of slightly non-minimum phase systems and systems for which relative degree is not well defined can be produced using state feedback and coordinate changes. With parameter uncertainty, a parameter identifier is used that continuously adjusts the parameter estimates on line based on observation error. The **certainty equivalence principle** suggests that these parameter estimates that are converging to their true values may be employed to approximately linearize the nonlinear system asymptotically. In the next section we review the approximate **input-output linearization** technique for nonlinear systems. In section (5.3), we present the main results of this chapter on observer-based adaptive approximate linearization. Simulation results for the familiar ball and beam experiment is presented in section (5.4).

5.2 Review of Approximate Input-Output Linearization

In this section we review the approximate output tracking technique and show that, by appropriate choice of vector fields close to the system vector fields, one can construct an **approximate** input-output linearized system using a nonlinear change of coordinates $z = \phi(x)$ and state feedback $u(x, v) = \alpha(x) + \beta(x) \cdot v$. For this approximate model, which unlike the original system has some **nice** properties, we can apply the known design techniques of asymptotic output tracking and regulation. The resulting controller will then achieve **approximate**

output tracking and regulation for the original system. Figure (5.1) shows the structure of the system approximation by regarding some of the nonlinear terms in the system as the perturbations of another nonlinear system.

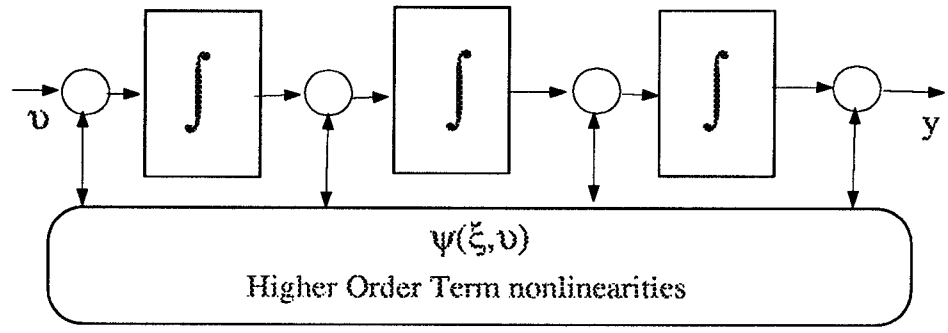

Figure 5.1: Approximate input-output linearization scheme

Consider the following nonlinear system:

$$\begin{aligned} \dot{x}(t) &= f(x) + g(x) \cdot u \\ y(t) &= h(x(t)) \end{aligned} \tag{5.1}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the input, $h \in \Re^p$ is the output with $f(0) = 0, h(0) = 0$, with relative degree r **outside** an open neighborhood U_ϵ of a singular point x_s. A state x_s is called a singular point for output tracking if $a(x_s) \triangleq \mathcal{L}_g \mathcal{L}_f^{r-1} h(x_s) = 0$ [45]. For this system, when the control objective is output tracking, one seeks an input-output linearization scheme [51, 81] in order to linearize the input-output map $v \rightarrow y$. To simplify the notation consider the SISO case. After taking subsequent derivatives of the output until the control appears one gets:

$$y^{(r)} = \mathcal{L}_f^r h(x) + u \cdot \mathcal{L}_g \mathcal{L}_f^{r-1} h(x)$$

for some $r > 0$. Then if the relative degree r of this system is well-defined in a neighborhood U_ϵ (0), i.e. $\forall x \in U_\epsilon$ (0):

$$\begin{aligned} \mathcal{L}_g \mathcal{L}_f h_{i-1}(x)(x) &= 0 \quad \forall i < r \\ \mathcal{L}_g \mathcal{L}_f h_{r-1}(x)(x) &\neq 0 \end{aligned} \tag{5.2}$$

the input-output linearization is achieved by applying the following control law:

$$u = \frac{v - \mathcal{L}_f^r h(x)}{\mathcal{L}_g \mathcal{L}_f h_{r-1}(x)} \tag{5.3}$$

However if the nonlinear system (5.1) does not have a well-defined relative degree in a neighborhood of the nominal operating point of interest, control law (5.3) is not feasible. In this case, one can proceed with **approximate** input-output linearization scheme, introduced by Hauser [39], to seek a smooth function $\phi_1(x)$ that approximates output y:

$$y = h(x) = \phi_1(x) + \psi_0(x)$$

where ψ_0 is of second or higher order with respect to the equilibrium manifold. An **approximate** input-output linearized system is obtained by ignoring the second or higher order terms in subsequent Lie derivatives of the approximate output ϕ_1.

Definition 5.2.1 (Robust Relative Degree) A nonlinear system (5.1) has a **robust relative degree** of γ about $x = 0$ if there exists smooth functions $\phi_i(x)$, $i = 1, \ldots, \gamma$ such that:

$$\begin{aligned} h(x) &= \phi_1(x) + \psi_0(x) \\ \mathcal{L}_{f+gu} \phi_i(x) &= \phi_{i+1}(x) + \psi_i(x, u) \quad i = 1, \ldots, \gamma - 1 \\ \mathcal{L}_{f+gu} \phi_\gamma(x) &= \tilde{b}(x) + \tilde{a}(x) \cdot u + \psi_\gamma(x, u) \end{aligned} \tag{5.4}$$

where functions $\psi_i(x,u), i = 0,\ldots,\gamma$ are $O(x,u)^2$ and $\tilde{a}(x)$ is $O(1)$. Also $\psi_i(x,u) = O(x)^2 + O(x)^1 \cdot u$.

Definition 5.2.2 (Uniformly Higher Order) A function $\psi : \Re^n \times \Re \to \Re$ is said to be uniformly higher order on $U_\epsilon \times B_\sigma \subset \Re^n \times \Re, \epsilon > 0$, if for some $\sigma > 0$, there exists a monotone increasing function of $\epsilon, K(\epsilon)$ such that:

$$|\psi(x,u)| \leq \epsilon K(\epsilon)(|x| + |u|) \quad \forall x \in U_\epsilon \ , |u| \leq \sigma \tag{5.5}$$

Remark 5.2.1 As shown in [39], the robust relative degree of system (5.11) is equal to the relative degree of its Jacobian linearization. Moreover, the functions $\xi_i(x) = \mathcal{L}_{f(x)}^{i-1} h(x), i = 1,\ldots,\gamma$ are independent in a neighborhood of the equilibrium x_e.

The advantage of this scheme is that if the nonlinear system (5.1) does not have a well-defined relative degree but it is linearly controllable we can **approximate** (5.1) with an input-output linearized one. Suppose (5.1) has a singular point x_s **i.e.** $\mathcal{L}_g \mathcal{L}_f^{r-1} h(x_s) = 0$ [45], but that it has a robust relative degree of γ in U_ϵ (x_s). Consider the following two local diffeomorphism $\Phi(x)$ of $x \in \Re^n$:

$$\begin{aligned} (\xi^T, \eta^T)^T &= (\xi_i = \mathcal{L}_f^{i-1} h(x), i = 1,2,\ldots,r, \quad \eta_1,\ldots,\eta_{n-r})^T \\ (\tilde{\xi}^T, \tilde{\eta}^T)^T &= (\xi^T, \tilde{\xi}_i = \mathcal{L}_f^{i-1} h(x), i = r+1,\ldots,\gamma, \quad \tilde{\eta}_1,\ldots,\tilde{\eta}_{n-\gamma})^T \end{aligned} \tag{5.6}$$

We have for the **true system** when $x \in U_\epsilon(x_s)$:

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= \tilde{\xi}_{r+1} + \psi_{r-1}(x) \cdot u \\
\dot{\tilde{\xi}}_{r+1} &= \tilde{\xi}_{r+2} + \psi_r(x) \cdot u \\
&\vdots \\
\dot{\tilde{\xi}}_{\gamma-1} &= \tilde{\xi}_\gamma + \psi_{\gamma-2}(x) \cdot u \\
\dot{\tilde{\xi}}_\gamma &= \tilde{b}(x) + \tilde{a}(x) \cdot u \\
\dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta})
\end{aligned}
\tag{5.7}
$$

where $\psi_i(x)$ are $O(x)^1, \tilde{a}(x) \triangleq \mathcal{L}_g \mathcal{L}_f^{\gamma-1} h(x), \tilde{b}(x) \triangleq \mathcal{L}_f^\gamma h(x)$, and $\tilde{a}(x_s)$ is $O(1)$. The approximate system is:

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_r &= \tilde{\xi}_{r+1} \\
&\vdots \\
\dot{\tilde{\xi}}_{\gamma-1} &= \tilde{\xi}_\gamma \\
\dot{\tilde{\xi}}_\gamma &= \tilde{b}(x) + \tilde{a}(x) \cdot u \\
\dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta})
\end{aligned}
\tag{5.8}
$$

This represents an **approximate** input-output linearized description of the true system (5.1) obtained by neglecting some high order terms in some neighborhood U_ϵ of the singular state x_s (**i.e.** $x \in U_\epsilon(x_s)$). When system (5.1) is operating in U_ϵ , where (5.8) is a valid approximation, one may design a feedback control law to achieve approximate output tracking [39]. The control law will, in fact, be the exact tracking control law using the approximate description (5.8).

With the above notation in mind, we say (5.1) is **slightly non-minimum phase** if the true system, described by (5.7), is non-minimum phase but its apprximate linearization, described by (5.8) is minimum phase [40].

Approximate tracking is achieved by choosing the control law u:

$$u = \frac{1}{\tilde{a}(\xi, \eta)}[-\tilde{b}(\xi, \eta) + v] \tag{5.9}$$

with:

$$v = y_d^{(\gamma)} + \alpha_{\gamma-1}(y_d^{(\gamma-1)} - \tilde{\xi}_\gamma) + \ldots + \alpha_0(y_d - \tilde{\xi}_1) \tag{5.10}$$

where α_i are chosen so that $s^\gamma + \alpha_{\gamma-1}s^{\gamma-1} + \ldots + \alpha_0$ is a Hurwitz polynomial. Thus the control law u in (5.9) approximately linearizes the system (5.1) from input v to the output y up to the order ϵ (say $O(x, u)^2$).

Theorem 5.2.1 ([39]) Let U_ϵ be a family of operating envelopes and suppose that the zero dynamics of the approximate system are exponentially stable, $\tilde{q}$ is Lipschitz in $\tilde{\xi}$ and $\tilde{\eta}$ on $\Phi(U_\epsilon)$ for each ϵ, and the functions $\psi_i(x, u)$ are uniformly higher order on $U_\epsilon \times B_\sigma$. Then, for ϵ sufficiently small and for desired trajectories with sufficiently small values and derivatives $(y_d, \dot{y}_d, \ldots, y_d^{(\gamma)})$, the states of the closed loop system and control (5.9) will remain bounded and the tracking error will be $O(\epsilon)$.

The approximate feedback linearization result of theorem (5.2.1) is a clear design alternative to the more restrictive scheme of exact feedback linearization. These results have already been applied to the design of automatic flight control systems [40]. In the next section, we present an indirect adaptive tracking and adaptive regulation scheme for nonlinear systems that are **approximately** feedback linearizable in the sense of [38] and hence, are subject to milder involutivity restrictions, and are not necessarily minimum phase with a well-defined

(vector) relative degree as assumed in most current adaptive control strategies for nonlinear systems.

In this thesis, for notational consistency, we use $O(x,u)^\rho$ to denote a uniformly higher order function of the form $O(x)^\rho + O(x)^{\rho-1} \cdot u$.

5.3 Adaptive Control

Consider a SISO nonlinear system of the form (5.1) under parameter uncertainty:

$$\begin{aligned} \dot{x}(t) &= f(x,\theta) + g(x,\theta) \cdot u \\ y(t) &= h(x,\theta) \end{aligned} \tag{5.11}$$

with relative degree r **outside** an open neighborhood U_ϵ of a singular point x_s and **robust relative degree** γ in $U_\epsilon(x_s)$. Further, assume $f(x), g(x)$ and $h(x)$ have the form:

$$\begin{aligned} f(x,\theta) &= \sum_{i=1}^{n} \theta_i^1 \cdot f_i(x) \\ g(x,\theta) &= \sum_{i=1}^{m} \theta_i^2 \cdot g_i(x) \\ h(x,\theta) &= \sum_{i=1}^{l} \theta_i^3 \cdot h_i(x) \end{aligned} \tag{5.12}$$

with θ^1, θ^2 and θ^3 vectors of unknown parameters and the $f_i(x), g_i(x)$, and $h_i(x)$ known functions. The estimates of these functions are given by:

$$\begin{aligned} \hat{f}(x) &= \sum_{i=1}^{n} \hat{\theta}_i^1 \cdot f_i(x) \\ \hat{g}(x) &= \sum_{i=1}^{m} \hat{\theta}_i^2 \cdot g_i(x) \\ \hat{h}(x) &= \sum_{i=1}^{l} \hat{\theta}_i^3 \cdot h_i(x) \end{aligned} \tag{5.13}$$

where $\hat{\theta}^i_j$ are the estimates of the unknown parameters θ^i_j. Now let's replace the control law (5.9) by:

$$u_{ad} = \frac{1}{L_g \widehat{L_f^{\gamma-1}} h}[-\widehat{L_f^{\gamma}} h(\xi, \eta) + v_{ad}] \tag{5.14}$$

with:

$$v_{ad} = y_d^{(\gamma)} + \alpha_{\gamma-1}(y_d^{(\gamma-1)} - \hat{\xi}_\gamma) + \ldots + \alpha_0(y_d - \hat{\xi}_1) \tag{5.15}$$

where α_i are chosen as before and $\tilde{\xi}_{i-1} = L_f^i h$ are replaced by their estimates $\widehat{L_f^i h}$:

$$\begin{aligned} \hat{\xi}_i = \widehat{L_f^{i-1}} h &\triangleq L_{\hat{f}}^{i-1} \hat{h} \\ L_g \widehat{L_f^{\gamma-1}} h &\triangleq L_{\hat{g}} L_{\hat{f}}^{\gamma-1} \hat{h} \end{aligned} \tag{5.16}$$

As in [88], since these estimates are not linear in the unknown parameters θ_i, we define each of the parameter products to be a new parameter. For example:

$$\widehat{L_f h} = \sum_{i=1}^{l} \sum_{j=1}^{n} \theta_i^3 \theta_j^1 L_{f_j} h_i$$

and we let $\Theta \in \Re^p$ be the large p-dimensional vector of all multilinear parameter products: $\theta_i^1, \theta_j^2, \theta_k^3, \theta_i^1 \theta_j^2, \ldots$. The vector containing all the estimates is denoted by $\hat{\Theta} \in \Re^p$ with $\Phi \triangleq \Theta - \hat{\Theta}$ representing the parameter error. Due to the indirect nature of our approach, this overparametrization does not increase the complexity of the closed loop system since a parameter identifier is to be used to estimate the unknown parameters θ^i_j. The parameter vector Θ is, however, constructed here in order to show the stability of the resulting adaptive system. Using the control law (5.14) in (5.8) yields:

$$\begin{aligned} \dot{\tilde{\xi}}_\gamma &= L_f^\gamma h + \left[L_g L_f^{\gamma-1} h - L_g \widehat{L_f^{\gamma-1}} h\right] \cdot u_{ad} - \widehat{L_f^\gamma h} + v_{ad} \\ &= \left[L_f^\gamma h - \widehat{L_f^\gamma h}\right] + \left[L_g L_f^{\gamma-1} h - L_g \widehat{L_f^{\gamma-1}} h\right] \cdot u_{ad} + v_{ad} \end{aligned} \tag{5.17}$$

Subtracting v in (5.10) from both sides gives:

$$\begin{aligned} e^{(\gamma)} + \alpha_{\gamma-1} e^{(\gamma-1)} + \ldots \alpha_0 e &= \left[L_g L_f^{\gamma-1} h - L_g \widehat{L_f^{\gamma-1}} h \right] \cdot u_{ad} + \left[L_f^{\gamma} h - \widehat{L_f^{\gamma}} h \right] \\ &+ \alpha_{\gamma-1} \left(L_f^{\gamma-1} h - \widehat{L_f^{\gamma-1}} h \right) + \ldots + \alpha_1 \left(L_f h - \widehat{L_f h} \right) \\ &= \Phi^T \cdot w(x, u_{ad}(x)) \end{aligned} \tag{5.18}$$

where: $w^T \triangleq \left[L_{g_i} L_{f_j}^{\gamma-1} h_k u_{ad}(x) | \ldots | L_{f_j} h_k \right]$.

Therefore, in the closed loop, for the approximate system, we have in compact form:

$$\begin{aligned} \dot{e} &= Ae + W^T(x, u_{ad}(x)) \cdot \Phi \\ \dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta}) \end{aligned} \tag{5.19}$$

where A is a Hurwitz matrix and note that if $\phi \triangleq \theta - \hat{\theta} \to B_\epsilon$ as $t \to \infty$, then $\Phi \to B_\epsilon$ as $t \to \infty$.

To estimate the unknown parameters, we consider an observer-based identifier proposed in [69, 62, 98]. First, we rewrite (5.11) as:

$$\begin{aligned} \dot{x} &= (f_1 \ldots f_n \mid g_1 u \ldots g_m u) \cdot \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} \\ &\triangleq Z^T(x, u_{ad}(x)) \cdot \theta \end{aligned} \tag{5.20}$$

Consider the following identifier system:

$$\begin{aligned} \dot{\hat{x}} &= \hat{A} \cdot (\hat{x} - x) + Z^T(x, u_{ad}(x)) \cdot \hat{\theta} \\ \dot{\hat{\theta}} &= -Z(x, u) \cdot P \cdot (\hat{x} - x) \end{aligned} \tag{5.21}$$

where $\hat{A}$ is a Hurwitz matrix, $\hat{x}$ is the observer state, x is the plant state in (5.11), and $P > 0$ is a solution to the Lyapunov equation:

$$\hat{A}^T P + P\hat{A} = -\lambda \cdot I$$

with $\lambda > 0$. We assume all the states x in (5.11) are available and hence $\hat{x}$ and $\hat{\theta}$ are given by (5.21). We also assume θ is a vector of constant but unknown parameters. Then:

$$\begin{aligned} \dot{\hat{e}} &= \hat{A} \cdot \hat{e} + Z^T(x,u) \cdot \phi \\ \dot{\phi} &= -Z(x,u) \cdot P \cdot \hat{e} \end{aligned} \tag{5.22}$$

is the observer error system where $\hat{e} \triangleq \hat{x} - x$ is the observer state error and $\phi = \hat{\theta} - \theta$ is the parameter error.

Properties of the observer-based identifier in (5.22) are [87, 98]:

i. $\phi \in \mathcal{L}_\infty$

ii. with $\hat{e}(0) = 0$, $\phi(t) \leq \phi(0) \;\; \forall t \geq 0$

iii. $\hat{e} \in \mathcal{L}_\infty \cap \mathcal{L}_2$

iv. if $Z^T(x, u_{ad})$ is bounded (in particular if x and u are bounded) then $\hat{e} \in \mathcal{L}_\infty$ and $\hat{e} \to 0$ as $t \to \infty$.

v. $\hat{e}$ and ϕ converge exponentially to zero if $Z(x,u)$ is sufficiently rich, **i.e.**, $\exists\, \delta_1, \delta_2, \sigma > 0$ such that $\forall t$:

$$\delta_1 I \leq \int_t^{t+\sigma} Z Z^T d\tau \leq \delta_2 I \tag{5.23}$$

However, since $Z(x,u)$ is a function of state x, the above condition can not be verified explicitly ahead of time.

We are now ready to state the main theorem on approximate tracking for slightly non-minimum phase systems under parameter uncertainty when identifier input is sufficiently rich. Figure (5.2) shows the control system design of the observer-based adaptive control scheme of this chapter.

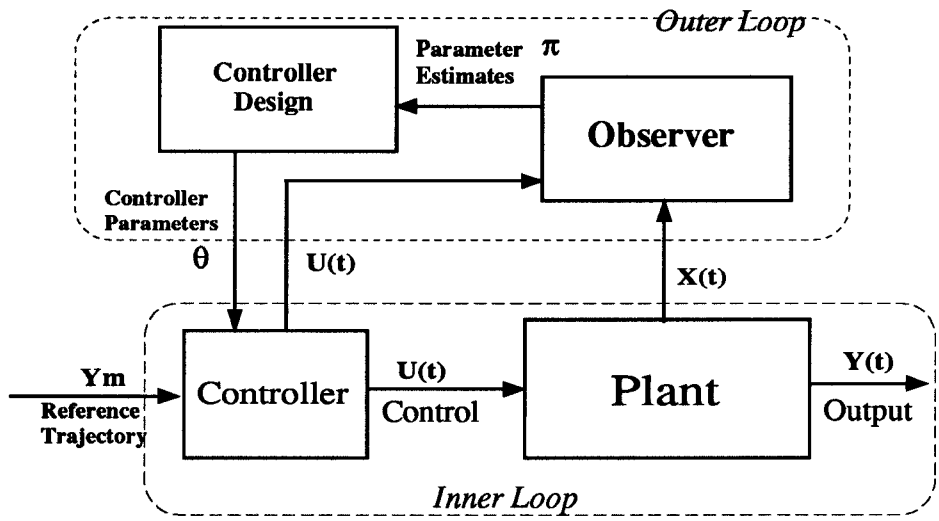

Figure 5.2: Observer-based adaptive control structure for asymptotic output tracking

Theorem 5.3.1 (Adaptive Tracking) Assume that:

i. the reference trajectory and its first $\gamma-1$ derivatives (i.e.,$y_d, y_d^{(1)}, \ldots, y_d^{(\gamma-1)}$) are bounded,

ii. the vector fields f, g, and h in (5.11) are unknown but may be parametrized linearly in unknown parameters in the form (5.12) where vector fields f_i, g_i, and h_i are known functions of x,

iii. the zero dynamics of the approximate system (5.8) are locally exponentially stable and $\tilde{q}(\tilde{\xi}, \tilde{\eta})$ is locally Lipschitz in $\tilde{\xi}$ and $\tilde{\eta}$,

iv. the functions $\psi(x)u_{ad}(x)$ in (5.7) and $w(x, u_{ad}(x))$ are locally Lipschitz continuous,

v. $\phi \to B_\epsilon(0)$ as $t \to \infty$ (for example, for observer-based identifier (5.22), $Z(x, u_{ad})$ in (5.20) is sufficiently rich),

Then, for ϵ sufficiently small, the states x are bounded and the tracking error is of order ϵ; i.e.,

$$|y - y_d| \leq k\epsilon$$

for some $k < \infty$.

Proof. Using the adaptive approximate control law $u_{ad}(x)$ in (5.14), the error equation with $\phi = \hat{\theta}(t) - \theta$ and $e = \tilde{\xi} - Y_d$ is:

$$\begin{pmatrix} e_1 \\ \vdots \\ e_\gamma \end{pmatrix} = \begin{pmatrix} \tilde{\xi}_1 \\ \vdots \\ \tilde{\xi}_\gamma \end{pmatrix} - \begin{pmatrix} y_d \\ \vdots \\ y_d^{(\gamma-1)} \end{pmatrix}$$

$$e_1^{(\gamma)} + \alpha_{\gamma-1} e_1^{(\gamma-1)} + \ldots \alpha_0 e_1 = w^T(x, u_{ad}) \cdot \Phi$$

The true error system is given by:

$$\begin{bmatrix} \dot{e}_1 \\ \vdots \\ \dot{e}_r \\ \vdots \\ \dot{e}_{\gamma-1} \\ \dot{e}_\gamma \end{bmatrix} = \begin{bmatrix} 0 & 1 & \ldots & 0 & \ldots & 0 \\ & & \ddots & & & \\ \vdots & & & 1 & & \vdots \\ & & & & \ddots & \\ 0 & & & & & 1 \\ -\alpha_0 & -\alpha_1 & \ldots & -\alpha_{r+1} & \ldots & -\alpha_{\gamma-1} \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ \vdots \\ e_r \\ \vdots \\ e_{\gamma-1} \\ e_\gamma \end{bmatrix}$$

$$+\epsilon \begin{bmatrix} 0 \\ \vdots \\ \psi_{r-1}(x) \\ \vdots \\ \psi_{\gamma-2} \\ 0 \end{bmatrix} u_{ad}(x) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ w^T(x,u) \end{bmatrix} \cdot \phi \tag{5.24}$$

Hence, (5.11), with the adaptive approximate tracking $u_{ad}(x)$ may be expressed in the following compact form:

$$\begin{aligned} \dot{e} &= A \cdot e + \epsilon \Psi(x) \cdot u_{ad}(x) + W^T(x,u) \cdot \phi \\ \dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta}) \end{aligned} \tag{5.25}$$

From (i):

$$|\tilde{\xi}| \leq |e| + b_d \tag{5.26}$$

for some b_d. From (iii), a converse Lyapunov theorem assures the existence of a Lyapunov function $v_2(\tilde{\eta})$ for the system:

$$\dot{\tilde{\eta}} = \tilde{q}(0, \tilde{\eta})$$

such that:

$$\begin{aligned} k_1|\tilde{\eta}|^2 \leq \quad v_2(\tilde{\eta}) \quad &\leq k_2|\tilde{\eta}|^2 \\ \frac{\partial v_2}{\partial \tilde{\eta}} \tilde{q}(0, \tilde{\eta}) \quad &\leq \quad -k_3|\tilde{\eta}|^2 \\ \left| \frac{\partial v_2}{\partial \tilde{\eta}} \right| \quad &\leq \quad k_4|\tilde{\eta}| \end{aligned} \tag{5.27}$$

for some positive constants k_1, k_2, k_3, and k_4.

Since x is a local diffeomorphism of $(\tilde{\xi}, \tilde{\eta})$:

$$\begin{aligned} |x| &\leq l_x(|\tilde{\xi}| + |\tilde{\eta}|) \\ &\leq l_x(|e| + b_d + |\tilde{\eta}|) \end{aligned} \tag{5.28}$$

From assumption (iv), and (v):

$$\begin{aligned} |\phi(t)| &\leq \rho \\ |2PW^T(x, u_{ad}) \cdot \phi| &\leq (l_w|x| + b_w) \cdot \phi \leq l_w l_x(|e| + b_{dw} + |\tilde{\eta}|) \cdot |\phi| \\ &\leq l_w l_x(|e| + b_{dw} + |\tilde{\eta}|) \cdot \rho \end{aligned} \tag{5.29}$$

where $b_{dw} \triangleq b_d + \frac{1}{l_w l_x} \cdot b_w$.

From (iii) and (iv), since $\tilde{q}(\tilde{\xi}, \tilde{\eta})$ and $\psi(x)u_{ad}(x)$ are locally Lipschitz with $\psi(0)u_{ad}(0) = 0$:

$$\begin{aligned} |\tilde{q}(\tilde{\xi}_1, \tilde{\eta}_1) - \tilde{q}(\tilde{\xi}_2, \tilde{\eta}_2)| &\leq l_q(|\tilde{\xi}_1 - \tilde{\xi}2| + |\tilde{\eta}_1 - \tilde{\eta}_2|) \\ |2P\psi(x)u_{ad}(x)| &\leq l_u|x| \end{aligned} \tag{5.30}$$

Also:

$$\begin{aligned} \frac{\partial v_2}{\partial \tilde{\eta}} \tilde{q}(\tilde{\xi}, \tilde{\eta}) &= \frac{\partial v_2}{\partial \tilde{\eta}} \tilde{q}(0, \tilde{\eta}) + \frac{\partial v_2}{\partial \tilde{\eta}} (\tilde{q}(\tilde{\xi}, \tilde{\eta}) - \tilde{q}(0, \tilde{\eta})) \\ &\leq -k_3|\tilde{\eta}|^2 + k_4 l_q |\tilde{\eta}(|\epsilon| + b_d) \end{aligned} \tag{5.31}$$

In order to show that e and $\tilde{\eta}$ are bounded consider the following Lyapunov candidate function for system (5.25):

$$\begin{aligned} V(e, \tilde{\eta}) &= e^T P e + \mu v_2(\tilde{\eta}) \\ A^T P + PA &= -I \end{aligned} \tag{5.32}$$

where $P > 0$, and $\mu > 0$ to be determined later.

Taking the derivative of V along the trajectories of (5.25), we have:

$$
\begin{aligned}
\dot{V} &= -|e|^2 + 2\epsilon e^T P\psi(x)u_{ad}(x) + 2e^T PW^T\phi + \mu\frac{\partial v_2}{\partial\tilde{\eta}}\tilde{q}(\tilde{\xi},\tilde{\eta}) \\
&\leq -|e|^2 + \epsilon|e|l_u l_x(|e| + b_d + |\tilde{\eta}|) + |e|l_w l_x(|e| + b_{dw} + |\tilde{\eta}|)|\phi| \\
&\quad +\mu(-k_3|\tilde{\eta}|^2 + k_4 l_q|\tilde{\eta}|(|e| + b_d)) \\
&\leq -\left(\tfrac{|e|}{2} - \epsilon l_u l_x b_d\right)^2 + (\epsilon l_u l_x b_d)^2 - \left(\tfrac{|e|}{2} - l_w l_x b_{dw}|\phi|\right)^2 + (l_w l_x b_{dw}|\phi|)^2 \\
&\quad -\left(\tfrac{|e|}{2} - (l_w l_x|\phi| + \epsilon l_u l_x + \mu k_4 l_q)\tilde{\eta}\right)^2 + (l_w l_x|\phi| + \epsilon l_u l_x + \mu k_4 l_q)^2|\tilde{\eta}|^2 \\
&\quad -\mu k_3\left(\tfrac{|\tilde{\eta}|}{2} - \tfrac{k_4 l_q b_d}{k_3}\right)^2 + \mu\tfrac{(k_4 l_q b_d)^2}{k_3} - (\tfrac{1}{4} - \epsilon l_u l_x - l_w l_x|\phi|)|e|^2 - \tfrac{3}{4}\mu k_3|\tilde{\eta}|^2 \\
&\leq -(\tfrac{1}{4} - \epsilon l_u l_x - l_w l_x|\phi|)|e|^2 - \left((\tfrac{3}{4}\mu k_3 - (l_w l_x|\phi| + \epsilon l_u l_x + \mu k_4 l_q)^2\right)|\tilde{\eta}|^2 \\
&\quad +(\epsilon l_u l_x b_d)^2 + (l_w l_x b_{dw}|\phi|)^2 + \mu\tfrac{(k_4 l_q b_d)^2}{k_3}
\end{aligned}
\tag{5.33}
$$

Define:

$$
\mu_0 \triangleq \frac{k_3}{4(l_w l_x + k_4 l_q + l_u l_x)} \tag{5.34}
$$

For $\mu \leq \mu_0$ and $\epsilon \leq min(\mu, \frac{1}{8[l_u l_x + l_w l_x]})$ and $|\phi| \leq \epsilon$, we have:

$$
\dot{V} \leq -\frac{|e|^2}{8} - \frac{\mu k_3|\tilde{\eta}|^2}{2} + (\epsilon l_u l_x b_d)^2 + (l_w l_x b_{dw}|\phi|)^2 + \mu\frac{(k_4 l_q b_d)^2}{k_3}
$$

Note that $|\phi| < \rho\forall t$, and from assumption (v), we can assume that there exists $T > 0$, such that $|\phi| \leq \epsilon$ for all $t \geq T$. Thus, for all $t \geq T, \dot{V} < 0$ whenever $|\tilde{\eta}|$ or $|e|$ is large which implies that $|\tilde{\eta}|$ and $|e|$ and hence, $|\tilde{\eta}|$ and $|x|$, are bounded.

Now using the continuity of $\psi(x)u_{ad}(x)$ and $W^T(x, u_{ad})\phi$, and boundedness

of x, we see that

$$\begin{aligned}\dot{e} &= A \cdot e + W^T(x, u_{ad})\phi(t) + \epsilon\psi(x)u_{ad}(x) \\ &= A \cdot e + \begin{bmatrix} 0 \\ \vdots \\ \epsilon\psi_r u_{ad}(x) \\ \vdots \\ \epsilon\psi_{\gamma-1} u_{ad}(x) \\ w^T(x, u_{ad})\phi(t) \end{bmatrix}\end{aligned} \tag{5.35}$$

is an exponentially stable linear system driven by a bounded input that approaches an order ϵ input asymptotically. Therefore, we conclude that the tracking error, e, converges to a ball of order ϵ. Adaptive stabilization clearly follows. One important special case is when the robust relative degree, γ, is equal to n, **i.e.** the approximate system has no zero dynamics but the true system is non-minimum phase. In this case, the true closed-loop system is exponentially stable. The following theorem summarizes this result:

Theorem 5.3.2 (Adaptive Regulation) Suppose that the the approximate system (5.8) has no zero dynamics, **i.e.** (5.11) has robust relative degree, γ, equal to n, $\psi(x)u_{ad}(x)$ and $w(x, u_{ad})$ are locally Lipschitz in x with $\psi(0)u_{ad}(0) = 0$, unknown parameters θ appear linearly in f, g, and h, and $\phi \to B_\epsilon(0)$ as $t \to \infty$. Then, the adaptive control law u_{ad} in (5.14) exponentially stablizes (5.11) with ϵ and $\phi(0)$ sufficiently small.

Proof. Using control law:

$$u_{ad} = \frac{1}{L_g \widehat{L_f^{n-1}} h(x)} \left[-\widehat{L_f^n} h(x) - \alpha_{n-1}\hat{\xi}_n - \ldots - \alpha_0 \hat{\xi}_1 \right]$$

in (5.11) yields in compact form:

$$\dot{\tilde{\xi}} = A \cdot \tilde{\xi} + \epsilon \Psi(x) \cdot u_{ad}(x) + W^T(x, u_{ad}) \cdot \phi \tag{5.36}$$

with $A, \Psi(x)$, and $W^T(x, u)$ as before. Choose the following Lyapunov candidate function:

$$V(\tilde{\xi}) = \tilde{\xi}^T P \tilde{\xi} \tag{5.37}$$

with $P > 0, \lambda > 0$, such that $A^T P + PA = -\lambda \cdot I$. Then, using the bounds similar to those in the proof of theorem (5.3.1), we have:

$$\begin{aligned} |x| &\leq l_x |\tilde{\xi}| \\ |\phi(t)| &\leq \rho \\ |2PW^T(x, u_{ad}) \cdot \phi| &\leq l_w \cdot |x| \cdot \rho \\ |2P\psi(x) u_{ad}| &\leq l_u |x| \end{aligned} \tag{5.38}$$

The derivative of $V(\tilde{\xi})$ along the solution trajectories of (5.36) is:

$$\begin{aligned} \dot{V} &= -\lambda |\tilde{\xi}|^2 + 2\epsilon \tilde{\xi}^T P \psi(x) u_{ad}(x) + 2\tilde{\xi}^T P W^T \phi \\ &\leq -(\lambda - \epsilon l_x l_u - l_w l_x \rho) |\tilde{\xi}|^2 \end{aligned} \tag{5.39}$$

Hence, $\dot{V}$ is a negative definite for ϵ and $\phi(0)$ sufficiently small, and consequently, (5.36) is exponentially stable.

The above design scheme may easily be generalized to the multi-input multi-output (MIMO) case where due to the presence of small terms, the decoupling matrix is almost singular. In this case, approximate linearization is achieved using dynamic extension algorithm [51, 81], and with some modifications, the stability analysis presented in this section can be shown to be true.

5.4 The Ball and Beam Example

In this section, to demonstrate the adaptive scheme developed in this paper and compare its performance with non-adaptive control, we consider the ball and beam experiment, depicted in Figure (5.3), with uncertainty in the mass M of the ball. We first review the controller form derived from the second approximation presented in [39]. Neglecting the angular velocity of the ball, the

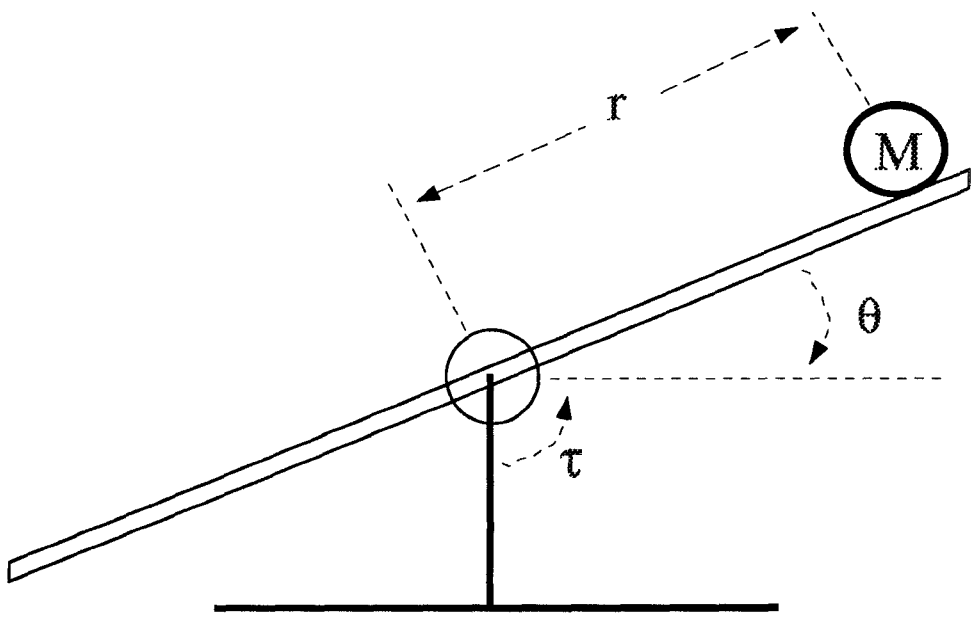

Figure 5.3: The Ball and Beam Experiment

equations of motion are given by:

$$\begin{aligned} \left(\frac{J_b}{R^2} + M\right)\ddot{r} + MgSin\theta - Mr\dot{\theta}^2 = 0 \\ (Mr^2 + J + J_b)\ddot{\theta} + 2Mr\dot{r}\dot{\theta} + MgrCos\theta = \tau \end{aligned} \tag{5.40}$$

where M is the mass of the ball, J is the moment of inertia of the beam, J_b is the moment of inertia of the ball, R is the radius of the ball, g is the acceleration of gravity, θ is the beam angle, r is the position of the ball, and τ is the torque applied to the beam.

With exact knowledge of all parameters, a change of coordinates in the input space is possible by applying a torque in the form of [39]:

$$\tau = 2Mr\dot{r}\dot{\theta} + Mgrcos\theta + (Mr^2 + J + J_b)u \tag{5.41}$$

where u is a new input. The resulting state-space description is:

$$\begin{aligned} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} &= \begin{bmatrix} x_2 \\ B(x_1x_4^2 - gsinx_3) \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= x_1 \end{aligned} \tag{5.42}$$

where $B \triangleq M/(\frac{J_b}{R^2}+M)$, $y \triangleq r$, $x = (x_1, x_2, x_3, x_4)^T \triangleq (r, \dot{r}, \theta, \dot{\theta})^T$. The objective is to track a desired trajectory $y_d(t)$.

With $\tilde{\xi}_i = \mathcal{L}_f^{i-1}h(x), i = 1, \ldots, 4$, we have:

$$\begin{aligned} \dot{\tilde{\xi}}_1 &= x_2 \quad = \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 &= -Bgsinx_3 + Bx_1x_4^2 \quad = \tilde{\xi}_3 \\ \dot{\tilde{\xi}}_3 &= -Bgx_4cosx_3 + Bx_2x_4^2 + 2Bx_1x_4u \quad = \tilde{\xi}_4 + \psi_3(x,u) \\ \dot{\tilde{\xi}}_4 &= B^2x_1x_4^4 + Bg(1-B)x_4^2sinx_3 + (-Bgcosx_3 + 2Bx_2x_4)u \\ &= \tilde{b}(x) + \tilde{a}(x)u \end{aligned} \tag{5.43}$$

where the origin is the singular state, **i.e.** $a(0) = 0$ with $a(x) \triangleq 2Bx_1x_4$. In this case the neglected nonlinearity is $\psi_3(x,u) = 2Bx_1x_4u$ which is of order ϵ in a neighborhood U_ϵ of the singular state 0. The resulting tracking control law is given by equations (5.9) and (5.10):

$$u = \frac{1}{\tilde{a}(\xi)}\left[-\tilde{b}(\xi) + y_d^{(4)} + \alpha_4(y_d^{(3)} - \tilde{\xi}_4) + \ldots + \alpha_0(y_d - \tilde{\xi}_1)\right] \tag{5.44}$$

where α_i are chosen so that $s^4 + \alpha_3 s^3 + \ldots + \alpha_0$ is a Hurwitz polynomial. This control law achieves approximate output tracking of a desired trajectory $y_d(t)$ up to order ϵ.

When M is not exactly known, control laws (5.44) and (5.41) can not be implemented and (5.42) is no longer a valid description. In this case, we construct an adaptive controller, as developed in the last section, that achieves approximate tracking under parameter uncertainty in mass M of the ball. Although parameter M does not appear linearly in (5.40), we can still proceed with our design scheme by reparametrizing the system as shown below. This is possible mainly due to the indirect nature of our adaptive controller. First, we use the observer-based identifier of (5.21) to estimate parameter B:

$$\begin{aligned} \dot{\hat{x}}_2 &= -\sigma(\hat{x}_2 - x_2) + (x_1 x_4^2 - g sin x_3)\hat{B} \\ &= -\sigma\hat{e} + Z^T(x)\hat{B} \\ \dot{\hat{B}} &= -Z^T(x)\hat{e} \\ \hat{x}_2(0) &= x_2(0) \quad = 0 \end{aligned} \tag{5.45}$$

Then substitute $\hat{M} = \frac{J_b \hat{B}}{R^2(1-\hat{B})}$ in (5.41) which gives:

$$\tau_{ad} = 2\hat{M} r \dot{r} \dot{\theta} + \hat{M} g r cos\theta + (\hat{M} r^2 + J + J_b) u_{ad} \tag{5.46}$$

From (5.40), the new state-space description is:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= B(x_1 x_4^2 - g sin x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{1}{M x_1^2 + J + J_b}\left[(\hat{M} x_1^2 + J + J_b) u_{ad} + 2(\hat{M} - M) x_1 x_2 x_4 + (\hat{M} - M) g x_1 cos x_3\right] \\ &= f_4(x, \hat{M}, u_{ad}) \\ y &= x_1 \end{aligned} \tag{5.47}$$

Let $\tilde{\xi}_1 = x_1$. Then, choosing $\tilde{\xi}_i$ at each step as in (5.43) gives:

$$\begin{aligned}
\dot{\tilde{\xi}}_1 &= x_2 = \tilde{\xi}_2 \\
\dot{\tilde{\xi}}_2 &= -Bgsinx_3 + Bx_1x_4^2 = \tilde{\xi}_3 \\
\dot{\tilde{\xi}}_3 &= -Bgx_4cosx_3 + Bx_2x_4^2 + 2Bx_1x_4f_4(\cdot) = \tilde{\xi}_4 + \psi_3(x, u_{ad})
\end{aligned}$$

$$\begin{aligned}
\dot{\tilde{\xi}}_4 &= B^2x_1x_4^4 + Bg(1-B)x_4^2sinx_3 + (-Bgcosx_3 + 2Bx_2x_4)f_4(\cdot) \\
&= B^2x_1x_4^4 + Bg(1-B)x_4^2sinx_3 + \tfrac{\hat{M}-M}{Mx_1^2+J+J_b}(-Bgcosx_3 + 2Bx_2x_4) \\
&\quad \cdot(2x_1x_2x_4 + gx_1cosx_3) + \ (-Bgcosx_3 + 2Bx_2x_4)\tfrac{\hat{M}x_1^2+J+J_b}{Mx_1^2+J+J_b} \cdot u_{ad} \\
&\triangleq \tilde{b}(x) + \tilde{a}(x)u_{ad}
\end{aligned} \tag{5.48}$$

where $\tilde{a}(0) \neq 0$. From (5.14), the resulting adaptive control law is:

$$u_{ad}(x) = \frac{1}{\hat{\tilde{a}}(x)} \cdot [-\hat{\tilde{b}}(x) + v_{ad}] \tag{5.49}$$

where:

$$\begin{aligned}
\hat{\tilde{a}}(x) &= 2\hat{B}x_2x_4 - \hat{B}gcosx_3 \\
\hat{\tilde{b}}(x) &= \hat{B}^2x_1x_4^4 + \hat{B}g(1-\hat{B})x_4^2sinx_3 \\
v_{ad} &= y_d^{(4)} + \alpha_4(y_d^{(3)} - \hat{\xi}_4) + \ldots + \alpha_0(y_d - \hat{\xi}_1)
\end{aligned} \tag{5.50}$$

For simulation we used $y_d(t) = 3cos\frac{\pi t}{5}, x(0) = (2.9, 0, 0.1698, 0)^T$, $\sigma = 1.5$, $M = 0.05kg, R = 0.01m, J = 0.02kgm^2, J_b = 2 \times 10^{-6}kgm^2$, and $g = 9.81m/s^2$. All closed-loop poles were placed at -5. Figures (5.4) and (5.5) show the performance of the adaptive controller with 80% uncertainty in M and initial error $0.1m$. The parameter B converged to the correct value 0.7143 in less than two seconds. The error $|r(t) - y_d|$ was driven to almost zero with maximum magnitude within $\epsilon = 4 \times 10^{-4}m$ ball and the neglected nonlinearity $|\psi_3(x, u_{ad})|$ was within 0.01 ball. The performance of the non-adaptive controller for 25% uncertainty in M is shown in figures (5.6) and (5.7). The output error was about 30

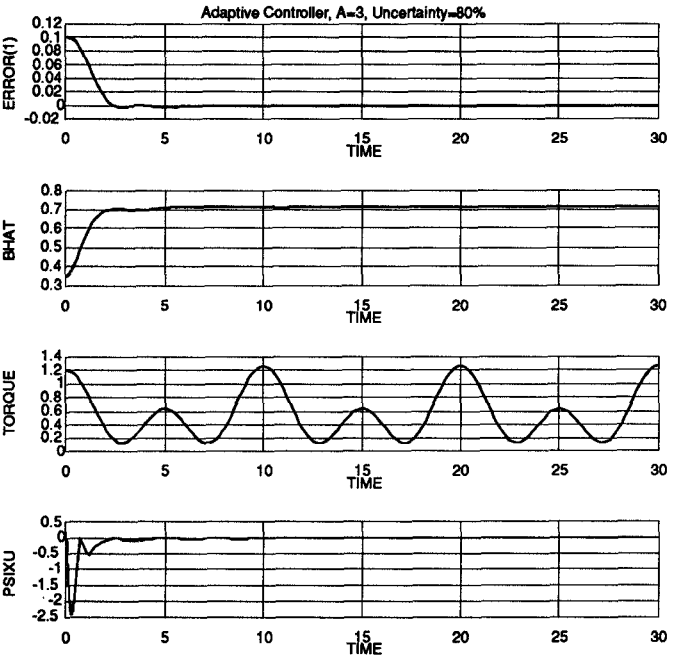

Figure 5.4: Adaptive Controller: error trajectory with $e(0) = 0.1m$, parameter estimate $\hat{B}$ with initial 80% uncertainty in mass M of the ball, applied torque, and neglected nonlinearity $\psi_3(x, u)$,

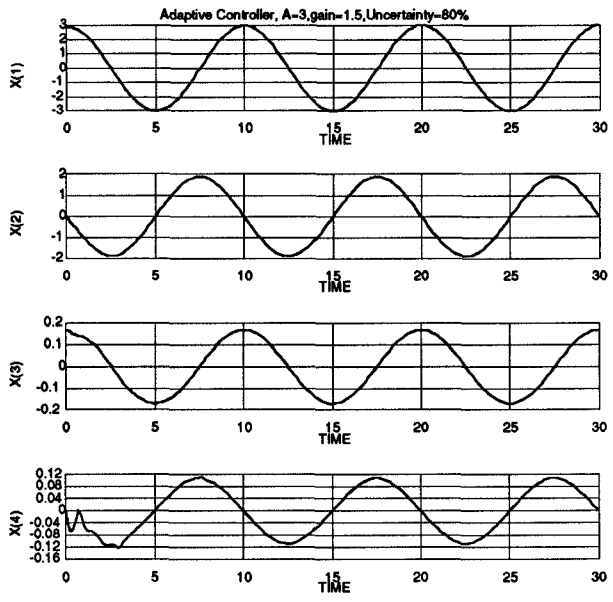

Figure 5.5: Adaptive Controller: state trajectories $x_i(t)$.

times more, around $0.015m$, than previous simulation even with less parameter uncertainty. The neglected nonlinearity $|\psi_3(x,u)|$ was much higher in this case, around 0.6, making the approximate linearization result hard to apply. The non-adaptive controller became unstable with 50% uncertainty in M. Clearly, the adaptive controller significantly improved the tracking by stabilizing the system, driving the error closer to zero, and providing a robust approximate feedback linearization for the controller design when nonlinearities can not be canceled due to the lack of exact knowledge of the system parameters.

5.5 Conclusions

We have presented an adaptive approximate tracking result using an approximate input-output linearization for nonlinear systems that do not have a well defined relative degree at a point of interest (singular state). This scheme is also

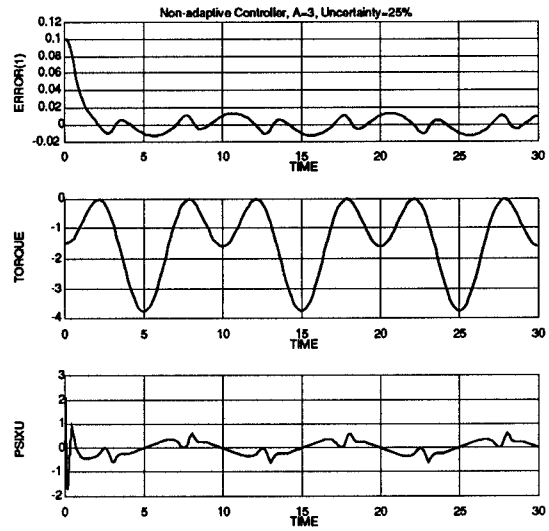

Figure 5.6: Non-adaptive Controller: error trajectory with $e(0) = 0.1m$, applied torque, and neglected nonlinearity $\psi_3(x, u)$, with 25% uncertainty in mass M of the ball.

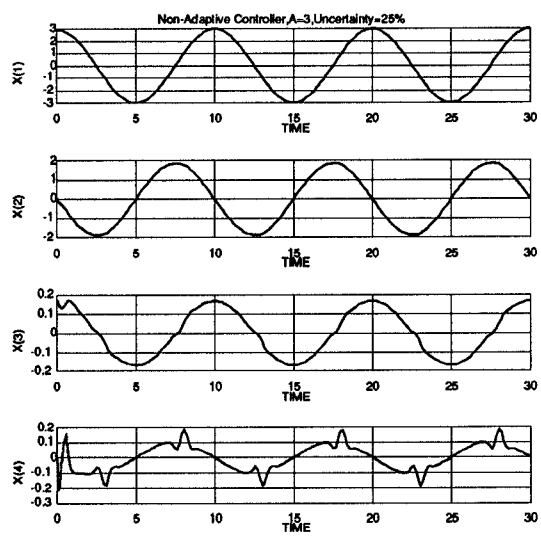

Figure 5.7: Non-adaptive Controller: state trajectories $x_i(t)$.

applicable to slightly non-minimum phase systems. It is shown that the adaptive controller can achieve output tracking of reasonable trajectories with small error. Our approach was based on certainty equivalence principle with an assumption of parameter identifier convergence. Simulation results were presented for an undergraduate control laboratory experiment, the ball and beam example, discussed in [39]. Simulation results show that under parameter uncertainty in the mass of the ball, our adaptive controller provides good tracking with stability while the non-adaptive controller results in an unstable closed-loop system.

Chapter 6

Adaptive Control of Nonlinear Systems via Approximate Linearization

6.1 Introduction

In this chapter we present a **direct** adaptive control scheme for nonlinear systems that fail to meet the restrictive regularity conditions of the current nonlinear adaptive control schemes in the literature based on feedback state or input-output linearization. We provide an adaptive **approximate** tracking scheme for systems that do not have a well-defined (vector) relative degree, nor can they achieve a vector relative degree through the dynamic inversion or extension of chapter (4). The results are also applicable to **slightly** non-minimum phase nonlinear systems with unknown parameters in their dynamics.

Over the last decade, geometric nonlinear control theory has provided powerful tools for systematic design of nonlinear feedback systems [51, 81]. Most of the available methods for nonlinear adaptive control system design, [78, 88, 97,

57, 98, 12, 56] among others, are based on linearizing the input-output response of the nonlinear system or exact state linearization using a coordinate change $z = T(x)$ and a state feedback [48, 51, 81]. Major limitations to these approaches come from the fact that they require certain regularity conditions such as involutivity, existence of a (vector) relative degree, minimum phase property, linearity in the unknown parameters, and conic growth conditions for the nonlinear terms present in system dynamics. Most of these conditions are necessary due to the deterministic part of the design problem in **exact** (partial) state or input-output feedback linearization.

Alternatively, there have been several successful approaches that aim to **approximately** linearize a nonlinear system by relaxing one or more of these restrictions. Krener [64, 65], Karahan [59], and co-workers [67, 58] introduced the approximate linearization approach as an alternative to exact state linearization. This approach has recently been successfully applied to chemical reactor control problem for non-involutive systems [29] Using Taylor expansion, a nonlinear system can always be approximated by a linear system to first degree. However, the idea here is to approximate a nonlinear system up to the highest degree feasible (computation defines the best). This will increase the validity of the approximation to the highest order possible with small error terms, causing minimal performance degradation that may be ignored in some neighborhood of the equilibrium. In the extreme case, there is no error term and exact state linearization is achieved. Other schemes include **extended linearization** introduced by Baumann and Rugh [10] and Rugh [86], **pseudolinearization** by Champetier and Reboulet [25, 84], and recently by Wang and Rugh [102], and **uniform system approximation** by Hauser [38] and co-workers [39]. A survey on general

applicability and properties of these approximate linearization approaches versus exact linearization techniques for chemical reactor control was reported in [30]. In [40, 39], it was shown that one can approximately (input-output) linearize a nonlinear system and design a stable **approximate** tracking controller under much weaker conditions than those needed for tracking design based on **exact** feedback linearization schemes. Using these results, we attempt to extend parameter adaptive schemes developed for feedback linearizable systems to **approximate** linearizable ones and hence, avoid several restrictions that limit the general applicability of current nonlinear adaptive schemes. We prove that the adaptive design scheme results in an asymptotically stable closed loop system and show that the controller can achieve adaptive tracking of reasonable trajectories with bounds on the tracking error. We also present a state regulation scheme based on state approximate linearization. This scheme is applicable to a large class of nonlinear systems that are not necessarily feedback linearizable. In particular, we present a systematic adaptive regulation technique for **linearly** controllable uncertain nonlinear systems where unknown parameters do not necessarily appear linearly in their dynamics. The usefulness of our approach is illustrated on several "benchmark" examples that are not feedback linearizable systems. In the next chapter, we use our adaptive approximate tracking results for the design of flight control systems and apply the techniques developed in this chapter to a simplified model of an aircraft which is a slightly non-minimum phase system.

We first review the approximate state linearization technique for nonlinear systems. The input-output counterpart was reviewed in the last chapter. In section (6.3), we present the design procedures for adaptive input-output ap-

proximate linearization and tracking. Adaptive regulation using approximate state linearization is addressed in section (6.4), and a systematic design procedure for adaptive **quadratic** linearization is presented. In section (6.5), we look at some "benchmark" examples of adaptive control design where the systems are not feedback linearizable and can not be transformed into the so-called "parametric-pure-feedback form" [78, 88, 97, 57, 98, 12, 56].

6.2 Review of Approximate State Linearization

Consider the following nonlinear system:

$$\dot{x}(t) = f(x) + \sum_{1}^{m} g_i(x) \cdot u_i \tag{6.1}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the input, and $f(0) = 0$. Krener [64] gave necessary and sufficient conditions for the existence of transformations:

$$\begin{aligned} z &= z(x) \\ v &= v(x,u) = \alpha(x) + \beta(x) \cdot u \end{aligned} \tag{6.2}$$

which transforms the nonlinear system (6.1) into an **approximate** linear system:

$$\begin{aligned} \dot{z} &= Az + Bv + O(x,u)^{\rho+1} \\ A &= \tfrac{\partial f}{\partial x}(0) \ , \ B = g(0) \end{aligned} \tag{6.3}$$

with an error of order $\rho + 1$, where $\rho \geq 1$ is the order of approximation[1]. This nonlinear transformation can then be followed by linear transformations of the

[1]Recall that a function $\psi(z)$ is $O(z)^n$ if $lim_{|z|\to 0} \frac{|\psi(z)|}{|z|^n}$ exists and is not zero. $O(z)^0$ is referred to as $O(1)$.

states to obtain any canonical form representation of (6.3), such as the Brunovsky form, at the cost of losing the physical significance of the states.

Given the nonlinear system (6.1) define the following distributions:

$$D^k = C^\infty \; span\{ad_f^i g_j : \; 0 \leq i < k, j = 1, \ldots, m\} \tag{6.4}$$

Definition 6.2.1 (Order ρ basis-[64]) Distribution D has an order ρ local basis around 0 if there exist vector fields $X_1, \ldots, X_d$ which are linearly independent at 0 and such that for every $Y \in D$ there exists functions c_i such that:

$$Y = \sum_{i=1}^{d} c_i X_i + O(x)^{\rho+1} \tag{6.5}$$

where the integer d is the order ρ dimension of D at 0.

Definition 6.2.2 (Order ρ involutive-[64]) Distribution D is said to be order ρ involutive at 0 if there exist functions c_k^{ij} such that:

$$[X_i, X_j] = \sum_{k=1}^{d} c_k^{ij} X_k + O(x)^\rho \tag{6.6}$$

Theorem 6.2.1 ([64]) The nonlinear system (6.1) can be transformed into an order ρ linear system (6.3) where (A, B) is a controllable pair with controllability indices $k_1 \geq \ldots \geq k_n$ iff:

(i). Distribution D^k has an order ρ local basis at 0 consisting of:

$$\left\{ad_f^l g_j : 0 \leq l < min(k_j, k); j = 1, \ldots, m\right\}$$

(ii). $D^{k_j - 1}$ is order ρ involutive at 0 for $j = 1, \ldots, m$.

Note that it is always possible to find such transformation where $\rho = 1$ using the Taylor expansion.

By neglecting the high order terms in some neighborhood U_ϵ of the operating point x_e we obtain the approximate description of (6.1):

$$\dot{\xi} = F\xi + G\mu \tag{6.7}$$

where F and G are some linear transformations of A and B in (6.3). When

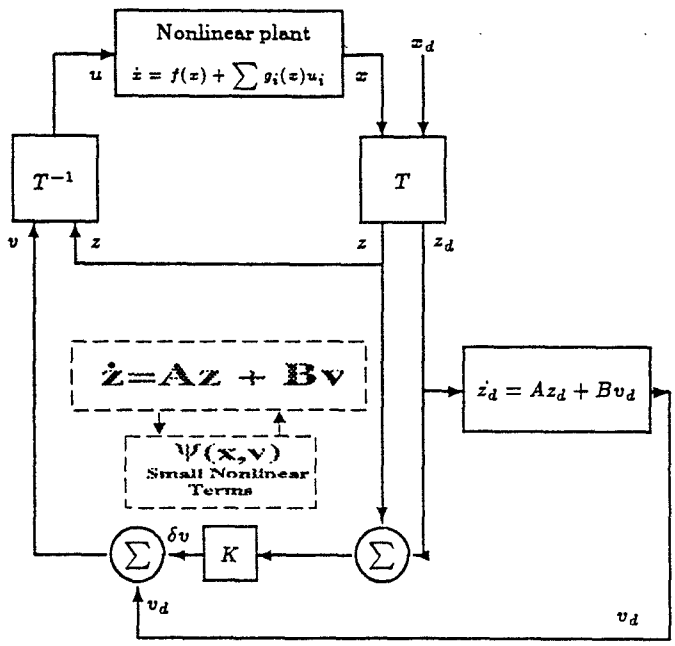

Figure 6.1: Adaptive control design structure via approximate linearization

system (6.1) is operating in U_ϵ , where (6.7) is a valid approximation, one may utilize linear feedback control theory to **approximately** achieve any desired design objective. In this chapter, we are interested in this technique for adaptive regulation and model reference adaptive control. Figure (6.1) shows the structure of this design approach. Our adaptive output tracking scheme is, however, based on approximate input-output linearization reviewed in section (5.2).

The approximate feedback linearization results of theorems (5.2.1) and (6.2.1) are clear design alternatives to the more restrictive schemes of exact feedback

linearization approach. These results have already been applied to the design of chemical engineering systems [29, 30], and automatic flight control systems [40]. In the next two sections, we present a direct adaptive tracking and adaptive regulation scheme for nonlinear systems that are **approximately** feedback linearizable in the sense of [64, 38] and hence, are subject to milder involutivity restrictions, and are not necessarily minimum phase with a well-defined (vector) relative degree as assumed in most current adaptive control strategies for nonlinear systems.

6.3 Adaptive Tracking

Consider a SISO nonlinear system of the form (6.1) under parameter uncertainty:

$$\begin{aligned} \dot{x}(t) &= f(x,\theta) + g(x,\theta)\cdot u \\ y(t) &= h(x,\theta) \end{aligned} \tag{6.8}$$

where $x \in \Re^n$ is the state, $u \in \Re$ is the input, $y \in \Re$ is the output, $\theta = [\theta_1, \theta_2, \cdots, \theta_p]^T$ is the vector of unknown constant parameters, f, g, and h are smooth functions on $\Re^n$. We assume (6.8) has relative degree r around the equilibrium x_e, but not necessarily a well defined relative degree at x_e. We further assume system (6.8) has a **robust relative degree** γ in $U_\epsilon(x_e)$:

Assumption 6.3.1 (Relative Degree) System (6.8) has a robust relative degree of γ on $U_\epsilon(x_e)$, an open neighborhood of the equilibrium point x_e. i.e. $\forall x \in U_\epsilon(x_e)$, $\forall \hat{\theta} \in U_\sigma(\theta)$:

$$\begin{aligned} &\mathcal{L}_{g(x)}\mathcal{L}^i_{f(x)}h(x) = 0 && i = 0, \cdots, r-2 \\ &\mathcal{L}_{g(x)}\mathcal{L}^j_{f(x)}h(x) \ are\ of\ order\ \epsilon && j = r-1, \cdots, \gamma-2 \\ &\mathcal{L}_{g(x)}\mathcal{L}^{\gamma-1}_{f(x)}h(x) \neq 0 \end{aligned}$$

where $0 < r < \gamma$ is the relative degree of (6.8) outside $U_\epsilon(x_e)$ but not necessarily well defined at every point inside $U_\epsilon(x_e)$. Moreover, terms of order ϵ could be either $O(x)^2$ or small bounded terms when $x \in U_\epsilon(x_e)$.

Assumption 6.3.2 (Linear Parameter Dependence) The vector fields f and g in (6.8) are unknown but may be parametrized linearly in unknown parameters θ:

$$\begin{aligned} f(x,\theta) &= \sum_{i=1}^{p} \theta_i \cdot f_i(x) \\ g(x,\theta) &= \sum_{i=1}^{p} \theta_i \cdot g_i(x) \end{aligned} \tag{6.9}$$

where vector fields f_i, and g_i are known functions of x.

By the Frobenius theorem, there exist $n-\gamma$ functions $\eta_i(x,\theta)$ such that $\mathcal{L}_{g(x,\theta)}\eta_i(x,\theta) = 0$. The resulting local diffeomorphism of $x \in \Re^n$ is:

$$(\xi^T, \eta^T)^T = (\xi_i = \mathcal{L}^{i-1}_{f(x,\theta)} h(x), i = 1, 2, \ldots, r, \ldots, \gamma, \quad \eta_1, \ldots, \eta_{n-\gamma})^T \tag{6.10}$$

transforms the system (6.8) to an approximate input-output linearized system given in (2.7). However, this transformation can not be used directly in the design scheme since it depends on unknown parameters θ. We replace the transformation ξ with its estimate $\hat{\xi}$ by replacing all unknown parameters θ_i appearing in ξ by their estimates $\hat{\theta}_i$:

$$(\hat{\xi}^T, \eta^T)^T = (\hat{\xi}_i = \mathcal{L}^{i-1}_{f(x,\hat{\theta})} h(x), i = 1, 2, \ldots, r, \ldots, \gamma, \quad \eta_1, \ldots, \eta_{n-\gamma})^T \tag{6.11}$$

The dynamics of (6.8) under this (time-varying) transformation along the solution trajectories of (6.8) is:

$$
\begin{aligned}
\dot{\hat{\xi}}_1 &= \mathcal{L}_{f(x,\theta)}h(x) \\
\dot{\hat{\xi}}_2 &= \mathcal{L}_{f(x,\theta)}\mathcal{L}_{f(x,\hat{\theta})}h(x) + \frac{\partial \hat{\xi}_2(x,\hat{\theta})}{\partial \hat{\theta}} \cdot \dot{\hat{\theta}} \\
&\vdots \\
\dot{\hat{\xi}}_{r-1} &= \mathcal{L}_{f(x,\theta)}\mathcal{L}^{r-2}_{f(x,\hat{\theta})}h(x) + \frac{\partial \hat{\xi}_{r-1}(x,\hat{\theta})}{\partial \hat{\theta}} \cdot \dot{\hat{\theta}} \\
\dot{\hat{\xi}}_r &= \mathcal{L}_{f(x,\theta)}\mathcal{L}^{r-1}_{f(x,\hat{\theta})}h(x) + \mathcal{L}_{g(x,\theta)}\mathcal{L}^{r-1}_{f(x,\hat{\theta})}h(x) \cdot u + \frac{\partial \hat{\xi}_r(x,\hat{\theta})}{\partial \hat{\theta}} \cdot \dot{\hat{\theta}} \\
\dot{\hat{\xi}}_{r+1} &= \mathcal{L}_{f(x,\theta)}\mathcal{L}^{r}_{f(x,\hat{\theta})}h(x) + \mathcal{L}_{g(x,\theta)}\mathcal{L}^{r}_{f(x,\hat{\theta})}h(x) \cdot u + \frac{\partial \hat{\xi}_r(x,\hat{\theta})}{\partial \hat{\theta}} \cdot \dot{\hat{\theta}} \\
&\vdots \\
\dot{\hat{\xi}}_{\gamma-1} &= \mathcal{L}_{f(x,\theta)}\mathcal{L}^{\gamma-2}_{f(x,\hat{\theta})}h(x) + \mathcal{L}_{g(x,\theta)}\mathcal{L}^{\gamma-2}_{f(x,\hat{\theta})}h(x) \cdot u + \frac{\partial \hat{\xi}_{\gamma-1}(x,\hat{\theta})}{\partial \hat{\theta}} \cdot \dot{\hat{\theta}} \\
\dot{\hat{\xi}}_{\gamma} &= \mathcal{L}_{f(x,\theta)}\mathcal{L}^{\gamma-1}_{f(x,\hat{\theta})}h(x) + \mathcal{L}_{g(x,\theta)}\mathcal{L}^{\gamma-1}_{f(x,\hat{\theta})}h(x) \cdot u + \frac{\partial \hat{\xi}_{\gamma}(x,\hat{\theta})}{\partial \hat{\theta}} \cdot \dot{\hat{\theta}} \\
\dot{\eta} &= q(\hat{\xi}, \eta)
\end{aligned}
\tag{6.12}
$$

From assumption (6.3.2), we have:

$$
\begin{aligned}
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + \sum_{i=1}^{p} (\theta_i - \hat{\theta}_i) \cdot \mathcal{L}_{f_i(x)}h(x) \\
\dot{\hat{\xi}}_2 &= \hat{\xi}_3 + \sum_{i=1}^{p} (\theta_i - \hat{\theta}_i) \cdot \mathcal{L}_{f_i(x)}\mathcal{L}_{f(x,\hat{\theta})}h(x) + \frac{\partial \hat{\xi}_2(x,\hat{\theta})}{\partial \hat{\theta}} \cdot \dot{\hat{\theta}} \\
&\vdots \\
\dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + \sum_{i=1}^{p} (\theta_i - \hat{\theta}_i) \cdot \mathcal{L}_{f_i(x)}\mathcal{L}^{i-1}_{f(x,\hat{\theta})}h(x) \\
&\quad + \mathcal{L}_{g(x,\theta)}\mathcal{L}^{i-1}_{f(x,\hat{\theta})}h(x) \cdot u + \frac{\partial \hat{\xi}_i(x,\hat{\theta})}{\partial \hat{\theta}} \cdot \dot{\hat{\theta}} \quad i = r, \ldots, \gamma - 1
\end{aligned}
\tag{6.13}
$$

From assumption (6.3.1), and applying the control law:

$$
u_{ad} = \frac{1}{\mathcal{L}_{g(x,\hat{\theta})}\mathcal{L}^{\gamma-1}_{f(x,\hat{\theta})}h(x)}[-\mathcal{L}^{\gamma}_{f(x,\hat{\theta})}h(x) + v_{ad}] \tag{6.14}
$$

with:

$$
v_{ad} = y_m^{(\gamma)} + \alpha_{\gamma-1}(y_m^{(\gamma-1)} - \hat{\xi}_\gamma) + \ldots + \alpha_0(y_m - \hat{\xi}_1) \tag{6.15}
$$

and α_i chosen such that $s^\gamma + \alpha_{\gamma-1}s^{\gamma-1} + \ldots + \alpha_0$ is a Hurwitz polynomial, we can rewrite (6.12) in a more compact form with $\phi = \theta - \hat{\theta}$:

$$\begin{aligned}
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + w_1(x,\hat{\theta}) \cdot \Phi \\
\dot{\hat{\xi}}_2 &= \hat{\xi}_2 + w_2(x,\hat{\theta}) \cdot \Phi + \frac{\partial \hat{\xi}_2(x,\hat{\theta})}{\partial \hat{\theta}} \cdot \dot{\hat{\theta}} \\
&\vdots \\
\dot{\hat{\xi}}_r &= \hat{\xi}_{r+1} + w_r(x,\hat{\theta}) \cdot \Phi + \frac{\partial \hat{\xi}_r(x,\hat{\theta})}{\partial \hat{\theta}} \cdot \dot{\hat{\theta}} + \psi_r(x,\theta,u) \\
&\vdots \\
\dot{\hat{\xi}}_{\gamma-1} &= \hat{\xi}_\gamma + w_{\gamma-1}(x,\hat{\theta}) \cdot \Phi + \frac{\partial \hat{\xi}_{\gamma-1}(x,\hat{\theta})}{\partial \hat{\theta}} \cdot \dot{\hat{\theta}} + \psi_{\gamma-1}(x,\theta,u) \\
\dot{\hat{\xi}}_\gamma &= v_{ad} + w_\gamma(x,\hat{\theta},u_{ad}) \cdot \Phi + \frac{\partial \hat{\xi}_\gamma(x,\hat{\theta})}{\partial \hat{\theta}} \cdot \dot{\hat{\theta}} \\
\dot{\eta} &= q(\hat{\xi},\eta)
\end{aligned} \tag{6.16}$$

where:

$$w_\gamma(x,\hat{\theta},u) \cdot \Phi = \sum_{i=1}^{p} (\theta_i - \hat{\theta}_i) \cdot \left[\mathcal{L}_{f_i(x)}\mathcal{L}^{\gamma-1}_{f(x,\hat{\theta})}h(x) + u \cdot \mathcal{L}_{g_i(x)}\mathcal{L}^{\gamma-1}_{f(x,\hat{\theta})}h(x)\right]$$

and $\psi_i(x,u) = \mathcal{L}_{g(x,\theta)}\mathcal{L}^{i-1}_{f(x,\hat{\theta})}h(x) \cdot u$. Finally:

$$\begin{aligned}
\dot{\hat{\xi}} &= A \cdot \hat{\xi} + B \cdot v + W \cdot \Phi + M \cdot \dot{\hat{\theta}} + \Psi(x,u) \\
\dot{\eta} &= q(\hat{\xi},\eta)
\end{aligned} \tag{6.17}$$

where:

$$A = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 1 \\ 0 & 0 & \ldots & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, W = \begin{bmatrix} w_1 \\ \vdots \\ \\ w_\gamma \end{bmatrix} \tag{6.18}$$

$$\Psi(x,u) = [0,\ldots,\psi_r(x),\ldots,\psi_{\gamma-1},0]^T \quad , \quad M = \frac{\partial \hat{\xi}(x,\hat{\theta})}{\partial \hat{\theta}}$$

The design objective is to force the output y of system (6.8) to asymptotically track a known reference signal y_m. For this, the control law and the parameter update law must be independent of unknown parameters θ and initial conditions $\hat{\xi}(0)$. Moreover, all the closed-loop signals must remain bounded. The error signal e is defined as:

$$e_i = \hat{\xi}_i - y_m^{(i-1)} \qquad i = 1, \ldots, \gamma \tag{6.19}$$

with $e_1 = \hat{\xi}_1 - y_m = y - y_m$. Therefore, for approximate tracking, we require $e_1(t) \to B_\epsilon(0)$ as $t \to \infty$.

Assumption 6.3.3 (Reference Signal) The reference trajectory $y_m(t)$ and its first γ derivatives are bounded. i.e $|y_m^{(i)}| \leq b_m \quad i = 0, 1, \ldots, \gamma$ for some $b_m > 0$.

Remark 6.3.1 Often, as in model reference adaptive control, the control objective is to force the states ξ to track the states ξ_m of an asymptotically stable linear reference model with a relative degree equal to that of system (6.8):

$$\dot{\xi_m} = A_m \cdot \xi_m + b_m \cdot r$$

where A_m and b_m are in controllable canonical form and $r(t)$ is a bounded reference input. In this case the error may be defined as: $e = \hat{\xi} - \xi_m$ with $v = \sum_{i=1}^{\gamma} \alpha_i \hat{\xi}_i + r(t)$ replacing (6.15).

To determine a parameter update law $\dot{\hat{\theta}}$ that assures the stability of the closed-loop system we first construct a regressor like equation from (6.17) by cancelling M using an auxiliary system in the adaptive loop and factoring θ in $\Psi(x, u)$. Let $\bar{\xi}$ be a new signal generated as the solution trajectory of the

following state observer system which is a modified version of the system used in [83, 3] and the semi-indirect adaptive scheme of [98]:

$$\begin{aligned} \dot{\bar{\xi}} &= A \cdot \hat{\xi} + B \cdot v_{ad} + M \cdot \dot{\hat{\theta}} + \hat{\Psi}(x, u, \hat{\theta}) + \bar{A} \cdot (\bar{\xi} - \hat{\xi}) \\ \bar{\xi}(0) &= \bar{\xi}_0 = \hat{\xi}(0) \end{aligned} \tag{6.20}$$

where $\hat{\xi} \in \Re^{\gamma}$ from (6.11), M as in (6.17), $\hat{\Psi}(x, u, \hat{\theta})$ is an estimate of $\Psi(x, u)$ in (6.18) evaluated at $\theta = \hat{\theta}$, and:

$$\bar{A} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 1 \\ -\alpha_0 & -\alpha_1 & \dots & -\alpha_{\gamma-1} \end{bmatrix}$$

with α_i chosen as in control (6.15) and $\dot{\hat{\theta}}$ still to be determined. In (6.20), if $\hat{\xi}(0)$ is not available $\bar{\xi}_0$ is set to an estimate of $\hat{\xi}(0)$. Let's define the augmented error $s(t)$ as:

$$s = \bar{\xi} - \hat{\xi} \tag{6.21}$$

where $\hat{\xi}$ is defined by (6.11) and its dynamics is given by (6.17) subject to control $v = v_{ad}$ as in (6.15). Observe that $s(t)$ satisfies:

$$\begin{aligned} \dot{s} &= \bar{A} \cdot s - W \cdot \Phi + \hat{\Psi}(x, u, \hat{\theta}) - \Psi(x, u, \theta) \\ &= \bar{A} \cdot s + W_2 \cdot \Phi \end{aligned} \tag{6.22}$$

where $W_2(x, u)$ is formed by factoring parameters θ in $\Psi(\cdot)$ and regrouping all parameter dependent nonlinearities.

The following assumption is needed to provide the internal stability of the plant:

Assumption 6.3.4 (Zero Dynamics) The Zero Dynamics of the approximate input-output linearized system (2.7), or equivalently (6.17), are locally exponentially stable with q locally Lipschitz in $\hat{\xi}$ and η.

Remark 6.3.2 The zero dynamics of the approximate system are a subsystem of the zero dynamics of the true system. Assumption (6.3.4) does not ask for the **plant** (6.8) to be minimum phase which is required in most adaptive control design schemes (e.g. [98, 12, 56, 88, 97]). In contrast, our scheme can handle **slightly** non-minimum phase systems which by definition have minimum phase **approximate** linearization.

We are now ready to state the main result of this section. The following theorem provides a parameter update law that guarantees adaptive approximate tracking and gives an upper bound on the tracking error.

Theorem 6.3.1 (Adaptive Approximate Tracking) Consider the system of (6.8) satisfying robust relative degree assumption (6.3.1) and the zero dynamics assumption (6.3.4) with the vector fields f and g parameterized as in assumption (6.3.2). Suppose that the system (6.17) is formed and assume that $\Psi(x, u_{ad}), W(x, u_{ad}, \hat{\theta})$ and $M(x, u_{ad}, \hat{\theta})$ are locally Lipschitz continuous. Then, given a reference trajectory y_m satisfying assumption (6.3.3) with sufficiently small b_m, it follows that for ϵ sufficiently small the control law u_{ad} in (6.14) achieves adaptive approximate tracking of order ϵ; i.e.,

$$|y - y_m| \leq k\epsilon$$

for some $k < \infty$, with the parameter update law:

$$\dot{\hat{\theta}} = -\Omega \cdot W_2^T \cdot P \cdot s \tag{6.23}$$

Furthermore, all the signals in the resulting closed loop adaptive system remain bounded.

Proof. Consider the following Lyapunov candidate function for the system (6.22):

$$V(s,\phi) = s^T P s + \Phi^T \Omega^{-1} \Phi \tag{6.24}$$

where Ω is a constant diagonal gain matrix and $P = P^T$ is the positive definite solution to the Lyapunov equation $\bar{A}^T P + P\bar{A} = -\lambda \cdot I$ with $\lambda > 0$, and $\bar{A}$ asymptotically stable as in (6.20). The derivative of V along the solution trajectories of (6.22) is:

$$\dot{V} = -\lambda s^T \cdot s + 2s^T P W_2^T \Phi + 2\Phi^T \Omega^{-1} \dot{\Phi}$$

From (6.23), we have:

$$\dot{V} = -\lambda |s|^2 \leq 0$$

Hence:

$$s(t) \in \mathcal{L}_\infty \cap \mathcal{L}_2 \tag{6.25}$$

To establish a bound on $|s(t)|$ and $|\Phi(t)|$, let $\lambda_{min}(P) > 0$ be the minimum eigenvalue of P in (6.24). Then $\forall t \geq 0$:

$$\begin{aligned} \lambda_{min}(P) \cdot |s(t)|^2 &\leq s^T(t) P s(t) \leq V(s(t), \Phi(t)) \leq V(s(0), \Phi(0)) \\ &\leq \tfrac{1}{g_{min}} \cdot |\Phi(0)|^2 + \lambda_{max}(P) \cdot |S_0|^2 \end{aligned}$$

where $g_{min} > 0$ is the minimum gain entry in Ω. Hence:

$$\begin{aligned} |s(t)| &\leq \lambda_s \\ |\Phi(t)| &\leq \lambda_\phi \end{aligned} \tag{6.26}$$

where λ_ϕ and λ_s are some positive constants with magnitudes depend on the error in initial estimates $\hat{\theta}(0)$ and $\bar{\xi}_0$ of θ and $\hat{\xi}(0)$. Note that if $\hat{\xi}(0)$ is available we have $s(0) = 0, \lambda_s = (g_{min} \cdot \lambda_{min}(P))^{-1/2} \cdot |\Phi(0)|$, and $\lambda_\phi = (g_{max}/g_{min})^{1/2} \cdot |\Phi(0)|$. Hence, with the update law (6.23), $\hat{\theta}(t)$ remains bounded. Next we need to show

that $s(t) \to 0$ as $t \to \infty$. A sufficient condition for this is $\dot{s}(t) \in \mathcal{L}_\infty$. We also need to show boundedness of states x and $u_{ad}(x)$ so that $W_2(x, u, \hat{\theta})$ in (6.22) and (6.23) remains bounded. This will be shown next together with the approximate tracking requirement in (6.19).

The tracking error signal e defined in (6.19) satisfies the following differential equation:

$$\begin{aligned} \dot{e} &= \bar{A} \cdot e + W^T \cdot \Phi + M \cdot \dot{\hat{\theta}} + \Psi(x, u) \\ &= \bar{A} \cdot e + W^T \cdot \Phi - M \cdot \Omega \cdot W_2^T \cdot P \cdot s + \Psi(x, u) \end{aligned} \tag{6.27}$$

To show $e_1 \to B_\epsilon(0)$, consider the total error defined as:

$$r \triangleq e + s \tag{6.28}$$

or equivalently $r_i = \bar{\xi}_i - y_m^{(i-1)}$, and note that from (6.27) and (6.22), $r(t)$ satisfies the following differential equation:

$$\begin{aligned} \dot{r} &= \bar{A} \cdot r + M \cdot \dot{\hat{\theta}} + \hat{\Psi}(x, u, \hat{\theta}) \\ &= \bar{A} \cdot r - M \cdot \Omega \cdot W_2^T \cdot P \cdot s + \hat{\Psi}(x, u, \hat{\theta}) \end{aligned} \tag{6.29}$$

Remark 6.3.3 Equation (6.29) may be interpreted as a linear time-varying filter under small perturbation $\Psi(\cdot)$ with bounded input $s(t)$ (from (6.25)) and subject to the internal dynamics: $\dot{\eta} = q(x, \theta)$ driven by r. Let's define the output of this filter, from (6.28), as:

$$\begin{aligned} \dot{r} &= \bar{A} \cdot r - \left[M \cdot \Omega \cdot W_2^T \cdot P\right] \cdot s(t) + \hat{\Psi}(x, u, \hat{\theta}) \\ e(t) &= r - s(t) \\ \dot{\eta} &= q(x, \theta) = q(\hat{\xi}, \eta, \theta) \end{aligned} \tag{6.30}$$

Next we will analyze the stability properties of this filter in order to show $e_1 \to B_\epsilon(0)$. More specifically we establish $e(t)$ as the output of an asymptotically stable linear filter (6.30) with stable internal dynamics $q(x, \theta)$. This

requires that x is bounded, or equivalently, $\hat{\xi}$ and η are bounded. To show $\hat{\xi}$ and η are bounded, we will first show that r and η are bounded using a suitable Lyapunov candidate function for (6.30):

$$V(r,\eta) = r^T \bar{P} r + \mu v_2(\eta) \tag{6.31}$$

where $\mu > 0$ is a constant to be determined later, $\bar{P} > 0$ is such that $\bar{A}^T \bar{P} + \bar{P}\bar{A} = -I$, and $v_2(\eta)$ is a Lyapunov function for the system $\dot{\eta} = q(0,\eta)$. From assumption (6.3.4), a converse Lyapunov argument assures the existence of v_2 with following properties:

$$\begin{aligned} k_1|\eta|^2 \leq \quad & v_2(\eta) \quad \leq k_2|\eta|^2 \\ \tfrac{\partial v_2}{\partial \eta} q(0,\eta) \quad & \leq \quad -k_3|\eta|^2 \\ \left|\tfrac{\partial v_2}{\partial \eta}\right| \quad & \leq \quad k_4|\eta| \end{aligned} \tag{6.32}$$

for some positive constants k_1, k_2, k_3, and k_4. The time derivative of $V(r,\eta)$ along the solution trajectories of (6.30) is:

$$\dot{V} = -|r|^2 - 2r^T \bar{P}[M \cdot \Omega \cdot W_2^T \cdot P] \cdot s + 2r^T \bar{P}\hat{\Psi} + \mu \frac{\partial v_2}{\partial \eta} q(\hat{\xi},\eta) \tag{6.33}$$

From (6.19) and assumption (6.3.3) we have:

$$|\hat{\xi}| \leq |e| + b_m \tag{6.34}$$

and from the definition of r in (6.28):

$$|e| \leq |r| + |s| \tag{6.35}$$

Since x is a local diffeomorphism of $(\hat{\xi},\eta)$, we have:

$$\begin{aligned} |x| \quad & \leq \quad l_x(|\hat{\xi}| + |\eta|) \\ & \leq \quad l_x(|e| + b_m + |\eta|) \end{aligned} \tag{6.36}$$

Since W_2 is assumed locally Lipschitz continuous we have:

$$|2\bar{P}[M \cdot \Omega \cdot W_2^T \cdot P]| \leq l_W|x| + c_1 \tag{6.37}$$

where $c_1 > 0$ is a constant.

Because $\Psi(x,u)$ is $O^2(x,u)$, we have for some constants $l_\epsilon > 0$ and $\delta > 0$:

$$\begin{aligned} |2\bar{P}\Psi(x,u)| &\leq l_\epsilon|x|^2 \quad \forall x : |x| \leq \epsilon, |u| \leq \delta \\ &\leq l_\epsilon\epsilon|x| \end{aligned} \tag{6.38}$$

Note that since u is a function of $x(t)$ and $y_m(t)$, δ depends on ϵ and b_m. This immediately suggests that $\alpha_i s$ in (6.15) should be chosen such that the assigned poles are not too far left. Otherwise, the resulting higher control magnitude $|u|$ will push the state $|x|$ outside its approximating region.

From assumption (6.3.4):

$$|q(\hat{\xi},\eta) - q(0,\eta)| \leq l_q|\hat{\xi}|$$

Hence:

$$\begin{aligned} \tfrac{\partial v_2}{\partial \eta} q(\hat{\xi},\eta) &= \tfrac{\partial v_2}{\partial \eta} q(0,\eta) + \tfrac{\partial v_2}{\partial \eta}(q(\hat{\xi},\eta) - q(0,\eta)) \\ &\leq -k_3|\eta|^2 + k_4 l_q|\eta|(|e| + b_m) \end{aligned} \tag{6.39}$$

Substituting above inequalities in (6.33) yields:

$$\begin{aligned} \dot{V} &\leq -|r|^2 + |r|(l_W|x| + c_1)|s| + \epsilon l_\epsilon|r||x| \\ &\quad +\mu(-k_3|\eta|^2 + k_4 l_q|\eta|(|e| + b_m)) \\ &\leq -|r|^2 + l_x l_W|r|(|r| + |s| + b_m + |\eta|)|s| \\ &\quad +c_1|r||s| + \epsilon l_\epsilon|r||x| + \mu(-k_3|\eta|^2 + k_4 l_q|\eta|(|r| + |s| + b_m)) \end{aligned}$$

Let $c_2 \triangleq b_m + \frac{c_1}{l_x l_W}$, then:

$$\begin{aligned}
\dot{V} \leq & -\left(\tfrac{|r|}{2} - l_W l_x(|s| + c_2)|s|\right)^2 + (l_W l_x(|s| + c_2))^2|s|^2 \\
& + \left(\tfrac{|r|}{2} - (l_W l_x|s| + \epsilon l_\epsilon l_x + \mu k_4 l_q)|\eta|\right)^2 + (l_W l_x|s| + \epsilon l_\epsilon l_x + \mu k_4 l_q)^2|\eta|^2 \\
& - \left(\tfrac{|r|}{2} - \epsilon l_\epsilon l_x(|s| + b_m)\right)^2 + (\epsilon l_\epsilon l_x(|s| + b_m))^2 \\
& -\mu k_3 \left(\tfrac{|\eta|}{2} - \tfrac{k_4}{k_3} l_q(|s| + b_m)\right)^2 + \tfrac{\mu}{k_3}(k_4 l_q(|s| + b_m))^2 \\
& -(1/4 - l_W l_x|s| - \epsilon l_\epsilon l_x)|r|^2 - \tfrac{3}{4}\mu k_3|\eta|^2 \\
\leq & -(1/4 - l_W l_x \lambda_s - \epsilon l_\epsilon l_x)|r|^2 - [\tfrac{3}{4}\mu k_3 - (l_W l_x \lambda_s + \epsilon l_\epsilon l_x + \mu k_4 l_q)^2]|\eta|^2 \\
& (l_W l_x(\lambda_s + c_2))^2 {\lambda_s}^2 + (\epsilon l_\epsilon l_x(\lambda_s + b_m))^2 + \tfrac{\mu}{k_3}(k_4 l_q(\lambda_s + b_m))^2
\end{aligned}$$

Define:

$$\mu \triangleq \frac{k_3}{4(l_W l_x + k_4 l_q + l_\epsilon l_x)^2} \tag{6.40}$$

Then for $\mu \leq \mu_0, \epsilon \leq min\left\{\frac{1}{8[l_W l_x + l_x l_\epsilon]}, \mu\right\}$, and $|\Phi(0)|$ and $|s(0)|$ small enough such that $\lambda_s \leq \epsilon$ we have:

$$\begin{aligned}
\dot{V} \leq & -\tfrac{|r|^2}{8} - \tfrac{\mu k_3}{2}|\eta|^2 + \tfrac{\mu}{k_3}(k_4 l_q(\lambda_s + b_m))^2 \\
& +(l_W l_x(\lambda_s + c_2))^2 {\lambda_s}^2 + (\epsilon l_\epsilon l_x(\lambda_s + b_m))^2
\end{aligned}$$

Hence when $|r|$ and $|\eta|$ are large $\dot{V} \leq 0$. Therefore, $|r|$ and $|\eta|$ are bounded which by (6.35) implies that $|e|$ is bounded. From (6.34), this shows that $|\hat{\xi}|$ is bounded and from (6.36), $|x|$ is bounded. This guarantees that M and W_2 are bounded which by (6.22) implies that $\dot{s}$ is bounded. This together with (6.25) implies that $s \to 0$ as $t \to \infty$. We have shown that (6.30) is an exponentially stable linear filter with stable internal dynamics $q(x, \theta)$ under bounded ϵ-order perturbation Ψ and input $s \to 0$. Hence, its output e converges to a ball of order ϵ. i.e. $|y - y_m| \leq k\epsilon$ for some constant k. This completes the proof. □

The adaptive design scheme developed in this section can also be applied to the multi-input multi-output (MIMO) nonlinear systems of the form:

$$\begin{aligned} \dot{x}(t) &= f(x,\theta) + \sum_{i=1}^{m} g_i(x,\theta) \cdot u_i \\ y_i(t) &= h_i(x) \qquad i = 1, \ldots, m \end{aligned} \tag{6.41}$$

where the vector relative degree $(r_1, \ldots, r_m)$ at some point of interest is ill-defined, or that the decoupling matrix A_r:

$$A_r = \begin{bmatrix} \mathcal{L}_{g_1}\mathcal{L}_f^{r_1-1}h_1(x) & \ldots & \mathcal{L}_{g_m}\mathcal{L}_f^{r_1-1}h_1(x) \\ \vdots & \vdots & \vdots \\ \mathcal{L}_{g_1}\mathcal{L}_f^{r_m-1}h_m(x) & \ldots & \mathcal{L}_{g_m}\mathcal{L}_f^{r_m-1}h_m(x) \end{bmatrix} \tag{6.42}$$

is almost singular due to the presence of small terms. In this case, with assumptions (6.3.1)-(6.3.4), we first apply the dynamic extension algorithm [51, 81] to the approximate model in order to get a nonsingular decoupling matrix A_γ in (6.42) with robust vector relative degree $(\gamma_1, \ldots, \gamma_m)$. The same procedure as in the SISO case can then be applied to the resulting extended system. We demonstrate these procedures in the section V where we apply our adaptive design scheme to a simplified model of an aircraft which is slightly non-minimum phase with an almost singular decoupling matrix.

6.4 Adaptive Regulation

When the objective is state regulation, the design procedure becomes simpler. In this case, we attempt to approximately (state) linearize the nonlinear system and then design a controller such that the closed loop system is asymptotically stable. The key here is to linearize the system to the highest order feasible.

Consider a nonlinear system with no output specified:

$$\dot{x}(t) \;=\; f(x,\theta) + \sum_{1}^{m} g_i(x,\theta)\cdot u_i \tag{6.43}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the input, $\theta = [\theta_1, \theta_2, \cdots, \theta_p]^T$ is the vector of unknown constant parameters, f and g_i are smooth functions on $\Re^n$ with $f(0,\theta) = 0$. The parameter update law for our adaptive regulation scheme is assumed to be of the following form:

$$\dot{\hat{\theta}} = g(x, \hat{\theta}, u) \tag{6.44}$$

where $g(\cdot)$ is $O(x,u)^{\rho_g}$ and will be determined later.

Lemma 6.4.1 Consider system (6.43) satisfying assumption (6.3.2). Let θ_0 be the unknown nominal value of θ and assume that $\forall\theta \in B_\sigma(\theta_0)$:

i. Distribution D^k has an order ρ_d local basis at 0 consisting of:

$$\left\{ad_f^l g_j : 0 \le l < min(k, k_j); j = 1, \ldots, m\right\}$$

for a set of $k_1, \ldots, k_m$, controllability indices.

ii. $D^{k_j - 1}$ is order ρ_d involutive at 0 for $j = 1, \ldots, m$.

Then there exists a local transformation $z = T(x,\hat{\theta}), T(0,\hat{\theta}) = 0$ and a nonlinear feedback $u_i(x,\hat{\theta})$ such that with the choice of the parameter update law in (6.44) the nonlinear system (6.43) is transformed into the following regressor form approximate linear system, with $\rho \triangleq min\{\rho_d, \rho_g\}$:

$$\dot{z}_j = \begin{cases} z_{j+1} \;+\; w_j^T(x,\hat{\theta})(\theta_0 - \hat{\theta}) + O(z,u)^{\rho+1} & if\ j \neq k_1 + \ldots + k_i \\ v_i \;+\; w_j^T(x,\hat{\theta},u)(\theta_0 - \hat{\theta}) + O(z,u)^{\rho+1} & if\ j = k_1 + \ldots + k_i \end{cases} \tag{6.45}$$

Proof It is clear from theorem (6.2.1) that for a fixed parameter θ and any $\rho_d > 0$, conditions (i) and (ii) are necessary and sufficient for the existence of a transformation $z = T(x, \theta)$ which transforms (6.43) into the following form:

$$\dot{z}_j = \begin{cases} z_{j+1} \quad + \quad O(x,u)^{\rho_d+1} & if\ j \neq k_1 + \ldots + k_i \\ v_i & if\ j = k_1 + \ldots + k_i \end{cases} \tag{6.46}$$

When parameter vector θ is not known and is replaced by an estimate $\hat{\theta}$ with an update law of the form (6.44), which is of order ρ_g, we get:

$$\dot{z}_j = \begin{cases} z_{j+1} \quad + \quad w_j^T(x,\hat{\theta})(\theta_0 - \hat{\theta}) + O(x,u)^{\rho_d+1} + \frac{\partial T(x,\hat{\theta})}{\partial \hat{\theta}} \cdot g(x,u,\hat{\theta}) \\ \qquad if\ j \neq k_1 + \ldots + k_i \\ v_i \quad + \quad w_j^T(x,\hat{\theta},u)(\theta_0 - \hat{\theta}) + \frac{\partial T(x,\hat{\theta})}{\partial \hat{\theta}} \cdot g(x,u,\hat{\theta}) \\ \qquad if\ j = k_1 + \ldots + k_i \end{cases} \tag{6.47}$$

where we used the fact that in (6.46): $z_{j+1} = \mathcal{L}_{f(x,\theta)} z_j \quad j \neq k_1 + \ldots + k_i$, and:

$$w_j^T = \left\{ \begin{array}{l} \left(\mathcal{L}_{f_1(x)} z_j, \ldots, \mathcal{L}_{f_p(x)} z_j\right) \qquad if\ j \neq k_1 + \ldots + k_i \\ \left(\mathcal{L}_{f_1(x)} z_j + \sum_{i=1}^{m} u_i \cdot \mathcal{L}_{g_{1,i}(x)} z_j\ , \ldots,\ \mathcal{L}_{f_p(x)} z_j + \sum_{i=1}^{m} u_i \cdot \mathcal{L}_{g_{p,i}(x)} z_j\right) \\ \qquad if\ j = k_1 + \ldots + k_i \end{array} \right\} \tag{6.48}$$

Finally, since $z = z(x)$ is a diffeomorphism with $z(0) = 0$, we have a term $O(x,u)^l$ if and only if it is $O(z,u)^l$. This together with the fact that $g(\cdot)$ is $O(x,u)^{\rho_g}$ results in (6.45). □.

The transformed system can be written as:

$$\dot{z} = A \cdot z + B \cdot v + W^T \cdot \Phi + O(z,u)^{\rho+1} \tag{6.49}$$

where W is a matrix with columns w_j defined in (6.48), and (A, B) are in Brunovsky form. The following control law can then be used to assign stable

poles:

$$v_j = \sum_{i=k_{j-1}+1}^{k_j} \alpha_{i,j} z_i \tag{6.50}$$

where $\alpha_{i,j}$ chosen such that $s^{k_j} + \alpha_{k_j-1,j} s^{k_j-1} + \ldots + \alpha_{1,j}$ is a Hurwitz polynomial. The resulting feedback system is of the form:

$$\dot{z} = \bar{A} \cdot z + W^T \cdot \Phi + O(z)^{\rho+1} \tag{6.51}$$

where $\bar{A}$ is an asymptotically stable matrix. Let P be the unique solution to the Lyapunov equation $\bar{A}^T P + P\bar{A} = -I$ which is guaranteed to exist. The following theorem provides the regulation counterpart of the tracking result in theorem (6.3.1).

Theorem 6.4.2 (Adaptive Regulation) Consider system (6.43) satisfying assumption (6.3.2) and conditions of lemma (6.4.1) on $U_\epsilon(0)$, an open neighborhood of the origin, with:

$$g(x, \hat{\theta}, u) = -\Omega \cdot W^T \cdot P \cdot z \tag{6.52}$$

Then there exist an open neighborhood $B(\theta_0)$ such that $\forall \hat{\theta}(0) \in B_\sigma(\theta_0)$ and ϵ sufficiently small, $\hat{\theta}$ remains bounded, $x \to 0$ as $t \to \infty$ and the equilibrium $x = 0, \hat{\theta} = \theta_0$ of the resulting closed-loop adaptive system is uniformly stable.

Proof Consider the following Lyapunov candidate function:

$$V(z, \phi) = z^T P z + \Phi^T \Omega^{-1} \Phi$$

where Ω is a constant diagonal gain matrix. The derivative of V along the solution trajectories of (6.51) is:

$$\begin{aligned} \dot{V} &= z^T(\bar{A}^T P + P\bar{A})z + 2z^T P W^T \Phi + 2z^T P \cdot O(z)^{\rho+1} + \Phi^T \Omega^{-1} \Phi \\ &\leq -z^T z + O(z)^{\rho+2} \end{aligned} \tag{6.53}$$

Since $\rho \geq 1$, by definition there exist a constant $l > 0$ such that: $O(z)^{\rho+2} \leq l|z|^{\rho+2} \leq l\epsilon^{\rho}|z|^2 \quad \forall z \in B_\epsilon(0)$. Hence:

$$\dot{V} \leq (-1 + l \cdot \epsilon^{\rho}) \cdot |z|^2 \tag{6.54}$$

which is negative semidefinite for:

$$\forall \epsilon \ s.t. \quad \epsilon^{\rho} < \epsilon_0 \triangleq \frac{1}{l} \tag{6.55}$$

This shows that for ϵ and $|\Phi(0)|$ sufficiently small, $z(t), \phi(t)$ and $x(t)$ remain bounded and $z \in \mathcal{L}_2$. Also, from (6.49), $\dot{z}$ is bounded and by Barbalat's Lemma $z(t) \to 0$ as $t \to \infty$. Hence, since $z = T(x)$ is a diffeomorphism on $B_\epsilon(0) \in U_\epsilon$ with $T(0) = 0$, $x(t) \to 0$ as $t \to \infty$. This proves that the regulation of state $x(t)$ is achieved for all initial conditions: $\Phi(0) \in B_\sigma(\theta_0), x(0) \in T^{-1}(B_\epsilon(0))$. It is also clear that the equilibrium:

$$x = 0 \quad , \quad \hat{\theta} = \theta_0$$

is uniformly stable. This completes the proof $\square$.

Remark 6.4.1 Due to the **approximate** nature of our nonlinear analysis, the feasibility domain of our adaptive scheme is generally **local**. We, however, note that in the proof of the above theorem, the feasibility domain obtained for regulating $x(t)$ depends monotonically on the order ρ of our approximation. This is true since for ϵ **small**, ϵ^{ρ} is a decreasing function of $\rho \geq 1$ and consequently (6.55) holds for a larger range of ϵ.

We now consider the case where $\rho = 2$ for a single input nonlinear system:

$$\dot{x}(t) \ = \ f(x, \theta) + g(x, \theta) \cdot u \tag{6.56}$$

Kang and Krener [58] proved that **any** linearly controllable nonlinear system is feedback linearizable to second degree by a dynamic state feedback. This is in contrast to the results of [27] showing that if a single input nonlinear system is not **exactly** feedback linearizable, then it is not linearizable by a dynamic state feedback. We next remove the linear parameter dependence assumption of theorem (6.4.2) and give a systematic design scheme for **adaptive quadratic regulation** which can be applied to any linearly controllable nonlinear system. The following definition and theorem is due to [58]:

Definition 6.4.1 (Quadratic Linearization) If we can find a dynamic state feedback for system (6.56) such that the resulting extended system is linearly controllable and it can be transformed into:

$$\dot{z} = Fz + Gv + O(z,v)^3 \tag{6.57}$$

by a change of coordinates:

$$\begin{bmatrix} x \\ \omega \end{bmatrix} = z + \psi^{[2]}(z) \tag{6.58}$$

where $\psi^{[2]}(z)$ is a polynomial vector field of order two, then system (6.56) is called quadratically linearizable by a dynamic state feedback.

Theorem 6.4.3 ([58]) Any linearly controllable system (6.56) is quadratically linearizable by an (n-1)-dimensional dynamic state feedback of the form:

$$\dot{\omega} = A\omega + Bv, \quad u = \omega_1 + \gamma^{[1]}(x,\omega) + \gamma^{[2]}(x,\omega) \tag{6.59}$$

where (A, B) is in Brunovsky form.

We now give a systematic design scheme to achieve quadratic regulation under parameter uncertainty:

Step One: Construct the second jet[2] of system (6.56) around the reference point 0. This will give an approximate system up to the second order:

$$\dot{x} = A(\bar{\theta})x + B(\bar{\theta})u + f^{[2]}(x,\bar{\theta}) + g^{[1]}(x,\bar{\theta}) \cdot u + O(x,u)^3 \tag{6.60}$$

where $f^{[2]}(\cdot)$ and $g^{[1]}(\cdot)$ are n-dimensional polynomial vector fields of order two and one in the components of x. Note that in this series expansion, all the unknown parameters θ of system (6.56) now appear, possibly after reparameterization to $\bar{\theta}$, **linearly** in the approximate model (6.60), i.e. in linear and quadratic terms: $Ax, Bu, f^{[2]}(x)$ and $g^{[1]}(x) \cdot u$ of system (6.56).

Step Two: Since (6.56) was assumed linearly controllable, the pair (A, B) is a controllable pair. Perform the following **linear** change of coordinates and **linear** state feedback:

$$\begin{aligned} z &= \mathbf{Est.}(T) \cdot x \ , \ T \triangleq \left[B_c, A_c B_c, \ldots, A_c^{n-1} B_c\right] \cdot \left[B, AB, \ldots, A^{n-1}B\right]^{-1} \\ u &= \gamma^{[1]}(z, \hat{\theta}) + \mu \end{aligned} \tag{6.61}$$

to transform (6.60) into:

$$\dot{z} = Fz + B_c\mu + f^{[2]}(z) + g^{[1]}(z) \cdot (\gamma^{[1]} + \mu) + W_1(x) \cdot \Phi + O(z,\mu)^3 \tag{6.62}$$

where $\phi = \bar{\theta} - \hat{\theta}$, (F, B_c) is in Brunovsky form, and $f^{[2]}(z), g^{[1]}(z)$ are of the form:

$$f_i^{[2]} = \sum_{j=1,k=1}^{n} a_{ijk} \cdot x_j x_k \ , \ g_i^{[1]} = \sum_{j=1}^{n} b_{ij} x_j \tag{6.63}$$

Since, in general, the transformation T depends on some unknown parameters $\bar{\theta}$, we used its estimate with parameter update laws still to be determined. Note that these parameter update laws will be some order 3 smooth functions in the state and have been included in $O(z,\mu)^3$ terms.

[2]Recall that the first kth terms in the Taylor expansion of a vector field is called a k-jet.

Step Three: Apply the following p-dimensional dynamic state feedback to (6.62):

$$\dot{\omega} = A\omega + Bv, \quad \mu = \omega_1 + \gamma^{[2]}(x, \omega, \hat{\theta}) \tag{6.64}$$

where (A, B) is in Brunovsky form with dimension $p \leq n$:

$$p \triangleq max\{j - i \ ; \ a_{ijk} \neq 0 \ , \ n + 1 - i \ ; \ b_{ij} \neq 0\} \tag{6.65}$$

where a_{ijk} and b_{ij} are as in (6.63). The resulting extended system is in the form:

$$\begin{aligned} \begin{bmatrix} \dot{z} \\ \dot{\omega} \end{bmatrix} &= A_{n+p} \cdot \begin{bmatrix} z \\ \omega \end{bmatrix} + B_{n+p} \cdot v + \begin{bmatrix} B_c \gamma^{[2]}(z, \omega) \\ 0 \end{bmatrix} + \\ &+ \begin{bmatrix} \bar{f}^{[2]}(z, \omega) \\ 0 \end{bmatrix} + \begin{bmatrix} \bar{W}_1(x, \omega) \\ 0 \end{bmatrix} \cdot \phi + O(z, \omega, v)^3 \end{aligned} \tag{6.66}$$

where $\bar{f}^{[2]}(z, \omega) = f^{[2]}(z) + g^{[1]}(z) \cdot (\gamma^{[1]}(z) + \omega_1)$.

Step Four: Consider the following change of coordinates:

$$\begin{aligned} \xi_1 &= z_1 \\ \xi_k &\triangleq \textit{Linear and quadratic parts of } \dot{z}_{k-1} \textit{ except terms containing } \phi \\ & \qquad\qquad\qquad\qquad\qquad\qquad 2 \leq k \leq n \end{aligned} \tag{6.67}$$

where we ignore all the terms containing $\dot{\phi}$ and $\dot{\hat{\theta}}$ since they will be of order 3. Also:

$$\xi_k = \omega_{k-n} \qquad n + 1 \leq k \leq n + p \tag{6.68}$$

The resulting system is transformed to:

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 + w_1(x,\omega)\phi \\
\dot{\xi}_2 &= \xi_3 + w_2(x,\omega)\phi + O(\xi,\mu)^3 \\
&\vdots \\
\dot{\xi}_{n-1} &= \xi_n + w_{n-1}(x,\omega)\phi + O(\xi,\mu)^3 \\
\dot{\xi}_n &= \omega_1 + \gamma^{[2]}(\xi) + \psi^{[2]}(x,\omega_1,\ldots,\omega_n) + w_n(x,\omega)\phi + O(\xi,\mu)^3
\end{aligned} \tag{6.69}$$

where $\psi^{[2]}(x,\omega_1,\ldots,\omega_n)$ is a homogeneous polynomial of second degree. Let:

$$\gamma^{[2]}(x) \triangleq [-\psi^{[2]}(x,\omega_1,\ldots,\omega_n)] \tag{6.70}$$

then we have:

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 + w_1(x,\omega)\phi \\
\dot{\xi}_2 &= \xi_3 + w_2(x,\omega)\phi + O(\xi,v)^3 \\
&\vdots \\
\dot{\xi}_{n-1} &= \xi_n + w_{n-1}(x,\omega)\phi + O(\xi,v)^3 \\
\dot{\xi}_n &= \xi_{n+1} + w_n(x,\omega)\phi + O(\xi,v)^3 \\
\dot{\xi}_{n+1} &= \xi_{n+2} \\
&\vdots \\
\dot{\xi}_{n+p} &= v
\end{aligned} \tag{6.71}$$

which can be rewritten in the following compact form:

$$\dot{\xi} = F\xi + Gv + W(x,\omega,u)\cdot\phi + O(\xi,v)^3 \tag{6.72}$$

where (F,G) is in the $n+p$ dimensional Brunovsky form.

Step Five: Choose the control law v and the parameter update law as:

$$\begin{aligned}
v &= -K\cdot\xi \\
\dot{\hat{\theta}} &= -\Omega\cdot W^T(x,\omega,u)\cdot P\cdot\xi
\end{aligned} \tag{6.73}$$

where K is such that $F_c \triangleq (F - G \cdot K) < 0$, P is the solution to $F_c^T P + P F_c = -I$, and Ω is a constant $n \times n$ diagonal gain matrix.

Remark 6.4.2 The procedures used to obtain the transformation $\xi(x)$ are similar to those of [58], where all parameters are assumed known and the system is assumed to be in the **extended quadratic controller normal form**. Here, with an additional integrator, we avoid solving the set of linear equations [58]:

$$\begin{aligned} [Fz + G(\omega_1 + \gamma^{[1]}(z,\omega)) \quad , \quad \phi^{[2]}(z,\omega)\,] \; + \; \tfrac{\partial \phi^{[2]}}{\partial \omega} \, A \, \omega \\ = \; G\gamma^{[2]}(z,\omega) + f^{[2]}(z) + g^{[1]}(z)(\omega_1 + \gamma^{[1]}(z,\omega)) \end{aligned} \tag{6.74}$$

where under parameter uncertainty a solution is not feasible.

The following theorem summarizes our results in this section on adaptive quadratic regulation. The adaptive quadratic model following can also be shown using the observer system (6.20).

Theorem 6.4.4 (adaptive quadratic regulation) For any linearly controllable nonlinear system (6.56) with unknown parameters θ, adaptive quadratic regulation can be achieved following the steps described above.

Proof The resulting closed-loop system can be written in the following compact form:

$$\dot{\xi} = F_c \xi + W(x,\omega,u) \cdot \phi + O(\xi, v)^3 \tag{6.75}$$

and an argument analogous to the one used in the proof of theorem (6.4.2) holds. Hence, for $|x(0)|$ and $|\Phi(0)|$ sufficiently small, $x(t)$ and $\hat{\theta}(t)$ remain bounded and $x(t) \to 0$ as $t \to \infty$. Moreover, the equilibrium:

$$x = 0 \quad , \quad \hat{\theta} = \theta_0$$

is uniformly stable. . □.

In the next section, we will demonstrate the usefulness of the above scheme with the help of a "benchmark" example of state regulation problem where the system is not feedback linearizable. We also illustrate the case where unknown parameters do not appear linearly in the system.

6.5 Simulations

In this section, we illustrate the features of our adaptive design scheme with the help of two "benchmark" examples. First, we consider adaptive regulation of a simple system that is not feedback linearizable and design an adaptive quadratic control law for quadratic linearization and regulation. We shall compare the performance of this controller with a non-adaptive quadratic controller and controller based on Jacobian linearization. In the second example, we discuss adaptive nonlinear regulation of a system with unknown parameters appearing nonlinearly in the dynamics.

For adaptive approximate tracking, in the next chapter, we consider a model of the Harrier aircraft studied in [40] under some parameter uncertainty in mass and moment of inertia and compare the performance of our adaptive tracking controller to the performance of the non-adaptive tracking controller.

6.5.1 Example 1

Consider a "benchmark" example of adaptive nonlinear regulation:

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta x_3^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u + x_1 x_3 \end{aligned} \tag{6.76}$$

which violates the conditions of [78, 88, 98, 57]. In fact, this system is not feedback linearizable. It is, however, linearly controllable and as shown in [58], it is feedback linearizable to second degree by a dynamic state feedback. The linear part of (6.76) is already in Brunovsky form. Applying steps 3 and 4 in section IV gives for the extended system (with a 2-dimensional dynamic state feedback):

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta x_3^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u + x_1 x_3 = x_4 + \gamma^{[2]}(x) + x_1 x_3 \\ \dot{x}_4 &= x_5 \\ \dot{x}_5 &= v \end{aligned} \tag{6.77}$$

Then:

$$\begin{aligned} \xi_1 &= x_1 \\ \xi_2 &= \textit{Lin. \& Quad. part of } \hat{\dot{\xi}}_1 = x_2 + \hat{\theta} x_3^2 \\ \xi_3 &= \textit{Lin. \& Quad. part of } \hat{\dot{\xi}}_2 = x_3 + 2\hat{\theta} x_3 x_4 \\ \xi_4 &= x_4 \\ \xi_5 &= x_5 \end{aligned} \tag{6.78}$$

$$\dot{\xi}_3 = x_4 + x_3 x_1 + \gamma^{[2]}(x) + 2\hat{\theta} x_4^2 + 2\hat{\theta} x_3 x_5 \tag{6.79}$$

Hence, $\gamma^{[2]}(x)$ may be chosen as:

$$\gamma^{[2]}(x) = -(x_3 x_1 + +2\hat{\theta} x_4^2 + 2\hat{\theta} x_3 x_5) \tag{6.80}$$

Therefore, the resulting system is transformed into:

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 + x_3^2 \cdot \phi \\
\dot{\xi}_2 &= \xi_3 + O(\xi, v)^3 \\
\dot{\xi}_3 &= \xi_4 + O(\xi, v)^3 \\
\dot{\xi}_4 &= \xi_5 \\
\dot{\xi}_5 &= v
\end{aligned} \tag{6.81}$$

which is in the form given by (6.72). Finally, let:

$$\begin{aligned}
v &= -K \cdot \xi \\
\dot{\hat{\theta}} &= -g \cdot W^T \cdot P \cdot \xi
\end{aligned} \tag{6.82}$$

where $W^T = [x_3^2, 0, 0, 0, 0]$, g is an adaptation gain, and K is the gain vector that specifies the closed-loop poles. We note that the poles should be chosen not too far left in the left half plane since this will increase the initial magnitude of the control v due to nonzero initial states. This could in turn introduce large perturbations in the system and the approximate model will no longer stays a valid approximation of the true system.

Figure (6.2) shows the response of the adaptive quadratic regulator (6.82) for the initial state $x(0) = [3, 0, 0]^T$, $\theta = 2$, and %25 uncertainty in the initial estimate of the parameter θ. The Jacobian approximation, in comparison, resulted in an unstable system even when there was no parameter uncertainty present in the system. The non-adaptive quadratic approximation resulted in an unstable system in the presense of this uncertainty. The simulations illustrate the usefulness of the adaptive quadratic control scheme proposed in section IV in providing a parameter robust control for regulation of nonlinear systems that violate the regularity conditions needed to apply adaptive schemes based on exact feedback linearization.

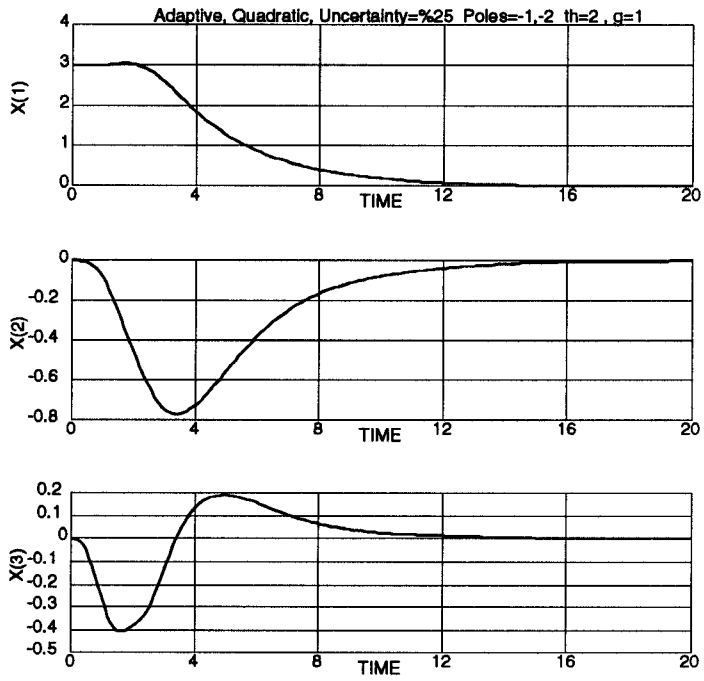

Figure 6.2: Example 1; state trajectories in response to the adaptive quadratic controller under %25 parameter uncertainty.

6.5.2 Example 2

Consider the following nonlinear system with unknown parameters θ_1 and θ_2:

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta_1 x_3^2 + e^{\theta_2 x_3} - 1 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u + x_1 x_3 \end{aligned} \tag{6.83}$$

which is not feedback linearizable but it is linearly controllable and hence, can be quadratically feedback linearized with dynamic state feedback. We now apply the design scheme of section (4). The approximate quadratic system of equation (6.60) is:

$$\dot{x} = \begin{bmatrix} 0 & 1 & \theta_2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} (\theta_1 + \frac{1}{2}\theta_2^2) x_3^2 \\ 0 \\ x_1 x_3 \end{bmatrix} + O(x)^3 \tag{6.84}$$

which has all eigenvalues at zero. The transformation T of (6.61) is ($z = est.\ (T) \cdot x$):

$$T = \begin{bmatrix} 1 & -\theta_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad z \triangleq \begin{bmatrix} 1 & -\hat{\theta}_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \tag{6.85}$$

which results in the following Brunovsky form quadratic system with $\bar{\theta}_1 \triangleq \theta_1 + 0.5\theta_2^2, \bar{\theta}_2 \triangleq \theta_2$:

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} \bar{\theta}_1 x_3^2 \\ 0 \\ x_1 x_3 \end{bmatrix} + \begin{bmatrix} x_3 \\ 0 \\ 0 \end{bmatrix} \cdot \phi_2 + O(x)^3 \tag{6.86}$$

Applying steps two and three, similar to the last example, suggests the following linearizing transformation and control law:

$$\begin{aligned}
\xi_1 &= z_1 = x_1 - \hat{\theta}_2 x_2 \\
\xi_2 &= z_2 + \hat{\theta}_1 x_3^2 \\
\xi_3 &= z_3 + 2\hat{\theta}_1 x_3 \omega_1 \\
\xi_4 &= \omega_1 \\
\xi_5 &= \omega_2 \\
u &= \omega_1 + \gamma^{[2]}(x, \omega) \\
\gamma^{[2]}(x, \omega) &= -2\hat{\theta}_1(\omega_1^2 + x_3\omega_2) - x_3 x_1
\end{aligned} \tag{6.87}$$

which transforms (6.83) into:

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 + (x_3^2, x_3) \cdot \phi + O(\xi, v)^3 \\
\dot{\xi}_2 &= \xi_3 + O(\xi, v)^3 \\
\dot{\xi}_3 &= \xi_4 + O(\xi, v)^3 \\
\dot{\xi}_4 &= \xi_5 \\
\dot{\xi}_5 &= v
\end{aligned} \tag{6.88}$$

with $\phi = (\bar{\theta}_1 - \hat{\theta}_1, \bar{\theta}_2 - \hat{\theta}_2)^T$. This system is in the form given in (6.72). Control law v and update law given in (6.82) can then be used here with slight modification in $W(x)$.

Figure (6.3) shows the response to the non-adaptive controller with $\theta_1 = 2, \theta_2 = 1$ for the initial state $x(0) = [1, 0.3, 0.3]^T$. In comparison, figure (6.4) shows the performance of the adaptive controller which provided a better result for a wider range of uncertainties in θ_1 and θ_2. The Jacobian linearization, however, resulted in an unstable response for this initial state even when there is no parameter uncertainty present.

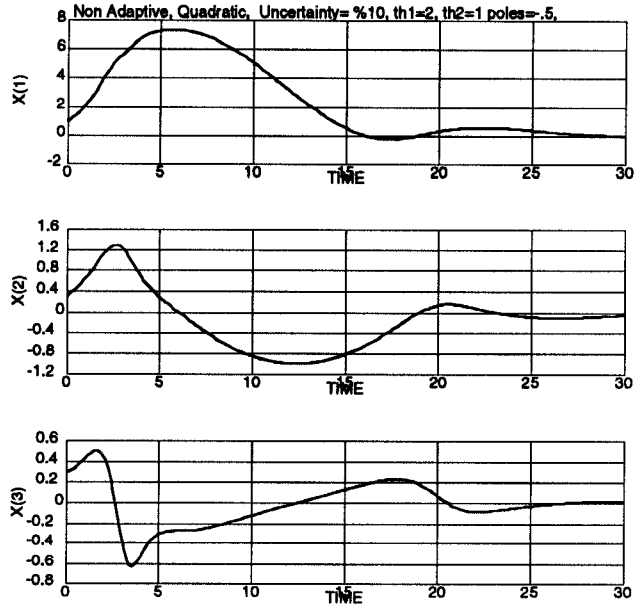

Figure 6.3: Example 3; state trajectories in response to the non-adaptive quadratic controller under %10 parameter uncertainty.

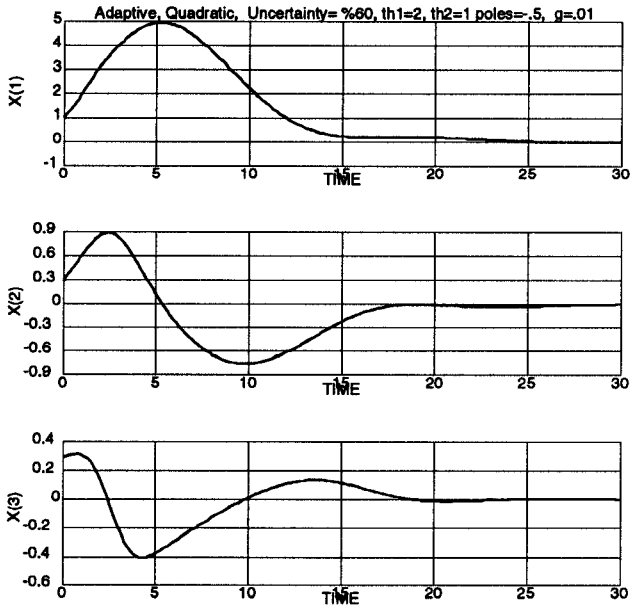

Figure 6.4: Example 3; state trajectories in response to the adaptive quadratic controller under %60 parameter uncertainty.

6.6 Conclusion

In this chapter we have presented an approach for the adaptive **approximate** feedback linearization of nonlinear systems under parameter uncertainty. The significance of this approach will be demonstrated in chapter 7 with its potential application to flight control systems where exact linearization approach fails and the non-adaptive controller produces undesirable performance. Compared to adaptive schemes that are based on exact state or input-output linearization, this approach avoids several restrictions such as involutivity, existence of a relative degree, and minimum phase property which are not often met in most complex engineering systems. In section 4, a systematic procedure for adaptive quadratic regulation of any linearly controlable nonlinear system was presented. The quadratic approximate model and the resulting parameter update laws are computed directly in terms of the Taylor series expansion and a simple change of coordinates, and can easily be generated by symbolic programming tools. It is also important to note that for the regulation the uncertain parameters in the system are not required to appear **linearly** in the system dynamics since they always appear linearly in the **approximate model** possibly after a reparametrization. The feasibility domain of this scheme is local, in general, around a nominal operating point. This limitation is intrinsic of the local nature of approximate feedback linearization technique. The broad applicability of this scheme coupled with its systematic approach motivate its use by control engineers in parameter robust control design for nonlinear systems.

Part III

Applications

Chapter 7

Applications to Flight Control Systems

7.1 Introduction

In this chapter, we illustrate the features of the adaptive design scheme developed in chapter 6. We consider a simplified model of the Harrier aircraft studied in [40] and compare the performance of our adaptive tracking controller to the performance of the non-adaptive tracking controller under parametric uncertainties in the mass and the moment of inertia of the aircraft. We will show that this system is very sensitive to the parameter uncertainties in its dynamics, and when these parameters are not exactly known for feedback control, the performance is very poor and unacceptable in practice. Motivated by this fact, we will construct an adaptive controller that is robust against this form of uncertainties and adapts to the parameter variations in the system. Using the results of the previous chapter, we shall drive the necessary parameter update laws that guarantee an acceptable tracking performance and closed loop stability. We shall see that due to the non-minimum phase nature of the aircraft dynamics, none of the existing adaptive nonlinear schemes, in particular [78, 88, 97, 57, 98, 12, 56], is

applicable to this situation. Therefore, the results of this chapter illustrates the significance of the adaptive nonlinear scheme developed in the last chapter.

This chapter is organized as follows: In section 7.2, we briefly review the dynamics of an aircraft and consider a simplified planar vertical takeoff and landing aircraft (PVTOL) that forms the basis of our analysis. Section 7.3 reviews the aircraft control design based on exact and approximate linearization technique. In section 7.4, we design an adaptive approximate tracking controller and parameter update laws for the PVTOL aircraft system. Section 7.5 presents the simulation results and discusses the features and the performance of the adaptive controller compare to the non-adaptive controller.

7.2 Aircraft Dynamics

The dynamics of an aircraft system can be viewed as a rigid body under the influence of a set of forces and torques. The equations of the motion for such a system can be written as:

$$\begin{aligned} m\ddot{r} &= R\, f_a + mg \\ J\dot{\omega}_a &= \tau_a + S(\omega_a) J \omega_a \\ \dot{R} &= -R\, S(\omega_a) \end{aligned} \tag{7.1}$$

where r is the position, R is the rotation matrix describing the angular position, $S(\omega)$ is a 3×3-matrix defined by:

$$S(\omega_a) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

ω_a is the angular velocity of the aircraft with respect to the axes in the aircraft, and f_a and τ are the force and torque acting on the aircraft expressed in the air-

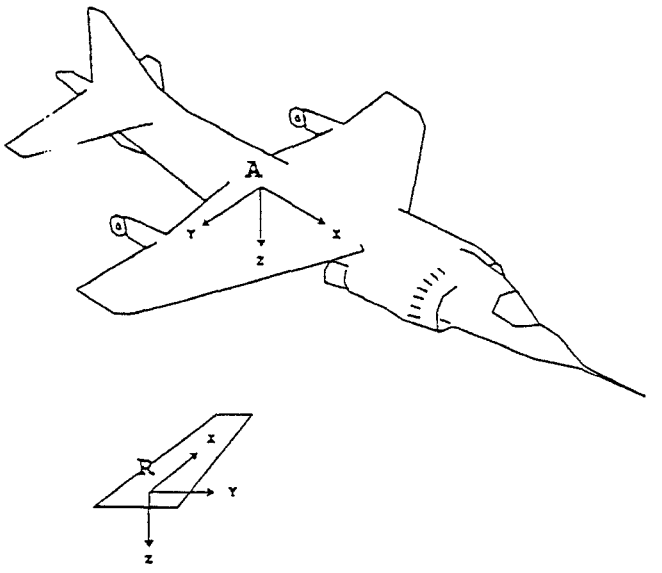

Figure 7.1: Aircraft Coordinate System

craft reference frame. The dynamic equations describing a six degree-of-freedom aircraft are typically complicated and depend on how the aircraft is controlled. The model considered here is the YAV-8B Harrier V/STOL (vertical/short take-off and landing) aircraft produced by McDonnell Aircraft Company [108, 80, 40] shown in figure (7.1). This aircraft uses a reaction control system (RSC) that provides three degree of freedom moment generation during jet-born. RSC jets typically create a force that is not perpendicular to the horizontal axis which results in a non-minimum phase system. We now consider a three degree-of-freedom simplified version of the Harrier aircraft, namely the planar VTOL used in [40], which is the restriction of the Harrier aircraft to operation in a vertical plane. The equations of motion for the prototype PVTOL (planar vertical takeoff and landing) aircraft, depicted in figure (7.2) are given by:

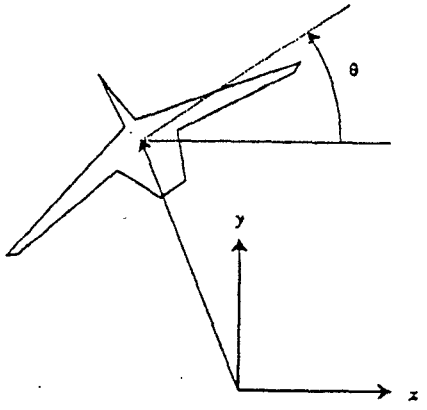

Figure 7.2: The Planer Vertical Takeoff and Landing system

$$
\begin{aligned}
m\ddot{x} &= -\sin\theta \cdot u_1 + \epsilon \cos\theta \cdot u_2 \\
m\ddot{y} &= \cos\theta \cdot u_1 + \epsilon \sin\theta \cdot u_2 + mg \\
J\ddot{\theta} &= u_2 \\
y_1 &= x \ , \ y_2 = y
\end{aligned}
\tag{7.2}
$$

where outputs x and y give the position of the aircraft center of mass, θ is the angle of the aircraft relative to the x-axis, m and J are the mass and the moment of inertia, u_1 and u_2 are the thrust and the rolling moment, g is the gravitational acceleration normalized to -1, and ϵ is a small coupling coefficient between the rolling moment u_2 and the lateral acceleration of the aircraft $\ddot{x}$ and $\ddot{y}$. The objective is to find a feedback law that decouples outputs x and y under some parameter uncertainty in m and J. We require x to track a smooth trajectory while y remains at zero.

7.3 Aircraft Control

7.3.1 Exact Input-Output Linearization by Feedback

The decoupling matrix for system (7.2) is:

$$\frac{1}{m}\begin{bmatrix} -\sin\theta & \epsilon\cos\theta \\ \cos\theta & \epsilon\sin\theta \end{bmatrix} \tag{7.3}$$

which is nonsingular with a small determinant $-\epsilon/m$. This will result in a relatively **large** decoupling control law:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = m\begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta/\epsilon & \sin\theta/\epsilon \end{bmatrix}\cdot\begin{bmatrix} v_1 \\ v_2+1 \end{bmatrix} \tag{7.4}$$

which input-output linearizes system (7.2):

$$\begin{aligned} \ddot{x} &= v_1 \\ \ddot{y} &= v_2 \\ \ddot{\theta} &= \tfrac{1}{\epsilon m J}(\sin\theta + \cos\theta\cdot v_1 + \sin\theta\cdot v_2) \end{aligned} \tag{7.5}$$

where v_1 and v_2 are the new inputs and can be designed such that x and y track any desired smooth trajectory. Unfortunately, since ϵ is small (typically around 0.01), the exact decoupling control (7.4) will require a **large** control effort and hence is not feasible in practice. Moreover, this control law produces **unstable** internal dynamics. From the last equation in (7.5), this causes the aircraft will either roll continuously or rock from side to side. In fact, the resulting input-output linearized system (7.5) is non-minimum phase since: (setting $x^{(i)}, y^{(i)}, u_1, u_2 = 0$ in (7.5))

$$\ddot{\theta} = \frac{1}{\epsilon m J}\sin\theta \tag{7.6}$$

is not asymptotically stable. Hence, the exact input-output linearization design approach fails. As a result, we can not meet our goal for **exact** output tracking. Next, we will try an approach that relies on some simplifications to the original model of the aircraft by ignoring some of the nonlinearities. It is important to note that these nonlinearities are not ignored completely at every step of the design but only at some key steps that assures the closed loop stability of the **true** system and **approximate** asymptotic tracking for the outputs.

7.3.2 Approximate Input-Output Linearization

Hauser [40] proposed a tracking controller design scheme for this system by first decoupling the y-output from ϵ and then ignoring the ϵ-dependent terms in the resulting system. The resulting approximate system overlooks the small coupling ϵ between rolling moments and lateral acceleration.

In order for the output y to be independent of the neglected nonlinear terms used in our approximation, we first remap the controls u_1 and u_2. This is mainly done to provide **exact** tracking of the altitude (y), important since PVTOL aircraft are designed to be maneuvered close to the ground. We then attempt to design a controller for **approximate** tracking in x-direction.

Let $\tilde{u}_1$ and $\tilde{u}_2$ be new controls such that:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{pmatrix} 1 & -\epsilon \tan\theta \\ 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} \tag{7.7}$$

then (7.2) can be rewritten as:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{1}{m} \cdot \begin{bmatrix} -\sin\theta & \epsilon/\cos\theta \\ \cos\theta & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} \tag{7.8}$$

$$\ddot{\theta} = \frac{1}{J} \cdot \tilde{u}_2$$

The approximate system is then given by (set: $\epsilon = 0$):

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{1}{m} \cdot \begin{bmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} \tag{7.9}$$

$$\ddot{\theta} = \frac{1}{J} \cdot \tilde{u}_2$$

which has a singular decoupling matrix. Therefore, we first need to apply the dynamic extension algorithm where we consider $\tilde{u}_1$ as a new state. The resulting extended approximate system, depicted in figure (7.3), is:

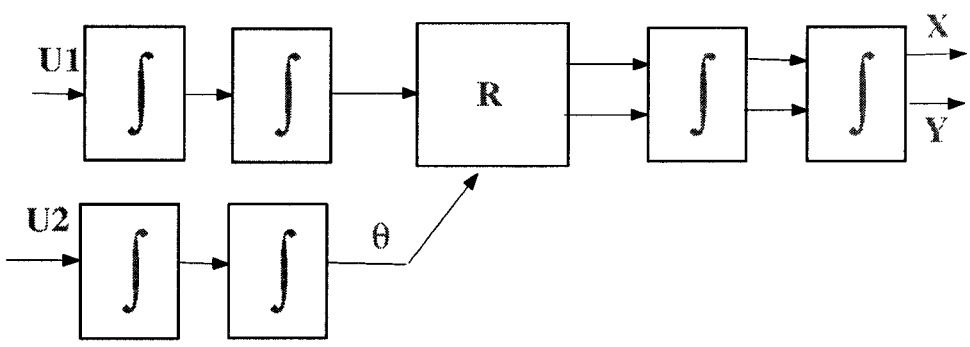

Figure 7.3: Block diagram of the PVTOL control system

$$
\begin{aligned}
\dot{\eta} &= \begin{bmatrix} \eta_2 \\ -\frac{1}{m}\eta_7 \sin \eta_5 \\ \eta_4 \\ -1 + \frac{1}{m}\eta_7 \cos \eta_5 \\ \eta_6 \\ 0 \\ \eta_8 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{J} \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} w \\ \tilde{u}_2 \end{bmatrix} \\
y &= \begin{bmatrix} \eta_1 \\ \eta_3 \end{bmatrix}
\end{aligned}
\tag{7.10}
$$

where $\eta \triangleq (x, \dot{x}, y, \dot{y}, \theta, \dot{\theta}, \tilde{u}_1, \dot{\tilde{u}}_1)^T$. This approximate system has a vector relative degree $(r_1, r_2) = (4, 4)$ and since $r_1 + r_2 = n = 8$, the approximate system (7.10) is a minimum phase system. Using this approximate model, we construct a tracking controller that is based on the exact input-output linearization approach. It can be shown that when this controller is applied to the **true** system (7.2), the tracking objective is achieved **approximately** [40]. Figure (7.4) shows the performance of this controller when applied to the true system. It is clear that when parameters m and J are exactly known and ϵ is small, close to what it normally is during aircraft operation, the closed loop system is stable and the performance penalty is very small.

One of the major restrictions to the direct application of this design procedure is the fact that it relies on the **exact** knowledge of all the system parameters in the aircraft dynamics. This assumption is not realistic in practice and, unfortunately, the performance of the resulting controller is often very sensitive

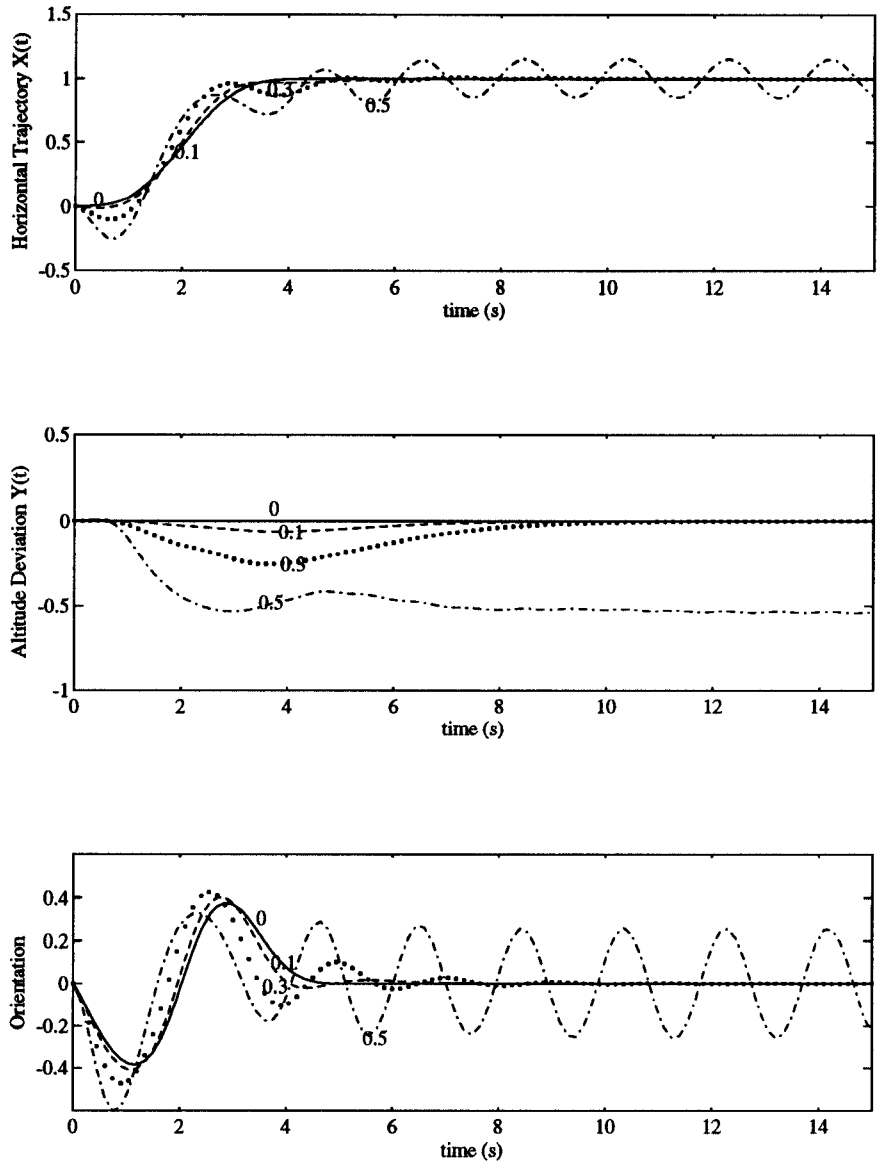

Figure 7.4: Response of the true PVTOL aircraft system under no parameter uncertainty to the approximate tracking control with ϵ ranging from 0 to 0.5.

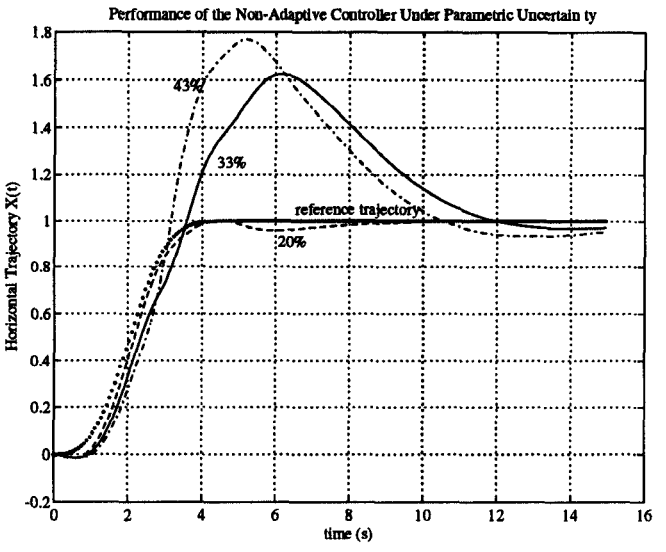

Figure 7.5: Response (x-direction) of the true PVTOL aircraft system under 20% to 43% parameter uncertainty in m and J, $\epsilon = 0.1$.

to parameter variations in the system, resulting in a poor performance when uncertainties are present. As shown in figures (7.5) and (7.6), under parameter uncertainty in mass m, which is common in aircraft due to the fuel consumption and load variation, the tracking controller of [40] results in an unacceptable behavior. This is particularly apparent in the y-output (i.e. aircraft altitude) shown in (7.6) where the altitude deviation is unacceptable for the vertical takeoff and landing maneuvers for which the PVTOL aircraft was designed. Therefore, it is of practical importance to design a controller that is robust against this form of uncertainty and can adapt to the parameter variations in the system dynamics. This approach is addressed in the next section.

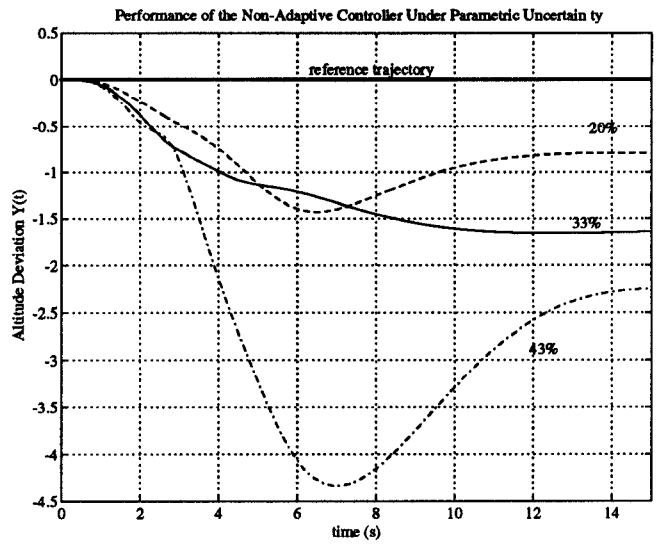

Figure 7.6: Response (y-direction) of the true PVTOL aircraft system under 20% to 43% parameter uncertainty in m and J, $\epsilon = 0.1$.

7.4 Adaptive Control Design for the PVTOL Aircraft

In this section, following the design procedures in section 6.3, we construct an adaptive approximate tracking controller for system (7.2) under parameter uncertainty in m and J.

Remark 7.4.1 The neglected nonlinearities in (7.9) are not $O(x,u)^2$ which is required for approximate linearization analysis. However, since ϵ is a small parameter, one may still apply the approximate linearization technique to achieve approximate tracking. In this case, the loss of performance is less for smaller values of ϵ, and as shown in [40], this approximation gives desirable results for ϵ up to around 0.3. The same argument holds for our adaptive tracking scheme.

With $\tilde{u}_1$ and $\tilde{u}_2$ defined as in (7.7), system (7.2) can be rewritten as:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \theta_1 \cdot \begin{bmatrix} -\sin\theta & \epsilon/\cos\theta \\ \cos\theta & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} \tag{7.11}$$

$$\ddot{\theta} = \theta_2 \cdot \tilde{u}_2$$

where for simplicity we have redefined the unknown parameters in (7.2) with the help of the following notation:

$$\theta_1 \triangleq 1/m \ , \ \theta_2 \triangleq 1/J \tag{7.12}$$

The **approximate** system is then given by:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \theta_1 \cdot \begin{bmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} \tag{7.13}$$

$$\ddot{\theta} = \theta_2 \cdot \tilde{u}_2$$

Application of the dynamic extension algorithm as in (7.10) results in the extended approximate system:

$$\dot{\eta} = \begin{bmatrix} \eta_2 \\ -\theta_1\eta_7\sin\eta_5 \\ \eta_4 \\ -1+\theta_1\eta_7\cos\eta_5 \\ \eta_6 \\ 0 \\ \eta_8 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \theta_2 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} w \\ \tilde{u}_2 \end{bmatrix} \tag{7.14}$$

$$y = \begin{bmatrix} \eta_1 \\ \eta_3 \end{bmatrix}$$

where $\eta \triangleq (x, \dot{x}, y, \dot{y}, \theta, \dot{\theta}, \tilde{u}_1, \dot{\tilde{u}}_1)^T$. The extended true system is given by:

$$\dot{\eta} = \begin{bmatrix} \eta_2 \\ -\theta_1\eta_7\sin\eta_5 \\ \eta_4 \\ -1+\theta_1\eta_7\cos\eta_5 \\ \eta_6 \\ 0 \\ \eta_8 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \epsilon\theta_1/\cos\eta_5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \theta_2 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} w \\ \tilde{u}_2 \end{bmatrix} \tag{7.15}$$

$$y = \begin{bmatrix} \eta_1 \\ \eta_3 \end{bmatrix}$$

Now, consider the following local diffeomorphism of η:

$$
\begin{aligned}
\xi_1 &= h_1(x) = \eta_1 \\
\xi_2 &= \mathcal{L}_f h_1(x) = \eta_2 \\
\xi_3 &= \mathcal{L}_f^2 h_1(x) = -\theta_1 \sin \eta_5 \eta_7 \\
\xi_4 &= \mathcal{L}_f^3 h_1(x) = -\theta_1 \eta_6 \eta_7 \cos \eta_5 - \theta_1 \eta_8 \sin \eta_5 \\
\xi_5 &= h_2(x) = \eta_3 \\
\xi_6 &= \mathcal{L}_f h_2(x) = \eta_4 \\
\xi_7 &= \mathcal{L}_f^2 h_2(x) = -1 + \theta_1 \eta_7 \cos \eta_5 \\
\xi_8 &= \mathcal{L}_f^3 h_2(x) = -\theta_1 \eta_6 \eta_7 \sin \eta_5 + \theta_1 \eta_8 \cos \eta_5
\end{aligned}
\tag{7.16}
$$

which transforms the approximate system (7.10) into the following input-output linearized system:

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 + \epsilon \theta_1 \tilde{u}_2 / \cos \eta_5 \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= \theta_1 \sin \eta_5 \eta_7 {\eta_6}^2 - 2\theta_1 \cos \eta_5 \eta_6 \eta_8 - \theta_1 \sin \eta_5 \cdot w - \theta_1 \theta_2 \cos \eta_5 \eta_7 \cdot \tilde{u}_2 \\
\dot{\xi}_5 &= \xi_6 \\
\dot{\xi}_6 &= \xi_7 \\
\dot{\xi}_7 &= \xi_8 \\
\dot{\xi}_8 &= \theta_1 \cos \eta_5 \eta_7 {\eta_6}^2 - 2\theta_1 \sin \eta_5 \eta_6 \eta_8 - \theta_1 \cos \eta_5 \cdot w - \theta_1 \theta_2 \sin \eta_5 \eta_7 \cdot \tilde{u}_2
\end{aligned}
\tag{7.17}
$$

Since transformation (7.16) depends on unknown parameters θ_i, we consider the following local diffeomorphism which is an estimate of the transformation in

(7.16):

$$
\begin{aligned}
\hat{\xi}_1 &= \eta_1 \\
\hat{\xi}_2 &= \eta_2 \\
\hat{\xi}_3 &= -\hat{\theta}_1 \sin \eta_5 \eta_7 \\
\hat{\xi}_4 &= -\hat{\theta}_1 \eta_6 \eta_7 \cos \eta_5 - \hat{\theta}_1 \eta_8 \sin \eta_5 \\
\hat{\xi}_5 &= \eta_3 \\
\hat{\xi}_6 &= \eta_4 \\
\hat{\xi}_7 &= -1 + \hat{\theta}_1 \eta_7 \cos \eta_5 \\
\hat{\xi}_8 &= -\hat{\theta}_1 \eta_6 \eta_7 \sin \eta_5 + \hat{\theta}_1 \eta_8 \cos \eta_5
\end{aligned}
\tag{7.18}
$$

where the update laws for $\hat{\theta}_1$ and $\hat{\theta}_2$ will be determined later. The equations describing the dynamics of $\hat{\xi}$ are given by:

$$
\dot{\hat{\xi}} = \begin{bmatrix} \eta_2 \\ -\theta_1 \sin \eta_5 \eta_7 \\ -\hat{\theta}_1 \eta_6 \eta_7 \cos \eta_5 - \hat{\theta}_1 \eta_8 \sin \eta_5 \\ \hat{\theta}_1 \sin \eta_5 \eta_7 \eta_6{}^2 - 2\hat{\theta}_1 \cos \eta_5 \eta_6 \eta_8 \\ \eta_4 \\ -1 + \theta_1 \eta_7 \cos \eta_5 \\ -\hat{\theta}_1 \eta_6 \eta_7 \sin \eta_5 + \hat{\theta}_1 \eta_8 \cos \eta_5 \\ -\hat{\theta}_1 \cos \eta_5 \eta_7 \eta_6{}^2 - 2\hat{\theta}_1 \sin \eta_5 \eta_6 \eta_8 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\hat{\theta}_1 \sin \eta_5 & -\hat{\theta}_1 \theta_2 \eta_7 \cos \eta_5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hat{\theta}_1 \cos \eta_5 & -\hat{\theta}_1 \theta_2 \eta_7 \sin \eta_5 \end{bmatrix} \cdot \begin{bmatrix} w \\ \tilde{u}_2 \end{bmatrix}
$$

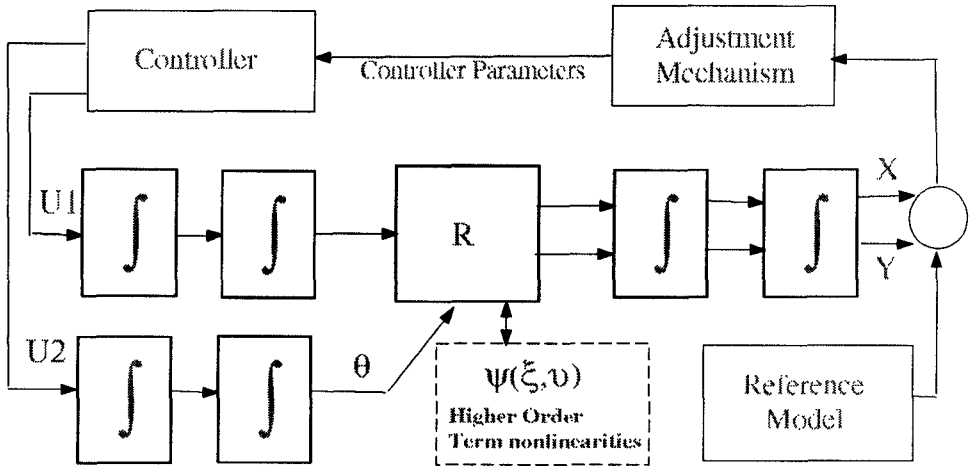

Figure 7.7: Block diagram of the PVTOL aircraft adaptive control system

$$+\begin{bmatrix} 0 \\ \epsilon\theta_1\tilde{u}_2/\cos\eta_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\eta_7\sin\eta_5 \\ -\eta_6\eta_7\cos\eta_5 - \eta_8\sin\eta_5 \\ 0 \\ 0 \\ \eta_7\cos\eta_5 \\ -\eta_6\eta_7\sin\eta_5 + \eta_8\cos\eta_5 \end{bmatrix} \cdot \dot{\hat{\theta}}_1 \tag{7.19}$$

The adaptive control structure for the PVTOL aircraft system is illustrated in figure(7.7). The approximate linearizing control law is given by:

$$\begin{bmatrix} w \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} -\hat{\theta}_1\sin\eta_5 & -\hat{\theta}_1\hat{\theta}_2\eta_7\cos\eta_5 \\ \hat{\theta}_1\cos\eta_5 & -\hat{\theta}_1\hat{\theta}_2\eta_7\sin\eta_5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \upsilon_x - \hat{\theta}_1\eta_7{\eta_6}^2\sin\eta_5 + 2\hat{\theta}_1\eta_6\eta_8\cos\eta_5 \\ \upsilon_y + \hat{\theta}_1\eta_7{\eta_6}^2\cos\eta_5 + 2\hat{\theta}_1\eta_6\eta_8\sin\eta_5 \end{bmatrix} \tag{7.20}$$

with:

$$\upsilon_x = x_m^{(4)} + \alpha_3(x_m^{(3)} - \hat{\xi}_4) + \ldots + \alpha_0(x_m - \hat{\xi}_1)$$

$$\upsilon_y = -\alpha_3\hat{\xi}_8 - \ldots - \alpha_0\hat{\xi}_5$$

and $\alpha_3 = 6, \alpha_2 = 13, \alpha_1 = 12, \alpha_0 = 4$ so that the resulting poles are at -1 and -2. The parameter update law is:

$$\begin{bmatrix} \dot{\hat{\theta}}_1 \\ \dot{\hat{\theta}}_2 \end{bmatrix} = - \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \cdot W_2^T \cdot P \cdot (\bar{\xi} - \hat{\xi}) \tag{7.21}$$

where g_1 and g_2 are adaptation gains, P is the solution to $\bar{A}^T P + P\bar{A} = -I$, $\bar{A}$ as in (5.23) with α_i given above, and W_2 and $\bar{\xi}$ are given by:

$$W_2^T = \begin{bmatrix} 0 & -\eta_7 \sin\eta_5 + \epsilon\tilde{u}_2/\cos\eta_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hat{\theta}_1\eta_7\tilde{u}_2\cos\eta_5 & 0 \\ & & \eta_7\cos\eta_5 & 0 & 0 \\ & & 0 & 0 & \hat{\theta}_1\eta_7\tilde{u}_2\sin\eta_5 \end{bmatrix} \tag{7.22}$$

and:

$$\dot{\bar{\xi}} = \begin{bmatrix} \eta_2 \\ -\hat{\theta}_1\eta_7\sin\eta_5 \\ -\hat{\theta}_1\eta_6\eta_7\cos\eta_5 - \hat{\theta}_1\eta_8\sin\eta_5 \\ \hat{\theta}_1\eta_7\eta_6{}^2\sin\eta_5 - 2\hat{\theta}_1\eta_6\eta_8\cos\eta_5 \\ \eta_4 \\ -1 + \hat{\theta}_1\eta_7\cos\eta_5 \\ -\hat{\theta}_1\eta_6\eta_7\sin\eta_5 + \hat{\theta}_1\eta_8\cos\eta_5 \\ -\hat{\theta}_1\eta_7\eta_6{}^2\cos\eta_5 - 2\hat{\theta}_1\eta_6\eta_8\sin\eta_5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\hat{\theta}_1\sin\eta_5 & -\hat{\theta}_1\hat{\theta}_2\eta_7\cos\eta_5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hat{\theta}_1\cos\eta_5 & -\hat{\theta}_1\hat{\theta}_2\eta_7\sin\eta_5 \end{bmatrix} \cdot \begin{bmatrix} w \\ \tilde{u}_2 \end{bmatrix}$$

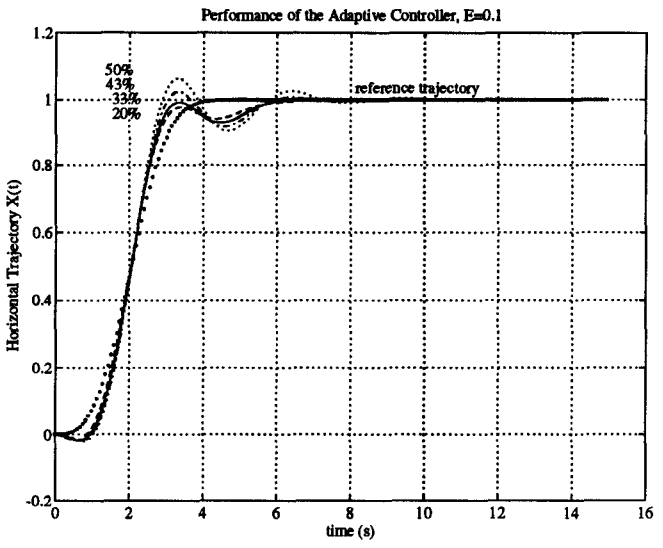

Figure 7.8: Response (x-direction) of the true PVTOL aircraft system under 20% to 50% parameter uncertainty in m and J, $\epsilon = 0.1$.

$$+ \begin{bmatrix} 0 \\ 0 \\ -\eta_7 \sin \eta_5 \\ -\eta_6\eta_7 \cos \eta_5 - \eta_8 \sin \eta_5 \\ 0 \\ 0 \\ \eta_7 \cos \eta_5 \\ -\eta_6\eta_7 \sin \eta_5 + \eta_8 \cos \eta_5 \end{bmatrix} \cdot \dot{\hat{\theta}}_1 + \begin{bmatrix} 0 \\ \epsilon\hat{\theta}_1 \tilde{u}_2 / \cos \eta_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \bar{A}(\bar{\xi} - \hat{\xi}) \tag{7.23}$$

with $\bar{\xi}(0) = 0$.

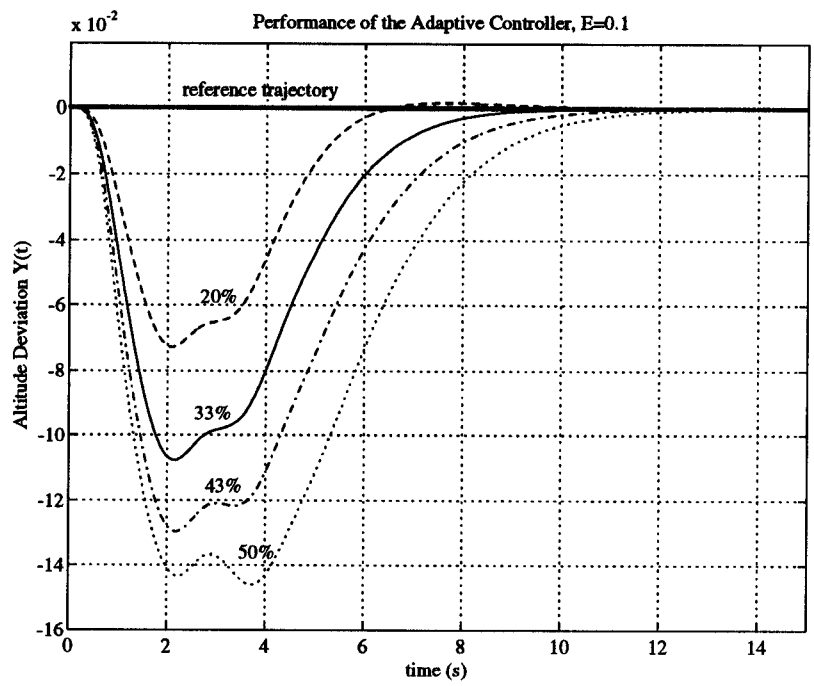

Figure 7.9: Response (y-direction) of the true PVTOL aircraft system under 20% to 50% parameter uncertainty in m and J, $\epsilon = 0.1$.

7.5 Simulation Results

For simulation, the initial conditions were chosen such that the system (7.14) is initially at rest, i.e. $\eta(0) = (0,0,0,0,0,0,1,0)$. Figures (7.5) and (7.6) show the performance of the (non-adaptive) nonlinear controller of [40] for $\epsilon = 0.1$ with uncertainties in M and J. While the tracking objective in the x-direction is achieved for small ϵ, the altitude (y) deviation is unacceptable for vertical takeoff and landing maneuvers. On the other hand, the decoupling is much improved with the adaptive controller as shown in figures (7.8) and (7.9). The altitude deviation is about %90 better and the convergence is much faster. The orientation of the PVTOL aircraft, for small ϵ, is the same in both adaptive and non-adaptive cases. Figures (7.10) through (7.13) compare the performance of the adaptive controller and the nonadaptive controller, the control amplitude, and parameter estimates for different values of parameter uncertainties in the system (20% and 50%) and $\epsilon = 0.1$. The case for $\epsilon = 0.3$ with 33% uncertainty is shown in figures (7.14) and (7.15). The simulations clearly demonstrate the advantage of the adaptive controller proposed in section III.

7.6 Conclusion

In this chapter we applied the adaptive scheme we developed in chapter 6 to an example of a system that is slightly non-minimum phase, namely, to the flight control of vertical takeoff and landing aircraft. We argued that since the system is non-minimum phase, none of the previously known techniques of adaptive control is applicable. This illustrates the significance of the approximation technique used to develop the theoretical frame work of our new adaptive control design

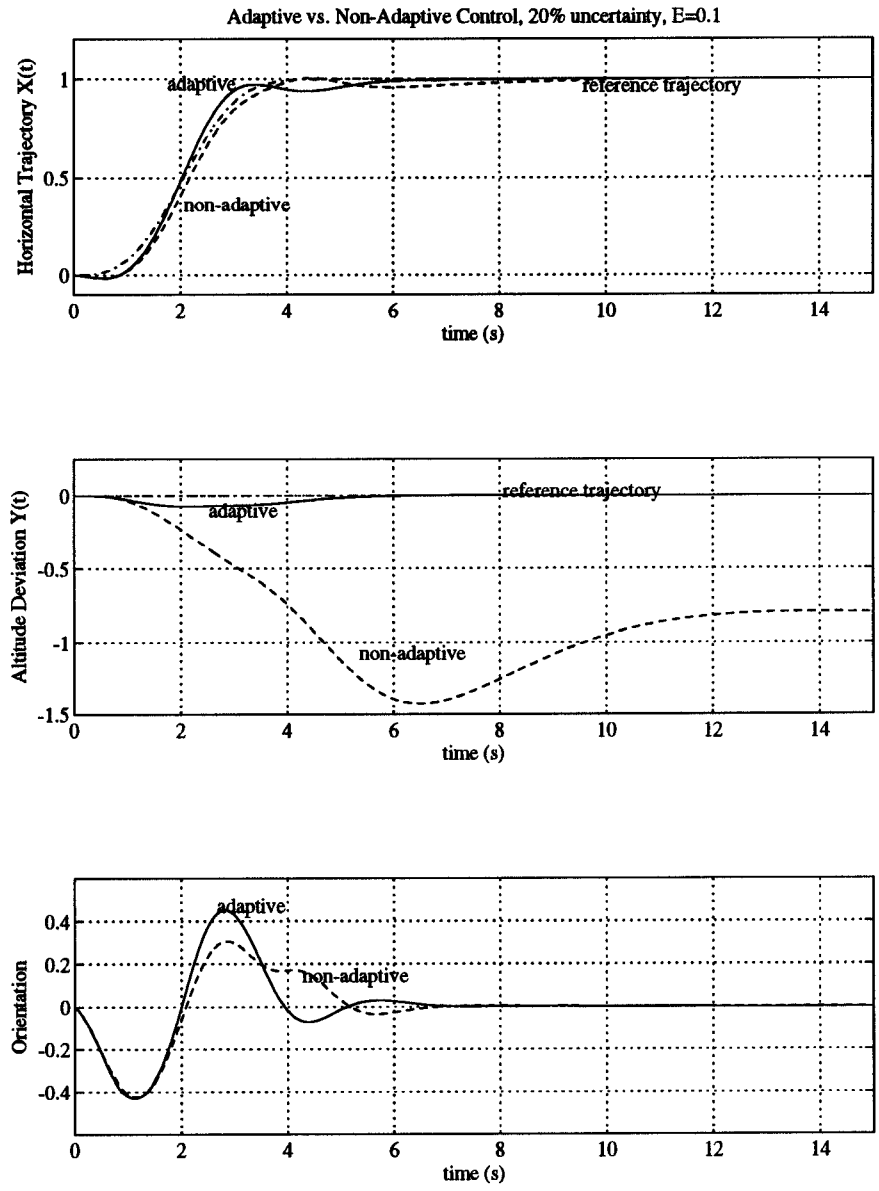

Figure 7.10: Response of the true PVTOL aircraft system under 20% parameter uncertainty in m and J, with $\epsilon = 0.1$.

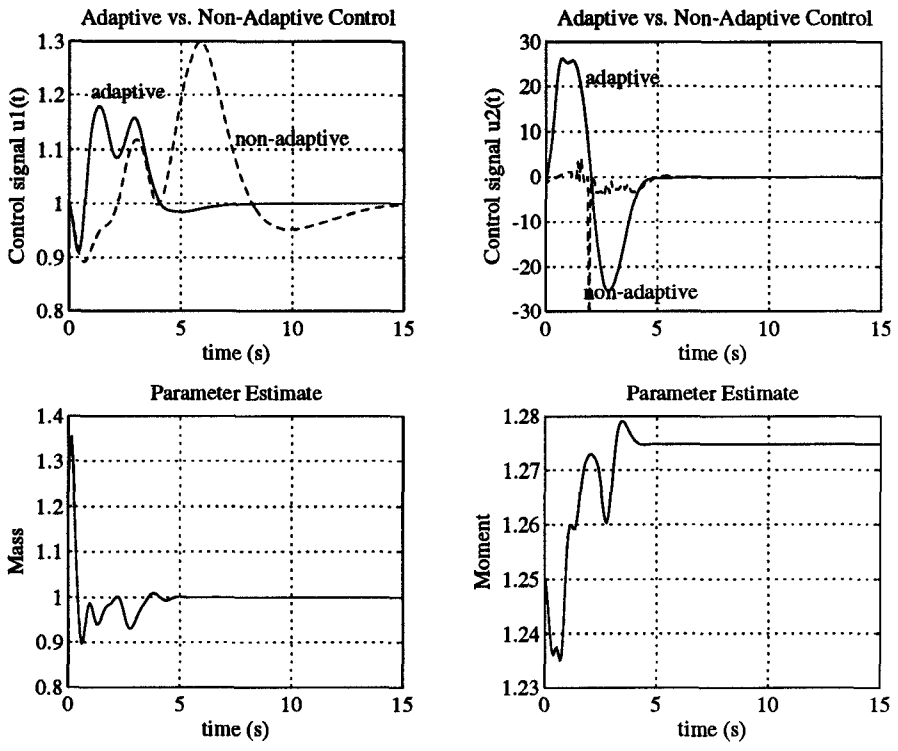

Figure 7.11: Control amplitude and parameter estimates in the aircraft control system with 20% parameter uncertainty in m and J, with $\epsilon = 0.1$.

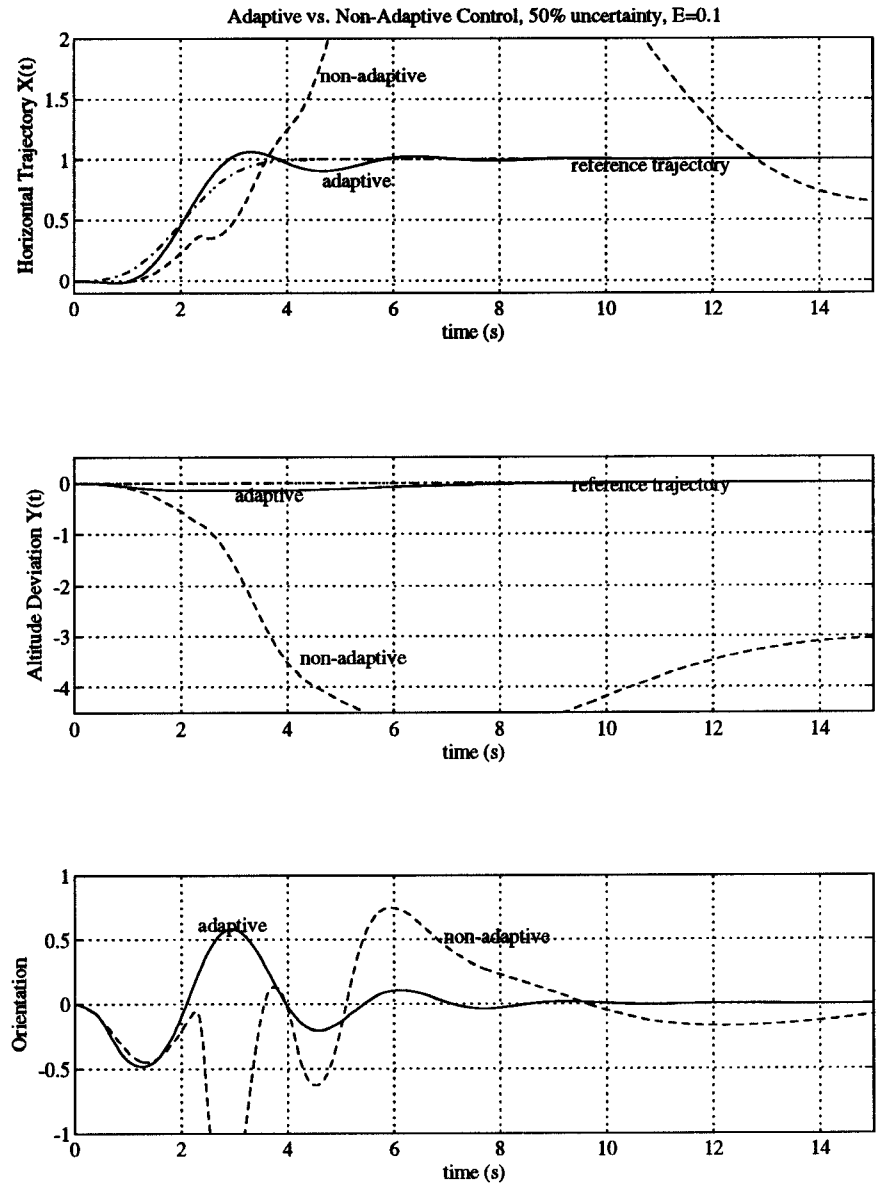

Figure 7.12: Response of the true PVTOL aircraft system under 50% parameter uncertainty in m and J, with $\epsilon = 0.1$.

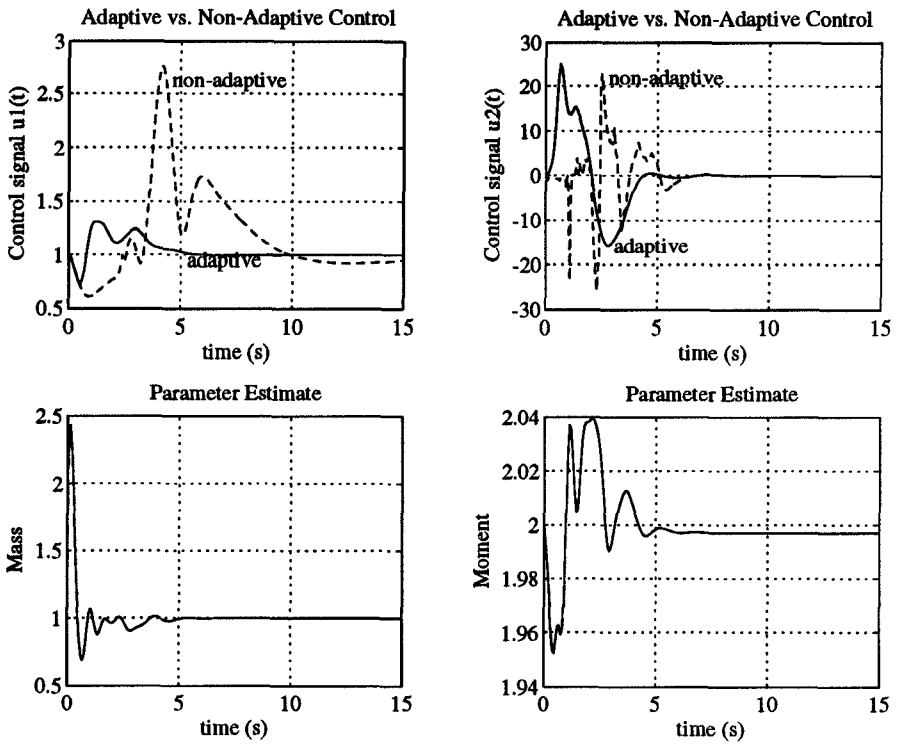

Figure 7.13: Control amplitude and parameter estimates in the aircraft control system with 50% parameter uncertainty in m and J, with $\epsilon = 0.1$.

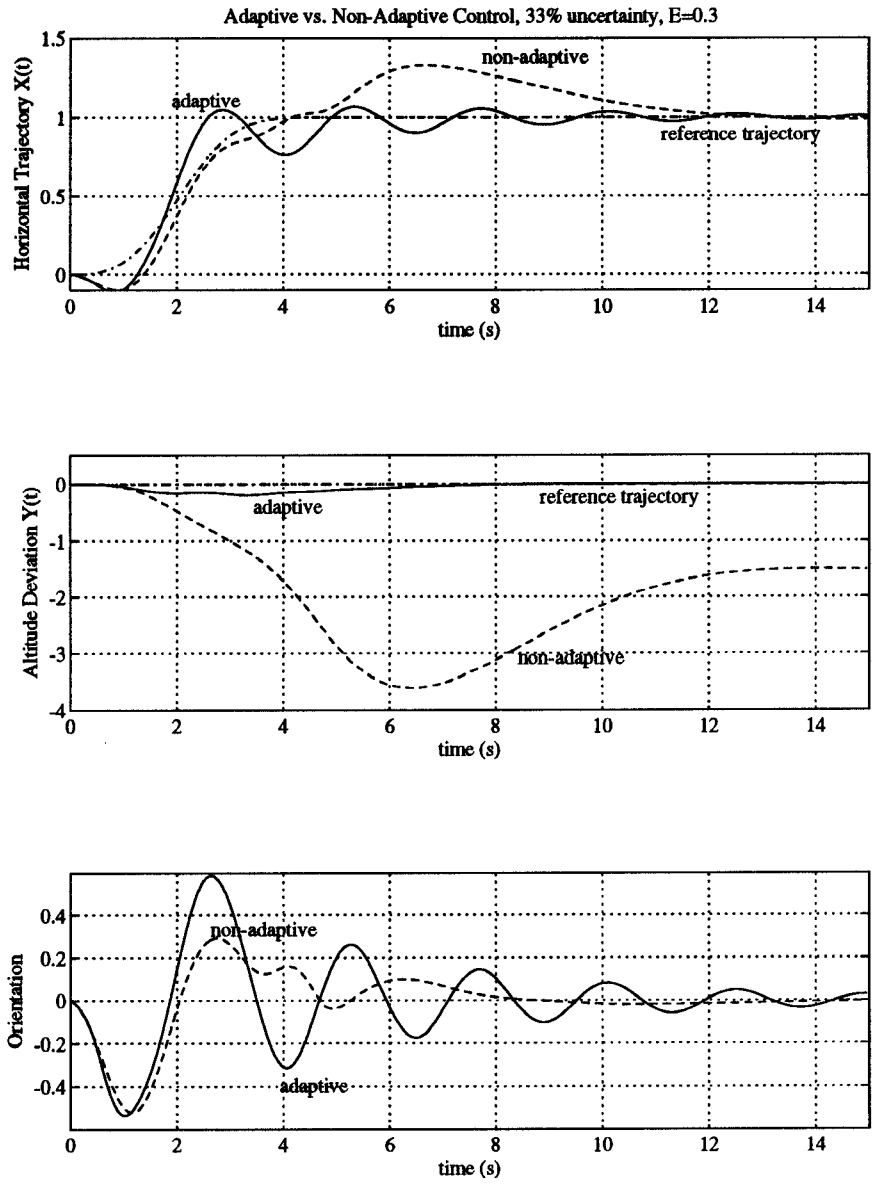

Figure 7.14: Response of the true PVTOL aircraft system under 33% parameter uncertainty in m and J, with $\epsilon = 0.3$.

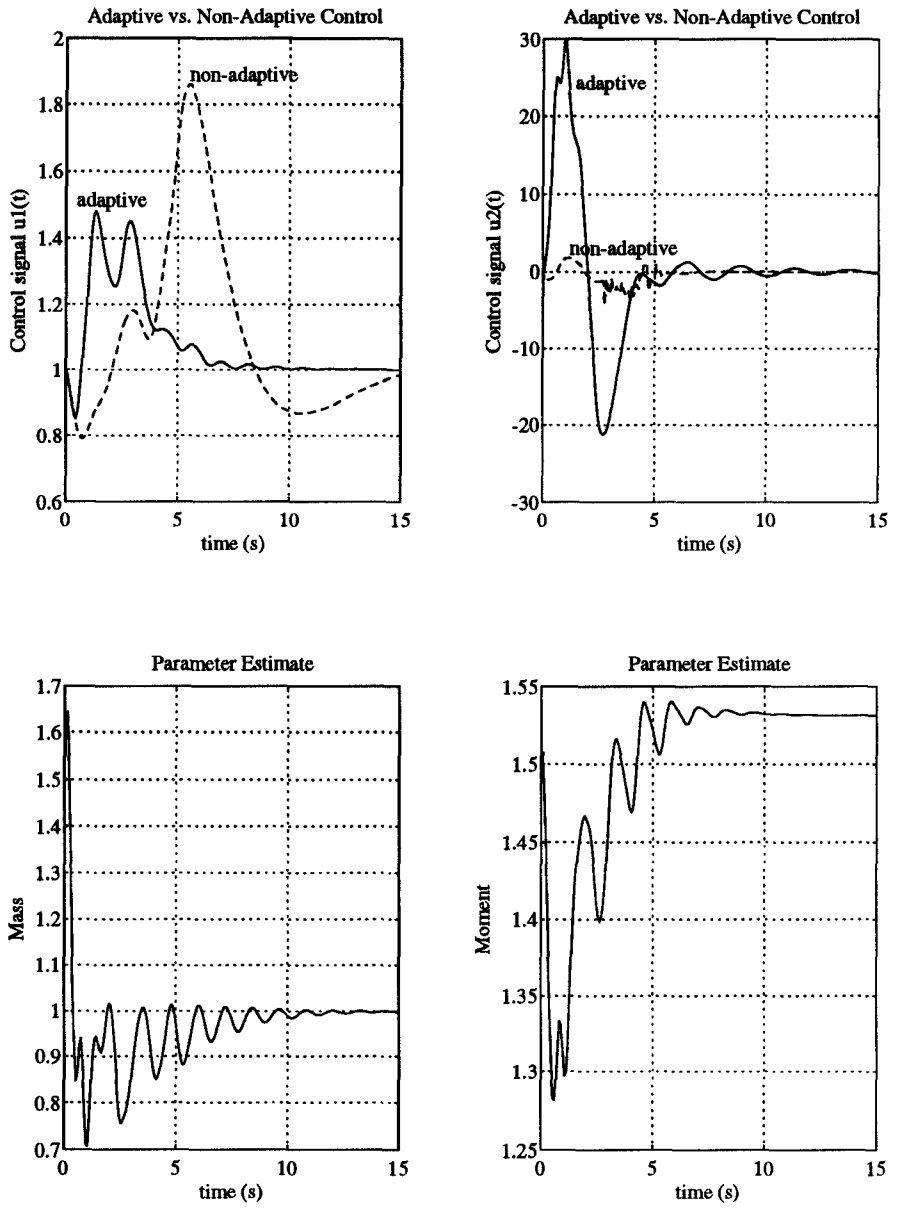

Figure 7.15: Control amplitude and parameter estimates in the aircraft control system with 33% parameter uncertainty in m and J, with $\epsilon = 0.3$.

scheme. We also compared the response of the PVTOL aircraft system, under some parametric uncertainty in the mass and the moment of inertia, to the adaptive and non-adaptive controller and showed that the adaptive controller significantly improves the performance.

Chapter 8

Computational Tools for Nonlinear and Adaptive Control

In this chapter we describe a prototype symbolic-numerical software system for analysis and design of multibody models for vehicle and robot subsystems. The system includes capabilities for automatic generation of model equations, for design of nonlinear tracking and adaptive control laws (as described in Chapter Four), and for generation of simulation codes (in C or Fortran) for performance evaluation. We illustrate the use of the tools by considering the design of an adaptive **magnetic levitation** control system and the design of nonlinear **active suspension** control systems for an **off-road** vehicle based on the Army HMMWV. In the second case we show that it is possible using adaptive, asymptotic tracking control laws to effectively isolate the sprung body dynamics (center of mass velocity and pitch rate) from the road disturbances and indeed control them independently. This permits substantial enhancements in vehicle performance in several areas including increased travel speed and platform stability, and decreased passenger absorbed power for better ride quality, among others.

This chapter is organized as follows: In the first part, we describe the com-

puter aided control system design tools developed for the implementation of the design techniques for nonlinear tracking/regulation and adaptive control. Our approach is based on differential geometric formulation of nonlinear control theory [51, 81] and adaptive nonlinear control scheme developed in Chapter Four. In section (8.2), we describe the class of models used for the design and analysis, and summarize the elements of nonlinear control design techniques we shall use. We then describe the implementation of these techniques with the integrated symbolic-numerical algorithm written in **Mathematica** and C. In section (8.3), we illustrate the use of this algorithm by considering the design of a simple magnetic levitation control system. We will generate both adaptive and non-adaptive control laws for regulating the position of a mass at a specified location in a gravitational field. In the adaptive case, we consider some uncertainty in the coil resistance R. In the second part of this chapter, section (8.4), we shall describe the design of a nonlinear adaptive active suspension control system for an off-road vehicle based on the Army HMMWV. We discuss the elements of a nonlinear active suspension system, controller design for the nonlinear model, followed by performance evaluation.

8.1 Introduction

The design of controlled multibody systems is assuming an increasingly important role in several areas of engineering. There has been a considerable amount of work on computational tools to support the development of models for multibody systems from first principles (see, e.g., the examples and references in [32, 50]). Multibody systems with embedded actuators are an especially important sub-

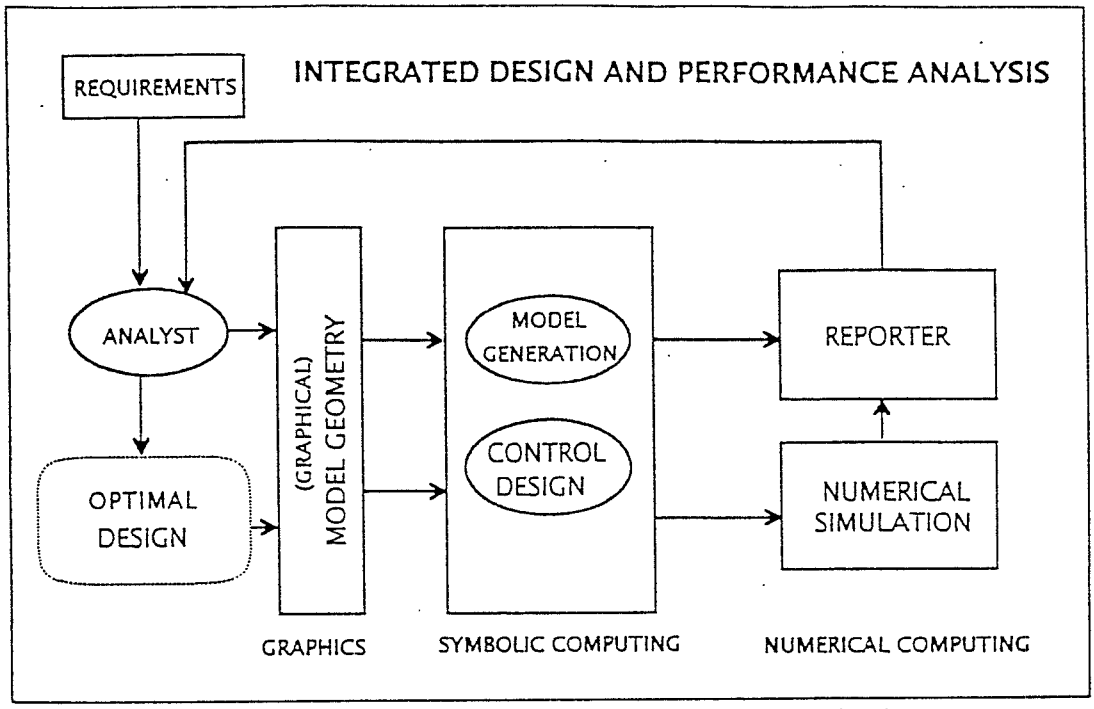

Figure 8.1: Integrated modeling and control system design.

class that includes multilink manipulators and vehicle subsystems. In developing such systems it is important to consider the **integrated design** of the multibody dynamics and the embedded actuator control laws. Our objective in this work is to contribute to the development of tools to support an integrated design process. The long term goal is an integrated design system as suggested in Figure 8.1. As indicated, our technical approach combines symbolic and numerical computing. Some early work on this idea is reported in [17].

While there has also been a large body of work on tools for the design and analysis of linear control systems, there has been much less work on tools for the design of nonlinear control systems. In this chapter we shall describe one approach to the synthesis of such tools starting from the geometric formulation of nonlinear control theory.

The language of much of the development of nonlinear control theory is dif-

ferential geometry. It not only provides a very useful framework for the analysis of nonlinear control systems; but it also permits the generalization of many constructions for linear control systems to the nonlinear case. However, differential geometric objects (Lie brackets, Lie derivatives, etc.) are not easily manipulated by hand. In this work we report on the development of a set of computer aided design tools for the implementation of the design techniques for nonlinear control systems. We use differential geometric tools for the analysis and design of nonlinear control systems, emphasizing the design of (adaptive) controllers for the (general) output tracking problem.

In 1987 O. Akhrif developed the first computational tools for the design of nonlinear control systems using symbolic computing (Macsyma) [2]. This work was inspired by the work of J.P. Quadrat and his colleagues on the use of Macysma (and Prolog) in the treatment of optimal stochastic control problems [26]. The work here builds on the tools developed by Akhrif. It includes new techniques for nonlinear adaptive control and performance evaluation by simulation.

8.2 Computational Algorithms for Nonlinear Control

In this section we shall review some of the basic geometric tools for design of control laws for tracking and regulation of nonlinear systems in the form

$$\begin{aligned} \dot{x} &= f(x) + \textstyle\sum_{i=1}^{m} u_i(t) g_i(x) \\ y &= h(x) \end{aligned} \tag{8.1}$$

where f, g, h are smooth, vector-valued functions of their arguments, $x \in \Re^n, y \in \Re^m$, and the control laws u_i may be chosen based on state (or output) feedback.

8.2.1 Exact State and Input-output Linearization

Consider a nonlinear system of the form (8.1) with no output specified. As stated in chapter Two, a necessary and sufficient condition for the existence of an **exactly linearizing** transformation in a neighborhood U of the origin ($f(0) = 0$) is the existence of a mapping T satisfying [48, 47]:

$$\begin{cases} < dT, ad_f^i(g) >= 0, & i = 0, 1, \ldots, n-2 \\ < dT, ad_f^{n-1}(g) \neq 0 \end{cases} \tag{8.2}$$

where:

$$ad_f^0(g) = g, \; ad_f^1(g) = [f, g], ad_f^k(g) = [f, ad_f^{k-1}(g)], \; k > 1 \tag{8.3}$$

Recall that using the Frobenius theorem [51], this condition is equivalent to

(i)

$$g, ad_f^1(g), \ldots, ad_f^{n-1}(g) \text{ are linearly independent} \tag{8.4}$$

(ii)

$$\text{The set of vector fields } \{g, ad_f^1(g), \ldots, ad_f^{n-2}(g)\} \text{ is involutive} \tag{8.5}$$

Recall that $[f, g]$ is the Lie Bracket of two vector fields, and $ad_f^k(g)$ is the kth order ad operator on two vector fields [51].

Once a transformation T is found, the new set of coordinates $(z_1, \ldots, z_n)^T$ is constructed in a straightforward fashion

$$\begin{aligned} z_1 &= T(x) \\ z_2 &= < dz_1, f > \\ &\vdots \\ z_n &= < dz_{n-1}, f > \\ v &= < dz_n, f > + < dz_n, g > u \end{aligned} \tag{8.6}$$

In the multi-input case, we obtain m subsystems of dimensions $k_1, \ldots, k_m$, respectively. The indices $k_1, \ldots, k_m$, being invariant under the transformations considered, as shown by Jacubczyck and Respondek [52], are the same as the Kronecker indices of the linearized system. We then obtain m sets of partial differential equations of the form (8.2). Hunt and Su also gave a procedure for computing such a transformation. This procedure involves constructing n sets of n ordinary differential equations from the set of partial differential equations (8.2). This is not always possible or easy to do. However, in some cases–the single input case or systems in block triangular form – the construction of the transformation is simple (see the algorithms in [2], based on the results of Brockett and Meyer).

Once the nonlinear system is transformed into a decoupled, controllable linear representation, standard linear design techniques, such as pole placement, can be used to design control laws for the equivalent linear system. Finally, the resulting control law is transformed back into the original coordinates to obtain the control law in terms of the original controls. This design in summarized in Figure 8.2.

To implement exact feedback linearization as a design algorithm we need tools to compute Lie derivatives and Lie brackets of vector fields, the ability to test the conditions expressed in (8.4),(8.5), and a procedure for finding the transformation in (8.2). The following are versions of some of these tools in **Mathematica.**[1]

[1]Computations similar to these were implemented by Akhrif in Macsyma, and applied to a variety of systems in [6].

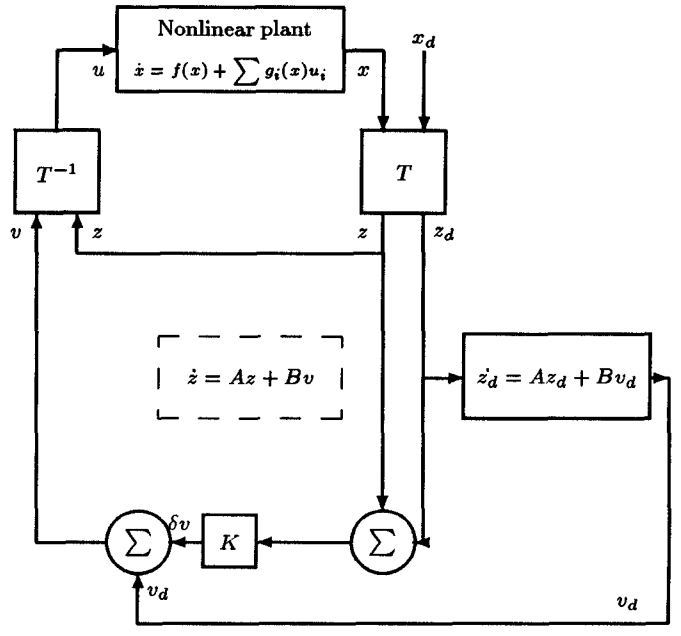

Figure 8.2: Control system design based on exact feedback linearization.

First, we define the gradient in a way which illustrates two very powerful features of **Mathematica**. The first line in the following is a usage statement associated with the help system in **Mathematica**. The second line is the computation of the gradient.

```
Grad::usage=
"Grad[f,varlist] computes the Grad of the function f
        with respect to the list of variables varlist."
Grad[f_,var_List]:=D[f,#]& /@ var
```

Mathematica supports a version of lambda functions as used in formal logic and Lisp [1, 106]. In the definition of `Grad`, the expression `D[f,#]&` is a formal (un-named) function. The symbol `D` stands for derivative; so `D[f,x]` is the derivative of `f` with respect to a (single) variable `x`. To compute the gradient

of a scalar function of a vector, we must compute its derivative with respect to each element of the vector. This is accomplished by "mapping" the operation "take the derivative of `f` with respect to a variable" (this is the meaning of the expression `D[f,#]&`). The symbol `&` stands for a "name" that one might assign to the function "take the derivative." However, since we will only use the formal function once, we do not need to name it. Similarly, we do not need to name the variable that is its argument, so the symbol `#` is used.

Arguments to function definitions in **Mathematica** are of the form

```
h[x_]:=x^2
```

which means any symbol substituted for the place holder `x_` is raised to the second power. The form `var_List` means the argument must be a list.

The symbol `/@` stands for the **Mathematica** operation `Map`; so we might have written the definition as

```
Grad[f_,var_List]:=Map[D[f,#]&,var]
```

The use of lambda functions and the capability to map functions over sets of arguments are powerful constructions which increase the expressive power of programs in **Mathematica**. `Map[]` is especially useful in avoiding procedural programming constructions [73]. The use of `Map[]` in the definition of `Grad[]` illustrates the capability of **Mathematica** to **treat functions as objects** like symbols or numbers and use them as arguments to other functions.

We use two lines (rules) to define the Jacobian of a function with respect to a vector. The first handles the case when the function is a vector function of a vector argument. The second handles the case of scalar functions (of vector arguments). These may be regarded as **rules** for the computation. **Mathematica**

uses a kind of pattern matching to find the case that applies.[2]

```
Jacob[f_List,var_List]:=Outer[D,f,var] /; VectorQ[f]
Jacob[f_,var_List]:=Grad[f,var]
```

`Outer[]` is a built-in **Mathematica** function which provides a generalized outer product. The test `VectorQ[f]` defined by the condition symbol "`/;`" verifies that `f` is a vector. If the test succeeds, this rule is used. If not, the next one is tested.

The next function illustrates the use of condition checking in **Mathematica** in more detail. The symbol `&&` is logical "and." In the first rule, we check that the functions are vector valued, that their lengths are the same, and that the lengths equal the length of the vector of variables. If this compound test succeeds, the rule is used.

```
LieBracket[f_List,g_List,var_List]:=
            (Jacob[g,var] . f -  Jacob[f,var] . g
     /; VectorQ[f] && VectorQ[g] && Length[f]==Length[g]==Length[var])

LieBracket[f_,g_,var_List]:=Jacob[g,var] f - Jacob[f,var] g
```

The next function illustrates the recursive power of the language to define the **Ad** operator. (We omit the vector cases.)

```
Ad::usage=
"Ad[f,g,varlist,n] computes the nth Adjoint of
 the functions f,g with respect to the variables varlist."
Ad[f,g,var,0]=g
```

[2]In the code examples that follow, we present selected components. In some cases additional code is required to complete the definition.

```
Ad[f,g,var,n]=LieBracket[f,Ad[f,g,var,n-1],var]
Ad[f,g,var]=Ad[f,g,var,1]"
```

Using these functions, we can express the Hunt-Su-Meyer condition (8.4,(8.5)) in **Mathematica** functions.

```
ControllabilityMatrix[f_,g_,var_List] :=Module[{k},
            Table[Ad[f,g,var,k],{k,0,Length[var]-1}]]

Controllable[f_,g_,var_List]:=
            If[Rank[ControllabilityMatrix[f,g,var]]==Length[var],
            True,False]

FeedbackLinearizable[f_,g_,var_List]:=Module[{cm,cm1,k},
            cm=Table[Ad[f,g,var,k],{k,0,Length[var]-1}]
            cm1=Drop[cm,-1];            (* drop last element *)
            If[Rank[cm]==Length[var]    (* system is controllable *)
               &&
            Involutive[cm1,var],True,False]]
```

The `Module[]` construction permits the use of local variables in the definition of functions. We use the **Mathematica** `Table[]` function to construct a set of derived vector fields, in this case the controllability matrix in (8.5). The function `Involutive[]` checks that this set of vector fields is involutive, that is, closed under the Lie Bracket.

```
Involutive[f_List,var_List]:=Module[{k,h,vec},
    k=Length[f];
    h=Table[LieBracket[f[[i]],f[[j]],var],{i,1,k},{j,i+1,k}];
    vec=Union[Flatten[h,1],f];
    If[Rank[vec]>Rank[f],False,True]]
```

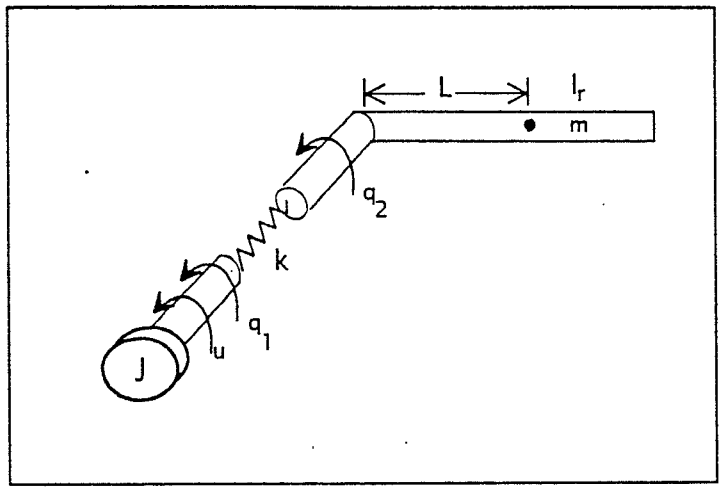

Figure 8.3: A controlled mechanism with a flexible joint.

In this expression the notation `f[[i]]` takes the ith element of the list (vector) `f`. `Union` and `Flatten` are functions for manipulating lists.

To illustrate the functions consider the equations of a mechanism with a flexible joint [77]

$$I\ddot{q}_1 + mg\ell\sin(q_1) + k(q_1 - q_2) = 0 \tag{8.7}$$

$$J\ddot{q}_2 - k(q_1 - q_2) = u$$

(See Figure 8.3.) The angles of the link and the motor shaft, q_1, q_2, are the state variables, and the applied torque u is the control. In state variable form (8.1) we identify $x = [q_1, \dot{q}_1, q_2, \dot{q}_2]^T$ and

$$f(x) = [x_2, -\frac{mg\ell}{I}\sin(x_1) - \frac{k}{I}(x_1 - x_2), x_4, \frac{k}{I}(x_1 - x_3)]^T$$

$$g(x) = [0, 0, 0, 1/J]^T, \qquad h(x) = x_1$$

In the following **Mathematica** script we compute the "controllability matrix" associated with this system and check the conditions for feedback lin-

earization. (We use **Ir** for the inertia of the shaft since I is a reserved symbol in **Mathematica**.)

```
In[3]:= <<examples.m


In[4]:= f[x]

                k (x1 - x3)     g L M Sin[x1]      k (x1 - x3)
Out[4]= {x2, -(-----------) - -------------, x4, -----------}
                     Ir               Ir                J

In[5]:= g[x]

                   1
Out[5]= {0, 0, 0, -}
                   J


In[6]:= ControllabilityMatrix[f[x],g[x],x]


                        k                k                1      k
Out[6]= {{0, 0, 0, -(------)}, {0, 0, ------, 0}, {0, -(-), 0, --},
                       Ir J             Ir J             J       2
                                                                J


      1       k
>    {-, 0, -(--), 0}}
      J        2
               J


In[6]:= Involutive[Table[Ad[f[x],g[x],x,k],{k,0,2}],x]
```

```
Out[6]= True
```

```
In[7]:= FeedbackLinearizable[f[x],g[x],x]
```

```
Out[7]= True
```

In the display of the controllability matrix, the syntax is to list the matrix by rows.

Since the **exact** feedback linearization problem above is not always solvable, a more general objective is to obtain **Partial Feedback Linearization (PFL)** of the input-output response by construction of an **inverse dynamic model.** The procedure is easily illustrated by consideration of the case with one input, $\dot{x} = f(x) + g(x)u$ and assume we have identified an output for this system:

$$y = h(x)$$

with h a smooth function mapping $\Re^n \to \Re$. Now if we differentiate y we obtain

$$\dot{y} = \frac{\partial h}{\partial x}(f(x) + g(x)u). \tag{8.8}$$

In the case that the scalar coefficient of u ($\frac{\partial h}{\partial x}g(x)$) is zero we differentiate again until a nonzero control coefficient appears. As mentioned in Chapter Two, once an output is specified, the number of required differentiations, called the **relative degree** of the associated system, is a fundamental system invariant which plays a role in constructing a system inverse and PFL.

The above construction can be made precise using the **Lie derivative** of the scalar function h with respect to the vector field f

$$L_f(h) = \langle dh, f \rangle := \frac{\partial h}{\partial x} f(x). \tag{8.9}$$

Higher order derivatives can be successively defined

$$L_f^k(h) = L_f(L_f^{k-1}(h)) := \langle dL_f^{k-1}(h), f\rangle. \tag{8.10}$$

Then we can write (8.4-(8.5)) as

$$\begin{aligned} \dot{y} &= \langle dh, f\rangle + \langle dh, g\rangle u \\ &= L_f(h) + L_g(h)u. \end{aligned}$$

If $L_g(h) = 0$ then we differentiate again to obtain

$$\begin{aligned} \ddot{y} &= \langle dL_f(h), f\rangle + \langle dL_f(h), g\rangle u \\ &= L_f^2(h) + L_g(L_f(h))u. \end{aligned} \tag{8.11}$$

If $L_g(L_f^{k-1}(h)) = 0$ for $k = 1, \ldots, r-1$, but $L_g(L_f^{r-1}(h)) \neq 0$ then the process terminates with

$$\frac{d^r y}{dt^r} = L_f^r(h) + L_g(L_f^{r-1}(h))u. \tag{8.12}$$

The system (8.12) can be effectively inverted by introducing a feedback transformation of the form

$$u = \frac{1}{L_g(L_f^{r-1}(h))}[v - L_f^r(h)] \tag{8.13}$$

which results in an input-output response from $v \to y$ given by

$$\frac{d^r y}{dt^r} = v,$$

a **linear** map.

The integer $r > 0$ can be viewed as a **relative order** for the nonlinear system (8.1) (single input single output case). Note that if we define new state coordinates $z \in \Re^r$ as

$$z_k = L_f^{k-1}(h), \quad k = 1, \ldots, r \tag{8.14}$$

for the r-dimensional nonlinear system (8.12), then the system model can be written in state space form as,

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \mathcal{A}(x) + \mathcal{B}(x)u \end{bmatrix} \tag{8.15}$$

where

$$\mathcal{A}(x) = L_f^r(h), \quad \mathcal{B}(x) = L_g(L_f^{r-1}(h)). \tag{8.16}$$

These coordinates express the original system in the **normal form** whose importance has been emphasized by Byrnes and Isidori [21, 51]. Note that if $r < n$, the original nonlinear system is not linearized completely. It is only (partially) input-output linearized resulting in an $n-r$-dimensional unobservable subsystem called **internal dynamics**.

The above procedure can be readily generalized to the multi-input/multi-output case (see Chapter Two). The control algorithm for the general MIMO case using partial feedback linearization is shown in Figure 8.4.

It is a simple matter to implement, in **Mathematica**, the Lie derivative and the other computations needed to construct the inverse system as defined above. We illustrate the simple scalar case. The MIMO case is demonstrated with the active suspension design later in this chapter.

```
LieDerivative[f_List,h_,var_List]:=(Inner[Times,Jacob[h,var],f]
                    /; Length[f]==Length[var]) (* Matrix f *)

LieDerivative[f_List,h_,var_List,0]:=h /; Length[f]==Length[var]
```

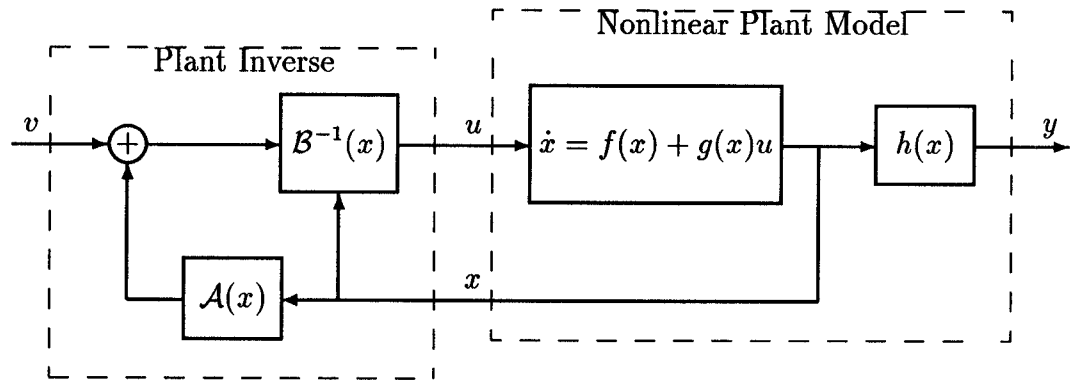

Figure 8.4: Input-Output Linearizing Feedback via Nonlinear Inverse Model

```
LieDerivative[f_List,h_,var_List,n_Integer]:=
        (LieDerivative[f,LieDerivative[f,h,var],var,n-1]
                                /; Length[f]==Length[var])
```

Note the use of the boundary condition for the zeroth order Lie Derivative, the use of the generalized inner product function `Inner[]`, and the use of recursion.

In the following **Mathematica** script we illustrate the use of these functions for the flexible joint mechanism described by (8.7) with output $h(x) = x_1 = q_1$

```
In[11]:= RelativeOrder[f[x],g[x],h[x],x]


Out[11]= 4


In[12]:= LieDerivative[f[x],h[x],x,2]
```

```
            k (x1 - x3)    g L M Sin[x1]
Out[12]= {-(-----------) - -------------}
                 Ir             Ir


In[13]:= NormalForm[f[x],g[x],h[x],x]
                                   k (x1 - x3)    g L M Sin[x1]
New Coordinates = {{x1}, {x2}, {-(-----------) - -------------},
                                        Ir              Ir


      k x4          k      g L M Cos[x1]
>    {---- + x2 (-(---) - -------------)}}
      Ir            Ir         Ir


In[14]:= InverseSystem[f[x],g[x],h[x],x]


The inverse system is


     dz/dt = A(z) + B(z) y(t)
      u(t) = C(z) + D(z) y(t)




              k (x1 - x3)    g L M Sin[x1]      Ir   k (x1 - x3)
A(z) = {x2, -(-----------) - -------------, x4, -- + -----------}
                  Ir              Ir            k         J
                 Ir
B(z) = {0, 0, 0, --}
```

k

We have also implemented more complex inversion algorithms based on the work of Hirschorn and Singh [42, 43, 90, 91], including algorithms using the dynamic extension property [51]. These are used in our nonlinear adaptive control algorithms. Other algorithms, such as approximate linearization techniques for approximate state and input-output feedback linearization, and adaptive approximate regulation and tracking, introduced in Chapters Five and Six, will be implemented next.

8.2.2 Adaptive Asymptotic Tracking Using Dynamic Inversion

The methods we have discussed so far are based on a precise model of the system dynamics. The sensitivity of the resulting closed loop system, especially disturbance decoupling and input/output linearization, to model uncertainty is a critical concern. Parameter adaptive control methods developed in previous chapters preserve the essential structure of the PFL design and are effective in reducing sensitivity to model uncertainty. We first briefly review the terminology used in the inversion algorithm of chapter four and describe the implementation of this algorithm into a computer code.

Consider again a nonlinear system (with m inputs and ℓ outputs with $\ell \leq m$) under parametric uncertainty:

$$\begin{aligned} \dot{x}(t) &= f(x,\theta) + \sum_{i=1}^{m} u_i \cdot g_i(x,\theta) \qquad x \in M \\ y(t) &= h(x(t),\theta) \end{aligned} \tag{8.17}$$

where θ represents the vector containing unknown parameters. Consider the

asymptotic tracking problem where we are interested to design a control law $u = (u_1, u_2, \ldots, u_m)$ to cause the output $y(t, u, x_0)$ of (8.17) to converge to a desired signal y_m as $t \to \infty$. In most applications one may know only nominal values for some or all of the parameters θ in the model (8.17). Parametric uncertainties, in most cases will result in performance degradation in the response of the system and may destabilize the closed-loop system. Our objective is therefore to solve the tracking problem under parametric uncertainty in the original system, i.e., in $f(x), G(x)$, and/or $h(x)$. As we showed in chapter four, one approach to design such a control law is based on Hirschorn-Singh dynamic inversion algorithm. This algorithm produces a sequence of systems associated with (8.17) derived by performing a series of operations such as differentiation, row ordering, and row reduction on the output $y(\cdot)$ of system (8.17). We skip the details of the algorithm. To summarize, associated with (8.17), one constructs a sequence of systems in the form:

$$\begin{aligned} \dot{x}(t) &= f(x,\theta) + G(x,\theta)u \qquad x \in M_k \\ z_k(t) &= C_k(x,\bar{\theta}) + D_k(x,\bar{\theta})u \end{aligned} \tag{8.18}$$

where $G(x) = (g_1, g_2, \ldots, g_m)$, $D_k(x)$ has all but the first r_k rows zero and has rank r_k for all $x \in M_k$, and $\bar{\theta}$ is a (possibly) new vector of unknown constants that is related to the original vector θ, but it may be of higher dimension. The tracking order β of the system (8.17) is the least positive integer k such that $r_k = \ell$ or $\beta = \infty$ if $r_k < \ell$ for all $k > 0$. Hence, $D_\beta(x)$ is an $\ell \times m$ matrix with rank ℓ ($\ell \leq m$) for all $x \in M_\beta$.

Therefore, if $\beta < \infty$, any given smooth function $y_m(\cdot)$ is **asymptotically functionally reproducible** by system β. The vector $z_k(t)$ is partitioned in the

form:

$$z_k(t) = \begin{bmatrix} \bar{z}_k(t) \\ \hat{z}_k(t) \end{bmatrix} = \begin{bmatrix} \bar{C}_k(x,\bar{\theta}) \\ \hat{C}_k(x,\bar{\theta}) \end{bmatrix} + \begin{bmatrix} D_{k1}(x,\bar{\theta}) \\ 0 \end{bmatrix} u(t) \tag{8.19}$$

where rank $D_{k1}(x,\bar{\theta}) = r_k$, for all $x \in M_k$, and $\bar{z}_k(t)$ and $\bar{C}_k(x)$ consist of the first r_k elements of $z_k(t)$ and C_k. The vector $z_k(t)$ can also be written in terms of the derivatives of the output $y(t,u,x_0)$ up to βth order:

$$z_k(t) = \begin{bmatrix} \bar{z}_k(t) \\ \hat{z}_k(t) \end{bmatrix} = \begin{bmatrix} H_k(x,\bar{\theta}) \\ J_k(x,\bar{\theta}) \end{bmatrix} Y_k(t) \tag{8.20}$$

where:

$$Y_k(t) = [(y^{(1)})^T, (y^{(2)})^T, \ldots, y^{(k)})^T]^T$$

and $J_\beta(x) = 0$. The system β is

$$\begin{aligned} \dot{x}(t) &= A(x,\theta) + B(x,\theta)u \\ z_\beta(t) &= C_\beta(x,\bar{\theta}) + D_\beta(x,\bar{\theta})u \end{aligned} \tag{8.21}$$

with

$$z_\beta(t) = H_\beta(x,\bar{\theta}) \cdot Y_\beta(t) \tag{8.22}$$

We can rewrite this as:

$$z_\beta(t) = N(x,\bar{\theta})\bar{Y} + M(x,\bar{\theta})\tilde{Y} \tag{8.23}$$

where n_i and N_i are the lowest and highest order derivatives of $y^{(i)}$ appearing in (8.22),

$$\begin{aligned} \bar{Y} &= \left[y_1^{(n_1)}, \ldots, y_\ell^{(n_l)}\right]^T \\ \tilde{Y} &= \left[y_1^{(n_1+1)}, \ldots, y_1^{(N_1)}, y_2^{(n_2+1)}, \ldots, y_\ell^{(N_\ell)}\right]^T \end{aligned} \tag{8.24}$$

and $N(x,\bar{\theta})$ is an $\ell \times \ell$ nonsingular matrix with determinant of $N(x,\bar{\theta}) = \pm 1$ for all $x \in M_\beta$.

A **Mathematica** code, `singh.m`, is developed [16] to perform the above steps and return the β system. The main functions are:

```
{Cbeta,DDbeta,Hbeta}=Singh[f,g,h,x,Internal->True];
{NMat,MMat,yN,yM,derlist}= H2NandM[Hbeta,l,Internal->True];
```

which takes as inputs the state variables x and system functions $f(x), g(x), h(x)$, in **symbolic** form, including the unknown parameters θ, and returns the matrices $C_\beta, D_\beta, H_\beta, N_\beta$ and M_β, and vectors: $\bar{Y}, \tilde{Y}$.

The control law is given in the following form (see (4.15) in chapter four):

$$u(t) = D_\beta^\dagger(x,\hat{\theta}) \cdot \left[-C_\beta(x,\hat{\theta}) + \hat{M}\tilde{y} + N(x,\hat{\theta}) \cdot K\right] \tag{8.25}$$

where $\hat{\theta}$ are estimates of $\bar{\theta}$ with updating laws:

$$\dot{\hat{\theta}} = -\Omega^{-1} \cdot W^T \cdot R \cdot \epsilon \tag{8.26}$$

with $\epsilon \triangleq [e_1, \dot{e}_1, \ldots, e^{(1)}{}_{n_1-1}, e_2, \ldots, e^{(\ell)}{}_{n_\ell-1}]^T$, and:

$$K = \begin{bmatrix} y_{m1}^{(n_1)} + \sum_{j=0}^{n_1-1} p_{1j}(y_{m1}^{(j)} - y_1^{(j)}) \\ \vdots \\ y_{m\ell}^{(n_\ell)} + \sum_{j=0}^{n_\ell-1} p_{\ell j}(y_{m\ell}^{(j)} - y_\ell^{(j)}) \end{bmatrix} \tag{8.27}$$

The control structure for asymptotic output tracking with parameter adaptation is shown in Figure 8.5.

The **Mathematica** code for adaptive asymptotic output tracking is too complex to list and explain here. The main function is

```
{rhstheta, control}=Adaptive[f_,g_,h_,x_,RefSig,AdGain,Theta];
```

which takes as inputs the state variables and functions defining the system, including labels for uncertain parameters (the list `theta_`), a reference signal

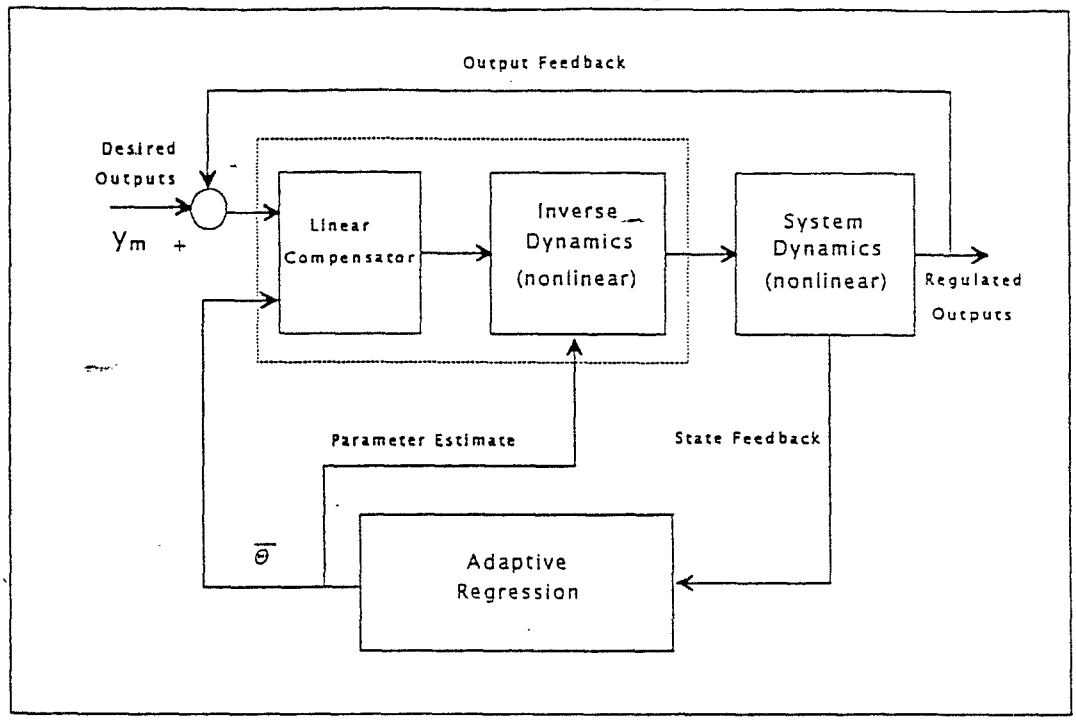

Figure 8.5: Control structure for asymptotic output tracking using a dynamic inverse and parameter adaptation.

`RefSig`, and a value for the gain in the adaptation process `AdGain` which corresponds to Ω in (8.26). (One can include other optional arguments.) It produces as outputs the adaptation law (8.26) for the parameters, the tracking control law (8.25) with the parameter **estimates** inserted, and the new state equations with the control law inserted.

The state equations and the parameter update laws may be passed to a module which solves them using either the built-in ODE solver in **Mathematica** or to a module which generates a conventional simulation code. We use the second option, using **Mathematica** to write a C code for the right hand sides of the differential equations (closed loop state equations and parameter update equations). This code is compiled and linked with a standard ODE solver. In our implementations the code is compiled and executed from **Mathematica**,

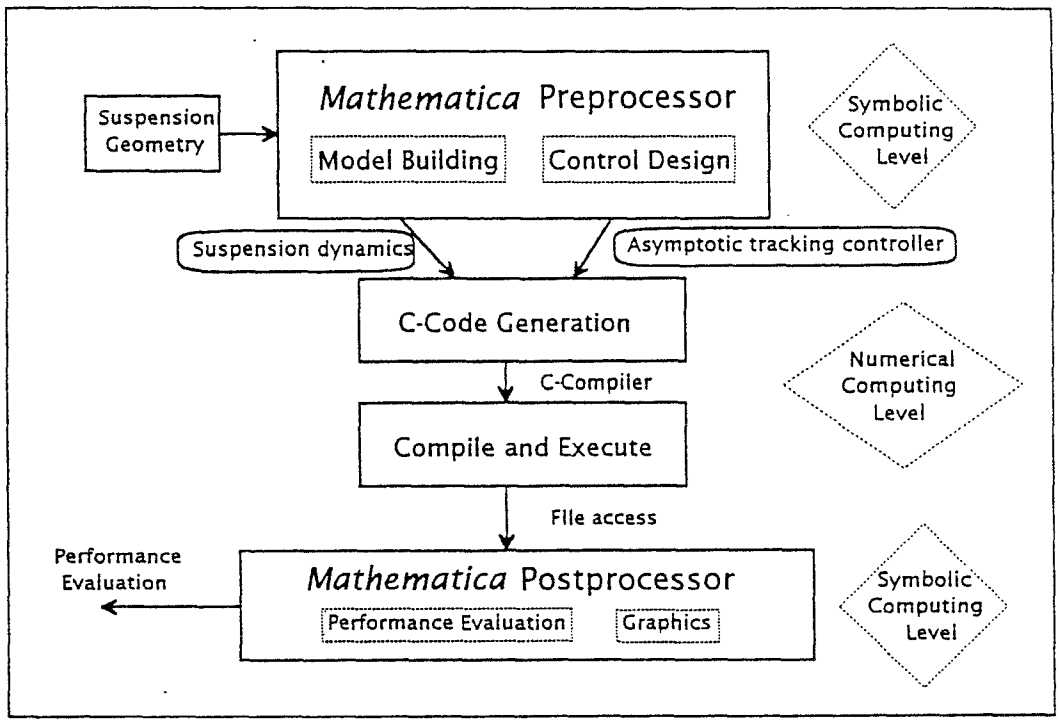

Figure 8.6: Integrated symbolic-numeric modeling and design system.

and the results are read in and displayed using the **Mathematica** graphics. The architecture is shown in Figure 8.6.

8.3 Design and Control of a Magnetic Levitation System

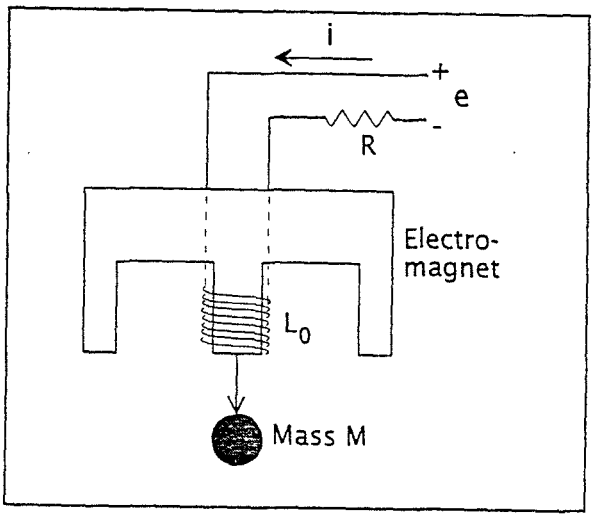

Figure 8.7: Magnetic levitation device.

As an example, we consider the simple magnetic levitation device shown in Figure 8.7. The objective is to control the input voltage e to the coil to maintain the ball at a specified position in a gravitational field. The system is highly nonlinear. It has been treated by Sugie, Simizu, and Imura using results from feedback linearization theory [95]. (See also [33, 89].) We will also consider this problem under some parameter uncertainty in the system. The dynamics are described by

$$M\ddot{x} = Mg + \frac{1}{2}i^2\frac{\partial L}{\partial x} \tag{8.28}$$

$$e = Ri + \frac{d}{dt}(Li)$$

where x is the gap length, M is the ball mass, g is gravity, i is the current, e is the input voltage, R is the resistance of the coil, and L is the inductance of the coil which depends on the location of the ball. The model for $L(x)$ is

$$L(x) = \frac{Q}{X + x} + L_0$$

for some constants Q, X, L_0. Defining the states $z = [x, \dot{x}, i]^T$ and control $u = e$, the model can be written in the abstract form

$$\dot{z} = \begin{bmatrix} z_2 \\ \alpha(z) \\ \beta(z) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \gamma(z) \end{bmatrix} u \tag{8.29}$$

with

$$\alpha(z) = g - \frac{Q z_3^2}{2M(X + z_1)^2}$$
$$\beta(z) = \frac{z_3(Q z_2 - R(X + z_1)^2}{Q(X + z_1) + L_0(X + z_1)^2}$$
$$\gamma(z) = \frac{(X + z_1)}{Q + L_0(X + z_1)}$$

Using the tools described in the last section we can easily check that the system can be exactly linearized by feedback, and written in normal form. In the transcript which follows we first show the states and equations defined by the symbols `xxx`, `fff`, `ggg`, `hhh`, then we show the new coordinates defined by the expression

$$y = \Phi(x) = \begin{bmatrix} L_f^0 h(x) \\ L_f^1 h(x) \\ L_f^2 h(x) \end{bmatrix}$$

or in **Mathematica**

```
newcoordinates:={
LieDerivative[fff,hhh,xxx,0],
LieDerivative[fff,hhh,xxx,1],
LieDerivative[fff,hhh,xxx,2]}
```

Next we show the input-output linearizing control law defined by the function

$$u = -\frac{\beta(x)}{\alpha(x)} + \frac{1}{\alpha(x)} v$$

with $\alpha(x) = L_g L_f^2 h(x)$ and $\beta(x) = L_f^3 h(x)$, or in **Mathematica**

```
alpha:=LieDerivative[ggg,LieDerivative[fff,hhh,xxx,2],xxx]

beta:=LieDerivative[fff,hhh,xxx,3]

feedbackcontrol:=Simplify[(-beta + newcontrol)/alpha]
```

Finally, we show the inverse coordinate change $x = \Phi^{-1}(y)$.

```
In[1]:= <<sugie.m

In[2]:= xxx

Out[2]= {x1, x2, x3}
```

```
In[3]:= fff
                                    2                    2
                         Q x3       (-(R (X + x1) ) + Q x2) x3
Out[3]= {x2, grav - -------------, -------------------------}
                                  2                          2
                     2 M (X + x1)   Q (X + x1) + LO (X + x1)
In[4]:= ggg
                     X + x1
Out[4]= {0, 0, ---------------}
                 Q + LO (X + x1)
In[5]:= hhh
Out[5]= x1
In[6]:= newcoordinates
                                  2
                              Q x3
Out[6]= {x1, x2, grav - -------------}
                                     2
                         2 M (X + x1)
In[7]:= feedbackcontrol
Out[7]= -((M (X + x1) (Q + LO X + LO x1)

                          2                     2            2
                   Q x2 x3      Q (-(R (X + x1) ) + Q x2) x3
> (newcontrol - ----------- + ------------------------------)) / (Q x3))
                          3             3
                 M (X + x1)    M (X + x1)  (Q + LO X + LO x1)

In[9]:= coordinatechange
```

```
                                    2                          2     2
Out[9]=x3 -> (Sqrt[M] Sqrt[2 grav X  + 4 grav X y1 + 2 grav y1  -2 X y3

                          2
>        - 4 X y1 y3 - 2 y1  y3]) / Sqrt[Q]
```

One example for the new control v for output regulation around some setpoint value x_{1_m} is the simple linear pole placement control (in new coordinates $y = \Phi(x)$): $v(y) = \alpha_0(y_1 - x_{1_m}) + \alpha_1 y_2 + \alpha_2 y_3$ where $x_{1_m}^{(i)}$ are zero. In **Mathematica**:

```
errors[[1]]=RefTraj[[1]]-x1;
newcontrol:=Inner[Times,
     Drop[CoefficientList[(s+pole1) (s+pole2) (s+pole3),{s}],-1],
            errors,Plus];
```

where `RefTraj[[1]]` is x_{1_m} and `pole1,pole2,pole3` are the three desired closed loop poles. Choosing the poles at 20,25,30 the above **Mathematica** script gives the `newcontrol` and the resulting `feedbackcontrol` as follows:

```
In[1]:= newcontrol
                                                        2
                                                     Q x3
Out[1]= 15000 (0.02 - x1) - 1850 x2 + 75 (-grav + -------------)
                                                            2
                                                  2 M (X + x1)
In[2]:= feedbackcontrol
Out[2]= -((M (X + x1) (Q + L0 X + L0 x1)
```

```
                                                  2
                                            Q x2 x3
>        (300. - 15000 x1 - 1850 x2 - ----------- +
                                                 3
                                          M (X + x1)

                      2            2                        2
        Q (-(R (X + x1) ) + Q x2) x3                    Q x3
>       ---------------------------- + 75 (-grav +-------------))) /
                 3                                            2
        M (X + x1)  (Q + L0 X + L0 x1)              2 M (X + x1)

>      (Q x3))
```

For simulation, the following values are used:

```
In[30]:= Values

Out[30]= {M ->0.357, R ->29.1, Q ->0.00324, X ->0.00537, grav ->9.8,

>    L0 -> 2.1}
```

In Figure (8.8) we show the state response of the system (8.28) depicted in Figure (8.7) to this `feedbackcontrol`. Note that the objective is to regulate the state x_1 at 0.02mm from some initial condition.

Now consider this system under some uncertainty in the coil resistance R. We have implemented this uncertainty by substituting R with $R + dR$ where dR is the new uncertain parameter to be estimated. In **Mathematica** a simple rule

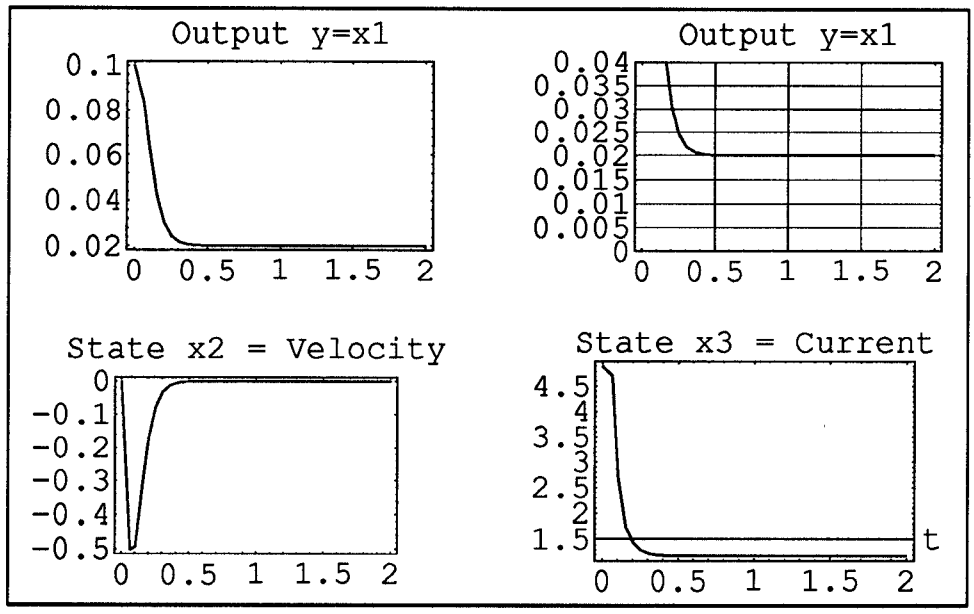

Figure 8.8: Response of the magnetic levitation system to the feedback linearizing control law under no parameter uncertainty

achieves the above substitution everywhere from the model to the control law. For example:

```
newfff:=fff/.{R -> dR+R}
```

The adaptive control law and update rule are too long to write here but a simplified and chopped (up to terms of order 10^-9) evaluated at their nominal values are as follows:

```
In[3]:= Chop[Simplify[Adaptivecontrol],10^-9]
Out[3]= {(1.30156 10   (1.11929 10    x1   + 0.0000207317 x1   +
                   4                 5         6                -7    2
>     0.00199202 x1   + 0.0872284 x1   + 1. x1   + 2.66272 10    x1   x2 +
```

```
                    3                    4                  5
>    0.0000471547 x1  x2 + 0.00416429 x1  x2 + 0.146667 x1  x2 +
                -9   2   2             -7   3   2
>    3.60183 10   x1  x3  + 2.23577 10   x1  x3  +
                                                  -8   3   2
                -9             3   2   1.17365 10   x1  x3
>    7.68307 10   thetabar1 x1  x3  - -------------------- -
                                                        2
                                          (0.00537 + x1)
                -6   4   2                  5   2
     1.03646 10   x1  x3    0.0000365042 x1  x3
>    -------------------- - -------------------- +
                    2                     2
       (0.00537 + x1)         (0.00537 + x1)
                -8   2      2            -8   3   (3)
>    1.61345 10   x1  x2 x3  + 6.2873 10   x1  y1   [t] +
                -6   4   (3)                    5   (3)
>    5.55238 10   x1  y1   [t] + 0.000195556 x1  y1   [t])) /
                   3
>    (0.00537 + x1)  x3)}

In[4]:=Simplify[rhstheta]

Out[4]= {(7.68307 10   (118.395 x1 x3  + 22047.5 x1  x3  +
                  6   3   2     4   2                  2
>      1.36856 10  x1  x3  + x1  x3  + 82.372 x1 x2 x3  +
```

```
                                                                  4
                  2      2              3      2   0.054822 x1 x3
>       15339.3 x1  x2 x3  + 952159. x1  x2 x3  - --------------- -
                                                                  2
                                                 (0.00537 + x1)
                  2   4              3   4
        10.2089 x1  x3    633.701 x1  x3                  2   (3)
>       --------------- - --------------- + 0.552213 x1 x3  y1   [t] +
                    2                 2
        (0.00537 + x1)    (0.00537 + x1)
                  2   2   (3)                 3   2   (3)
>       102.833 x1  x3  y1   [t] + 6383.17 x1  x3  y1   [t])) /


                   3                                  2
>  ((0.00537 + x1)  (0.000037122 + 0.0122829 x1 + x1 ))}
```

Note that the control depends on the estimate of dR represented here as **thetabar1**. The small coefficient for **thetabar1** shows that with the desired poles above, the closed loop system is not sensitive to uncertainty in dR since under no parameter uncertainty, the above control law would have been the **exactly** linearizing control law. Figures (8.9) and (8.10) show the response to the adaptive controller and the parameter estimate.

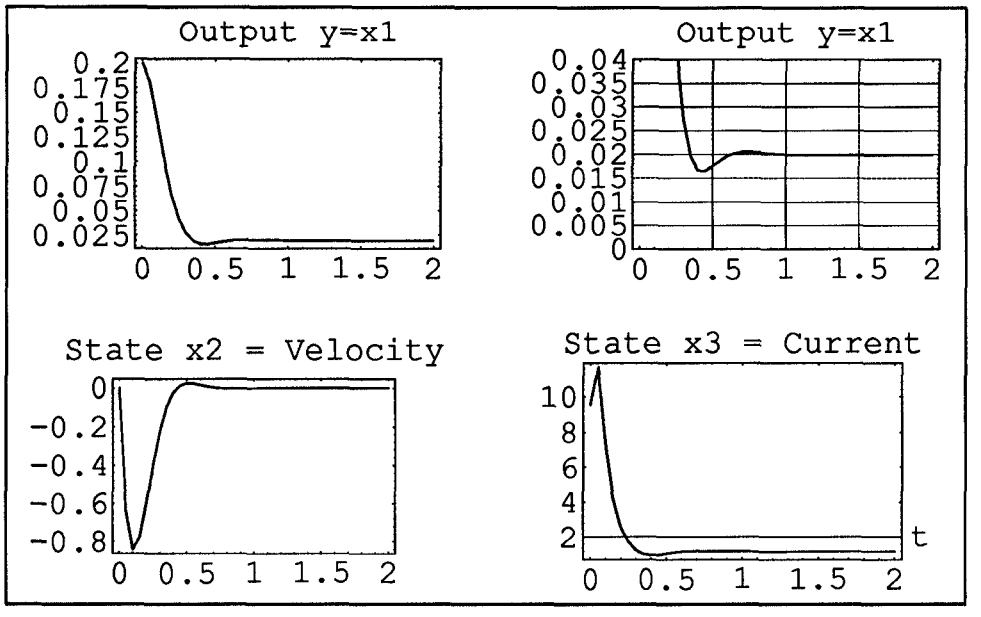

Figure 8.9: Response of the magnetic levitation system to the adaptive feedback linearizing control law under 20% parameter uncertainty in the coil resistance $R_{..}$

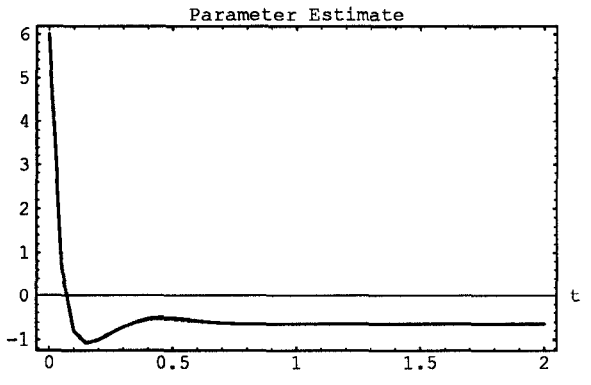

Figure 8.10: Parameter estimate for uncertainties in the coil resistance

8.4 Design of Active Suspensions

In this section, we discuss the design of nonlinear adaptive control algorithms for active suspensions for vehicles, focusing on a model of the Army High Mobility Multipurpose Wheeled Vehicle (HMMWV).

Vehicle suspensions are designed to provide effective isolation of the passenger and payload from road disturbances and to enhance vehicle stability and control (safety) during handling maneuvers. Performance of the suspension should be insensitive to parameter variations such as external loads and damping coefficients. However, the requirements necessary to achieve these different goals conflict and the traditional **passive** suspension design has always been a **compromise** between these objectives. While insensitivity to parameter variations and external loads requires a **stiff** suspension, road comfort requires a **soft** suspension [53]. On the other hand, good road handling and stability requires a suspension that is not too stiff nor too soft.

With increasing demand on vehicles suspensions ability to perform effectively over a wide range of performance requirements and road disturbances at higher vehicle speeds, it has become clear that there are various unavoidable performance limitations when only passive elements are used in the suspension design. Although there has been attempts to remove these limitations by allowing some suspension parameters to adjust, either manually or automatically, to the changes in the system characteristics, the improvement has been quite limited since the performance of any passive suspension system is inherently limited primarily due to the following reasons:

- Passive elements can only **store** or **dissipate** energy and are bound to be

active only in direct response to the motion of only few **local** variables.

- Suspension parameters have to cope with a wide range of operating conditions.
- A compromise must be made between **ride comfort** and **safety** (i.e. soft setting and stiff setting.)
- Passive forces can only be exercised for appropriate spring deflections and can not be provided **arbitrarily**.

Over the past twenty years there has been considerable interest in **active suspensions**. The design of active suspension systems for passenger vehicles has been investigated for more than 25 years [11]. In an active suspension system, the reaction to the applied forces are supplied by automatically controlled powered actuators. In addition, arbitrary forces can be continuously exercised between wheels and car body on demand by a control law which can effectively **supply** energy to parts of the suspension system and **remove** energy from other parts. The control law responses to various variables in the suspension system most of them **remotely** measured. Therefore, compared to a passive suspension, active suspension may appear **soft** for ride comfort and **stiff** for handling purposes. This can substantially improve the performance of the vehicle suspension system with respect to the objectives outlined above (i.e. ride, handling, and parameter variations.) The actuators are typically hydraulic with electronic controls and are usually combined with a passive spring/damper element in parallel (see figure (8.11).)

We have used the system shown in Figure 8.6 to support the design of active suspension control systems for **off-road** vehicles [16]. Most of the research on

active suspension system in the past has been directed to improve passenger comfort and road handling [9, 46, 71, 74, 85]. Less work has been done on design of active suspensions for off-road vehicles where the concerns are stability and high speed performance. In off-road conditions large (nonlinear) excursions take place in the suspension system. Active suspensions offer many specific advantages for military vehicles, including lowering ground clearance to reduce visibility, and increasing cross country speed, while maintaining a tolerance level of passenger comfort. Design of suspension to permit higher speeds at a constant absorbed power level is the principal concern in this study.

Section (8.4.1) gives the equations describing the dynamic behavior of a half-car model of a passenger vehicle suspension system and discusses the elements of an active suspension system. The nominal parameter values used in the linear case are based on the US Army HMMWV. In the nonlinear setting, we have assumed that the springs and damper elements are defined by smooth nonlinear functions of their arguments. For simulation purposes, a simple cubic nonlinearity was chosen. The results of the computer aided controller design for the nonlinear model is then presented in section (8.4.2), followed by performance evaluation.

8.4.1 Models of Active Suspension Systems

The HMMWV is a 4×4 wheeled vehicle with a gross vehicle weight of 7800lbs. and 8 inches of suspension travel. In this work we use the half-car model shown in Figure 8.11. This two-dimensional vehicle model includes vehicle pitch and vertical motions, as well as the vertical motions of the wheels. Roll motion is neglected. Nonlinear suspension spring and damper characteristics are included

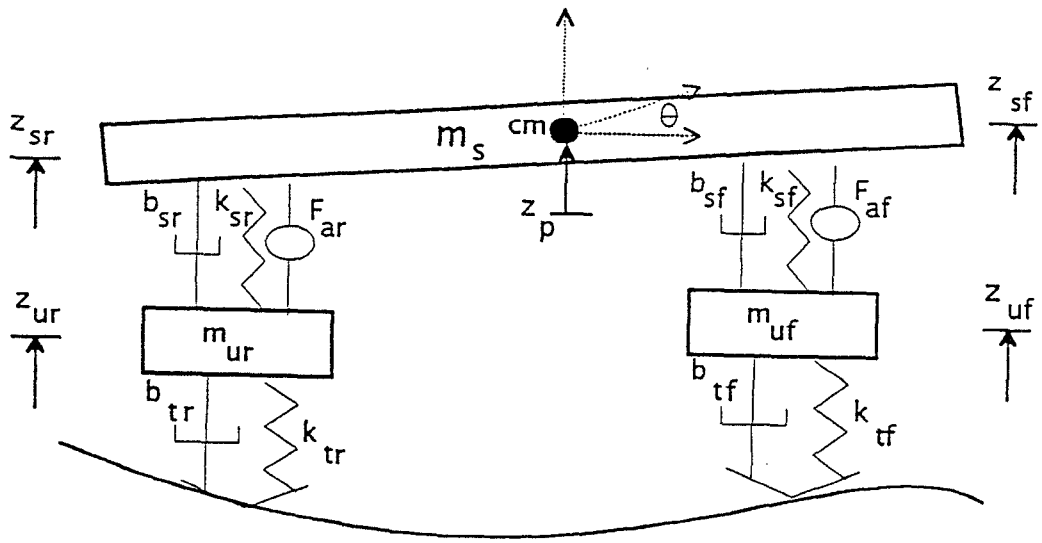

Figure 8.11: Half Car vehicle model.

in the model. We consider active suspension elements working in parallel with the passive elements. We assume that it is possible to exert active control forces at the two wheels in parallel with the passive suspension forces. Control laws are designed based on measurements of suspension displacements and velocities.

The model is as follows:

$$m_s\ddot{z}_p = F_{pf} + F_{af} + F_{pr} + F_{ar} - m_s g \tag{8.30}$$

$$j_0\ddot{\theta} = F_{pf}L_f + F_{af}L_f - F_{pr}L_r - F_{ar}L_r \tag{8.31}$$

$$m_{uf}\ddot{z}_{uf} = -F_{pf} - F_{af} + F_{tf} - m_{uf}g \tag{8.32}$$

$$m_{ur}\ddot{z}_{ur} = -F_{pr} - F_{ar} + F_{tr} - m_{ur}g \tag{8.33}$$

where

z_p = position of center of mass

z_{sf} = forward sprung mass point

$$
\begin{aligned}
z_{sr} &= \text{rear sprung mass point} \\
z_{uf} &= \text{forward unsprung mass position} \\
z_{ur} &= \text{rear unsprung mass position} \\
\theta &= \text{body pitch angle} \\
z_{rf} &= \text{forward road position amplitude} \\
z_{rr} &= \text{rear road position amplitude} \\
g &= \text{gravity}
\end{aligned}
\tag{8.34}
$$

and the passive forces acting on each mass are

$$
\begin{aligned}
F_{pf} &= -k_{sf}(z_{sf} - z_{uf}) - b_{sf}(\dot{z}_{sf} - \dot{z}_{uf}) \\
F_{pr} &= -k_{sr}(z_{sr} - z_{ur}) - b_{sr}(\dot{z}_{sr} - \dot{z}_{ur}) \\
F_{tf} &= k_{tf}(z_{rf} - z_{uf}) + b_{tf}(\dot{z}_{rf} - \dot{z}_{uf}) \\
F_{tr} &= k_{tr}(z_{rr} - z_{ur}) + b_{tr}(\dot{z}_{rr} - \dot{z}_{ur})
\end{aligned}
\tag{8.35}
$$

We assume that the springs and damper elements $k_{sf}(\cdot), k_{sr}(\cdot), b_{sf}(\cdot), b_{sr}(\cdot)$ are defined by smooth nonlinear functions of their arguments. Thus, the forces generated by the springs are of the form

$$
F_j^k = k_j(\Delta x) \tag{8.36}
$$

where $j = sf, sr, tf, tr$ and $k_j(\cdot)$ is a smooth (analytic) function of its argument. For example, a simple choice is the cubic function $F^k = k_1(\Delta x) + k_2(\Delta x)^3$.

The nominal parameter values (linear case) used in this study are based on the US Army HMMWV and are given in Table 8.1.

We **assume that the pitch angle of the vehicle remains small**, which gives the additional conditions

$$
z_{sf} = z_p + L_f \sin(\theta) \approx z_p + L_f \theta
$$

g	=	32.16	gravity ft/sec^2
m_s	=	7077.0/(2 g)	7077.0 lbs is the sprung weight of the HMMWV
j_0	=	3995.0/2.0	Pitching moment
m_{uf}	=	868.0/(4 g)	868.0 lbs is the unsprung weight of the HMMWV
m_{ur}	=	868.0/(4 g)	rear unsprung mass
b_{sf}	=	628.0	front damper midpoint value
k_{sf}	=	2760.0	front spring midpoint value
b_{sr}	=	749.0	rear damper midpoint value
k_{sr}	=	4380.0	rear spring midpoint value
k_{tf}	=	12381.0	stiffness of front tire
b_{tf}	=	15.0	estimated damping of front tire
k_{tr}	=	14015.0	stiffness of rear tire
b_{tr}	=	15.0	estimated damping of rear tire
L_f	=	5.7667	distance (ft.) of center of mass from front end
L_r	=	5.0667	distance (ft.) of center of mass from rear end
L_v	=	$L_f + L_r$	length (ft.) of the vehicle

Table 8.1: Nominal parameter values for the HMMWV.

$$z_{sr} = z_p - L_r \sin(\theta) \approx z_p - L_r\theta$$

The suspension system active controls are

$$F_{af}, F_{ar} = \text{ active forces at the front and rear wheels}$$

These forces are generated by actuators which may have dynamics and be subject to nonlinear effects like saturation. In this work, we also studied a suspension model where each actuator is modeled by a first order filter and a tangent hyperbolic saturation form.

We use the state variables:

$$\begin{aligned}
x_1 &= \dot{z}_p \text{ velocity of center of mass} \\
x_2 &= \dot{z}_{sr} \text{ velocity of rear sprung mass point} \\
x_3 &= \dot{z}_{uf} \text{ velocity of front unsprung mass} \\
x_4 &= \dot{z}_{ur} \text{ velocity of rear unsprung mass} \\
x_5 &= z_{sf} - z_{uf} \text{ front rattle space} \\
x_6 &= z_{sr} - z_{ur} \text{ rear rattle space} \\
x_7 &= z_{uf} - z_{rf} \text{ front tire deflection} \\
x_8 &= z_{ur} - z_{rr} \text{ rear tire deflection}
\end{aligned} \tag{8.37}$$

This system may be written in the abstract form

$$\dot{z} = f(z) + \sum_{i=1}^{n_c} g_i(z)u_i(t) + \tilde{R}(t) - \tilde{G} \tag{8.38}$$

where $n_c = 2$ is the number of controls, the state vector is $z = [x_1, \ldots, x_8]^T$, and the coefficient functions $f(z), g_i(z)$ are derived from the second order model (8.30)-(8.35) in the usual way. The terms $\tilde{R}(t), \tilde{G}$ are the road and gravity inputs, respectively.

Defining a set of outputs $y = h(z)$, our goal is to design control laws to isolate or "decouple" selected outputs from the the road disturbances. For example, choosing the velocity of the body center of mass and the pitch angle rate as outputs, we can define control laws using the theory of input/output linearization to completely isolate the outputs from the road disturbances.

8.4.2 Design and Performance Evaluation

In this section we present some typical cases illustrating the use of asymptotic output tracking (adaptive) control laws for active feedback control of active suspension elements. The results presented illustrate the capabilities of the control algorithms.

We present results in two categories. First we give a case which shows the decoupling performance of the asymptotic output tracking control laws when there are no parametric uncertainties in the suspension system. We show that the designated outputs, e.g., sprung body mass position and pitch angle, can be commanded to track a prescribed trajectory independently of the road profile. Second, we present several cases in which there are parametric uncertainties present, including uncertainties in the values of the spring and damper constants at each of the wheels.

Tracking Control with No Parameter Uncertainty: First we consider a simple case when the parameter values including the masses, inertias, spring and damper constants, and center of mass position are known. The nominal

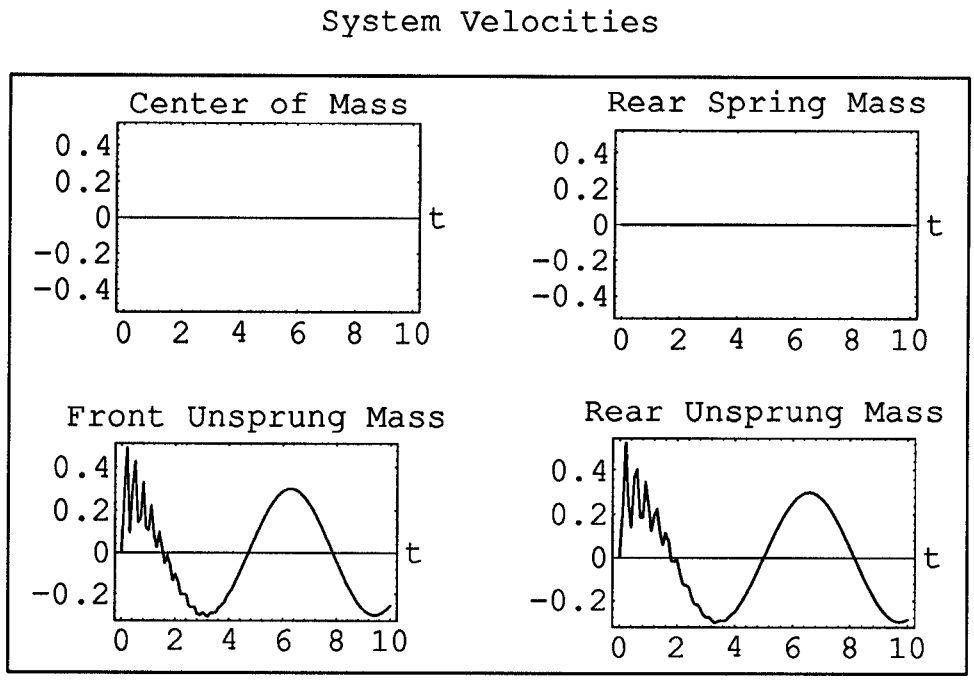

Figure 8.12: Response of HMMWV half car model traveling over a sinusoidal road with the sprung mass stationary.

parameter values are those in Table 8.1.[3]

In Figure 8.12 we show the response of the half car model of the HMMWV (Figure 8.11) traveling over a single frequency sinusoidal road. (The spring and damper elements are nonlinear as in (8.36).) In this case there are two active control elements. We can, therefore, decouple two selected outputs. In the figure we have commanded the velocity of the center of mass ($\dot{z}_p(t)$) and the velocity of the rear hull point ($\dot{z}_{sr}(t)$) to track zero; hence, the trajectories are trivial. Note the high frequency oscillations in the unsprung mass velocities. These indicate the effects of the increased stiffness in the effective suspension. In Figure 8.13 we

[3]The dimensions of all the variables shown in the graphs in this section are in feet or feet/sec. etc.

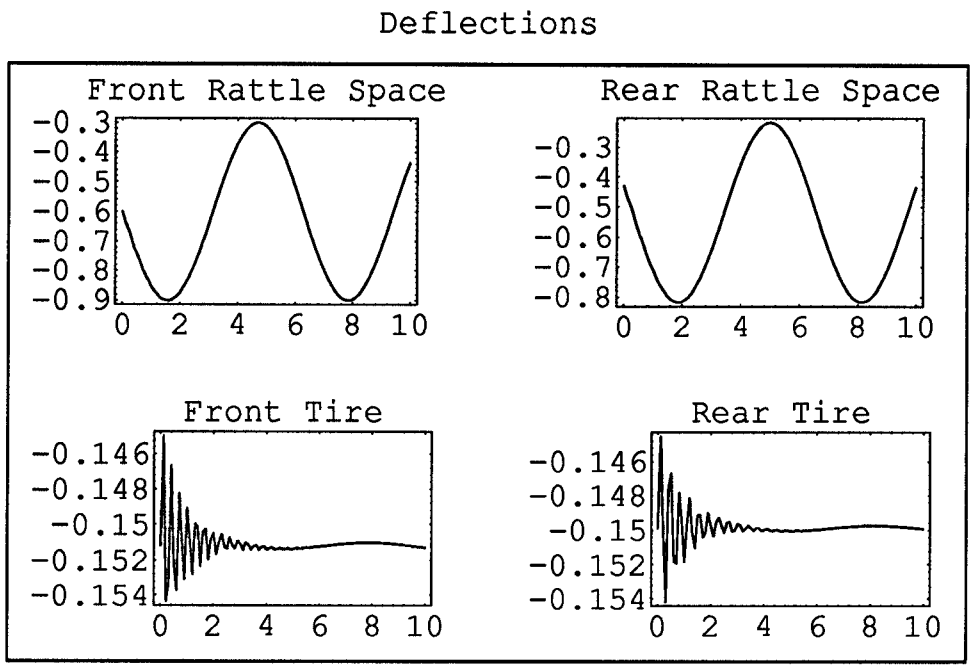

Figure 8.13: Rattle space response for HMMWV half car model traveling over a sinusoidal road with the sprung mass stationary.

show the rattle space response and the tire deflections. Note the low amplitudes of the high frequency components of the tire deflections.[4]

These cases clearly indicate the capability of the tracking control laws based on partial feedback linearization to isolate selected outputs from selected disturbances. Note that active control is required to achieve the decoupling and tracking. In Figures (8.14) and (8.15) we show the response of the half car model of the HMMWV to a sinusoidal road when there are uncertainties in the spring constants and the same controller (nonadaptive) as before is used. Acceleration of the center of mass is shown in Figure (8.16).

[4]Our simulation software includes automatic scaling of graphs to produce the full plot range.

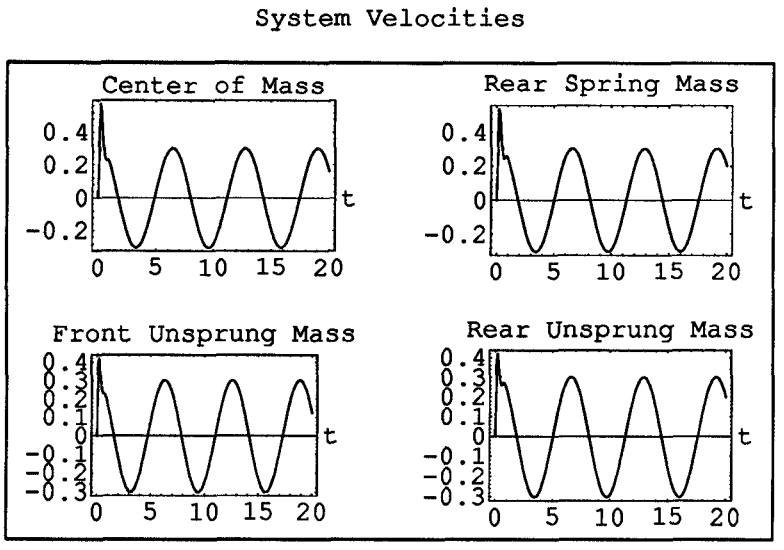

Figure 8.14: Velocities in HMMWV 1/2 car model under parameter uncertainty in k_{sf} and k_{sr}.

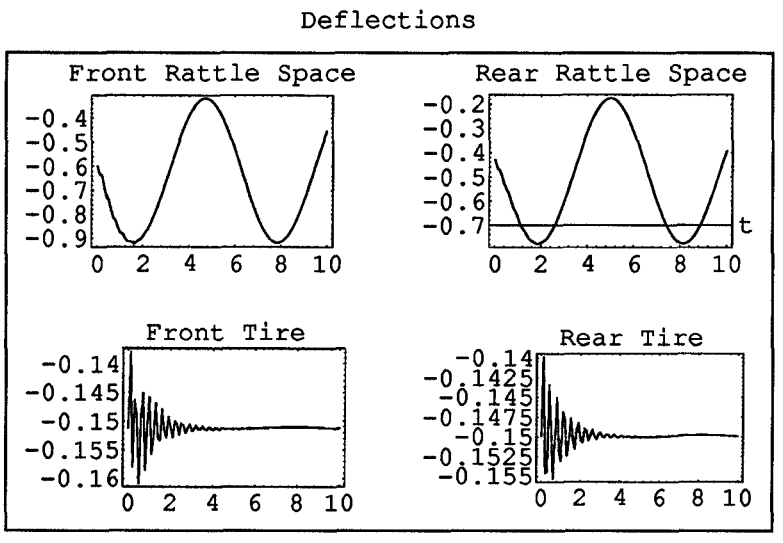

Figure 8.15: Deflections in HMMWV 1/2 car model under parameter uncertainty in k_{sf} and k_{sr}.

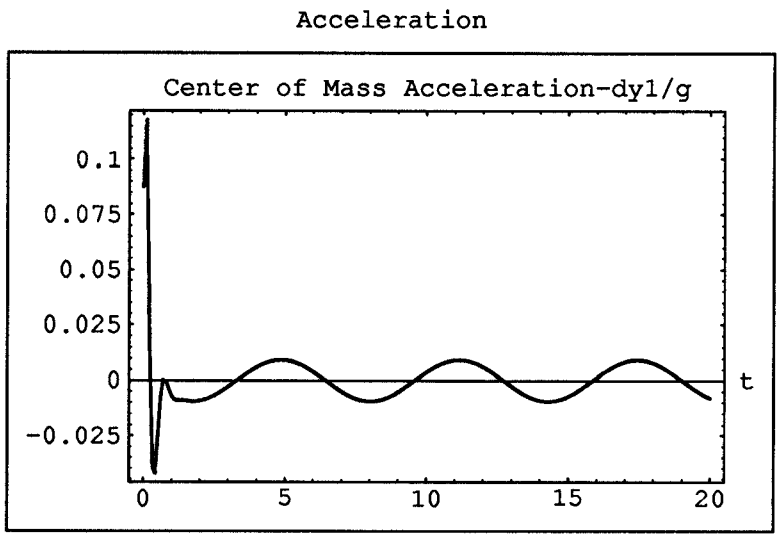

Figure 8.16: Center of mass acceleration for HMMWV 1/2 car model under parameter uncertainty.

Tracking with Adaptive Control: Now we present several cases which illustrate the performance of the asymptotic tracking control with parameter adaptation. In each case we use the HMMWV 1/2 car model (8.30)-(8.35). In Figure 8.17 we show the response of the half car model to a sinusoidal road when there are uncertainties in the spring constants. Specifically, the two active controls are tasked to isolate the body center of mass velocity and pitch rate – as "regulated outputs" – from the road disturbances, that is, to track zero as a reference signal. The control law also includes a loop to estimate the two uncertain parameters. The control law is the asymptotic output tracking law (8.25) with the parameter update given by (8.26):

```
In[1]:= control

                                                 9   3
Out[2]= {1654.93 + 464.258 x1 + 6.13873 10  x1  - 350.85 x2 -
                 9   2                 9      2            8   3
>  9.80307 10  x1  x2 + 5.21825 10  x1 x2 - 9.25904 10  x2 - 628. x3
```

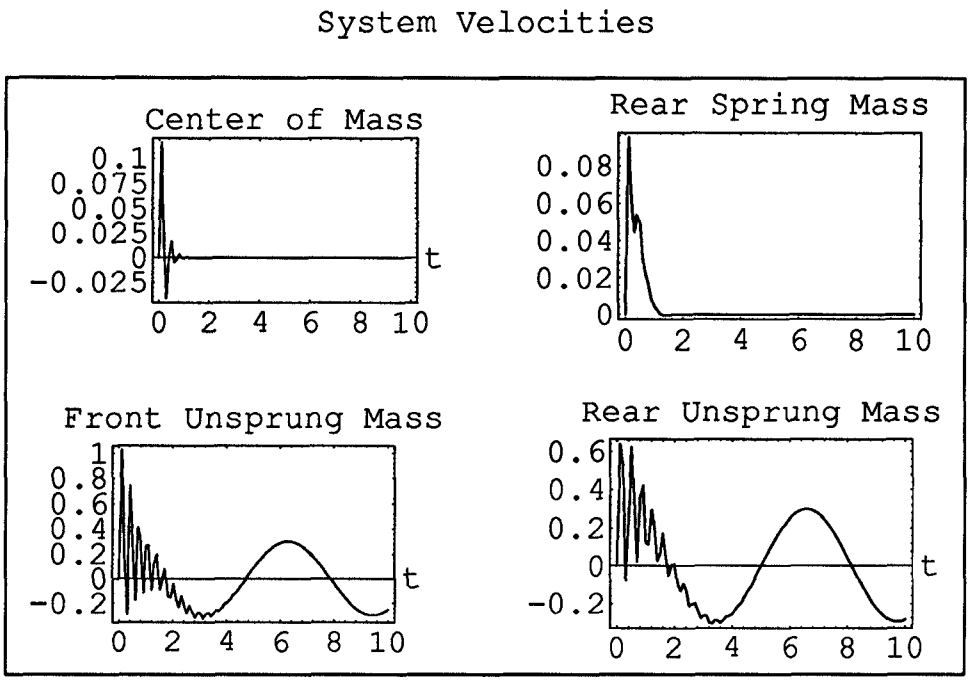

Figure 8.17: Velocities in HMMWV 1/2 car model with tracking and adaptive control.

```
                9   2                  9                        9   2
> -8.61311 10  x1  x3 + 9.16965 10  x1 x2 x3 - 2.44054 10  x2  x3 +
                9      2              9      2           8   3
> 4.02829 10  x1 x3  - 2.14429 10  x2 x3  - 6.28 10  x3  -
                -7   2                  -7      2              -7   3
> 3.57628 10   x2  x4 + 3.57628 10   x2 x4  - 1.78814 10   x4  +
                          3                    3
> 2760. x5 + 27600. x5  + 10. thetabar1 x5 ,
                                                  -7   2
> 1883.57 - 221.775 x1 + 385.088 x2 - 9.53674 10   x1  x2 -
```

```
                -7       2            8   3              -7
>  4.76837 10    x1 x2  + 7.49 10  x2  + 9.53674 10    x1 x2 x3 -
                 -7   2                 -7       2
>  2.38419 10    x2  x3 + 2.38419 10    x1 x3  - 749. x4 -
              9   2                9       2           8   3
>  2.247 10  x2  x4 + 2.247 10  x2 x4  - 7.49 10  x4  + 4380. x6 +
               3                    3
>  43800. x6  + 10. thetabar2 x6 }
```

The high frequency components in the unsprung mass velocities are caused by the initial conditions and the initial "stiffness" introduced into the system. It is clear from the deflections shown in Figure 8.18 that the effects of these high frequency components is very small.

In Figure 8.19 we show the spring constant estimates. The spring constants used in this simulation are

$$k_{sf} = 2760.0 + dksf \quad \text{front spring midpoint value}$$
$$k_{sr} = 4380.0 + dksr \quad \text{rear spring midpoint value}$$

with $dksf, dksr$ deviations of the spring constants (midpoint values) from nominal. The adaptive controller estimates $dksf$, which has the value 300 (about 10% change) in this simulation. Clearly, the adaptive loop performs well. Figure (8.20) shows the outputs. Active Controls are plotted in Figure (8.21), center of mass acceleration is plotted in Figure (8.22)

In Figures (8.23) and (8.24) we show the parameter estimates and output tracks in a case when X both the spring and damper constants are uncertain. The correct values are

$$dksf = 0.0, \qquad dksr = -10.0, \qquad dbsf = -50.0, \qquad dbsr = -100.0$$

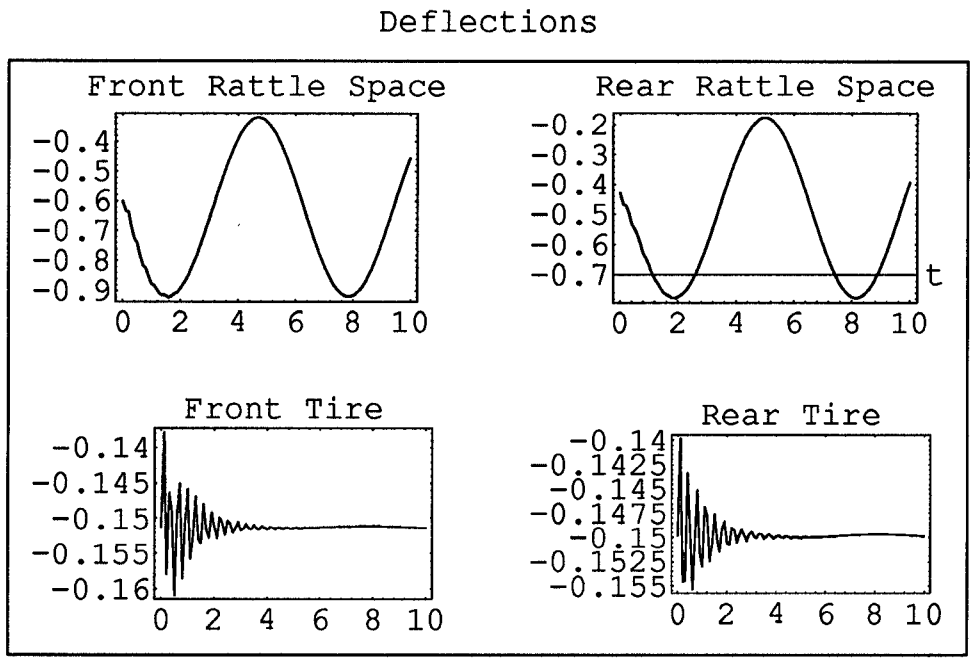

Figure 8.18: Deflections in HMMWV 1/2 car model with tracking and adaptive control.

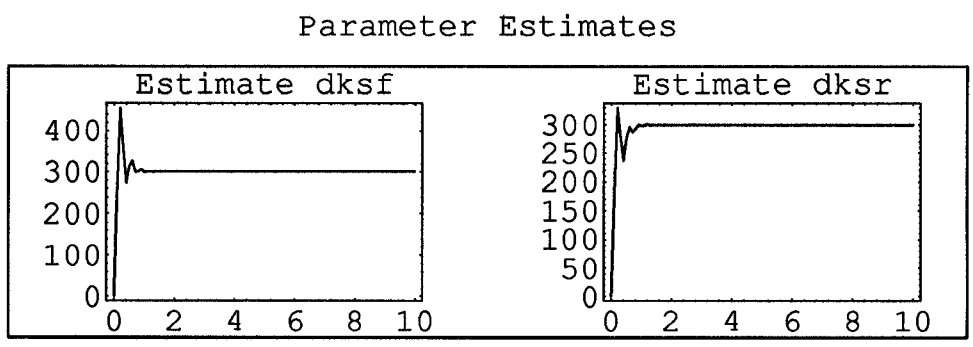

Figure 8.19: Spring constant estimates in HMMWV 1/2 car model with tracking and adaptive control.

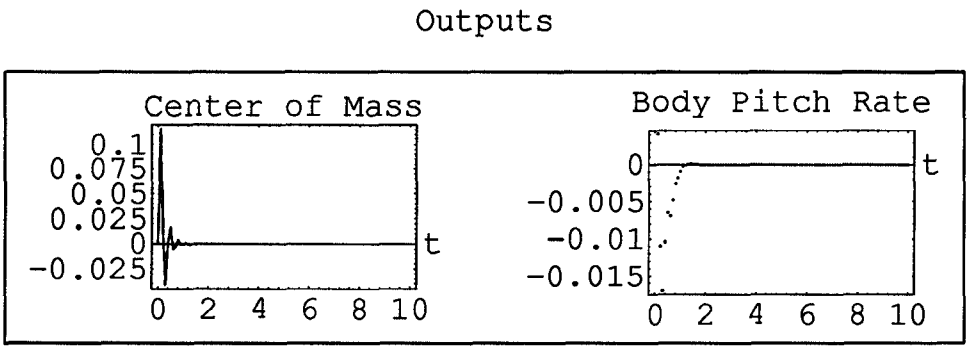

Figure 8.20: System outputs: center of mass velocity and body pitch rate for HMMWV 1/2 car model with tracking and adaptive control.

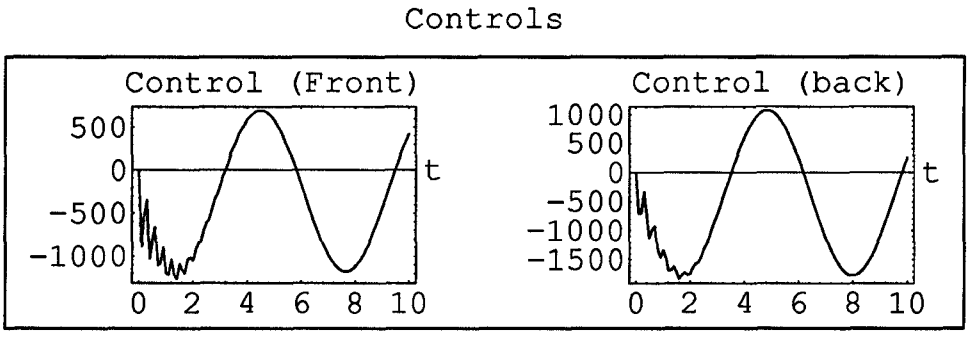

Figure 8.21: Control laws for front and rear, HMMWV 1/2 car model with tracking and adaptive control.

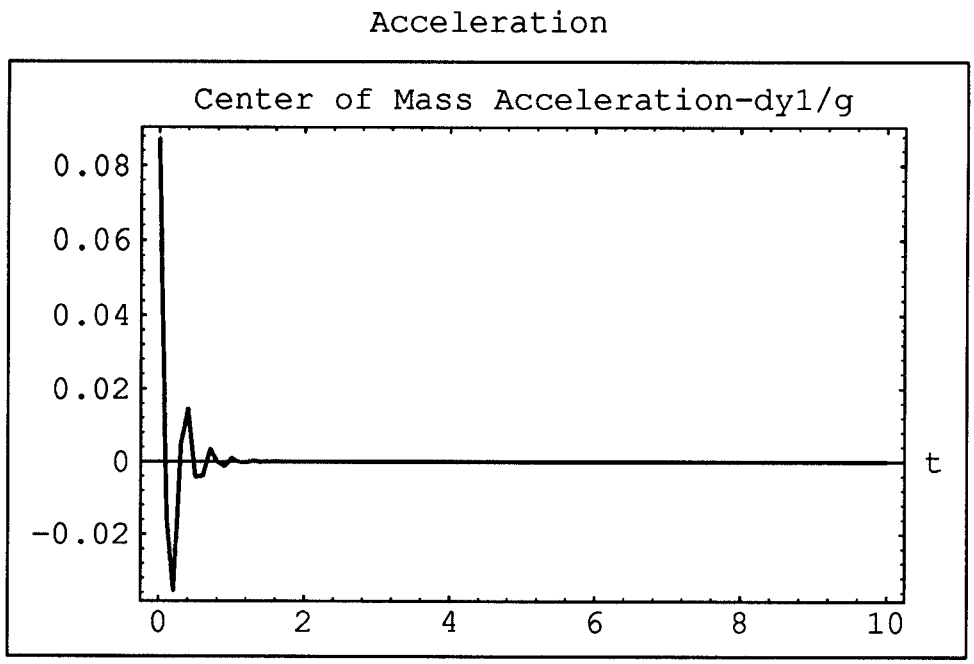

Figure 8.22: Center of mass acceleration for HMMWV 1/2 car model with tracking and adaptive control.

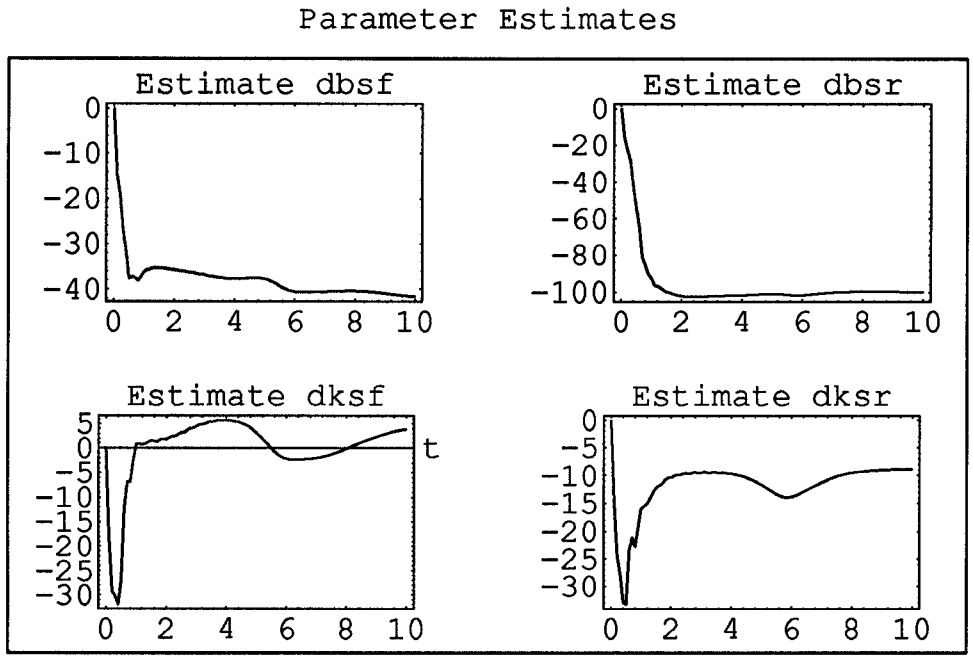

Figure 8.23: Parameter estimates for uncertainties in spring and damper constants.

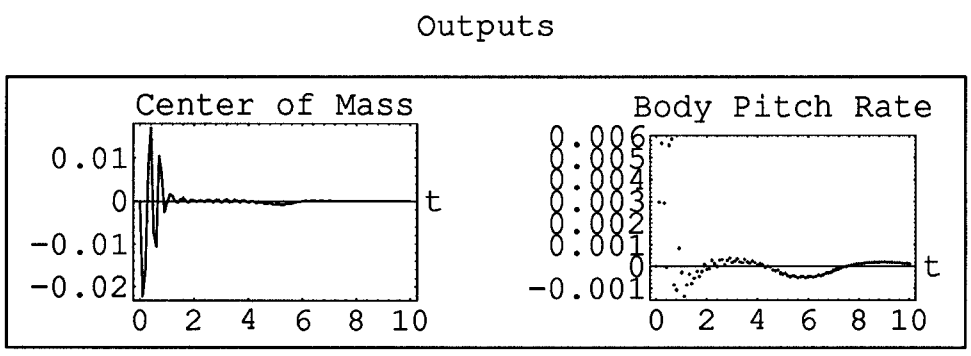

Figure 8.24: Output tracks for uncertainties in spring and damper constants.

The damper values are substantial errors in the nominal values $b_{sf} = 628.0, b_{sr} = 749.0$. Taking note of the small sizes of the variations in the output tracks (from zero) tracking performance of the regulated outputs is satisfactory. (The disturbance is a sinusoidal road.)

8.5 Conclusions

The examples in this chapter illustrate the potential of an integrated symbolic-numerical computing tool for the computer aided design and evaluation of nonlinear control systems. The suspension system example involves synthesis of the model equations (from a Lagrangian), computation of the PFL control law, computation of the parameter adaptation law, generation of a numerical code (in C) to simulate the closed loop adaptive control, compilation of the code (inside **Mathematica**), execution and post processing of the results. This cycle takes less than one minute on an IBM compatible 486 PC or a NeXT computer. The differential equations involve 8 states and 4 parameter update differential equations. (10 states taking actuator dynamics into account.) Other parameters such as the location of the center of mass may also be added at the cost of a more complex control law. Note that other nonlinearities may easily be added to the model for analysis and design. The procedures used in this software are very systematic and can cope with a variety of complex nonlinearities in the model typically too difficult to work with by design engineers.

We illustrated the use of this tool to design a nonlinear adaptive controller for active suspensions for off-road vehicles, focusing on a model of an Army wheeled vehicle (HMMWV). We discussed the performance of this controller and showed

that the body center of mass velocity (and acceleration) and pitch rate can be isolated from the road disturbances. Simulations show that under parameter uncertainty in the front and rear spring constants (around 10%), and both spring and damper constants (around 10%), adaptive controller performs well. The road profile in this study was a single frequency sinusoid with amplitude 0.3 feet and frequency 1.0 radian at speeds 25-50 mph.

While more complex examples will require more computing resources, we believe that the ease with which one can couple symbolic and numerical computing will prove invaluable in extending the design technology available today.

Chapter 9

Conclusions and Future research

In this dissertation, we demonstrated that many restrictive assumptions required in current nonlinear adaptive control literature for adaptive tracking and regulation can be softened using approximate linearization techniques. As a result, it is possible to achieve reasonable tracking performance under parameter uncertainty in nonlinear dynamics for a large class of nonlinear systems. For the problem of adaptive output tracking, we were able to design adaptive controllers for nonlinear systems that are slightly non-minimum phase and do not have a well defined (vector) relative degree. Recall that for non-minimum phase systems, exact tracking is not feasible. It is important to note that while the controller was designed using the approximate (minimum phase) system, the adaptive loop was constructed using the true system in order to avoid any parameter drift typically caused by dynamic uncertainty in the system. For the adaptive regulation/stabilization problem, we were able to remove the long standing assumption on linearity of the unknown parameters in the nonlinear dynamics. We also replaced the involutivity assumption, necessary for exact full or partial linearization technique, with a ρ-order feedback linearization. In all cases, we

proved that the design schemes result in an asymptotically stable closed loop system and showed that the adaptive controller can achieve output tracking of reasonable reference trajectories with bounds on the tracking error.

We illustrated the significance of the adaptive approximate linearization approach by considering its potential application to the design of flight control system, focusing on PVTOL Harrier aircraft. We demonstrated that the non-adaptive controller produces undesirable performance when there is parameter uncertainty (mass and moment of inertia) present in the system. For such systems, standard (non-adaptive) controllers are not robust against modeling errors arising from parametric uncertainty in the model. We argued that since this system has an unstable inverse, in particular unstable zero-dynamics, none of the previously known techniques of adaptive control is applicable. We showed that the adaptive controller significantly improves the performance by providing about 90% better signal tracking performance.

For the case where a vector relative degree can be achieved via dynamic inverse algorithm of Hirschorn and Singh, we constructed an adaptive right-inverse to serve as an prefilter to the original system under parameter uncertainty. The inverse system was then used to generate a control signal necessary to achieve output tracking or regulation. Application to outer loop control of an aircraft demonstrated this technique.

As a step to bring nonlinear and adaptive control theory into a practical control design strategy, we described a prototype symbolic-numerical software system for analysis and design of nonlinear models. Such a system has a good potential to effectively "translate" the mathematically oriented "language" of nonlinear and adaptive system analysis into a graphical "language" easily under-

standable to engineering practitioners. This software system has capabilities for modeling complex nonlinear systems, for design of nonlinear tracking/regulation and adaptive control laws for such models, and for generation of simulation codes (in C) for performance evaluations. As an illustration, we used this tool for the design of nonlinear adaptive active suspension control systems for an off-road vehicle based on the Army High Mobility Wheeled Vehicle. Using this software system, we showed that it is possible to design adaptive asymptotic tracking control laws to effectively isolate the sprung body dynamics from the road disturbances. We also considered the design of an adaptive magnetic levitation control system under parameter uncertainty in the coil resistance.

Much interesting work remains to be done. Several interesting applications have recently been investigated in the nonlinear setting. There are many unresolved theoretical issues in adaptive control. For example, there are no good results on analysis of convergence rates in nonlinear adaptive control. It is likely that interesting developments will be made using averaging techniques. Existing adaptation schemes for trajectory tracking generally require linear parametrization of the unknown parameters. There is no measure to compare the robustness of various adaptive schemes. Very little work has been done on dealing with measurement noise and dynamic uncertainties (high-frequency unmodeled dynamics) in the nonlinear setting. Peaking phenomenon in nonlinear systems and how to counteract peaking in adaptive setting needs to be explored further. Adaptive output feedback control and adaptive state observer design for nonlinear systems are relatively new areas with very interesting potential for practical use. Most of the available techniques use highly restrictive assumptions and important developments are still to follow. Adaptive, robust, and observer-based backstepping

techniques of Kokotovic and co-workers have proved extremely useful, and it is very likely that more developments will follow from their use. Approximate techniques in nonlinear geometric control developed by Krener have also been very useful to obtain a more practical design methodology and remove many regulatory assumptions that restrict the use of the current theory. Also, the field of adaptive nonlinear control theory has basically been built around the results of differential geometric control theory. Combining these results with other areas of nonlinear and engineering system analysis would be helpful. Other open areas of research in this field includes adaptive optimization, semi-global adaptive tracking, and adaptive switching controllers.

There are also many important problems arisen in applications. For example, the issue of how to initialize an adaptive system and reduce the transient response is still unclear. The problem of how to design effective safeguards and how to efficiently interconnect several different algorithms are still open. Much work needs to be done on implementing the existing theoretical results into real-world applications. We feel that integrated symbolic-numerical software systems for automatic generation of the adaptive control laws and update rules for complex systems are very instrumental in closing the current gap between the theory and practice of adaptive nonlinear control.

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