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**Local Inversion of the Radon Transform in
Even Dimensions Using Wavelets**

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Local inversion of the Radon transform in even dimensions using wavelets

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Abstract. We use the theory of the continuous wavelet transform to derive inversion formulas for the Radon transform on $L^1 \cap L^2(\mathbf{R}^d)$. These inversion formulas turn out to be local in even dimensions in the following sense. In order to recover a function f from its Radon transform in a ball of radius $R > 0$ about a point x to within error ϵ , we can find $\alpha(\epsilon) > 0$ such that this can be accomplished by knowing the projections of f only on lines passing through a ball of radius $R + \alpha(\epsilon)$ about x . We give explicit a priori estimates on the error in the L^2 and L^∞ norms.

0. Introduction.

Given a function f defined on \mathbf{R}^d , its *Radon transform*, Rf , is defined by

$$R_\theta f(s) = \int_{\theta^\perp} f(s\theta + y) dy,$$

where $\theta \in S^{d-1}$ and $s \in \mathbf{R}$. $Rf(\theta, s)$ is the integral of f on the hyperplane in \mathbf{R}^d defined by $\{x: \langle x, \theta \rangle = s\}$. The *backprojection operator* is given by

$$R^\# g(x) = \int_{S^{d-1}} g(\theta, \langle x, \theta \rangle) d\theta,$$

where $x \in \mathbf{R}^d$ and g is defined on $S^{d-1} \times \mathbf{R}$, which may be identified with the set of hyperplanes in \mathbf{R}^d . Then the identity

$$\frac{1}{2} R^\# I^{1-d} Rf = f,$$

holds where, for $\alpha \in \mathbf{R}$, I^α is the Riesz potential operator defined by

$$(I^\alpha f)^\wedge(\gamma) = |\gamma|^{-\alpha} \hat{f}(\gamma).$$

If d is odd, then I^{1-d} amounts to differentiation and so recovery of $f(x)$ requires only the projections of f on lines passing through a neighborhood of x . If d is even, this is not the case, e.g., [N].

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In practice, it is often useful to seek a function $e(t)$ on \mathbf{R} so that $E(x) = R^\#e(x)$ approximates the δ -distribution (which is the convolution identity). E is commonly referred to as a *point-spread* function. In this case, the *filtered backprojection* formula,

$$R^\#(e * Rf)(x) = E * f(x),$$

holds and is used to recover a good approximation to f . The point-spread function E is related to e by

$$e = \frac{1}{2}I^{1-d}RE.$$

If d is even, the application of the Riesz potential operator I^{1-d} can introduce singularities into \hat{E} and thereby nullify any good decay that may have been present in E . Therefore, in even dimensions, one cannot expect the function e to have good decay. Of course, if \hat{E} vanishes to high order at the origin, then e would still have good decay, but in this case, E is not a good approximation to the δ -distribution. An attempt to retain locality in even dimensions has been proposed by Smith et. al., and involves the recovery not of f but of Λf where Λ denotes the square root of the Laplacian. This technique is referred to as local tomography (see, e.g., [FKNRS], [K], [SWB]). In the case of local tomography, the formula

$$\Lambda f * E = f * \Lambda E = R^\#(e' * Rf),$$

where $e' = I^{-d}RE$, holds. Since d is even, I^{-d} amounts to differentiation so good decay in E , and even compact support, is retained by e' . Thus, the recovery of $E * \Lambda f$ near a point x requires only knowledge of Rf near $\langle x, \theta \rangle$ for each $\theta \in S^{d-1}$ and hence the projections of f on lines through a ball centered at x .

The purpose of this paper is to investigate the use of wavelets in performing local inversion of the Radon transform in even dimensions. Wavelets give an integral representation of arbitrary functions on \mathbf{R}^d in terms of a two-parameter family of basic functions whose Fourier transforms vanish to high degree at the origin. Combining this representation with filtered backprojection can give a local approximate representation of a function f from local information of its Radon transform. In this paper, we look to the continuous wavelet transform as a substitute for the Λ operator and demonstrate that one can recover the high-frequency (fine-scale) parts of f in a local fashion. In fact, the technique allows for full recovery of f but at the cost of locality, as is to be expected. The tradeoff between partial but local recovery and non-local full recovery is explored from the wavelet point of view. Specifically, we ask to what extent a function in \mathbf{R}^d , d even, can be recovered in a ball of radius $R > 0$ from measurements of its projections on lines passing through a larger ball of radius $R + \alpha$, some $\alpha > 0$. We give explicit a priori error estimates on the recovered function in the L^2 and uniform norms. The application of wavelet and Gabor transforms, and their discrete analogues to the inversion of the Radon transform has been explored in [W], and in the wavelet case, this has been investigated in depth by, e.g., [H], [KS]. The use of wavelets in the local inversion problem has been studied in [DO] in which the

authors show that one needs only global measurements for a few angles, together with full local measurements, in order to get full local reconstruction.

In Section 1, we present some basic properties of the Radon transform. The proofs of these well-known results can be found in, e.g., [N]. In Section 2, we present some basic properties of the continuous wavelet transform. Most of these are well-known and for more details, the reader is referred to, e.g., [D], [FJW]. We do, however, include some error estimates for the inversion formulas in the L^2 and L^∞ cases. While these estimates are elementary, they do not seem to have appeared as such in the literature. In Section 3, we invert the Radon transform by means of the continuous wavelet transform. We give a formula for a d -dimensional wavelet, Ψ , which corresponds to a given one-dimensional wavelet, ψ , such that expanding $R_\theta f$ for each fixed θ , with respect to ψ gives the expansion of f with respect to Ψ . We investigate decay and smoothness properties of each wavelet. We give an inversion formula for the Radon transform in the non-local case and also its local analogue which is obtained for free. Finally, we show how to find the one-dimensional wavelet corresponding to a given d -dimensional wavelet, and investigate decay and smoothness properties of each. We conclude with some examples.

In the remainder of this section, we specify the notation used in this paper. L^p , $1 \leq p \leq \infty$ denotes the usual Lebesgue spaces on \mathbf{R}^d , $d \geq 1$. Unless otherwise specified, all integrals are over \mathbf{R}^d , $d \geq 1$. The Fourier transform of a function in $L^1(\mathbf{R}^d)$, $d \geq 1$ is defined by

$$\hat{f}(\xi) = \int f(x) e^{-2\pi i \langle x, \xi \rangle} dx.$$

If $f \in L^2(\mathbf{R}^d)$, then \hat{f} is defined as a limit in the usual way, and in all other cases, the Fourier transform is to be interpreted in the sense of distributions. $\hat{\mathbf{R}}^d (= \mathbf{R}^d)$ is the dual group of \mathbf{R}^d and represents the “space of frequencies.” The space $A(\mathbf{R}^d)$, $d \geq 1$ is defined by

$$A(\mathbf{R}^d) = \{f: \hat{f} \in L^1(\hat{\mathbf{R}}^d)\}$$

with norm $\|f\|_{A(\mathbf{R}^d)} = \int |\hat{f}(\xi)| d\xi$. The convolution product of $f \in L^1$ and $g \in L^p$ is defined by $f * g(x) = \int f(y)g(x-y) dy$ and is a well-defined function in L^p . The formula $(f * g)^\wedge = \hat{f}\hat{g}$ holds. Also, $\tilde{f}(x) = \overline{f(-x)}$.

A function f defined on \mathbf{R}^d , $d \geq 2$ is radial if there exists a function f_0 defined on $[0, \infty)$ such that $f(x) = f_0(|x|)$ for a.e. $x \in \mathbf{R}^d$. If $f \in L^1(\mathbf{R}^d)$ is radial, $\hat{f} \in A(\hat{\mathbf{R}}^d)$ is also radial. If in this case $\hat{f}(\xi) = F_0(|\xi|)$ and $f \in A(\mathbf{R}^d)$, then

$$(0.1) \quad f_0(r) = 2\pi r^{(2-d)/2} \int_0^\infty F_0(s) J_{(d-2)/2}(2\pi r s) s^{d/2} ds,$$

where, for $k > -1/2$, J_k is the Bessel function of order k defined by

$$(0.2) \quad J_k(t) = \frac{(t/2)^k}{\Gamma((2k+1)/2)\Gamma(1/2)} \int_{-1}^1 e^{its} (1-s^2)^{(2k-1)/2} ds,$$

(cf. [StW, Chapter IV, Section 3]). The following formulas hold for $k > -1/2$.

$$(0.3) \quad \frac{d}{dt}(t^k J_k(t)) = t^k J_{k-1}(t),$$

and

$$(0.4) \quad \frac{d}{dt}(t^{-k} J_k(t)) = -t^{-k} J_{k+1}(t),$$

and the asymptotic estimate

$$(0.5) \quad J_k(t) \sim \sqrt{\frac{2}{\pi t}} \cos(t - (k\pi/2) - (\pi/4)),$$

holds as $|t| \rightarrow \infty$. In particular, J_k is bounded for all $k > -1/2$.

1. The Radon Transform.

DEFINITION 1.1. Given $f \in \mathcal{S}(\mathbf{R}^d)$, we define the **Radon transform**, Rf of f by

$$Rf(\theta, s) = R_\theta f(s) = \int_{\theta^\perp} f(s\theta + y) dy,$$

where $\theta \in S^{d-1}$, $s \in \mathbf{R}$.

Since for each $f \in \mathcal{S}(\mathbf{R}^d)$, and $\theta \in S^{d-1}$,

$$\int_{\mathbf{R}} |R_\theta f(s)| ds \leq \int_{\mathbf{R}} \int_{\theta^\perp} |f(s\theta + y)| dy ds = \int_{\mathbf{R}^d} |f(x)| dx,$$

R_θ extends to a continuous operator on $L^1(\mathbf{R}^d)$. Hence, R extends continuously to $L^1(\mathbf{R}^d)$.

DEFINITION 1.2. Given $h \in L^\infty(\mathbf{R})$, we define for each $\theta \in S^{d-1}$, the operator $R_\theta^\#$ by

$$R_\theta^\# h(x) = h(\langle x, \theta \rangle).$$

For $h \in L^\infty(S^{d-1} \times \mathbf{R})$, we define the operator $R^\#$ by

$$R^\# h(x) = \int_{S^{d-1}} h(\theta, \langle x, \theta \rangle) d\theta.$$

Note that $R^\# h \in L^\infty(\mathbf{R}^d)$ if $h \in L^\infty(S^{d-1} \times \mathbf{R})$.

Note that given $f \in L^1(\mathbf{R}^d)$ and $h \in L^\infty(\mathbf{R})$,

$$\begin{aligned} \int_{\mathbf{R}} R_\theta f(s) h(s) ds &= \int_{\mathbf{R}} \int_{\theta^\perp} f(s\theta + y) dy h(s) ds \\ &= \int_{\mathbf{R}^d} f(x) h(\langle x, \theta \rangle) dx = \int_{\mathbf{R}^d} f(x) R_\theta^\# h(x) dx. \end{aligned}$$

Also, for $f \in L^1(\mathbf{R}^d)$, and $h \in L^\infty(S^{d-1} \times \mathbf{R})$, integrating the above over S^{d-1} gives

$$\int_{S^{d-1}} \int_{\mathbf{R}} Rf(\theta, s) h(\theta, s) ds d\theta = \int_{\mathbf{R}^d} f(x) R^\# h(x) dx.$$

In this sense, $R_\theta^\#$ and $R^\#$ are the formal adjoints of R_θ and R .

We now collect some basic properties of the Radon transform whose proofs can be found in any standard text on the subject, e.g., [N].

PROPOSITION 1.3. Let $f, g \in L^1(\mathbf{R}^d)$. Then for a.e. $\theta \in S^{d-1}$ and $s \in \mathbf{R}$,

$$R_\theta(f * g)(s) = R_\theta f * R_\theta g(s),$$

where the convolution on the left is in \mathbf{R}^d and that on the right is in \mathbf{R} .

PROOF: Suppose that $f, g \in L^1(\mathbf{R}^d)$.

$$\begin{aligned} R_\theta(f * g)(s) &= \int_{\theta^\perp} \int_{\mathbf{R}^d} f(x) g(s\theta + y - x) dx dy \\ &= \int_{\mathbf{R}} \int_{\theta^\perp} f(\tau\theta + x') \int_{\theta^\perp} g((s - \tau)\theta + y - x') dy d\tau dx' \\ &= \int_{\mathbf{R}} R_\theta f(\tau) R_\theta g(s - \tau) d\tau = R_\theta f * R_\theta g(s). \blacksquare \end{aligned}$$

PROPOSITION 1.4. Let $f \in L^1(\mathbf{R}^d)$, $g \in L^\infty(\mathbf{R})$. Then for each $\theta \in S^{d-1}$,

$$(1.4.1) \quad (R_\theta^\# g) * f = R_\theta^\# (g * R_\theta f)$$

where the convolution on the left is in \mathbf{R}^d and that on the right is in \mathbf{R} .

If $g \in L^\infty(S^{d-1} \times \mathbf{R})$, then

$$(1.4.2) \quad R^\# g * f = R^\# (g * Rf).$$

PROOF: Assume first that $f \in \mathcal{S}(\mathbf{R}^d)$. For $\theta \in S^{d-1}$ fixed, let $y = \tau\theta + y'$ and $x = s\theta + x'$. Then

$$\begin{aligned} \int_{\mathbf{R}^d} g(\langle y, \theta \rangle) f(x - y) dy &= \int_{\theta^\perp} \int_{\mathbf{R}} g(\tau) f((s - \tau)\theta + (x' - y')) d\tau dy' \\ &= \int_{\mathbf{R}} g(\tau) \int_{\theta^\perp} f((s - \tau)\theta + (x' - y')) dy' d\tau \\ &= \int_{\mathbf{R}} g(\tau) R_\theta f(s - \tau) ds = g * R_\theta f(\langle x, \theta \rangle). \end{aligned}$$

This proves (1.4.1). Integrating the above formula over $\theta \in S^{d-1}$ gives (1.4.2) for $f \in \mathcal{S}(\mathbf{R}^d)$. Since convolution by $g \in L^\infty$ is a continuous operator on L^1 , and since $R_\theta^\#$ and $R^\#$ are continuous operators on $L^\infty(\mathbf{R})$ and $L^\infty(S^{d-1} \times \mathbf{R})$, respectively, (1.4.2) holds for all $f \in L^1(\mathbf{R}^d)$. ■

PROPOSITION 1.5. (THE FOURIER SLICE THEOREM) Let $f \in L^1(\mathbf{R}^d)$. Then for $\theta \in S^{d-1}$, $\gamma \in \hat{\mathbf{R}}$,

$$(1.5.1) \quad (R_\theta f)^\wedge(\gamma) = \hat{f}(\gamma\theta).$$

PROPOSITION 1.6. Let $f \in L^1 \cap A(\mathbf{R}^d)$. Then

$$(1.6.1) \quad f = \frac{1}{2} R^\# I^{1-d} R f.$$

PROOF: See [N].

2. The Continuous Wavelet Transform.

DEFINITION 2.1. A radial function $\Psi \in L^2(\mathbf{R}^d)$ is *admissible* provided that

$$\int_0^\infty \frac{|\hat{\Psi}(r)|^2}{r} dr < \infty.$$

A pair of radial functions $\Psi, \Phi \in L^2(\mathbf{R}^d)$ form an *admissible wavelet pair* provided that each is admissible and

$$\int_0^\infty \frac{\overline{\hat{\Psi}(r)} \hat{\Phi}(r)}{r} dr = 1.$$

DEFINITION 2.2. Let $\Psi \in L^2(\mathbf{R}^d)$ be radial and admissible. For $a > 0$, $b \in \mathbf{R}^d$ define

$$\Psi_{ab}(x) = a^{-d/2} \Psi\left(\frac{x-b}{a}\right).$$

Denote Ψ_{a0} by $\tilde{\Psi}_a$. The d -dimensional wavelet transform of $f \in L^2(\mathbf{R}^d)$ with basic wavelet Ψ is defined by

$$W^{(d)}(\Psi, f)(a, b) = \int f(x) \overline{\Psi_{ab}(x)} dx = (f * \tilde{\Psi}_a)(b).$$

Given an admissible wavelet pair, $\Psi, \Phi \in L^2(\mathbf{R}^d)$, any $f \in L^2(\mathbf{R}^d)$ can be recovered from its wavelet transform by means of the Calderon reproducing formula. We reproduce that formula here together with some estimates on convergence.

LEMMA 2.3. Let $\Psi, \Phi \in L^1 \cap L^2(\mathbf{R}^d)$ be radial and an admissible wavelet pair. Given $f \in L^2(\mathbf{R}^d)$, $0 < \epsilon < \delta < \infty$, define

$$(2.3.1) \quad f^{\epsilon, \delta}(x) = \int_{\epsilon}^{\delta} \int W^{(d)}(\Psi, f)(a, b) \Phi_{ab}(x) db \frac{da}{a^{d+1}}.$$

Then $f^{\epsilon, \delta} \in L^2(\mathbf{R}^d)$ and

$$(2.3.2) \quad (f^{\epsilon, \delta})^\wedge(\xi) = \hat{f}(\xi) \int_{\epsilon}^{\delta} \frac{\widehat{\Psi}(a|\xi|) \widehat{\Phi}(a|\xi|)}{a} da.$$

Moreover, if $f \in L^1 \cap A(\mathbf{R}^d)$ then $f^{\epsilon, \delta} \in L^1 \cap A(\mathbf{R}^d)$.

PROOF: Assume $f \in L^1(\mathbf{R}^d)$. Then,

$$f^{\epsilon, \delta}(x) = \int_{\epsilon}^{\delta} f * \tilde{\Psi}_a * \Phi_a(x) \frac{da}{a^{d+1}}.$$

Since $f \in L^1(\mathbf{R}^d)$,

$$\begin{aligned} \int |f^{\epsilon, \delta}(x)| dx &\leq \int_{\epsilon}^{\delta} \int |f * \tilde{\Psi}_a * \Phi_a(x)| dx \frac{da}{a^{d+1}} \\ &\leq \|f\|_1 \int_{\epsilon}^{\delta} \|\Psi_a\|_1 \|\Phi_a\|_1 \frac{da}{a^{d+1}} \\ &= \|f\|_1 \|\Psi\|_1 \|\Phi\|_1 \int_{\epsilon}^{\delta} \frac{da}{a} \\ &= \log\left(\frac{\delta}{\epsilon}\right) \|f\|_1 \|\Psi\|_1 \|\Phi\|_1 < \infty. \end{aligned}$$

Therefore,

$$\begin{aligned}
(f^{\epsilon, \delta})^\wedge(\xi) &= \int_\epsilon^\delta \hat{f}(\xi) \overline{\hat{\Psi}(a|\xi|)} \hat{\Phi}(a|\xi|) a^d \frac{da}{a^{d+1}} \\
&= \hat{f}(\xi) \int_\epsilon^\delta \frac{\overline{\hat{\Psi}(a\xi)} \hat{\Phi}(a\xi)}{a} da \\
&= \hat{f}(\xi) \int_\epsilon^\delta \frac{\overline{\hat{\Psi}(a|\xi|)} \hat{\Phi}(a|\xi|)}{a} da,
\end{aligned}$$

since Ψ and Φ are radial and $a > 0$.

Now,

$$\begin{aligned}
&\int_\epsilon^\delta \frac{|\hat{\Psi}(a|\xi|)| |\hat{\Phi}(a|\xi|)}{a} da \\
&\leq \left(\int_\epsilon^\delta \frac{|\hat{\Psi}(a|\xi|)|^2}{a} da \right)^{1/2} \left(\int_\epsilon^\delta \frac{|\hat{\Phi}(a|\xi|)|^2}{a} da \right)^{1/2} \\
&\leq \left(\int_0^\infty \frac{|\hat{\Psi}(a)|^2}{a} da \right)^{1/2} \left(\int_0^\infty \frac{|\hat{\Phi}(a)|^2}{a} da \right)^{1/2} < \infty.
\end{aligned}$$

Therefore, if $f \in L^1 \cap A(\mathbf{R}^d)$, so is $f^{\epsilon, \delta}$, and a simple limiting argument shows that if $f \in L^2$, so is $f^{\epsilon, \delta}$. ■

LEMMA 2.4. Let $\Psi, \Phi \in L^1 \cap L^2(\mathbf{R}^d)$ be radial and an admissible wavelet pair. If $f \in L^2(\mathbf{R}^d)$, then

$$(2.4.1) \quad \lim_{\epsilon \rightarrow 0, \delta \rightarrow \infty} \|f - f^{\epsilon, \delta}\|_2 = 0.$$

If $f \in L^1 \cap A(\mathbf{R}^d)$ then

$$(2.4.2) \quad \lim_{\epsilon \rightarrow 0, \delta \rightarrow \infty} \|f - f^{\epsilon, \delta}\|_\infty = 0.$$

PROOF: Combining Plancherel's formula with (2.3.2),

$$\begin{aligned}
\|f - f^{\epsilon, \delta}\|_2^2 &= \|\hat{f} - (f^{\epsilon, \delta})^\wedge\|_2^2 \\
&= \int |\hat{f}(\xi)|^2 \left(1 - \int_\epsilon^\delta \frac{\overline{\hat{\Psi}(a|\xi|)} \hat{\Phi}(a|\xi|)}{a} da \right)^2 d\xi.
\end{aligned}$$

Equation (2.4.1) follows from the Dominated Convergence Theorem.

If $f \in L^1 \cap A(\mathbf{R}^d)$ then by (2.3.2),

$$\begin{aligned} \|f - f^{\epsilon, \delta}\|_{\infty} &\leq \|\hat{f} - (f^{\epsilon, \delta})^{\wedge}\|_1 \\ &= \int |\hat{f}(\xi)| \left(1 - \int_{\epsilon}^{\delta} \frac{\overline{\hat{\Psi}(a|\xi|)} \hat{\Phi}(a|\xi|)}{a} da\right) d\xi. \end{aligned}$$

Equation (2.4.2) follows from the Dominated Convergence Theorem. ■

A useful paradigm for the recovery of a function $f \in L^2(\mathbf{R}^d)$ from its wavelet transform involves taking

$$\hat{\Psi}(\xi) = \hat{\Phi}(\xi) = (\ln 2)^{-1/2} \chi_{\{\xi: 1 \leq |\xi| \leq 2\}}(\xi).$$

In this case (2.3.2) becomes

$$(f^{\epsilon, \delta})^{\wedge}(\xi) = \hat{f}(\xi) \int_{\epsilon}^{\delta} \frac{|\hat{\Psi}(a|\xi|)|^2}{a} da = \hat{f}(\xi) \frac{1}{\ln 2} \int_{[1/|\xi|, 2/|\xi|] \cap [\epsilon, \delta]} \frac{da}{a}.$$

From this, it follows that

$$(f^{\epsilon, \delta})^{\wedge}(\xi) = \begin{cases} 0, & \text{if } |\xi| \leq 1/\delta \\ \hat{f}(\xi) \ln(\delta|\xi|), & \text{if } 1/\delta \leq |\xi| \leq 2/\delta \\ \hat{f}(\xi), & \text{if } 2/\delta \leq |\xi| \leq 1/\epsilon \\ \hat{f}(\xi) \ln(2/\epsilon|\xi|), & \text{if } 1/\epsilon \leq |\xi| \leq 2/\epsilon \\ 0, & \text{if } 2/\epsilon \leq |\xi|. \end{cases}$$

This example suggests that as $\delta \rightarrow \infty$, the lower frequencies of f are recovered, and as $\epsilon \rightarrow 0$, the high frequencies are recovered. Similar remarks hold true for more general admissible wavelet pairs, Ψ and Φ .

We are ultimately interested in the local recovery of f from local wavelet transform data (which will be recovered from local Radon transform data). The complete recovery of low frequency components of f is not possible from local wavelet transform data, therefore, we desire more accurate estimates on the convergence of $f^{\epsilon, \delta}$ to f as $\delta \rightarrow \infty$. The following Lemma gives a priori estimates on this convergence.

LEMMA 2.5. Let Ψ and Φ be radial and an admissible wavelet pair. Let

$$(2.5.1) \quad \int_0^{\infty} \frac{|\hat{\Psi}(r)|^2}{r} dr = B_1, \quad \text{and} \quad \int_0^{\infty} \frac{|\hat{\Phi}(r)|^2}{r} dr = B_2.$$

Given $\eta > 0$, let $A > 0$ be such that

$$\int_A^{\infty} \frac{|\hat{\Psi}(r)|^2}{r} dr < \eta, \quad \text{and} \quad \int_A^{\infty} \frac{|\hat{\Phi}(r)|^2}{r} dr < \eta.$$

If $f \in L^2(\mathbf{R}^d)$, then

$$(2.5.2) \quad \|f - f^{\epsilon, \delta}\|_2 \leq c_0(\epsilon) + \eta \|f\|_2 + c_1(f, \delta)$$

where

$$c_1(f, \delta)^2 = B_1 B_2 \int_{|\xi| \leq A/\delta} |\hat{f}(\xi)|^2 d\xi$$

and $c_0(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. If $f \in L^1 \cap A(\mathbf{R}^d)$ then

$$(2.5.3) \quad \|f - f^{\epsilon, \delta}\|_\infty \leq c'_0(\epsilon) + \eta \|f\|_{A(\mathbf{R}^d)} + (B_1 B_2)^{1/2} c_2(f, \delta) \delta^{-d}$$

where

$$\lim_{\delta \rightarrow \infty} c_2(f, \delta) = A^d |\hat{f}(0)|$$

and $c'_0(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. In fact,

$$(2.5.4) \quad c_2(f, \delta) = \delta^d \int_{|\xi| \leq A/\delta} |\hat{f}(\xi)| d\xi.$$

PROOF: If $f \in L^2(\mathbf{R}^d)$, then by (2.3.2) and Plancherel's formula,

$$\begin{aligned} \|f - f^{\epsilon, \delta}\|_2^2 &= \int |\hat{f}(\xi)|^2 \left(1 - \int_\epsilon^\delta \frac{\overline{\hat{\Psi}(a|\xi|)} \hat{\Phi}(a|\xi|)}{a} da\right)^2 d\xi \\ &= \int |\hat{f}(\xi)|^2 \left(\int_0^\epsilon \frac{\overline{\hat{\Psi}(a|\xi|)} \hat{\Phi}(a|\xi|)}{a} da + \int_\delta^\infty \frac{\overline{\hat{\Psi}(a|\xi|)} \hat{\Phi}(a|\xi|)}{a} da\right)^2 d\xi \\ &= \int |\hat{f}(\xi)|^2 \left(\int_0^{\epsilon|\xi|} \frac{\overline{\hat{\Psi}(r)} \hat{\Phi}(r)}{r} dr + \int_{\delta|\xi|}^\infty \frac{\overline{\hat{\Psi}(r)} \hat{\Phi}(r)}{r} dr\right)^2 d\xi. \end{aligned}$$

Therefore,

$$\begin{aligned} \|f - f^{\epsilon, \delta}\|_2 &\leq \left(\int |\hat{f}(\xi)|^2 \left(\int_0^{\epsilon|\xi|} \frac{\overline{\hat{\Psi}(r)} \hat{\Phi}(r)}{r} dr\right)^2 d\xi\right)^{1/2} \\ &\quad + \left(\int |\hat{f}(\xi)|^2 \left(\int_{\delta|\xi|}^\infty \frac{\overline{\hat{\Psi}(r)} \hat{\Phi}(r)}{r} dr\right)^2 d\xi\right)^{1/2} = I + II. \end{aligned}$$

By the Dominated Convergence Theorem, $I = o(1)$ as $\epsilon \rightarrow 0$, and,

$$\begin{aligned} II &\leq \left(\int |\hat{f}(\xi)|^2 \left(\int_{\delta|\xi|}^A \frac{\overline{\hat{\Psi}(r)} \hat{\Phi}(r)}{r} dr\right)^2 d\xi\right)^{1/2} \\ &\quad + \left(\int |\hat{f}(\xi)|^2 \left(\int_A^\infty \frac{\overline{\hat{\Psi}(r)} \hat{\Phi}(r)}{r} dr\right)^2 d\xi\right)^{1/2} = II_1 + II_2. \end{aligned}$$

Now,

$$II_2^2 \leq \int |\hat{f}(\xi)|^2 d\xi \int_A \frac{|\hat{\Psi}(r)|^2}{r} dr \int_A \frac{|\hat{\Phi}(r)|^2}{r} dr < \eta^2 \|f\|_2^2,$$

and

$$\begin{aligned} II_1^2 &\leq \int |\hat{f}(\xi)|^2 \int_{\delta|\xi|}^A \frac{|\hat{\Psi}(r)|^2}{r} \int_{\delta|\xi|}^A \frac{|\hat{\Phi}(s)|^2}{s} ds dr d\xi \\ &= \int_0^A \frac{|\hat{\Psi}(r)|^2}{r} \int_0^A \frac{|\hat{\Phi}(s)|^2}{s} \int_{|\xi| \leq \min(r,s)/\delta} |\hat{f}(\xi)|^2 ds dr d\xi \\ &\leq \int_0^A \frac{|\hat{\Psi}(r)|^2}{r} dr \int_0^A \frac{|\hat{\Phi}(s)|^2}{s} ds \int_{|\xi| \leq A/\delta} |\hat{f}(\xi)|^2 d\xi. \end{aligned}$$

From this (2.5.2) follows.

Now, if $f \in L^1 \cap A(\mathbf{R}^d)$,

$$\begin{aligned} \|f - f^{\epsilon, \delta}\|_\infty &\leq \int |\hat{f}(\xi)| \int_0^{\epsilon|\xi|} \frac{\overline{\hat{\Psi}(r)} \hat{\Phi}(r)}{r} dr d\xi \\ &\quad + \int |\hat{f}(\xi)| \int_{\delta|\xi|}^\infty \frac{\overline{\hat{\Psi}(r)} \hat{\Phi}(r)}{r} dr d\xi = I + II. \end{aligned}$$

By the Dominated Convergence Theorem, $I = o(1)$ as $\epsilon \rightarrow 0$, and,

$$\begin{aligned} II &= \int |\hat{f}(\xi)| \int_{\delta|\xi|}^A \frac{\overline{\hat{\Psi}(r)} \hat{\Phi}(r)}{r} dr d\xi \\ &\quad + \int |\hat{f}(\xi)| \int_A \frac{\overline{\hat{\Psi}(r)} \hat{\Phi}(r)}{r} dr d\xi = II_1 + II_2. \end{aligned}$$

Now,

$$II_2 \leq \left(\int_A \frac{|\hat{\Psi}(r)|^2}{r} dr \int_A \frac{|\hat{\Phi}(r)|^2}{r} dr \right)^{1/2} \int |\hat{f}(\xi)| d\xi < \eta \|f\|_{A(\mathbf{R}^d)},$$

and

$$\begin{aligned} II_1 &= \int_0^A \frac{\overline{\hat{\Psi}(r)} \hat{\Phi}(r)}{r} \int_{|\xi| \leq r/\delta} |\hat{f}(\xi)| dr d\xi \\ &\leq \left(\int_0^A \frac{|\hat{\Psi}(r)|^2}{r} dr \int_0^A \frac{|\hat{\Phi}(s)|^2}{s} ds \right)^{1/2} \int_{|\xi| \leq A/\delta} |\hat{f}(\xi)| d\xi. \end{aligned}$$

Since \hat{f} is continuous,

$$\lim_{\delta \rightarrow \infty} \frac{\delta^d}{A^d} \int_{|\xi| \leq A/\delta} |\hat{f}(\xi)| d\xi = |\hat{f}(0)|,$$

and from this, (2.5.2) and (2.5.4) follow. ■

3. Inversion of the Radon transform using wavelets.

In this section, we derive some inversion formulas for the Radon transform based on the wavelet transform. They are based on the fact that the Riesz operator I^{1-d} preserves the essential character of a wavelet. For convenience, we assume our wavelets are radial.

LEMMA 3.1. Let $\psi \in L^1 \cap L^2(\mathbf{R})$ be even and real-valued. For $d \geq 2$, let $n = (d-2)/2$ and suppose that there exists an integer $M \geq d+1$ such that $\hat{\psi}$ has M continuous derivatives, and for $k = 0, 1, \dots, M$,

$$(3.1.1) \quad \left| \frac{d^k}{d\gamma^k} \hat{\psi}(\gamma) \right| \leq C(1 + |\gamma|)^{-2},$$

and

$$(3.1.2) \quad \frac{d^k}{d\gamma^k} \hat{\psi}(0) = 0.$$

Define $\Psi(x)$ on \mathbf{R}^d by

$$(3.1.3) \quad \Psi(x) = 4\pi|x|^{-n} \int_0^\infty \hat{\psi}(\gamma) \gamma^{-n} J_n(2\pi\gamma|x|) d\gamma,$$

where J_n is the Bessel function of order n . Then $\Psi \in L^1 \cap A(\mathbf{R}^d)$ is a real-valued, radial, admissible function satisfying

$$(3.1.4) \quad R^\# \psi = \Psi,$$

$$(3.1.5) \quad |\Psi(x)| \leq C(1 + |x|)^{-M}, \text{ and,}$$

$$(3.1.6) \quad |\hat{\Psi}(\xi)| \leq C(1 + |\xi|)^{-d-1}.$$

PROOF: By (3.1.1), $\hat{\psi}(\gamma)$ is integrable and, by (0.2), $(|x|\gamma)^{-n} J_n(2\pi\gamma|x|)$ is a bounded function. The integral (3.1.3) is therefore absolutely convergent and defines a bounded, real-valued, radial function.

We first prove (3.1.5). Define

$$f_0(\gamma) = \hat{\psi}(\gamma)\gamma^{-n},$$

and for $j \in \mathbf{N}$, define

$$f_{2j+1}(\gamma) = \gamma^{n+1} \frac{d}{d\gamma}(\gamma^{-n-1} f_{2j}(\gamma)); \quad f_{2j}(\gamma) = \gamma^{-n} \frac{d}{d\gamma}(\gamma^n f_{2j-1}(\gamma)).$$

Then for $k = 0, 1, \dots, M$,

$$(3.1.7) \quad f_k(\gamma) = \sum_{m=0}^k c_{m,k} \frac{d^m}{d\gamma^m} \hat{\psi}(\gamma) \gamma^{m-k-n},$$

for some constants $c_{m,k}$. To see this, note that (3.1.7) holds for $k = 0$ and $k = 1$. Now suppose it holds for $k < 2j$, some $j > 0$. Then

$$\begin{aligned} f_{2j}(\gamma) &= \gamma^{-n} \frac{d}{d\gamma} \left(\gamma^n \sum_{m=0}^{2j-1} c_{m,2j-1} \frac{d^m}{d\gamma^m} \hat{\psi}(\gamma) \gamma^{m-2j+1-n} \right) \\ &= \gamma^{-n} \sum_{m=0}^{2j-1} (m-2j+1) c_{m,2j-1} \frac{d^m}{d\gamma^m} \hat{\psi}(\gamma) \gamma^{m-2j} \\ &\quad + \gamma^{-n} \sum_{m=0}^{2j-1} c_{m,2j-1} \frac{d^{m+1}}{d\gamma^{m+1}} \hat{\psi}(\gamma) \gamma^{m-2j+1} \\ &= \sum_{m=0}^{2j} c_{m,2j} \frac{d^m}{d\gamma^m} \hat{\psi}(\gamma) \gamma^{m-2j-n}. \end{aligned}$$

Also,

$$\begin{aligned} f_{2j+1}(\gamma) &= \gamma^{n+1} \frac{d}{d\gamma} \left(\gamma^{-n-1} \sum_{m=0}^{2j} c_{m,2j} \frac{d^m}{d\gamma^m} \hat{\psi}(\gamma) \gamma^{m-2j-n} \right) \\ &= \gamma^{n+1} \sum_{m=0}^{2j} c_{m,2j} (m-2j-2n-1) \frac{d^m}{d\gamma^m} \hat{\psi}(\gamma) \gamma^{m-2j-2n-2} \\ &\quad + \gamma^{n+1} \sum_{m=0}^{2j} c_{m,2j} \frac{d^{m+1}}{d\gamma^{m+1}} \hat{\psi}(\gamma) \gamma^{m-2j-2n-1} \\ &= \sum_{m=0}^{2j+1} c_{m,2j+1} \frac{d^m}{d\gamma^m} \hat{\psi}(\gamma) \gamma^{m-(2j+1)-n}. \end{aligned}$$

Now, if $2j + 1 \leq M$, then

$$(3.1.8) \quad \Psi(x) = \frac{(-1)^{j+1} 4\pi}{(2\pi|x|)^{2j+1}} |x|^{-n} \int_0^\infty f_{2j+1}(\gamma) J_{n+1}(2\pi\gamma|x|) d\gamma,$$

and if $2j \leq M$,

$$(3.1.9) \quad \Psi(x) = \frac{(-1)^j 4\pi}{(2\pi|x|)^{2j}} |x|^{-n} \int_0^\infty f_{2j}(\gamma) J_n(2\pi\gamma|x|) d\gamma.$$

Note that (3.1.3) is just (3.1.9) with $j = 0$. To see that (3.1.8) holds if $j = 0$, integrate (3.1.3) by parts. By (0.3),

$$\begin{aligned} \Psi(x) &= 4\pi|x|^{-n} \int_0^\infty \gamma^{-n-1} f_0(\gamma) \gamma^{n+1} J_n(2\pi\gamma|x|) d\gamma \\ &= \frac{4\pi|x|^{-n}}{(2\pi|x|)} \left(\gamma^{n+1} f_0(\gamma) \gamma^{-n-1} J_{n+1}(2\pi\gamma|x|) \Big|_0^\infty \right. \\ &\quad \left. - \int_0^\infty \frac{d}{d\gamma} (\gamma^{-n-1} f_0(\gamma)) \gamma^{n+1} J_{n+1}(2\pi\gamma|x|) d\gamma \right). \end{aligned}$$

Note that for each fixed x , $\gamma^{-n-1} J_{n+1}(2\pi\gamma|x|)$ is bounded for γ in a neighborhood of zero and infinity, by (0.2). Also, $\gamma^{n+1} f_0(\gamma) = \gamma \hat{\psi}(\gamma)$ so that

$$\lim_{\gamma \rightarrow 0} \gamma^{n+1} f_0(\gamma) = 0$$

and by (3.1.1),

$$\lim_{\gamma \rightarrow \infty} \gamma^{n+1} f_0(\gamma) = 0.$$

Hence, the boundary terms in the integration by parts formula vanish and

$$\begin{aligned} \Psi(x) &= \frac{-4\pi}{(2\pi|x|)} |x|^{-n} \int_0^\infty \frac{d}{d\gamma} (\gamma^{-n-1} f_0(\gamma)) \gamma^{n+1} J_{n+1}(2\pi\gamma|x|) d\gamma \\ &= \frac{-4\pi}{(2\pi|x|)} |x|^{-n} \int_0^\infty f_1(\gamma) J_{n+1}(2\pi\gamma|x|) d\gamma. \end{aligned}$$

Therefore, (3.1.8) holds when $j = 0$. Assume that (3.1.8) holds for $2j - 1$ for some $j > 0$ with $2j \leq M$. Now, integrating by parts and applying (0.4) gives

$$\begin{aligned} \Psi(x) &= \frac{4\pi}{(2\pi|x|)^{2j-1}} |x|^{-n} \int_0^\infty \gamma^n f_{2j-1}(\gamma) (-\gamma^{-n} J_{n+1}(2\pi\gamma|x|)) d\gamma \\ &= \frac{4\pi}{(2\pi|x|)^{2j}} |x|^{-n} \left(\gamma^n f_{2j-1}(\gamma) \gamma^{-n} J_n(2\pi\gamma|x|) \Big|_0^\infty \right. \\ &\quad \left. - \int_0^\infty \frac{d}{d\gamma} (\gamma^n f_{2j-1}(\gamma)) \gamma^{-n} J_n(2\pi\gamma|x|) d\gamma \right). \end{aligned}$$

Note that for each fixed x , $\gamma^{-n} J_n(2\pi\gamma|x|)$ is bounded for γ in a neighborhood of zero and infinity, by (0.2). Also, by (3.1.7)

$$\gamma^n f_{2j-1}(\gamma) = \sum_{m=0}^{2j-1} c_{m,2j-1} \frac{d^m}{d\gamma^m} \hat{\psi}(\gamma) \gamma^{m-2j+1}.$$

Since $2j \leq M$ and by (3.1.2), and L'Hôpital's rule,

$$\lim_{\gamma \rightarrow 0} \gamma^{m-2j+1} \frac{d^m}{d\gamma^m} \hat{\psi}(\gamma) = \lim_{\gamma \rightarrow 0} \frac{d^{2j-1}}{d\gamma^{2j-1}} \hat{\psi}(\gamma) = 0,$$

and by (3.1.1),

$$\lim_{\gamma \rightarrow \infty} \gamma^{m-2j+1} \frac{d^m}{d\gamma^m} \hat{\psi}(\gamma) = 0.$$

Hence, the boundary terms in the integration by parts formula vanish and

$$\begin{aligned} \Psi(x) &= \frac{-4\pi}{(2\pi|x|)^{2j}} |x|^{-n} \int_0^\infty \frac{d}{d\gamma} (\gamma^n f_{2j-1}(\gamma)) \gamma^{-n} J_n(2\pi\gamma|x|) d\gamma \\ &= \frac{-4\pi}{(2\pi|x|)^{2j}} |x|^{-n} \int_0^\infty f_{2j}(\gamma) J_n(2\pi\gamma|x|) d\gamma, \end{aligned}$$

which is (3.1.9). A similar argument proves (3.1.8) for $2j + 1 \leq M$.

Finally, there is a constant $C > 0$ independent of $|x| \geq 1$ such that

$$(3.1.10) \quad |x|^{-n} \int_0^\infty f_{2j}(\gamma) J_n(2\pi\gamma|x|) d\gamma \leq C,$$

and

$$(3.1.11) \quad |x|^{-n} \int_0^\infty f_{2j+1}(\gamma) J_{n+1}(2\pi\gamma|x|) d\gamma \leq C.$$

To prove (3.1.10), note that

$$\begin{aligned} |x|^{-n} \int_0^\infty f_{2j}(\gamma) J_n(2\pi\gamma|x|) d\gamma \\ = \int_0^\infty \gamma^n f_{2j}(\gamma) (\gamma|x|)^{-n} J_n(2\pi\gamma|x|) d\gamma. \end{aligned}$$

Now,

$$\gamma^n f_{2j}(\gamma) = \sum_{m=0}^{2j} c_{m,2j} \frac{d^m}{d\gamma^m} \hat{\psi}(\gamma) \gamma^{m-2j},$$

and for each m , $\frac{d^m}{d\gamma^m}\hat{\psi}(\gamma)$ is integrable and $\frac{d^m}{d\gamma^m}\hat{\psi}(\gamma)\gamma^{m-2j}$ is bounded in a neighborhood of the origin. Therefore, $\gamma^n f_{2j}(\gamma)$ is integrable. Also, since, by (0.2)

$$(\gamma|x|)^{-n} J_n(2\pi\gamma|x|) d\gamma$$

is bounded independently of $|x| \geq 1$, (3.1.10) follows.

To prove (3.1.11), note that

$$\begin{aligned} |x|^{-n} \int_0^\infty f_{2j+1}(\gamma) J_{n+1}(2\pi\gamma|x|) d\gamma \\ &= |x|^{-n} \int_0^{1/|x|} f_{2j+1}(\gamma) J_{n+1}(2\pi\gamma|x|) d\gamma \\ &\quad + |x|^{-n} \int_{1/|x|}^\infty f_{2j+1}(\gamma) J_{n+1}(2\pi\gamma|x|) d\gamma \\ &= I + II. \end{aligned}$$

Now, if $\gamma \leq 1/|x|$ then $\gamma|x| \leq 1$ and there is a constant c_1 independent of $|x|$ such that

$$J_{n+1}(2\pi\gamma|x|) \leq c_1(\gamma|x|)^{n+1}.$$

Thus,

$$\begin{aligned} I &\leq c_1 |x|^{-n} \int_0^{1/|x|} \gamma^n f_{2j+1}(\gamma) (\gamma|x|)^{n+1} \gamma^{-n} d\gamma \\ &\leq c_1 \int_0^{1/|x|} \gamma^n f_{2j+1}(\gamma) (\gamma|x|) d\gamma \\ &\leq c_1 \int_0^1 \gamma^n f_{2j+1}(\gamma) d\gamma \leq C, \end{aligned}$$

since $\gamma^n f_{2j+1}(\gamma)$ is bounded near $\gamma = 0$. Now, if $\gamma \geq 1/|x|$ then $\gamma|x| \geq 1$ and there is a constant c_2 independent of $|x|$ such that

$$J_{n+1}(2\pi\gamma|x|) \leq c_2.$$

Also, if $\gamma \geq 1/|x|$ then $(\gamma|x|)^{-n} \leq 1$. Thus,

$$\begin{aligned} II &\leq c_2 |x|^{-n} \int_{1/|x|}^\infty \gamma^n f_{2j+1}(\gamma) \gamma^{-n} d\gamma \\ &\leq c_2 \int_{1/|x|}^\infty \gamma^n f_{2j+1}(\gamma) d\gamma \\ &\leq c_2 \int_0^\infty \gamma^n f_{2j+1}(\gamma) d\gamma \leq C. \end{aligned}$$

This gives (3.1.11). Combining (3.1.9) with (3.1.10) or (3.1.8) with (3.1.11) depending on whether M is even or odd, (3.1.5) follows. Since $M \geq d + 1$ this says in particular that $\Psi \in L^1(\mathbf{R}^d)$.

Writing (3.1.3) as

$$\Psi(x) = 4\pi|x|^{(2-d)/2} \int_0^\infty \hat{\psi}(\gamma)\gamma^{1-d} J_{(d-2)/2}(2\pi\gamma|x|)\gamma^{d/2} d\gamma,$$

and observing that since the radial function on $\hat{\mathbf{R}}^d$ defined by $F(\xi) = \hat{\psi}(|\xi|)|\xi|^{1-d}$ is integrable, then by (0.1),

$$(3.1.12) \quad \hat{\Psi}(\xi) = 2\hat{\psi}(|\xi|)|\xi|^{1-d},$$

for $\xi \in \hat{\mathbf{R}}^d$. Since Ψ is radial, $R\Psi$ is independent of $\theta \in S^{d-1}$. By (1.5.1), and the fact that $\hat{\psi}$ is even,

$$(R\Psi)^\wedge(\gamma) = \hat{\Psi}(|\gamma|) = 2\hat{\psi}(\gamma)|\gamma|^{1-d}.$$

Hence,

$$(3.1.13) \quad \hat{\psi}(\gamma) = \frac{1}{2}|\gamma|^{d-1}(R\Psi)^\wedge(\gamma),$$

or,

$$\psi = \frac{1}{2}I^{1-d}R\Psi.$$

By (1.6.1), $R^\# \psi(x) = \Psi(x)$ and (3.1.4) is established.

Equation (3.1.6) follows from (3.1.1) with $k = 0$ and (3.1.12). Finally, $\Psi \in L^1 \cap A(\mathbf{R}^d)$ by (3.1.5) and (3.1.6) and

$$\int_0^\infty \frac{|\Psi(r)|^2}{r} dr = 4 \int_0^\infty \frac{|\hat{\psi}(r)|^2}{r^{2d-1}} dr < \infty,$$

so Ψ is admissible. ■

In order to invert the Radon transform, we need the following simple lemma.

LEMMA 3.2. Given $g \in L^1(\mathbf{R}^d)$, $a > 0$, $\theta \in S^{d-1}$,

$$R(g_a)(\theta, s) = a^{(d-1)/2}(Rg)_a(\theta, s)$$

where $g_a(x) = a^{-d/2}g(x/a)$, $x \in \mathbf{R}^d$.

PROOF:

$$\begin{aligned}
a^{-d/2} \int_{\theta^\perp} g(a^{-1}(s\theta + y)) dy &= a^{-d/2} \int_{\theta^\perp} g(a^{-1}s\theta + a^{-1}y) dy \\
&= a^{d/2} a^{-1} \int_{\theta^\perp} g(a^{-1}s\theta + y) dy \\
&= a^{(d-1)/2} D_a R_\theta g(s). \blacksquare
\end{aligned}$$

LEMMA 3.3. Let $\psi \in L^1 \cap L^2(\mathbf{R})$ be an even, real-valued, admissible function satisfying (3.1.1) and (3.1.2). Let $\Psi(x)$ be defined by (3.1.3). Then for any $f \in L^1 \cap L^2(\mathbf{R}^d)$,

$$(3.3.1) \quad W^{(d)}(\Psi, f)(a, b) = a^{(1-d)/2} \int_{S^{d-1}} W^{(1)}(\psi, R_\theta f)(a, \langle b, \theta \rangle) d\theta.$$

PROOF: Extend $\psi(t)$ to $S^{d-1} \times \mathbf{R}$ by $\psi(\theta, t) = \psi(t)$ for all $\theta \in S^{d-1}$. Then by (3.1.4), $R^\# \psi(x) = \Psi(x)$. Note that by (3.1.13),

$$\begin{aligned}
(\psi_a)^\wedge(\gamma) &= a^{1/2} \hat{\psi}(a\gamma) = \frac{1}{2} a^{1/2} |a\gamma|^{d-1} (R\Psi)^\wedge(a\gamma) \\
&= \frac{1}{2} a^{1/2} a^{d-1} |\gamma|^{d-1} a^{-d/2} (R\Psi_a)^\wedge(\gamma) \\
&= \frac{1}{2} a^{(d-1)/2} |\gamma|^{d-1} (R\Psi_a)^\wedge(\gamma).
\end{aligned}$$

Thus, $\psi_a = \frac{1}{2} a^{(d-1)/2} I^{1-d} R\Psi_a$, and so

$$a^{(1-d)/2} R^\# \psi_a = \Psi_a.$$

Since $\psi \in L^\infty(S^{d-1} \times \mathbf{R})$ and $\Psi \in L^1(\mathbf{R}^d)$, by (1.4.2),

$$\begin{aligned}
W^{(d)}(\Psi, f)(a, b) &= f * \tilde{\Psi}_a(b) = a^{(1-d)/2} (f * R^\# \tilde{\psi}_a)(b) \\
&= a^{(1-d)/2} R^\# (Rf * \tilde{\psi}_a)(b) = a^{(1-d)/2} \int_{S^{d-1}} (R_\theta f * \tilde{\psi}_a)(\langle b, \theta \rangle) d\theta \\
&= a^{(1-d)/2} \int_{S^{d-1}} W^{(1)}(\psi, R_\theta f)(a, \langle b, \theta \rangle) d\theta. \blacksquare
\end{aligned}$$

This gives immediately the following.

COROLLARY 3.4. Let $\psi \in L^1 \cap L^2(\mathbf{R})$ be even, real-valued, admissible, and satisfy (3.1.1) and (3.1.2). Let Ψ be given by (3.1.3), and let $\Phi \in L^1 \cap L^2(\mathbf{R}^d)$ be real-valued, radial and

admissible such that Ψ, Φ is an admissible wavelet pair. If $f \in L^1 \cap L^2(\mathbf{R}^d)$, $0 < \epsilon < \delta < \infty$ then

$$(3.4.1) \quad f^{\epsilon, \delta}(x) = \int_{S^{d-1}} \int_{\epsilon}^{\delta} a^{(1-d)/2} \int W^{(1)}(\psi, R_{\theta} f)(a, \langle b, \theta \rangle) \Phi_{ab}(x) db \frac{da}{a^{d+1}} d\theta.$$

PROOF: By (3.3.1) and (2.3.1),

$$f^{\epsilon, \delta}(x) = \int_{\epsilon}^{\delta} a^{(1-d)/2} \int \int_{S^{d-1}} W^{(1)}(\psi, R_{\theta} f)(a, \langle b, \theta \rangle) \Phi_{ab}(x) db \frac{da}{a^{d+1}} d\theta$$

where the integral converges absolutely.

To see the absolute convergence of the integral, note that,

$$\begin{aligned} & \int |W^{(1)}(\psi, R_{\theta} f)(a, \langle b, \theta \rangle)| |\Phi_a(x - b)| db \\ &= \int_{\theta^{\perp}} \int_{\mathbf{R}} |R_{\theta} f * \psi_a(\beta)| |\Phi_a((s - \beta)\theta + (x' - b'))| d\beta db', \end{aligned}$$

where $b = \beta\theta + b'$, and $x = s\theta + x'$,

$$\leq \int_{\theta^{\perp}} \|R_{\theta} f\|_1 \|\psi_a\|_{\infty} \int_{\mathbf{R}} |\Phi_a(\tau\theta + (x' - b'))| d\tau db' \leq \|f\|_1 \|\psi\|_{\infty} \|\Phi\|_1 a^{(d-1)/2}.$$

Thus,

$$\begin{aligned} & \int_{\epsilon}^{\delta} a^{(1-d)/2} \int_{S^{d-1}} \int |W^{(1)}(\psi, R_{\theta} f)(a, \langle b, \theta \rangle)| |\Phi_{ab}(x)| db \frac{da}{a^{d+1}} d\theta \\ & \leq \|f\|_1 \|\psi\|_{\infty} \|\Phi\|_1 \int_{\epsilon}^{\delta} \int_{S^{d-1}} \frac{d\theta}{a^{d+1}} \\ & = \|f\|_1 \|\psi\|_{\infty} \|\Phi\|_1 |S^{d-1}| (\delta^d - \epsilon^d) / d < \infty. \end{aligned}$$

Hence we may interchange the order of integration at will and (3.4.1) follows. \blacksquare

From this inversion formula, we get for free a ‘‘local’’ inversion formula which says in effect that to recover f to a given accuracy in a ball of radius $R > 0$ about a point $x_0 \in \mathbf{R}^d$, it is sufficient to know only those projections of f on lines passing through a ball of radius $R + \alpha$ about x_0 for some $\alpha > 0$. The greater the accuracy desired, the greater α must be.

THEOREM 3.5. Let $\psi \in L^1 \cap L^2(\mathbf{R})$ be even, real-valued, admissible, and satisfy (3.1.1) and (3.1.2). Let Ψ be defined by (3.1.3) and let $\Phi \in L^1 \cap L^2(\mathbf{R}^d)$ be real-valued, radial, $\text{supp } \Phi \subseteq B(0, 1)$, and such that Ψ, Φ is an admissible wavelet pair. For $\alpha > 0$, $\theta \in S^{d-1}$, and $f \in L^1 \cap L^2$, define

$$W_{\alpha}^{(1)}(\psi, R_{\theta} f)(a, s) = W^{(1)}(\psi, R_{\theta} f)(a, s) \chi_{[-\alpha, \alpha]}(s)$$

and for $R > 0$, $0 < \epsilon < \delta < \infty$, define

$$(3.5.1) \quad f_R^{\epsilon, \delta}(x) = \int_{S^{d-1}} \int_{\epsilon}^{\delta} a^{(1-d)/2} \int_{|b| \leq R+\delta} W_{R+\delta}^{(1)}(\psi, R_{\theta} f)(a, \langle b, \theta \rangle) \Phi_{ab}(x) db \frac{da}{a^{d+1}} d\theta.$$

Then $f_R^{\epsilon, \delta}(x) = f^{\epsilon, \delta}(x)$ for $|x| \leq R$.

PROOF: As in Corollary 3.4, we may interchange the order of integration in (3.5.1) at will. Also, for any $f \in L^1 \cap L^2(\mathbf{R}^d)$, by (3.3.1),

$$W^{(d)}(\Psi, f)(a, b) = a^{(1-d)/2} \int_{S^{d-1}} W^{(1)}(\psi, R_{\theta} f)(a, \langle b, \theta \rangle) d\theta.$$

Now, for $\theta \in S^{d-1}$ and $|b| \leq R + \delta$, $|\langle b, \theta \rangle| \leq R + \delta$. Therefore, for $|b| \leq R + \delta$,

$$W^{(1)}(\psi, R_{\theta} f)(a, \langle b, \theta \rangle) = W_{R+\delta}^{(1)}(\psi, R_{\theta} f)(a, \langle b, \theta \rangle).$$

Now, if $|x| \leq R$ and $a \in [\epsilon, \delta]$ then $\text{supp } \Phi_a \subseteq B(0, \delta)$ and if $|x| \leq R$ and $|x - b| < \delta$, then $|b| \leq R + \delta$. Thus, for $|x| \leq R$ fixed,

$$\begin{aligned} & \int_{|b| \leq R+\delta} W^{(d)}(\Psi, f)(a, b) \Phi_{ab}(x) db \\ &= \int_{|b| \leq R+\delta} W^{(d)}(\Psi, f)(a, b) \Phi_a(x - b) db \\ &= \int_{|x-b| \leq \delta} W^{(d)}(\Psi, f)(a, b) \Phi_a(x - b) db \\ &= \int W^{(d)}(\Psi, f)(a, b) \Phi_a(x - b) db. \end{aligned}$$

Therefore, for $|x| \leq R$,

$$\begin{aligned} f^{\epsilon, \delta}(x) &= \int_{\epsilon}^{\delta} \int W^{(d)}(\Psi, f)(a, b) \Phi_{ab}(x) db \frac{da}{a^{d+1}} \\ &= \int_{\epsilon}^{\delta} \int_{|b| \leq R+\delta} W^{(d)}(\Psi, f)(a, b) \Phi_{ab}(x) db \frac{da}{a^{d+1}} \\ &= \int_{\epsilon}^{\delta} \int_{|b| \leq R+\delta} a^{(1-d)/2} \int_{S^{d-1}} W_{R+\delta}^{(1)}(\psi, R_{\theta} f)(a, \langle b, \theta \rangle) d\theta \Phi_{ab}(x) db \frac{da}{a^{d+1}} \\ &= \int_{S^{d-1}} \int_{\epsilon}^{\delta} a^{(1-d)/2} \int_{|b| \leq R+\delta} W_{R+\delta}^{(1)}(\psi, R_{\theta} f)(a, \langle b, \theta \rangle) \Phi_{ab}(x) db \frac{da}{a^{d+1}} d\theta \\ &= f_R^{\epsilon, \delta}(x). \blacksquare \end{aligned}$$

REMARK 3.6. a. Note that under the assumption that $\text{supp } \psi \subseteq [-1, 1]$, computing $W_{R+\delta}^{(1)}(\psi, R_\theta f)(a, s)$ for $0 < a \leq \delta$ requires for each $\theta \in S^{d-1}$ the projections of f along lines through a ball of radius $R + 2\delta$ about the origin.

b. Applying Lemma 2.5 to Theorem 3.5, the following estimates hold.

Given $\eta > 0$, there exists $A > 0$ such that for all $R > 0$ and $f \in L^1 \cap L^2(\mathbf{R}^d)$,

$$(3.6.1) \quad \left(\int_{B(0,R)} |f(x) - f_R^{\epsilon, \delta}(x)|^2 dx \right)^{1/2} = c_0(\epsilon) + \eta \|f\|_2 + (B_1 B_2)^{1/2} \left(\int_{|\xi| \leq A/\delta} |\hat{f}(\xi)|^2 d\xi \right)^{1/2},$$

where $c_0(\epsilon)$ is given by (2.5.2), and B_1, B_2 are given by (2.5.1). Also, for all $f \in L^1 \cap A(\mathbf{R}^d)$,

$$(3.6.2) \quad \sup_{|x| \leq R} |f(x) - f_R^{\epsilon, \delta}(x)| = c'_0(\epsilon) + \eta \|f\|_{A(\mathbf{R}^d)} + (B_1 B_2)^{1/2} c_2(f, \delta) \delta^{-d},$$

where $c'_0(\epsilon)$ is given by (2.5.2) and

$$\lim_{\delta \rightarrow \infty} c_2(f, \delta) = A^d |\hat{f}(0)|.$$

We now derive a formula for computing the one-dimensional wavelet ψ given a radial, real-valued, admissible function Ψ on \mathbf{R}^d . The formula is dual to (3.1.3).

LEMMA 3.7. Let $\Psi \in L^1 \cap L^2(\mathbf{R}^d)$, $d \geq 2$, be radial, real-valued, admissible, and suppose that there exists an integer $M \geq d$ such that $\hat{\Psi}$ has continuous derivatives up to order M ,

$$(3.7.1) \quad \left| \frac{d^k}{dr^k} \hat{\Psi}(r) \right| \leq C(1 + |r|)^{-d-1},$$

for $k = 0, 1, \dots, M$, and

$$(3.7.2) \quad \frac{d^k}{dr^k} \hat{\Psi}(0) = 0,$$

for $k = 0, 1, \dots, M - d$.

Define $\psi(t)$ on \mathbf{R} by

$$(3.7.3) \quad \psi(t) = \int_0^\infty \hat{\Psi}(\gamma) \gamma^{d-1} \cos(2\pi\gamma t) d\gamma.$$

Then $\psi \in L^1 \cap A(\mathbf{R})$ is a real-valued, even, admissible function satisfying

$$(3.7.4) \quad R^\# \psi = \Psi,$$

$$(3.7.5) \quad |\psi(t)| \leq C(1 + |t|)^{-M}, \quad \text{and,}$$

$$(3.7.6) \quad |\hat{\psi}(\gamma)| \leq C(1 + |\gamma|)^{-2}.$$

PROOF: By (3.7.1), $\gamma^{d-1}\hat{\Psi}(\gamma)$ is a well-defined integrable function on $\hat{\mathbf{R}}$, so $\psi(t)$ is well-defined. Now, by (3.7.3),

$$\psi(t) = \frac{1}{2} \int_{-\infty}^{\infty} |\gamma|^{d-1} \hat{\Psi}(\gamma) e^{2\pi i \gamma t} d\gamma,$$

so that

$$\hat{\psi}(\gamma) = \frac{1}{2} |\gamma|^{d-1} \hat{\Psi}(\gamma) = \frac{1}{2} |\gamma|^{d-1} (R\Psi)^\wedge(\gamma).$$

Thus, $\psi = (1/2) I^{1-d} R\Psi$ so that $R^\# \psi = \Psi$. This establishes (3.7.4).

By (3.7.1) with $k = 0$,

$$|\hat{\psi}(\gamma)| = |\gamma|^{d-1} |\hat{\Psi}(\gamma)| \leq C(1 + |\gamma|)^{-d-1} (1 + |\gamma|)^{d-1} = C(1 + |\gamma|)^{-2},$$

which is (3.7.6).

To see (3.7.5), observe that,

$$\frac{d^k}{d\gamma^k} (\gamma^{d-1} \hat{\Psi}(\gamma)) = \sum_{j=0}^{\min(k, d-1)} c_j \gamma^{d-1-j} \frac{d^{k-j}}{d\gamma^{k-j}} \hat{\Psi}(\gamma),$$

so that, by (3.7.2),

$$\lim_{\gamma \rightarrow 0} \frac{d^k}{d\gamma^k} (\gamma^{d-1} \hat{\Psi}(\gamma)) = 0,$$

for $k = 0, 1, \dots, M$, and by (3.7.1),

$$|\gamma|^{d-1-j} \left| \frac{d^{k-j}}{d\gamma^{k-j}} \hat{\Psi}(\gamma) \right| \leq C(1 + |\gamma|)^{-2-j},$$

for $j \leq \min(k, d-1)$ and $k = 0, 1, \dots, M$, so that,

$$\lim_{\gamma \rightarrow \infty} \frac{d^{k-j}}{d\gamma^{k-j}} (\gamma^{d-1} \hat{\Psi}(\gamma)) = 0,$$

and, moreover, $d^k/d\gamma^k (\gamma^{d-1} \hat{\Psi}(\gamma))$ is integrable over \mathbf{R} for $k = 0, 1, \dots, M$. Therefore, integrating (3.7.3) by parts M times with $|t| \geq 1$,

$$\psi(t) = \frac{-2}{(2\pi t)^M} \int_0^\infty \frac{d^M}{d\gamma^M} (\gamma^{d-1} \hat{\Psi}(\gamma)) \left\{ \begin{array}{l} \sin(2\pi\gamma t), \quad M \text{ odd} \\ \cos(2\pi\gamma t), \quad M \text{ even} \end{array} \right\} d\gamma.$$

Thus,

$$|\psi(t)| \leq C(1 + |t|)^{-M},$$

which establishes (3.7.5). Combining (3.7.5), (3.7.6), and the fact that $d \geq 2$ implies that $\psi \in L^1 \cap A(\mathbf{R})$.

Finally, to see that ψ is admissible, note that

$$\int_0^\infty \frac{|\hat{\psi}(r)|^2}{r} dr = \frac{1}{4} \int_0^\infty |\hat{\Psi}(r)|^2 r^{2d-3} dr \leq C \int_0^\infty (1+r)^{-5} dr < \infty. \quad \blacksquare$$

We conclude this section with two examples.

EXAMPLE 3.8. An important kernel in image processing is given by

$$\Psi(x) = \Delta^N(e^{-\pi|x|^2}),$$

where $x \in \mathbf{R}^2$, Δ denotes the Laplacian and $N \geq 1$ is an integer. Kernels of this type are used in image compression [BA] and edge detection [C] algorithms.

Clearly, Ψ is radial and in fact,

$$\hat{\Psi}(\xi) = (2\pi i)^{2N} |\xi|^{2N} e^{-\pi|\xi|^2} = (-4\pi^2)^N |\xi|^{2N} e^{-\pi|\xi|^2}.$$

Thus,

$$\hat{\Psi}(r) = (-4\pi^2)^N r^{2N} e^{-\pi r^2}.$$

Clearly $\hat{\Psi} \in C^\infty(\hat{\mathbf{R}})$ and for any integer $M \geq 2$, we can find a constant C_M such that for $k = 0, 1, \dots, M$,

$$\left| \frac{d^k}{dr^k} \hat{\Psi}(r) \right| \leq C_M (1 + |r|)^{-3}.$$

Letting $M = 2N + 1$, $d^k/dr^k \hat{\Psi}(0) = 0$ for $k = 0, 1, \dots, M - 2$. Defining $\psi(t)$ by (3.7.3), $\psi \in L^1 \cap L^2(\mathbf{R})$ is even, admissible, real-valued and satisfies

- (a) $R^\# \psi(x) = \Delta^N(e^{-\pi|x|^2})$,
- (b) $|\psi(t)| \leq C(1 + |t|)^{-2N-1}$, and,
- (c) $|\hat{\psi}(\gamma)| \leq C(1 + |\gamma|)^{-2}$.

EXAMPLE 3.9. Given an integer $m \geq 0$, consider the one-dimensional m^{th} order B -wavelet, $\psi_m(t)$, defined by

$$\psi_m(t) = \sum_n q_{n+2m-1} N_m(2t - n),$$

where

$$q_n = \frac{(-1)^n}{2^{m-1}} \sum_{l=0}^m \binom{m}{l} N_{2m}(n+1-l),$$

and N_m is the order m box spline defined by

$$N_m(t) = \chi_{[0,1]} * \chi_{[0,1]} * \cdots * \chi_{[0,1]}(t),$$

where the convolution product is taken m times. This wavelet is a time-shifted version of the one described in [Ch1, Sec. 6.2].

The wavelet ψ_m , and related constructions of spline wavelets, e.g., [Ch2] have been extensively studied in connection with symmetric non-orthogonal multiresolution analyses of $L^2(\mathbf{R})$. The wavelet, ψ_m , is piecewise polynomial, even for m even, odd for m odd, and supported in $[-m+1/2, m-1/2]$.

Defining

$$Q_m(\gamma) = \frac{1}{2} \sum_n q_n e^{-2\pi i n \gamma},$$

then

$$Q_m(\gamma) = \left(\frac{1 - e^{-2\pi i \gamma}}{2} \right)^m \sum_{k=0}^{2m-2} N_{2m}(k+1) (-1)^k e^{-2\pi i k \gamma},$$

and

$$\begin{aligned} \hat{\psi}_m(\gamma) &= e^{2\pi i(2m-1)\gamma/2} Q_m(\gamma/2) \hat{N}_m(\gamma/2) \\ &= i^m \sin^m(\pi\gamma/2) \left(\frac{2 \sin(\pi\gamma/2)}{\pi\gamma} \right)^m \sum_{k=-m+1}^{m-1} N_{2m}(k+m) (-1)^{k+1} e^{-2\pi i k \gamma/2}. \end{aligned}$$

Since $\sin^m(\pi\gamma)$ has a zero of order m at $\gamma = 0$, we conclude that

$$\frac{d^k}{d\gamma^k} \hat{\psi}_m(0) = 0,$$

for $k = 0, 1, \dots, m-1$. Since ψ_m is compactly supported, $\hat{\psi}_m$ is infinitely differentiable and for each integer $k \geq 0$, there is a constant $C > 0$ such that

$$\left| \frac{d^k}{d\gamma^k} \hat{\psi}_m(\gamma) \right| \leq C(1+|\gamma|)^{-m}.$$

Taking then m even and $m \geq 4$, ψ_m satisfies all of the hypotheses of Lemma 3.1 with $d = 2$ and $M = m - 1$. Therefore, defining Ψ by (3.1.3), $\Psi \in L^1 \cap L^2(\mathbf{R}^2)$ is radial, admissible, real-valued and satisfies

- (a) $R^\# \psi_m(x) = \Psi(x)$,
- (b) $|\Psi(x)| \leq C(1+|x|)^{-m+1}$, and
- (c) $|\hat{\Psi}(\xi)| \leq C(1+|\xi|)^{-3}$.

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