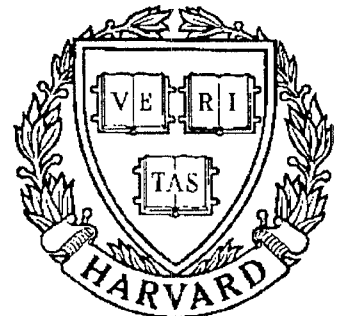


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The Interception of Spread-Spectrum Waveforms with the Amplitude Distribution Functions

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THE INTERCEPTION OF SPREAD-SPECTRUM WAVEFORMS WITH
THE AMPLITUDE DISTRIBUTION FUNCTIONS

William E. Snelling and Evaggelos Geraniotis

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THE INTERCEPTION OF SPREAD SPECTRUM WAVEFORMS WITH THE AMPLITUDE DISTRIBUTION FUNCTION

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ABSTRACT

Within the research effort related to unfriendly detection and interception of secure communications, an innovative concept called the Amplitude Distribution Function (ADF) is used to construct a detector that is an enhancement to the radiometer. The ADF is introduced and shown to be roughly the average probability distribution of a random process. The significance of ADF in the is that, under most spreading modulations, e.g. phase and frequency, the ADF is invariant. This suggests that a detector built around the ADF idea would be robust and of general purpose.

To develop the ADF methodology, a mathematical foundation is laid consisting of a sequence of definitions, lemmas, and theorems, an outline of which is included in the paper. The most significant result is that the ADF of signal plus noise is the convolution of the ADF of signal and the ADF of noise taken separately. These ideas are applicable through the definition of the Amplitude Moment Statistic (AMS), a statistical transform that converges to the moment generating function of the ADF. Hence, the AMS is the vehicle for indirectly estimating the ADF from observations. For the particular problem of detecting a modulated sinusoid in stationary Gaussian noise, a detector is developed around the AMS. The detector's performance is analyzed, compared with that of a radiometer, and shown superior for small (≈ 10) time-bandwidth products.

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1 BACKGROUND AND MOTIVATION

The unfriendly detection and interception of secure communications is a topic of much current research. Secure communications usually involve some variety of spread spectrum modulation, whose purpose is to add ambiguity or “randomness” to the communication waveform as a measure against unintended detection or interception. The usual procedure for randomizing the waveform is the pseudo-random variation of transmission times (time hopping), phases (direct sequence), or frequencies (frequency hopping). A method to detect a large class of spread-spectrum waveforms described in this report.

The use of the Amplitude Distribution Function (ADF) for detection is a new idea with potentially numerous and diverse applications. Although this report focuses solely on the detection of spread spectrum waveforms, the general ADF technique can be applied to related areas such as radar or sonar detection. The central idea of the technique is that the ADF of a spread-spectrum signal is invariant over a large class of modulations. This report demonstrates how to exploit this, not by estimating the ADF directly, but by estimating its characteristic function, a technique that works well even for small signal levels.

There are previous works [1,2,3,4] that, in essence, use the ADF, but none has given a precise definition and mathematical development such as those offered here. Moreover, there is apparently no reference that directly uses the ADF idea for detection.

The ADF indicates the time fraction that a waveform is below a given amplitude, much like a probability distribution function measures the probability that a random variable is below a given value. Previous researchers have used this concept but failed to give a precise definition of the ADF as it applies to both deterministic and stochastic signals. For signal $X(t)$, the definition is

$$F_X(a) = \lim_{T \rightarrow \infty} \frac{1}{T} \mathcal{E}\{\mathcal{L}[t : X(t) \leq a, 0 \leq t < T]\} \quad (1)$$

where \mathcal{L} is the set function giving length. Figure 1 illustrates that the ADF is simply the time fraction that $X(t)$ is below a given threshold a . With this definition it has been proved that, under very general conditions, the ADF of signal plus noise is the convolution of the signal ADF and the

noise ADF individually. This result would not have been possible without a definition that applied to both deterministic and stochastic signals.

The amplitude density function (adf) is the density, if it exists, implied by the ADF. There is, of course, a corresponding convolutional relationship between the adf of signal and noise and the individual adfs of signal and noise. An example makes this convolutional relationship concrete and hints at the potential of the ADF in signal detection. Figure 2 shows the ADF of a modulated sine wave (the signal), the ADF of noise, and the ADF of signal plus noise. The main point here is that the ADF of a sine wave is invariant under most phase and frequency modulations, but these are exactly the modulations used to thwart a potential interceptor. Therefore, the most typical spreading modulations will not degrade the performance of an ADF detector.

This original idea allows the application of the amplitude techniques to signal detection. Suppose a noisy signal $X(t)$, observed over the interval T , is transformed into the function $\hat{F}_X(\omega)$ by

$$\hat{F}_X(\omega) = \frac{1}{T} \int_0^T e^{j\omega X(t)} dt. \quad (2)$$

It is shown that, for large T , $\hat{F}_X(\omega)$ converges directly to the moment generating function of ADF. For this reason, this is called the “Amplitude Moment Statistic” (AMS). Furthermore, it is shown that samples of the AMS are approximately jointly Gaussian, to which well known optimal detection techniques apply.

In summary, the approach consists of defining the ADF precisely and proving the existence of an intuitive relationship between signal and noise; i.e., that of convolution. By transforming the observed waveform, a random process is generated that converges directly to the moment generating function of the ADF and to which standard detection techniques apply.

2 MATHEMATICAL TOOLS FOR THE ADF

The ADF, as defined here, is a new concept and thus needs a firm mathematical foundation. This section defines the ADF in a way that applies equally well to deterministic and to stochastic signals.

This basic definition is extended to include the concepts of a joint ADF and the notion of amplitude independence, a notion analogous to independence in probability. By way of a sequence of lemmas and theorems, two significant results are established. The first is the already promised result that the ADF of signal plus noise is the convolution of the signal ADF with the noise ADF. This is proved under the very general constraint that the second derivative of the noise autocorrelation exists and is finite at time difference zero. The concept of expected amplitude is introduced leading to the remaining result which links the ADF to the instantaneous probability distribution of the stochastic process.

2.1 DEFINITION OF ADF

The discussion begins with a precise definition of the ADF.

Definition 1 *The Amplitude Distribution Function (ADF), written $F_X(a)$ for a stochastic process $X(t)$, is*

$$F_X(a) = \lim_{T \rightarrow \infty} \frac{1}{T} \mathcal{E} \{ \mathcal{L} [t : X(t) \leq a, 0 \leq t < T] \} \quad (3)$$

where \mathcal{L} is a set function giving length. Additionally, the limit must exist for all a .

If the signal is deterministic, the definition of the ADF reduces to

$$F_X(a) = \lim_{T \rightarrow \infty} \frac{1}{T} \{ \mathcal{L} [t : X(t) \leq a, 0 \leq t < T] \}. \quad (4)$$

The ADF is not a distribution function in the strict sense of the word, because its extreme values are not necessarily one or zero, and it may not be right-continuous. For example, the function

$$S(t) = \begin{cases} t & \text{for } 0 < t \bmod 1 \leq 1/2 \\ -t & \text{for other } t \end{cases} \quad (5)$$

has ADF identically equal to 1/2. Consider also the function

$$S(t) = \begin{cases} 1 & \text{for } 0 < t \bmod 1 \leq 1/2 \\ -1 & \text{for other } t. \end{cases} \quad (6)$$

Its ADF is

$$F_S(a) = \begin{cases} 0 & \text{for } \infty < a < -1 \\ 1/2 & \text{for } -1 \leq a \leq 1 \\ 1 & \text{for } 1 < a \end{cases} \quad (7)$$

which is right-continuous at -1 and left-continuous at 1 . The ADF, not necessarily being a true distribution, creates problems in situations that require an ADF-induced measure; for instance, the Lebesgue-Stieltjes integral. In these cases, the right-continuous extension of the ADF is used and defined as

$$F^*(a) = \lim_{x \rightarrow a^+} F(x). \quad (8)$$

2.2 JOINT ADF AND AMPLITUDE INDEPENDENCE

Analogous to the joint probability function of random variables, there exists a joint ADF between stochastic processes, defined as follows:

Definition 2 *The joint ADF, written $F_{X,Y}(a, b)$ for stochastic processes $X(t)$ and $Y(t)$, is*

$$F_{X,Y}(a, b) = \lim_{T \rightarrow \infty} \frac{1}{T} \mathcal{E} \{ \mathcal{L} [t : X(t) \leq a \text{ and } Y(t) \leq b, 0 \leq t < T] \}. \quad (9)$$

This definition will be used to define the following concept of amplitude independence, analogous to that of independence between random variables.

Definition 3 *Two stochastic processes $X(t)$ and $Y(t)$ are amplitude independent if their joint ADF is the product of the ADF for each process. In other words,*

$$F_{X,Y}(a, b) = F_X(a)F_Y(b). \quad (10)$$

2.3 ADF OF STATIONARY GAUSSIAN NOISE

A relationship is now shown between the ADF of stationary Gaussian noise and its instantaneous probability distribution, thus enabling the establishment of more directly applicable results.

Lemma 1 *Let $Y(t)$ be a stationary, zero-mean, Gaussian process with autocorrelation $R(t)$, such that $-R''(0) < \infty$; then, for any measurable set A ,*

$$\mathcal{E} \{ \mathcal{L} [t : Y(t) \leq a, t \in A] \} = \Phi \left(\frac{a}{\sigma_0} \right) \mathcal{L} A \quad (11)$$

where Φ is the distribution function of a standard Gaussian random variable and $\sigma_0 = \sqrt{R(0)}$.

The proof for Lemma 1 is included in Appendix A.

This lemma means that the average time that the noise process is below the threshold a on the set A is equal to the percent of time that the noise process is below a at any single point times the length of the set A . This result implies that the ADF of stationary Gaussian noise is identical to its instantaneous distribution.

With the help of the previous results, amplitude independence can now be proved between a deterministic signal and stationary Gaussian noise, whose autocorrelation has finite second derivative at time difference zero. This result will be necessary to prove the convolutional relationship between signal and additive noise.

Theorem 1 *Let $S(t)$ be a deterministic signal and let $N(t)$ be a stationary, zero-mean, Gaussian process with autocorrelation $R(t)$ such that $-R''(0) < \infty$; then, $S(t)$ and $N(t)$ are amplitude independent. Stated symbolically,*

$$F_{S,N}(a, b) = F_S(a)F_N(b). \quad (12)$$

2.3.1 Proof

By Lemma 1,

$$\begin{aligned} \mathcal{E} \{ \mathcal{L} [t : S(t) \leq a \text{ and } N(t) \leq b, 0 \leq t < T] \} \\ = \mathcal{E} \{ \mathcal{L} [t : N(t) \leq b, t \in S_a^{-1} \cap [0, T]] \} \end{aligned} \quad (13)$$

$$= F_N(b) \mathcal{L} [S_a^{-1} \cap [0, T]] \quad (14)$$

where $S_a^{-1} = [t : S(t) \leq a]$. Hence the joint ADF of $S(t)$ and $N(t)$ is

$$F_{S,N}(a, b) = F_N(b) \lim_{T \rightarrow \infty} \frac{1}{T} \mathcal{L} [S_a^{-1} \cap [0, T]] \quad (15)$$

$$= F_S(a) F_N(b) \quad (16)$$

implying the amplitude independence of $S(t)$ and $N(t)$.

2.4 ADF OF ADDITIVE AMPLITUDE-INDEPENDENT SIGNALS

The most important result of this section can be proved; i.e., that of convolution between the ADFs of signal and additive noise. The idea of deconvolution and the deconvolution statistic rests firmly on this result.

Theorem 2 *Let $S(t)$ and $N(t)$ be amplitude independent and let either F_N or F_S be continuous; then, the ADF of $Y(t) = S(t) + N(t)$ is*

$$F_Y(a) = \int_{-\infty}^{\infty} F_N(a - s) dF_S^*(s) \quad (17)$$

$$= \int_{-\infty}^{\infty} F_S(a - n) dF_N^*(n) \quad (18)$$

where F_S^* and F_N^* are the right-continuous extensions of F_S and F_N .

The proof is given in Appendix B.

2.5 EXPECTED AMPLITUDE

A notion analogous to the expected value of a random variable is expected amplitude, which is defined as follows:

Definition 4 *The expected amplitude, written $\overline{G(Y)}$ for a stochastic process $Y(t)$ and function G , is*

$$\overline{G(Y)} = \int_{-\infty}^{\infty} G(y) dF_Y(y) \quad (19)$$

provided that the integral exists.

From this definition follows that

Theorem 3 *Let the stochastic process $Y(t)$ have right-continuous ADF, F_Y , and then let function $G(y)$ be bounded and continuous almost everywhere with respect to the measure induced by F_Y ; then,*

$$\overline{G(Y)} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T G[Y(t)] dt. \quad (20)$$

The proof is included in Appendix C.

This theorem can now be used to prove a general statement about the ADF of signal with additive noise and its instantaneous distribution. Namely,

Lemma 2 *Let $Y(t) = S(t) + N(t)$ and let F_S be right-continuous; then, if either F_N or F_S is continuous, the ADF of $Y(t)$ is*

$$F_Y(a) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T F_N[a - S(t)] dt. \quad (21)$$

Alternatively, this result could have served as the definition of the ADF, but there would then be technical difficulties in determining the ADFs of purely deterministic signals.

3 AMPLITUDE MOMENT FUNCTION

3.1 RECAPITULATION

As shown in the previous section, the ADF of signal plus noise is the ADF of the signal convolved with the ADF of the noise. What was exactly shown is that

for $X(t) = S(t) + N(t)$, where $N(t)$ is a stationary Gaussian process with autocorrelation $R(t)$ satisfying $-R''(0) < \infty$ and where $S(t)$ is a deterministic signal with defined ADF; then $F_X = F_S * F_N$, where F_X , F_S , and F_N are the respective ADFs of $X(t)$, $S(t)$, and $N(t)$.

Also, the notion of expected amplitude was introduced and connected to time average of a non-linearity applied to the observed stochastic process. To apply these results on the construction of a detector, it will be assumed herein that the above restrictions are met and that noise, signal, and observations have densities defined as $f_X \triangleq dF_X(x)/dx$, $f_S \triangleq df_S/da$, and $f_N \triangleq dF_N/da$. These densities are called the amplitude density functions (adfs). For reasons explained later, the restriction is made that the noise autocorrelation is zero after some duration (i.e. $R(t) = 0$, for t greater than some T_1).

3.2 AMPLITUDE MOMENT FUNCTION AND STATISTIC

If the adf could somehow be measured or estimated, these results could be used to detect the presence of a signal. But to circumvent the problems of density estimation and map the detection problem into a domain with greater differentiation between signal-present and signal-not-present hypotheses, it was chosen instead to estimate the moment generating function of the adf. For this purpose, the Amplitude Moment Function (AMF) and Amplitude Moment Statistic (AMS) are introduced.

The AMS, written $\hat{F}_X\omega$, of the process $X(t)$ is defined as

$$\hat{F}_X(\omega) = \frac{1}{T} \int_0^T e^{j\omega X(t)} dt. \quad (22)$$

Its usefulness stems from the fact (proved below) that, for large T , it converges in probability to the moment generating function of the adf of $X(t)$. This limit defines the AMF.

3.3 STATISTICAL CHARACTERIZATION

To set up the detection problem, at least an asymptotic statistical characterization of the AMS is needed. Specifically, it will be shown that samples of the AMS are asymptotically jointly Gaussian for large T . Additionally, its asymptotic mean and variance will be shown.

As for the jointly Gaussian property, consider samples of the AMS

$$z_i = \frac{1}{T} \int_0^T e^{j\omega_i X(t)} dt \quad (23)$$

for some finite sequence $\{\omega_i\}_{i=1}^n$. To prove that the z_i s are jointly Gaussian, it is sufficient to prove that $\sum_i^n c_i z_i$ is Gaussian for arbitrary constants c_i . Rewrite

$$\sum_i^n c_i z_i = \sum_{j=1}^l I_{2j} + \sum_{j=1}^m I_{2j-1} \quad (24)$$

where

$$I_k = \frac{1}{T} \int_{(k-1)T_1}^{\max(T, kT_1)} \sum_{i=1}^n c_i e^{j\omega_i X(t)} dt \quad (25)$$

$$l = \left\lceil \frac{T}{2T_1} \right\rceil \quad (26)$$

$$m = \left\lceil \frac{T}{2T_1} + \frac{1}{2} \right\rceil. \quad (27)$$

It is easily seen that the I_{2j} s are independent and that the I_{2j-1} s are also independent. Hence, for large T , each sum $\sum_{j=1}^l I_{2j}$ and $\sum_{j=1}^m I_{2j-1}$ is individually asymptotically Gaussian. Furthermore, even though the sums are correlated, the overall sum is approximately Gaussian since each component sum is Gaussian. Hence, samples of the AMS are jointly Gaussian.

As for the mean,

$$\mu(x) \triangleq \mathcal{E} \left[\hat{F}_X(\omega) \right] \quad (28)$$

$$= \frac{1}{T} \mathcal{E} \left\{ \int_0^T e^{j\omega X(t)} \right\} \quad (29)$$

$$\approx F_X(\omega) \quad \text{for large } T \quad (30)$$

with the last step following from Theorem 3.

As for the variance,

$$\sigma^2(x, y) \triangleq \mathcal{E} \{ [\hat{F}_X(x) - \mu(x)][\hat{F}_X(y) - \mu(y)] \} \quad (31)$$

$$= \frac{1}{T^2} \int_0^T \int_0^T \left(e^{-\frac{1}{2}M^T \Sigma_1 M} - e^{-\frac{1}{2}M^T \Sigma_0 M} \right) ds dt \quad (32)$$

where

$$\mathbf{M} = \begin{bmatrix} v - S(s) \\ w - S(t) \end{bmatrix} \quad (33)$$

$$\boldsymbol{\Sigma}_1 = \begin{bmatrix} \sigma_0^2 & R(s-t) \\ R(s-t) & \sigma_0^2 \end{bmatrix} \quad (34)$$

$$\boldsymbol{\Sigma}_0 = \begin{bmatrix} \sigma_0^2 & 0 \\ 0 & \sigma_0^2 \end{bmatrix}. \quad (35)$$

It will be shown that the variance goes to zero as the integration time T goes to infinity from whence it follows that the detection statistic converges, in probability, to its mean. To this end, assume $T > T_1$, note that $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_0$ for $|s - t| > T_1$, and then rewrite

$$\sigma^2(x, y) = \frac{1}{T^2} \int \int_D \left(e^{-\frac{1}{2} \mathbf{M}^T \boldsymbol{\Sigma}_1 \mathbf{M}} - e^{-\frac{1}{2} \mathbf{M}^T \boldsymbol{\Sigma}_0 \mathbf{M}} \right) ds dt \quad (36)$$

where the region of integration within the square, $Q = [s, t : 0 \leq s \leq T, 0 \leq t \leq T]$, is

$$D = [s, t : |s - t| \leq T_1] \cap Q. \quad (37)$$

The area of D is less than $2TT_1$ and the integrand is absolutely bounded by 1. Hence,

$$|\sigma^2(x, y)| \leq \frac{2T_1}{T} \quad (38)$$

implying that $\sigma^2(x, y) \rightarrow 0$ for large T , as conjectured.

The fact that samples of the AMS are asymptotically jointly Gaussian and the expressions of its mean and variance will be necessary for the detector development described in Section 4.

4 DETECTION WITH THE AMF

Using the fact that samples of the amplitude moment statistic are asymptotically jointly Gaussian, a classical detector can be constructed that observes amplitude moments. This idea is developed by defining an observation model encompassing a large class of spread spectrum waveforms. For

these signals, samples of the amplitude moment statistic are considered upon which an optimum decision statistic is applied. To complete the discussion, a performance analysis complements the signal model definition and detector development.

4.1 OBSERVATION MODEL

The observation model is for a composite hypothesis problem. Specifically, given the observation $X(t)$, the problem is one of choosing between H_0 , which is the hypothesis that a spread spectrum waveform is not present, and $H_{\gamma'}$, which is the hypothesis that a waveform is present with SNR γ' greater than some minimum SNR γ (SNR is defined the signal to noise ratio). The model is precisely

$$\begin{array}{l} H_0 : X(t) = N(t) \\ \text{versus} \\ H_{\gamma'} : X(t) = S(t) + N(t) \end{array} \quad \gamma < \gamma' \quad (39)$$

where the spread-spectrum waveform $S(t)$ has mean square E' and where $N(t)$ is stationary colored Gaussian noise with variance σ_0^2 and autocorrelation function $R(t)$. The hypothesized SNR γ' is related to the other model parameters by $\gamma' = E'/\sigma_0^2$, whereas, similarly, the minimum SNR $\gamma = E/\sigma_0^2$.

4.2 DETECTOR SYNTHESIS

Before a decision is made, the amplitude moment statistic is applied to the observations (Figure 3) which after sampling forms the new detection problem of deciding between the presence and absence of a signal, given $\{z_i = \hat{F}_X(\omega_i)\}_{i=1}^n$. As shown earlier, the z_i s are jointly Gaussian and have means (Equation 30)

$$\mu_{\gamma'}(\omega_i) = \begin{cases} \hat{F}_N(\omega_i), & \text{signal absent} \\ \hat{F}_N(\omega_i)\hat{F}_S(\omega_i), & \text{signal present} \end{cases} \quad (40)$$

where

$$\hat{F}_N(\omega) = \frac{1}{\sqrt{2\pi N_0 W}} e^{-\frac{1}{2} N_0 W \omega^2}. \quad (41)$$

Additionally, the z_i s have covariances $\sigma_\gamma(\omega_i, \omega_j)$ defined by Equation 32. The jointly Gaussian samples along with the expressions for their means and variance constitutes a complete statistical description that allows classical methods to be used for designing a detector.

The detector design proceeds by assuming that both the noise and signal amplitudes are known, from which an optimum detection statistic follows. Later this unrealistic restriction will be relaxed by the assumption that the noise level is known or measured and the signal level is above that used in the detector's synthesis. This will be suboptimum in general, but in the important low-signal-level case, performance will approach the optimum.

Now, via the likelihood ratio [5] the optimal test statistic is defined as

$$L = \mathbf{E}_0^T \mathbf{R}_0^{-1} \mathbf{E}_0 - \mathbf{E}_\gamma^T \mathbf{R}_\gamma^{-1} \mathbf{E}_\gamma \quad (42)$$

where

$$\mathbf{R}_\gamma = \begin{bmatrix} \sigma_\gamma(\omega_1, \omega_1) & \cdots & \sigma_\gamma(\omega_1, \omega_n) \\ \vdots & & \vdots \\ \sigma_\gamma(\omega_n, \omega_1) & \cdots & \sigma_\gamma(\omega_n, \omega_n) \end{bmatrix} \quad (43)$$

and

$$\mathbf{E}_\gamma = \begin{bmatrix} z_i - \mu_\gamma(\omega_1) \\ \vdots \\ z_i - \mu_\gamma(\omega_n) \end{bmatrix}. \quad (44)$$

4.3 PERFORMANCE ANALYSIS

An exact analysis of the detector is very difficult, since the test statistic is a quadratic form. But, under conditions defined below, the test statistic proves to be asymptotically Gaussian; performance expressions follow, given the mean and variance of the test statistic.

To justify the asymptotic Gaussian assumption, it is noted that, since \mathbf{R}_γ is nonnegative-definite and symmetric, there exists a matrix $\mathbf{R}_\gamma^{\frac{1}{2}}$ such that $\mathbf{R}_\gamma^{\frac{T}{2}} \mathbf{R}_\gamma^{\frac{1}{2}} = \mathbf{R}_\gamma$. This fact and the diagonalization

of $\mathbf{R}_\gamma^{-\frac{T}{2}} \mathbf{R}^{-1} \mathbf{R}_\gamma^{-\frac{1}{2}} = \mathbf{T}_{\gamma',\gamma}^T \Lambda_{\gamma',\gamma} \mathbf{T}_{\gamma',\gamma}$ are used to rewrite the test statistic as

$$L = (\mathbf{G} + \mathbf{M}_{\gamma',0})^T \Lambda_{\gamma',0} (\mathbf{G} + \mathbf{M}_{\gamma',0}) - (\mathbf{G} + \mathbf{M}_{\gamma',\gamma})^T \Lambda_{\gamma',\gamma} (\mathbf{G} + \mathbf{M}_{\gamma',\gamma}) \quad (45)$$

where

$$\mathbf{G} = \begin{bmatrix} g_1 \\ \vdots \\ g_2 \end{bmatrix} \quad (46)$$

with $\{g_i\}$ independent, zero mean, and unity variance and where

$$\mathbf{M}_{\gamma',\gamma} \triangleq \begin{bmatrix} m_{1,\gamma',\gamma} \\ \vdots \\ m_{n,\gamma',\gamma} \end{bmatrix} = \mathbf{T}_{\gamma',\gamma}^T \mathbf{R}_{\gamma'}^{-\frac{1}{2}} \begin{bmatrix} \mu_{\gamma'}(\omega_1) - \mu_\gamma(\omega_1) \\ \vdots \\ \mu_{\gamma'}(\omega_n) - \mu_\gamma(\omega_n) \end{bmatrix}. \quad (47)$$

It is noted that $(\mathbf{G} + \mathbf{M}_{\gamma',\gamma})^T \Lambda_{\gamma',\gamma} (\mathbf{G} + \mathbf{M}_{\gamma',\gamma})$ is a sum of squares of independent Gaussian variables. Now, through application of the Berry-Esseen Theorem [6] this term is approximately Gaussian distributed with an error of no more than $4c/\sigma$ where

$$c = \max_i \lambda_{i,\gamma',\gamma} \frac{24m_{i,\gamma',\gamma}^2 + 7}{4m_{i,\gamma',\gamma}^2 + 2} \quad (48)$$

$$\sigma^2 = \sum_{i=1}^n \lambda_{i,\gamma',\gamma}^2 (4m_{i,\gamma',\gamma}^2 + 2). \quad (49)$$

For each particular detection problem and for each value of SNR, this error bound determines the validity of the Central Limit Theorem (CLT) argument. Assuming the bound is small, L itself must be approximately Gaussian, being the sum of two Gaussian distributed random variables. The distribution, and hence performance of L , is determined by its mean and variance as computed below. The mean is

$$\mathcal{M}_{\gamma',\gamma} = \sum_{i=1}^n \left[\lambda_{i,\gamma',0} (m_{i,\gamma',0}^2 + 1) - \lambda_{i,\gamma',\gamma} (m_{i,\gamma',\gamma}^2 + 1) \right] \quad (50)$$

whereas, the variance is

$$\mathcal{V}_{\gamma',\gamma} = \sum_{i=1}^n \left[2 \left(\lambda_{i,\gamma',0}^2 - \lambda_{i,\gamma',\gamma}^2 \right)^2 + 4 \left(m_{i,\gamma',0} \lambda_{i,\gamma',0} - m_{i,\gamma',\gamma} \lambda_{i,\gamma',\gamma} \right)^2 \right]. \quad (51)$$

Since the test statistic L has an approximately Gaussian distribution, the threshold v and probability of detection P_D , for a given probability of false alarm, follow as

$$v = \sqrt{\mathcal{V}_{0,\gamma}} \Phi^{-1}(1 - P_F) + \mathcal{M}_{0,\gamma} \quad (52)$$

and

$$P_D = 1 - \Phi \left(\frac{\sqrt{\mathcal{V}_{0,\gamma}} \Phi^{-1}(1 - P_F) - \mathcal{M}_{\gamma',\gamma} + \mathcal{M}_{0,\gamma}}{\sqrt{\mathcal{V}_{\gamma',\gamma}}} \right) \quad (53)$$

where $\Phi(x)$ is the distribution function of the standard Gaussian.

5 PERFORMANCE COMPARISONS

Section 4 showed how amplitude ideas can be used to construct a detector and also analytically shown how this detector performs. The following discussion numerically compares AMF-based detector performance with that of the radiometer.

The numerical analysis has several parameters that specify the detector and signal. With more explicit explanations to follow, they are:

f_0 signal frequency

P signal power

W spread spectrum bandwidth in hertz (hz)

$B = WT$ time-bandwidth product; i.e., W times the detector integration time, T ,
in seconds

γ expected SNR given as P/σ_0^2

γ' actual SNR given as P'/σ_0^2

N one half the number of AMS frequencies sampled

f_{\max} maximum AMS frequency sampled

f_{\min} minimum AMS frequency sampled

Now, the specific observations used are described. The signal has form

$$S(t) = \sqrt{2P'} \sin(2\pi f_0 t). \quad (54)$$

Even though the signal is not spread, any conclusions about relative performance between the ADF-based detector and the radiometer still follow because of the invariance of the ADF to most spreading modulations. It is assumed that the noise $N(t)$ has autocorrelation

$$R(t) = \sigma_0^2(1 - W|t|), \quad -1 < W|t| < 1 \quad (55)$$

where σ_0 is a standard deviation. This particular noise process is examined because it satisfies a sufficient requirement for the performance equations to be valid; specifically, the autocorrelation is identically zero after some time interval, but more important, to our choice, this process closely approximates the widely encountered process whose power is uniformly spread over W .

The detector itself is specified. For observation $X(t)$, the detector has AMS samples, $\{z_i = \widehat{F}_X(2\pi f_i)\}_{i=1}^{2N}$, taken at corresponding moment frequencies $\{f_i\}_{i=1}^{2N}$ where

$$f_i = -f_{\min} + (i - N - .5) \frac{f_{\max} - f_{\min}}{N} \quad 1 \leq i \leq N \quad (56)$$

and

$$f_i = f_{\min} + (i - N - .5) \frac{f_{\max} - f_{\min}}{N} \quad N < i \leq 2N \quad (57)$$

Several other parameters are needed to complete the specification of the detector. Two are the expected SNR γ and integration time T which is computable from the time-bandwidth product B . These two, in turn, specify the moments of the AMS samples when given Equations 40 and 32 and the AMF of a sine wave; namely,

$$\widehat{F}_S(\omega) = J_0(\sqrt{2P'}\omega) \quad (58)$$

with J_0 being the zeroth-order Bessel function of the first kind. The final specification, the test statistic given by Equation 42, is implied by the moments of the AMS samples.

For particular comparisons, the performance Equations 50, 51, 52, and 52 and the parameters posted in Table 1 were used. The results are pictured in Figures 4 and 5. The Receiver Operating

Characteristics (ROCs) show a significant gain in detection probabilities with the largest gain for the zero SNR case. This is expected, because the detector is specified under this case and is hence optimal or tuned for this signal level. Amplitude tuning is a unique feature of the AMF approach. The other ROCs show substantial performance gains even for diminishing SNR. This aspect coupled the relatively small time-bandwidth product of these results shows that AMF processing could be useful for predetection processing; i.e., accumulate intermediate results from the AMS over a sufficiently large time to yield useful detection probabilities. The probability of detection curves reinforce these conclusions in that they show the most significant gains over the zero SNR region and also show that, even though there is no performance gain (or even a small loss) for high SNRs the gains at smaller SNRs can prove useful in the accumulation scheme mentioned previously.

6 CONCLUSIONS

The ADF was introduced and shown to be roughly the average probability distribution of a random process. The significance of ADF in the world of spread spectrum interception is that, under most spreading modulations (e.g. phase and frequency), the ADF is invariant. Hence, a detector built around the ADF idea would be robust and general purpose.

To develop the ADF methodology, a mathematical foundation was outlined consisting of a sequence of definitions, lemmas, and theorems, the most significant of which was the fact that the ADF of signal plus noise is the convolution of the ADF of signal and the ADF of noise taken separately. These ideas were made applicable through the definition of the amplitude moment statistic, a statistical transform that converges to the moment generating function of the ADF. Hence, the ADF could be indirectly estimated from the observations.

For the particular problem of detecting a modulated sinusoid in stationary Gaussian noise, a detector was developed around the detection statistic. The detector's performance was analyzed and compared with that of a radiometer where a significant performance gain was concluded for small time-bandwidth products. Therefore, it is recommended that the radiometer be replaced

with the AMF processing in such detector structures as the channelized detector and others with small integration times.

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A PROOF OF LEMMA 1

The sets A , which are finite half open intervals, are considered first. Of these, it is necessary to consider only the interval $[0, T)$ since $Y(t)$ is stationary. Partition $A = [0, T)$ into n subintervals

$$B_i = \left[(i-1)\frac{T}{n}, i\frac{T}{n} \right) \text{ for } i = 1, \dots, n \quad (59)$$

and define the set $Y_a^{-1} = [t : Y(t) \leq a]$. Observing that the length of the set $Y_a^{-1} \cap A$ is the sum of the lengths of the sets $Y_a^{-1} \cap B_i$, one can write

$$\mathcal{E} \left[\mathcal{L}(Y_a^{-1} \cap A) \right] = \sum_{i=1}^n \mathcal{E} \left[\mathcal{L}(Y_a^{-1} \cap B_i) \right] \quad (60)$$

$$= n \mathcal{E} \left[\mathcal{L}(Y_a^{-1} \cap B_1) \right] \quad (61)$$

since $Y(t)$ is stationary. Now define the following three events:

C is the event that $Y(t) < a$, for some $t \in B_1$

D is the event that $Y(t) \leq a$, for all $t \in B_1$

E is the event that $Y(t)$ crosses a on B_1

Notice that

$$\mathcal{L}(Y_a^{-1} \cap B_1) \geq \frac{T}{n} I_D \quad (62)$$

where I_D is the indicator of the event D . To understand this relationship, consider the case that the sample path $Y(t)$ is in D , meaning that it is not above a during the entire interval B_1 . It follows that the amount of time that it is not above a [i.e. $\mathcal{L}(Y_a^{-1} \cap B_1)$] equals the length of B_1 , which is T/n . Upon taking expectations of this relationship,

$$\mathcal{E} \left[\mathcal{L}(Y_a^{-1} \cap B_1) \right] \geq \frac{T}{n} \mathcal{E}(I_D) \quad (63)$$

$$= \frac{T}{n} \Pr(D) \quad (64)$$

$$= \frac{T}{n} \{ \Pr[Y(0) \leq a] - \Pr[E \text{ and } (Y(0) \leq a)] \} \quad (65)$$

since the probability that $Y(t)$ is not above level a over the interval B_1 is exactly the probability that $Y(0)$ is not above a and, under this condition, $Y(t)$ does not cross a . Now, since $\Pr[E \text{ and } (Y(0) \leq a)]$ is less than or equal to $\Pr(E)$,

$$\mathcal{E} \left[\mathcal{L}(Y_a^{-1} \cap B_1) \right] \geq \frac{T}{n} \{ \Pr[Y(0) \leq a] - \Pr(E) \}. \quad (66)$$

In an analogous manner, a complementary inequality can be produced by observing that

$$\mathcal{L}(Y_a^{-1} \cap B_1) \leq \frac{T}{n} I_C. \quad (67)$$

This inequality follows by considering two cases. When the sample path $Y(t)$ is in C , meaning that it is below a sometime during the interval B_1 , the amount of time that it is not above a is not greater than the length of B_1 or T/n . Alternatively, whenever $Y(t)$ is not in C , it is not above a for zero time. Taking expectations of this inequality yields

$$\mathcal{E} \left[\mathcal{L}(Y_a^{-1} \cap B_1) \right] \leq \frac{T}{n} \mathcal{E}(I_C) \quad (68)$$

$$= \frac{T}{n} \Pr(C) \quad (69)$$

$$= \frac{T}{n} \{ \Pr[Y(0) \leq a] - \Pr[E \text{ and } (Y(0) \geq a)] \} \quad (70)$$

since the probability that $Y(t)$ is below level a for some time during the interval B_1 is exactly the probability that $Y(0)$ is not below a but crosses a during the interval B_1 plus the probability that $Y(0)$ is below a initially. Now, since $\Pr[E \text{ and } Y(0) \geq a]$ is less than or equal to $\Pr(E)$,

$$\mathcal{E} \left[\mathcal{L}(Y_a^{-1} \cap B_1) \right] \leq \frac{T}{n} \{ \Pr[Y(0) \leq a] - \Pr(E) \}. \quad (71)$$

Equations 66 and 71 applied to Equation 61 imply

$$T \{ \Pr[Y(0) \leq a] - \Pr(E) \} \leq \mathcal{E} \left[\mathcal{L}(Y_a^{-1} \cap B_1) \right] \leq T \{ \Pr[Y(0) \leq a] + \Pr(E) \} \quad (72)$$

thus

$$\mathcal{E} \left[\mathcal{L} \left(Y^{-1} \cap B_1 \right) \right] = \Pr[Y(0) \leq a]T \quad (73)$$

$$= \Phi \left(\frac{a}{\sigma_0} \right) \mathcal{L}A \quad (74)$$

if $\lim_{n \rightarrow \infty} \Pr(E) \rightarrow 0$.

To prove that $\lim_{n \rightarrow \infty} \Pr(E) \rightarrow 0$, we define process $N_a(t)$ is defined as the number of crossings of the threshold a by the process $Y(t)$ on the interval $[0, t)$. By Chebyshev's inequality,

$$\Pr(E) = \Pr \left[N_a \left(\frac{T}{n} \right) \geq 1 \right] \quad (75)$$

$$\leq \mathcal{E} \left[N_a \left(\frac{T}{n} \right) \right] \quad (76)$$

but from Karlin and Taylor [7]

$$\mathcal{E} \left[N_a \left(\frac{T}{n} \right) \right] = \frac{T}{n} \frac{\alpha}{\sigma_0} e^{-\frac{\alpha^2}{2\sigma_0}} \quad (77)$$

where $\alpha^2 = -R(0)''$. Upon letting $n \rightarrow \infty$, the last two equations imply $\Pr(E) \rightarrow 0$. Now that the result

$$\mathcal{E} \left[\mathcal{L}(Y_a^{-1} \cap A) \right] = \Phi \left(\frac{a}{\sigma_0} \right) \mathcal{L}A \quad (78)$$

has been proven for A , an interval, it can be extended to any finite set as follows. Let A be a set of finite length; then for any $\epsilon > 0$, there exists a finite set of intervals $\{L_i\}_{i=1}^l$ such that

$$\mathcal{L} \left[A - \bigcup_{i=1}^l L_i \right] \leq \frac{\epsilon}{2} \quad (79)$$

and

$$\mathcal{L} \left[\bigcup_{i=1}^l L_i - A \right] \leq \frac{\epsilon}{2} \quad (80)$$

(see [8]). Hence,

$$\sum_{i=1}^l \mathcal{L}(Y_a^{-1} \cap L_i) - \frac{\epsilon}{2} \leq \mathcal{L}(Y_a^{-1} \cap A) \leq \sum_{i=1}^l \mathcal{L}(Y_a^{-1} \cap L_i) + \frac{\epsilon}{2}. \quad (81)$$

Taking expectations and applying the result for intervals

$$\Phi \left(\frac{a}{\sigma_0} \right) \sum_{i=1}^l \mathcal{L}L_i - \frac{\epsilon}{2} \leq \mathcal{E} \left[\mathcal{L}(Y_a^{-1} \cap A) \right] \leq \Phi \left(\frac{a}{\sigma_0} \right) \sum_{i=1}^l \mathcal{L}L_i + \frac{\epsilon}{2}. \quad (82)$$

But,

$$\mathcal{L}A - \frac{\epsilon}{2} \leq \sum_{i=1}^l \mathcal{L}L_i \leq \mathcal{L}A + \frac{\epsilon}{2} \quad (83)$$

implying

$$\Phi\left(\frac{a}{\sigma_0}\right) \mathcal{L}A - \epsilon \leq \mathcal{E}\left[\mathcal{L}(Y_a^{-1} \cap A)\right] \leq \Phi\left(\frac{a}{\sigma_0}\right) \mathcal{L}A + \epsilon. \quad (84)$$

But, since ϵ is an arbitrary positive number, the result is a set A with finite length,

$$\mathcal{E}\left[\mathcal{L}(Y_a^{-1} \cap A)\right] = \Phi\left(\frac{a}{\sigma_0}\right) \mathcal{L}A. \quad (85)$$

To extend to the case in which A is not of finite length, write $A = \cup_{i=0}^{\infty} A_i$, where the A_i 's are disjoint and of finite length; then

$$\mathcal{E}\left[\mathcal{L}(Y_a^{-1} \cap A)\right] = \sum_{i=0}^{\infty} \mathcal{E}\left[\mathcal{L}(Y_a^{-1} \cap A_i)\right] \quad (86)$$

$$= \sum_{i=0}^{\infty} \Phi\left(\frac{a}{\sigma_0}\right) \mathcal{L}A_i \quad (87)$$

$$= \Phi\left(\frac{a}{\sigma_0}\right) \mathcal{L}A. \quad (88)$$

B PROOF OF THEOREM 2

Without loss of generality, assume F_S is continuous. To prove the result, compute

$$F_Y(a) = \lim_{T \rightarrow \infty} \frac{1}{T} \left\{ \mathcal{L}\left[Y_a^{-1} \cap [0, T)\right] \right\} \quad (89)$$

where

$$Y_a^{-1} = [t : S(t) + N(t) \leq a]. \quad (90)$$

Begin by selecting an integer m and constructing a partition of the real line, ($s_{-m^2} < s_{-m^2+1} < \dots < s_{m^2}$), where

$$s_i = \begin{cases} -\infty & \text{for } i = -m^2 \\ i/m & \text{for } -m^2 < i < m^2 \\ \infty & \text{for } i = m^2. \end{cases} \quad (91)$$

Continue by defining the sets

$$A_i^- = [t : N(t) \leq a - s_i \text{ and } s_{i-1} < S(t) \leq s_i] \quad (92)$$

$$A_i^+ = [t : N(t) \leq a - s_{i-1} \text{ and } s_{i-1} \leq S(t) < s_i] \quad (93)$$

for $i = -m^2 + 1, \dots, m^2$. Observe that $A_i^- \subset Y_a^{-1}$ because $S(t) + N(t) \leq a$ whenever $N(t) \leq a - s_i$ and $s_{i-1} < S(t) \leq s_i$. Hence,

$$\bigcup_{i=1}^n A_i^- \subset Y_a^{-1}. \quad (94)$$

Furthermore,

$$Y_a^{-1} \subset \bigcup_{i=1}^n A_i^+ \quad (95)$$

because, for any t where $N(t) + S(t) \leq a$, there exists an i in the range $-m^2 \leq i \leq m^2$, such that $s_{i-1} \leq S(t) < s_i$ implying $N(t) \leq a - s_i$. Because $\{A_i^-\}_{i=-m^2+1}^{m^2}$ and $\{A_i^+\}_{i=-m^2+1}^{m^2}$ are disjoint, Equations 94 and 95 imply that

$$\frac{1}{T} \mathcal{E} \left\{ \mathcal{L} \left[Y_a^{-1} \cap [0, T) \right] \right\} \geq \sum_{i=-m^2+1}^{m^2} \frac{1}{T} \mathcal{E} \left\{ \mathcal{L} \left[A_i^- \cap [0, T) \right] \right\} \quad (96)$$

$$\frac{1}{T} \mathcal{E} \left\{ \mathcal{L} \left[Y_a^{-1} \cap [0, T) \right] \right\} \leq \sum_{i=-m^2+1}^{m^2} \frac{1}{T} \mathcal{E} \left\{ \mathcal{L} \left[A_i^+ \cap [0, T) \right] \right\}. \quad (97)$$

Now notice that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathcal{E} \left\{ \mathcal{L} \left[Y_a^{-1} \cap [0, T) \right] \right\} \triangleq F_Y(a). \quad (98)$$

Because $S(t)$ and $N(t)$ are amplitude independent, F_S is continuous, and because $F_S(+\infty) \triangleq \lim_{s \rightarrow \infty} F_S(s)$ and $F_S(-\infty) \triangleq \lim_{s \rightarrow -\infty} F_S(s)$, it follows that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathcal{E} \left\{ \mathcal{L} \left[A_i^- \cap [0, T) \right] \right\} = F_N(a - s_i) [F_S(s_i) - F_S(s_{i-1})]. \quad (99)$$

Similarly,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathcal{E} \left\{ \mathcal{L} \left[A_i^+ \cap [0, T) \right] \right\} = F_N(a - s_{i-1}) [F_S(s_i) - F_S(s_{i-1})]. \quad (100)$$

By defining two particular step functions, these results with Equations 97 and 96 form a relationship between the ADF of $S(t) + N(t)$ and the integrals of the two step functions. The step functions are

$$F_N^-(a - s) \triangleq F_N(a - s_i) \quad (101)$$

$$F_N^+(a - s) \triangleq F_N(a - s_{i-1}) \quad (102)$$

whenever $s_{i-1} < s \leq s_i$. Upon passing T to ∞ , Equations 97 and 96 become, with the aid of the above definitions and Equations 98, 99, and 100,

$$\int_{-\infty}^{\infty} F_N^-(a-s) dF_S(s) \leq F_Y(a) \leq \int_{-\infty}^{\infty} F_N^+(a-s) dF_S(s). \quad (103)$$

Proceed by enlarging m and find that F_N^- and F_N^+ converge weakly to F_N that coupled with Equation 103, implies that

$$F_Y(a) = \int_{-\infty}^{\infty} F_N(a-s) dF_S(s) \quad (104)$$

$$= \int_{-\infty}^{\infty} F_N(a-s) dF_S^*(s) \quad (105)$$

where the last equation, following from the continuity of F_S , proves the first convolution. Note that the interchange between limit and integration is justified in the last operation because the integrals have essentially all mass within a bounded domain, upon which the integrand is bounded between zero and one. Note also that, if F_N is not continuous, F_N^+ or F_N^- may converge to some function that differs from F_N at a countable number of points, but luckily the hypothesis that F_S is continuous makes Equation 103 invariant to the limit value of F_N^- and F_N^+ at these problem points.

The complementary convolution is obtained by integrating the product measure $dF_S^*(s) \times dF_N^*(n)$ over the half plane $H \triangleq [s, n : s+n \leq a]$. Proceeding with the help of Fubini's theorem,

$$\int_H dF_S^*(s) \times dF_N^*(n) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{a-s} dF_N^*(n) \right] dF_S^*(s) \quad (106)$$

$$= \int_{-\infty}^{\infty} F_N^*(a-s) dF_S^*(s) \quad (107)$$

or, alternatively,

$$\int_H dF_S^*(s) \times dF_N^*(n) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{a-n} dF_S^*(s) \right] dF_N^*(n) \quad (108)$$

$$= \int_{-\infty}^{\infty} F_S^*(a-n) dF_N^*(n) \quad (109)$$

implying

$$\int_{-\infty}^{\infty} F_N^*(a-s) dF_S^*(s) = \int_{-\infty}^{\infty} F_S^*(a-n) dF_N^*(n). \quad (110)$$

The continuity of F_S and Equation 105 yields

$$F_Y(a) = \int_{-\infty}^{\infty} F_N^*(a-s) dF_S^*(s) \quad (111)$$

that, with Equation 110,

$$F_Y(a) = \int_{-\infty}^{\infty} F_S^*(a-n) dF_N^*(n) \quad (112)$$

$$= \int_{-\infty}^{\infty} F_S(a-n) dF_N^*(n) \quad (113)$$

by the continuity of F_S . Equation 113, the remaining convolutional relationship, is now proved.

C PROOF OF THEOREM 3

Begin by selecting an integer m and constructing a partition of the real line ($y_{-m^2} < y_{-m^2+1} < \dots < y_{m^2}$), where

$$y_i = \begin{cases} -\infty & \text{for } i = -m^2 \\ i/m & \text{for } -m^2 < i < m^2 \\ \infty & \text{for } i = m^2. \end{cases} \quad (114)$$

Continue by defining the sets

$$\Xi_i = [t : y_{i-1} < Y(t) \leq y_i] \quad (115)$$

for $i = -m^2 + 1, \dots, m^2$. The sets $\{\Xi_i\}$ partition the real line into disjoint subsets; therefore,

$$\int_{(0,T]} G[Y(t)] dt = \sum_{i=-m^2+1}^{m^2} \int_{\Xi_i \cap (0,T]} G[Y(t)] dt. \quad (116)$$

Furthermore,

$$\int_{(0,T]} G[Y(t)] dt \geq \sum_{i=-m^2+1}^{m^2} G_i^l \mathcal{L}\{\Xi_i \cap (0,T]\} \quad (117)$$

and

$$\int_{(0,T]} G[Y(t)] dt \leq \sum_{i=-m^2+1}^{m^2} G_i^u \mathcal{L}\{\Xi_i \cap (0,T]\} \quad (118)$$

where

$$G_i^l \triangleq \inf_{y \in (y_{i-1}, y_i]} G(y) \quad (119)$$

$$G_i^u \triangleq \sup_{y \in (y_{i-1}, y_i]} G(y). \quad (120)$$

Divide Equation 117 by T and enlarge T ; then,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T G[Y(t)] dt \geq \sum_{i=-m^2+1}^{m^2} G_i^l [F_Y(y_i) - F_Y(y_{i-1})] \quad (121)$$

after conveniently defining

$$F_Y(+\infty) \triangleq \lim_{y \rightarrow \infty} F_Y(y) \quad (122)$$

$$F_Y(-\infty) \triangleq \lim_{y \rightarrow -\infty} F_Y(y). \quad (123)$$

Operate similarly on Equation 118 to produce

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T G[Y(t)] dt \leq \sum_{i=-m^2+1}^{m^2} G_i^u [F_Y(y_i) - F_Y(y_{i-1})]. \quad (124)$$

To express Equations 121 and 124 in convolutional form, define the two step functions

$$G^-(y) \triangleq G_i^l \quad (125)$$

$$G^+(y) \triangleq G_i^u \quad (126)$$

whenever $y_{i-1} < y \leq y_i$. Since F_Y is assumed to be at least right-continuous, it follows from Equations 121 and 124 that

$$\int_{-\infty}^{\infty} G^-(y) dF_Y(y) \leq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T G[Y(t)] dt \quad (127)$$

and

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T G[Y(t)] dt \leq \int_{-\infty}^{\infty} G^+(y) dF_Y(y). \quad (128)$$

Finish the proof by enlarging m and find that G^- and G^+ converge weakly to G that, coupled with Equations 127 and 128, implies

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T G[Y(t)] dt = \int_{-\infty}^{\infty} G(y) dF_Y(y) \quad (129)$$

$$= \overline{G(Y)}. \quad (130)$$

Note that the interchange between limit and integration is justified in the last operation because the integrands are bounded.

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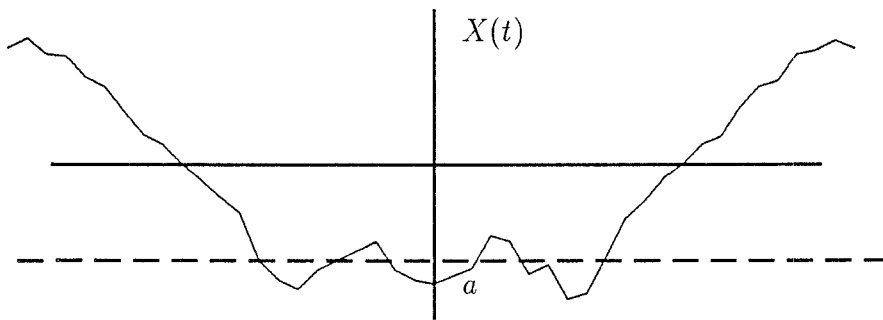
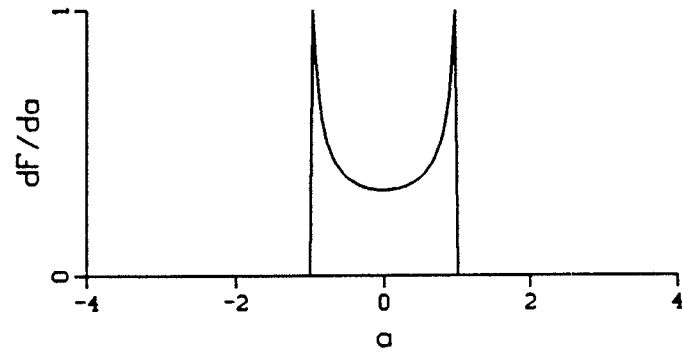
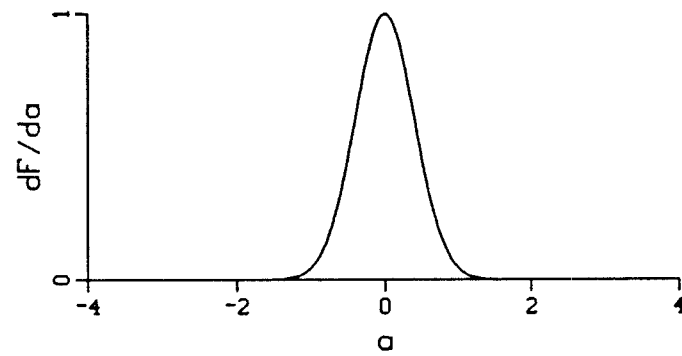


Figure 1: Definition of ADF

Signal



Noise



Signal plus Noise

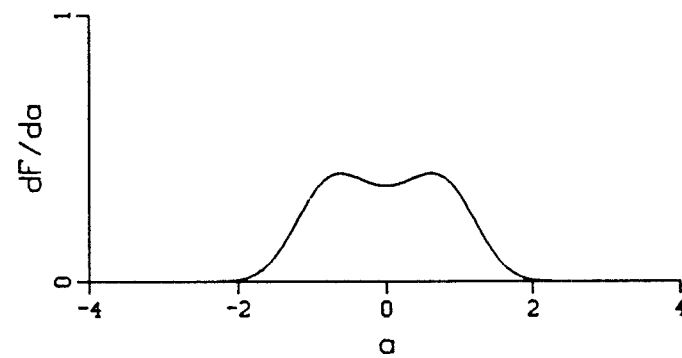


Figure 2: The adf of Modulated Sine Wave and Noise

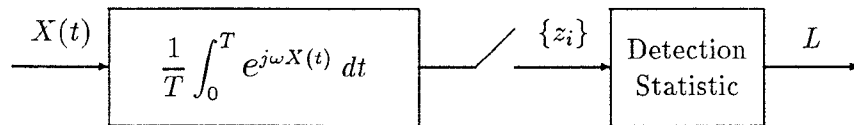


Figure 3: AMF-Based Detector

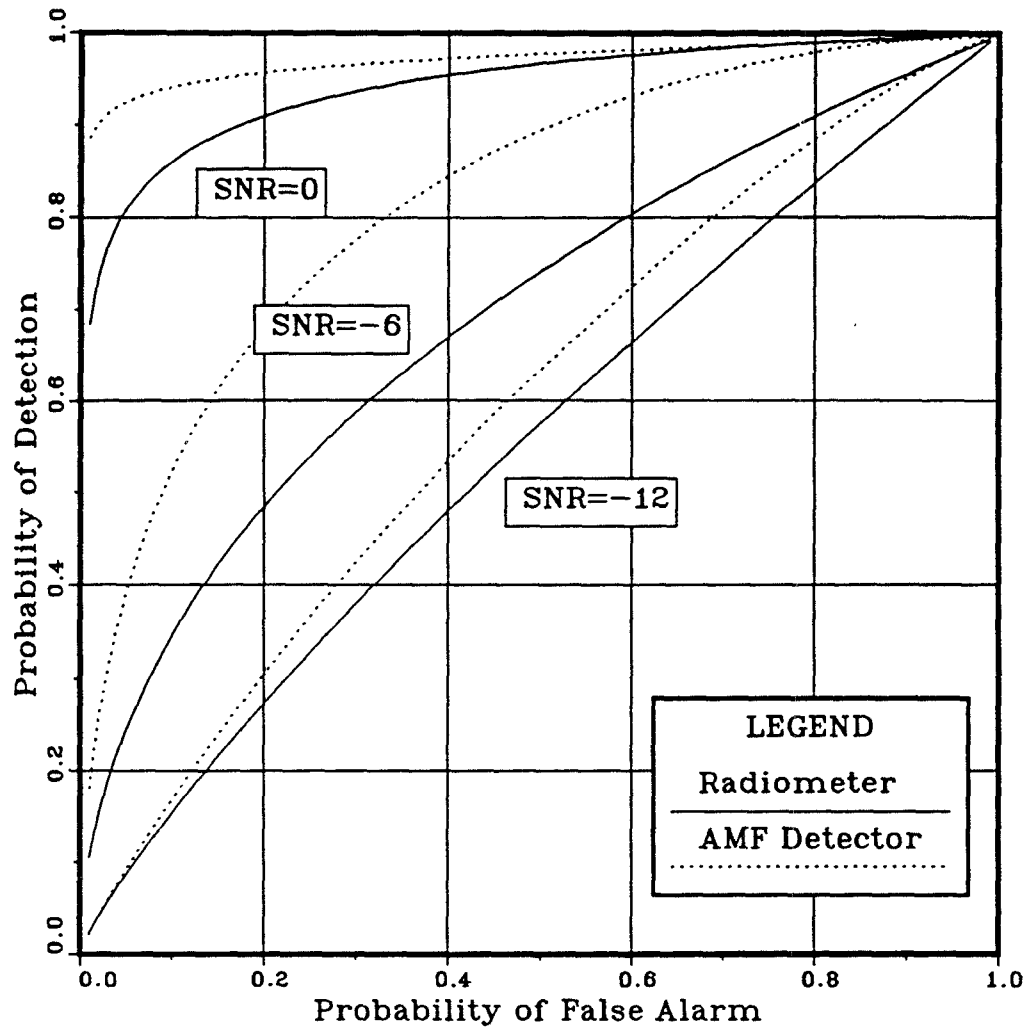


Figure 4: Comparisons of ROCs between AMS Detector and Radiometer

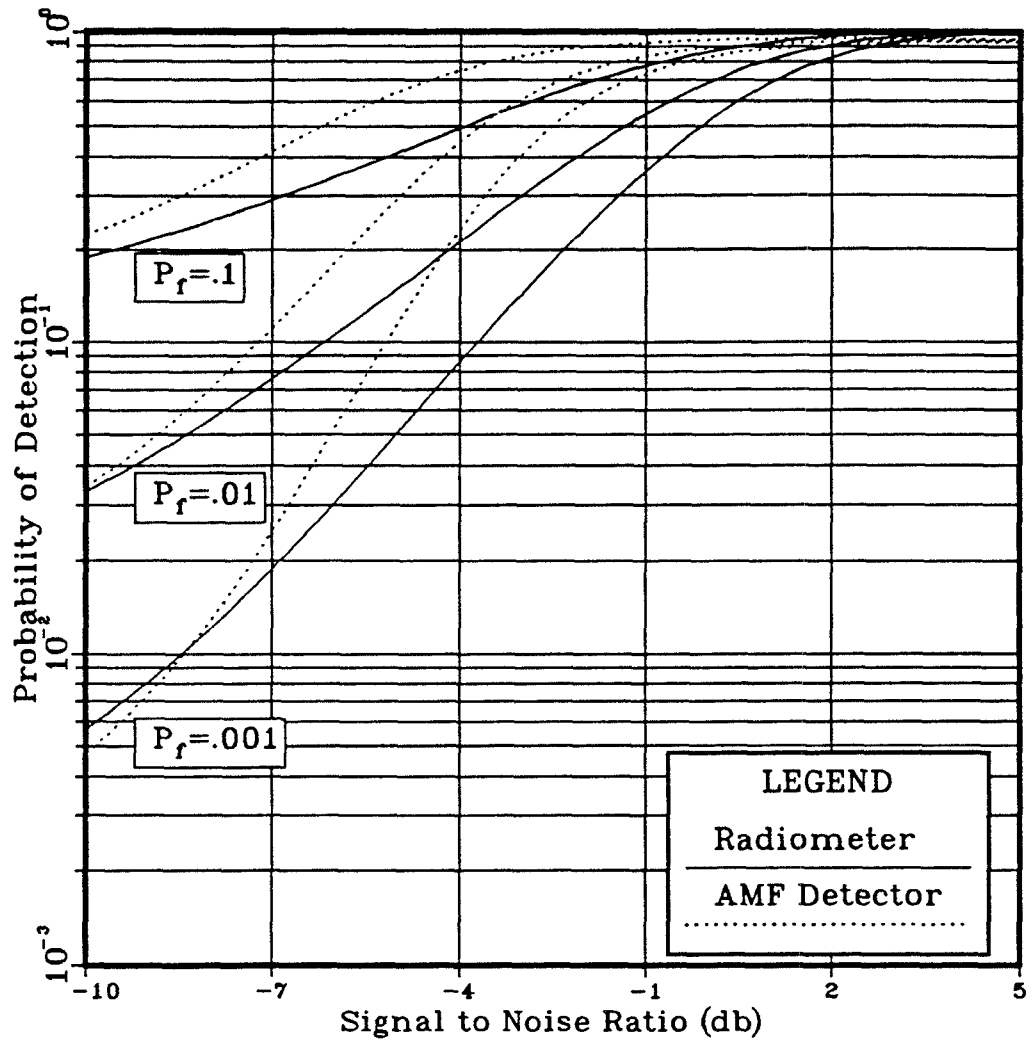


Figure 5: Comparisons of Detection Probabilities between AMS Detector and Radiometer

Table 1: Parameters used for Numerical Comparisons

f_{\max}	f_{\min}	N	W	B	σ_0^2	γ	f_0
1.43 hz	0 hz	5	1 hz	10	W	0 db	.5 hz