Technical Research Report

Systems Research Center

Supported by the
National Science Foundation
Engineering Research Center
Program (NSFD CD 8803012),
Industry and the University

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by M.K.H. Fan and A.L. Tits

TR 91-97

A Measure of Worst-Case H_∞ Performance and of Largest Acceptable Uncertainty*

Michael K.H. Fan
School of Electrical Engineering
Georgia Institute of Technology, Atlanta, GA 30332

André L. Tits
Department of Electrical Engineering and Systems Research Center
University of Maryland, College Park, MD 20742

Abstract

The structured singular value (SSV or μ) is known to be an effective tool for assessing robust performance of linear time-invariant models subject to structured uncertainty. Yet all a single μ analysis provides is a bound β on the uncertainty under which stability as well as H_∞ performance level of k/β are guaranteed, where k is preselectable. In this paper, we introduce a related quantity, denoted by ν which provides answers for the following questions: (i) given β, determine the smallest α with the property that, for any uncertainty bounded by β, an H_∞ performance level of α is guaranteed; (ii) conversely, given α, determine the largest β with the property that, again, for any uncertainty bounded by β, an H_∞ performance level of α is guaranteed. Properties of this quantity are established and approaches to its computation are investigated. Both unstructured uncertainty and structured uncertainty are considered.

Keywords: Robust control, robust performance, robust stability, structured singular value, structured uncertainty.

*Some of the results established in this paper were given without proof in [6]. This work was supported in part by the National Science Foundation's Engineering Research Centers Program: NSFD CDR-88-03012, by the NSF under Grants DMC-85-51515 and DMC-88-15996, and by Rockwell International.

0. Notation

I_k	Identity matrix of size $k \times k$
$O_{k_1 \times k_2}$	Zero matrix of size $k_1 \times k_2$
O_k	$O_{k \times k}$
$\|P\|_\infty$	H_∞ norm of stable transfer matrix P
$\overline{\sigma}(A)$	Largest singular value of matrix A
$\overline{\lambda}(A)$	Largest eigenvalue of Hermitian matrix A
$\overline{\lambda}(A, B)$	$\sup\{\lambda \in \mathbb{R} : \det(A - \lambda B) = 0\}$, for Hermitian A, B ($\overline{\lambda}(A, B)$ may be $\pm\infty$)
$\rho(A)$	Spectral radius of square matrix A
$\rho(A, B)$	$\sup\{\lvert\lambda\rvert : \det(A - \lambda B) = 0, \lambda \in \mathbb{C}\}$, ($\rho(A, B)$ may be $\pm\infty$)
x^H	Hermitian transpose of vector x
$\|x\|$	Euclidean norm of vector x
A^H	Hermitian transpose of matrix A
$A \geq 0$	Matrix A is positive semi-definite
∂B	Unit sphere in $\mathbb{C}^n$
$\mathbb{R}^+$	Set of nonnegative real numbers

1. Introduction

Consider a linear time-invariant model affected by uncertainty. It is by now well known (see, e.g., [4]) that in many cases of interest such a system can be represented in "feedback" form as in Fig. 1 (Δ(s) and $P(s)$ are both square systems). Here Δ represents the uncertainty and is typically block diagonal, each block corresponding to uncertainty affecting a specific subsystem. Both parametric and dynamic uncertainties can be accounted for. While the former give rise to real scalar blocks in Δ, the latter are often represented by H_∞-norm bounded linear time-invariant transfer functions. Under the assumption that the nominal model is internally stable, the overall system will be internally stable for all Δ of size (H_∞-norm) no more than 1, Δ having the specified structure, if and only if

$$\sup_\omega \mu(P_{11}(j\omega)) < 1$$

where μ is Doyle's structured singular value (SSV) for the given structure [3] (Small μ Theorem [4]; see also [11]). The SSV framework also permits to assess robust performance [4]. Namely, referring again to Fig. 1, suppose one desires to know whether the worst-case H_∞ performance is satisfactory, i.e., whether, for any structured Δ of size no more than 1, the H_∞-norm of the transfer function $F_u(P, \Delta)$ from exogenous (e.g., disturbance) signal u to error signal e is small, say, no larger than 1. It turns out that this will be the case if and only if the system of Fig. 2 is internally stable for all Δ of size no more than 1, Δ having the specified structure, and for all δ of size no more than 1;

equivalently, if and only if

$$\sup_{\omega} \tilde{\mu}(P(j\omega)) < 1$$

where $\tilde{\mu}$ is the structured singular value corresponding to the "augmented structure". Straightforward scaling then yields the following, given any $\beta > 0$: The system of Fig. 1 is stable for all structured Δ of size β or less, and the worst-case performance under such uncertainty is no more than $1/\beta$, if and only if

$$\sup_{\omega} \tilde{\mu}(P(j\omega)) < 1/\beta \, .$$

Thus an SSV analysis can answer the question

(Q1) what is the largest β, if any, such that, whenever the uncertainty has size β or less, (i) the system is stable and (ii) the worst-case H_∞ performance is better than $1/\beta$?

While this does provide some kind of "stability-and-performance margin", it may well happen that a good estimate of the actual uncertainty bound is available. In this case, assuming that the uncertainty bound has been normalized, a question of interest is whether the system is stable whenever the uncertainty has size less than 1, and if so, what is the worst case performance for this same uncertainty size. In other words:

(Q2) what is the smallest α, if any, such that, whenever the uncertainty has size 1 or less, (i) the system is stable and (ii) the worst-case H_∞ performance is better than α ?

Yet another question of possible interest is whether with no uncertainty, a given performance level, say 1, is achieved, and if so, how much uncertainty can be tolerated if this performance level is to be preserved, i.e.,[1]

(Q3) what is the largest β, if any, for which, whenever the uncertainty has size β or less, (i) the system is stable, and (ii) the worst-case H_∞ performance is better than 1.

The following approach to answering (Q2), via an (infinite) sequence of SSV analyses, comes to mind. Note that for any $\alpha > 0$, stability of the system in Fig. 1 is equivalent to stability of the system in Fig. 3, with $P^\alpha(s)$ given by

$$P^\alpha(s) = \begin{bmatrix} \alpha P_{11}(s) & \alpha P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix},$$

and that the transfer function from u to e is identical for both systems. It follows that the system of Fig. 1 is stable whenever $\|\Delta\|_\infty \leq 1$ (i.e., whenever $\|(1/\alpha)\Delta\|_\infty \leq 1/\alpha$), with worst-case performance better than α, if and only if

$$\sup_{\omega} \tilde{\mu}(P^\alpha(j\omega)) < \alpha \, . \tag{1.1}$$

[1]This question was suggested to us by Carl Nett.

Thus the answer to (Q2) is given by $\hat{\alpha}$, the infimum of those α satisfying (1.1). This suggests the fixed point iteration

$$\alpha_{i+1} = \sup_{\omega} \tilde{\mu}(P^{\alpha_i}(j\omega)), \quad \alpha_0 > 0.$$

A similar idea applies for (Q3) with $P_\alpha(s)$ replaced by

$$P_\alpha(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ \alpha P_{21}(s) & \alpha P_{22}(s) \end{bmatrix};$$

simply note that the system of Fig. 1 is equivalent to that of Fig. 4 and that its robust performance can be characterized via Fig. 5. It follows as a byproduct of the results obtained in this paper that these iterations do converge to the sought quantities.

The purpose of this paper is to introduce a quantity closely related to the structured singular value, but providing an answer to (Q2) (resp. (Q3)) in a *single analysis.* For simplicity of exposition, we first consider the case of two-block structures (performance block and single uncertainty block). Questions (Q2) and (Q3) are considered jointly. To facilitate this, the representation given on Fig. 6 (obtained by renumbering the blocks of $P(s)$) will be used in connection with (Q3). In Section 2 below we define the new function ν of a matrix and state theorems (related to the Small μ Theorem) that show its relation to (Q2) and (Q3). In Section 3, we discuss elementary properties of ν and in Section 4 we elucidate its correspondence with μ. In Section 5, we indicate how, for a complex matrix M, $\nu(M)$ can be efficiently computed. Finally, in Section 6, we state without proof extensions of the results of Sections 2 to 5 to the case of structured uncertainty. Some simple proofs are left out or given in the appendix. Throughout the paper, scalar functions, including value functions of optimization problems, take values in the extended real line $\mathbb{R} \cup \{\infty\}$.

2. A Measure of Robust Performance

Thus, for a complex $n \times n$ matrix M, consider the multiindex $\mathcal{K} = (k_1, k_2)$, k_1 and k_2 positive integers, with $k_1 + k_2 = n$. $\mathcal{K}$ are to be referred to as block structure; for the systems of Figs. 1 and 6, k_1 and k_2 are to be taken as the dimensions of blocks number 1 and 2 respectively. Below, we make use of the notation

$$\begin{aligned} \mathcal{D} &= \{\text{block diag } (dI_{k_1}, I_{k_2}) : d > 0\}, \\ \mathcal{U} &= \{\text{block diag } (U_1, U_2), U_i : k_i \times k_i, \text{ unitary}\}, \\ P_1 &= \text{block diag } (I_{k_1}, O_{k_2}),\ P_2 = \text{block diag } (O_{k_1}, I_{k_2}), \\ Q_1 &= \text{block row } (I_{k_1}, O_{k_1 \times k_2}),\ Q_2 = \text{block row } (O_{k_2 \times k_1}, I_{k_2}), \end{aligned}$$

and M is partitioned according to

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

with $M_{ij} : k_i \times k_j$.

Recall [3] that, for the given block structure, the structured singular value $\mu(M)$ of a complex matrix M is equal to zero if there is no $\Delta \in \mathcal{X}$ such that $\det(I + \Delta M) = 0$ and

$$\mu(M) = \left(\min_{\Delta \in \mathcal{X}} \{\overline{\sigma}(\Delta) : \det(I + \Delta M) = 0\}\right)^{-1}$$

otherwise, where $\mathcal{X}$ is a subspace of $\mathbb{C}^{n \times n}$ given by

$$\mathcal{X} = \left\{\text{block diag}(\Delta_1, \Delta_2) : \Delta_i \in \mathbb{C}^{k_i \times k_i}, i = 1, 2\right\} .$$

Consider now the related quantity $\nu(M)$, equal to zero if there is no $\Delta \in \mathcal{Y}$ such that $\det(I + \Delta M) = 0$ and given by

$$\nu(M) = \left(\min_{\Delta \in \mathcal{Y}} \{\overline{\sigma}(\Delta_2) : \det(I + \Delta M) = 0\}\right)^{-1}$$

(possibly ∞) otherwise, where $\mathcal{Y}$ is given by

$$\mathcal{Y} = \{\text{block diag}(\Delta_1, \Delta_2) : \Delta_i \in \mathbb{C}^{k_i \times k_i}, i = 1, 2, \overline{\sigma}(\Delta_1) \leq 1\}.$$

Note that, in the formula for $\nu(M)$, the size of Δ_1 is not minimized but merely kept below 1, reflecting the fact that the required uncertainty (resp. performance) bound on block 1 for (Q2) (resp. (Q3)) is fixed (equal to 1).

The following results, to be compared to the Small μ Theorem [4], follow. Here k_1 and k_2 are the dimensions of $P_{11}(s)$ and $P_{12}(s)$ respectively, $F_u(P, \Delta)$ is the transfer function from u to e on Fig. 1 and $F_\ell(P, \Delta)$ the transfer function from u to e on Fig. 6.

Theorem 2.1. Suppose $P \in H_\infty$ is internally stable and let $\alpha > 0$. Then the system depicted in Fig. 1 is well formed and internally stable for all $\Delta \in H_\infty, \|\Delta\|_\infty \leq 1$, and $\|F_u(P, \Delta)\|_\infty < \alpha$ for all such Δ if and only if $\sup_\omega \nu(P(j\omega)) < \alpha$. □

Theorem 2.2. Suppose $P \in H_\infty$ is internally stable and let $\beta > 0$. Then the system depicted in Fig. 6 is well formed and internally stable for all $\Delta \in H_\infty$, $\|\Delta\|_\infty \leq \beta$, and $\|F_\ell(P, \Delta)\|_\infty < 1$ for all such Δ if and only if $\sup_\omega \nu(P(j\omega)) < 1/\beta$. □

Thus (Q2) and (Q3) can each be answered by means of a single "ν" analysis.

3. Properties of ν

First, the discussion of Section 2 suggests that ν may be related to μ in some recursive way. This is indeed the case as stated in Proposition 3.1 below.

Proposition 3.1.

(a) Suppose $\nu(M) < \infty$. Then

$$\mu\left(\begin{bmatrix} \nu(M)I_{k_1} & 0 \\ 0 & I_{k_2} \end{bmatrix} M\right) = \nu(M)\ . \tag{3.1}$$

(b) Suppose $\mu(M) > \overline{\sigma}(M_{11})$. Then

$$\nu\left(\begin{bmatrix} \mu^{-1}(M)I_{k_1} & 0 \\ 0 & I_{k_2} \end{bmatrix} M\right) = \mu(M)\ . \tag{3.2}$$

□

The properties of ν listed in Proposition 3.2 below are to be compared to similar properties of μ given in [3].

Proposition 3.2.

(a) $\nu(M) < \infty$ if and only if $\overline{\sigma}(M_{11}) < 1$.

(b) $\nu(\alpha M) \geq |\alpha|\nu(M)$ for any $|\alpha| \geq 1$. (3.3)

(c) $\nu(\alpha M) \leq |\alpha|\nu(M)$ for any $|\alpha| \leq 1$. (3.4)

(d) $\nu(DMD^{-1}) = \nu(M)$ for any $D \in \mathcal{D}$. (3.5)

(e) $\nu(UM) = \nu(MU) = \nu(M)$ for any $U \in \mathcal{U}$. (3.6)

□

Lower and upper bounds for $\nu(M)$ can be readily obtained in terms of the largest singular values of the subblocks of M. This is shown in Proposition 3.3 below. The following two lemmas are used in proving it.

Lemma 3.1. (see, e.g., [2]) $\mu(M) \geq \max\{\overline{\sigma}(M_{11}), \overline{\sigma}(M_{22}), \sqrt{\overline{\sigma}(M_{12})\overline{\sigma}(M_{21})}\}$. □

Lemma 3.2. (see, e.g., [8])

$$\begin{aligned} \mu(M) &\leq \rho\left(\begin{bmatrix} \overline{\sigma}(M_{11}) & \overline{\sigma}(M_{12}) \\ \overline{\sigma}(M_{21}) & \overline{\sigma}(M_{22}) \end{bmatrix}\right) \\ &= \frac{\overline{\sigma}(M_{11}) + \overline{\sigma}(M_{22}) + \sqrt{(\overline{\sigma}(M_{11}) - \overline{\sigma}(M_{22}))^2 + 4\overline{\sigma}(M_{12})\overline{\sigma}(M_{21})}}{2} \end{aligned}$$

□

Proposition 3.3. Suppose that $\overline{\sigma}(M_{11}) < 1$. Then the following inequalities hold.

$$\max\{\overline{\sigma}(M_{22}), \overline{\sigma}(M_{12})\overline{\sigma}(M_{21})\} \leq \nu(M) \leq \overline{\sigma}(M_{22}) + \frac{\overline{\sigma}(M_{12})\overline{\sigma}(M_{21})}{1 - \overline{\sigma}(M_{11})}\ . \tag{3.7}$$

Proof. Let $\nu = \nu(M)$ and $m_{ij} = \overline{\sigma}(M_{ij})$, $i, j = 1, 2$. In view of (3.1) and Lemma 3.1, we have

$$\nu \geq \max\{\nu m_{11}, m_{22}, \sqrt{\nu m_{12} m_{21}}\}$$

which implies that $\nu \geq m_{22}$ and $\nu \geq m_{12}m_{21}$. Thus the first inequality in (3.7) holds. In view of (3.1) again and Lemma 3.2, we have

$$\nu \leq \frac{\nu m_{11} + m_{22} + \sqrt{(\nu m_{11} - m_{22})^2 + 4\nu m_{12} m_{21}}}{2} . \tag{3.8}$$

Then using the facts that $m_{11} < 1$ and $\nu \geq m_{22}$, it is straightforward to show that (3.8) reduces to the second inequality in (3.7). □

We will employ the following lemma in proving Proposition 3.4.

Lemma 3.3.

$$\begin{aligned} \mu(M) &= \max_{\substack{\lambda \in \partial B \\ \theta \geq 0}} \{\theta : \|P_i M x\| \geq \theta \|P_i x\|,\ i = 1, 2\} \\ &= \max_{\substack{x \in \partial B \\ \theta \geq 0}} \{\theta :\ \|P_i M x\| = \theta \|P_i x\|,\ i = 1, 2\} . \end{aligned}$$

Proof. The former was obtained in [7]. Next

$$\begin{aligned} \mu(M) &= \max_{x \in \partial B} \{\|Mx\| : \|P_i M x\| = \|Mx\| \cdot \|P_i x\|, i = 1, 2\} \\ &\leq \max_{\substack{x \in \partial B \\ \theta \geq 0}} \{\theta : \|P_i M x\| = \theta \|P_i x\|, i = 1, 2\} \\ &\leq \max_{\substack{x \in \partial B \\ \theta \geq 0}} \{\theta : \|P_i M x\| \geq \theta \|P_i x\|, i = 1, 2\} = \mu(M) \end{aligned}$$

where the first equality was obtained in [5]. This proves the second claim. □

Proposition 3.4. Suppose that $\nu(M) < \infty$. Then the following statements are true.

(a) Suppose that M has rank one. Write $M = uv^H$ for some $u, v \in \mathbb{C}^n$. Then

$$\nu(M) = \frac{\|P_2 u\| \|P_2 v\|}{1 - \|P_1 u\| \|P_1 v\|} . \tag{3.9}$$

(b)

$$\nu(M) = \max_{\substack{x \in \partial B \\ \theta \geq 0}} \{\theta : \|P_1 M x\| = \|P_1 x\|\ , \|P_2 M x\| = \theta \|P_2 x\|\} \tag{3.10}$$

$$= \max_{\substack{x \in \partial B \\ \theta \geq 0}} \{\theta : \|P_1 M x\| \geq \|P_1 x\|, \|P_2 M x\| \geq \theta \|P_2 x\|\} . \tag{3.11}$$

(c) $\nu(M) = \max_{U \in \mathcal{U}} \rho(UM - P_1, P_2)$.

(d) $\nu(\cdot)$ is continuous at M.

Proof.

(a) Let $\Delta = \text{diag}\{\Delta_1, \Delta_2\} \in \mathcal{Y}$, $u_i = Q_i u$ and $v_i = Q_i v$, $i = 1, 2$. Then

$$\det(I + \Delta M) = \det(I + \Delta u v^H) = 1 + v^H \Delta u = 1 + v_1^H \Delta_1 u_1 + v_2^H \Delta_2 u_2 \ . \tag{3.12}$$

It is straightforward to check that the smallest Δ_2, which makes (3.12) equal zero for some Δ_1 with $\overline{\sigma}(\Delta_1) \leq 1$, satisfies

$$1 - \|u_1\|\|v_1\| - \overline{\sigma}(\Delta_2)\|u_2\|\|v_2\| = 0 \ .$$

In view of the definition of $\nu(\cdot)$, (3.9) holds.

(b) We prove the first claim. The second claim can be proved similarly. We first show that the right hand side of (3.10) is well defined, i.e., that the feasible set is nonempty. In view of Lemma 3.3 and (3.1), we have

$$\nu(M) = \max_{\substack{x \in \partial B \\ \theta \geq 0}} \{\theta : \ \nu(M)\|P_1 M x\| = \theta\|P_1 x\|, \ \|P_2 M x\| = \theta\|P_2 x\|\} \ . \tag{3.13}$$

Thus, there exists x such that $(x, \nu(M))$ is feasible for (3.13). If $\nu(M) > 0$, it follows that $(x, \nu(M))$ is feasible for (3.10). If $\nu(M)$=0, a feasible point for (3.10) can be constructed by considering a sequence $\{M_k\} \rightarrow M$, with $\nu(M_k) > 0$, letting x_k be a maximizer for the problem of the form (3.10) for M_k, and extracting converging subsequences from $\{\nu(M_k)\}$ and $\{x_k\}$ (in view of Proposition 3.3, $\nu(M_k)$ is eventually bounded). Now let ν_1 denote the right hand side in (3.10). We first show $\nu(M) \leq \nu_1$. If $\nu(M) = 0$, the result holds trivially. Otherwise, it follows from feasibility of $(x, \nu(M))$ for some x. To show that $\nu(M) \geq \nu_1$, we let (x, θ) be feasible for (3.10) and prove that $\nu(M) \geq \theta$. If $\theta = 0$ this holds trivially. Thus assume $\theta > 0$. Then there exists $\Delta = \text{diag}(\Delta_1, \Delta_2) \in \mathcal{Y}$ such that $\overline{\sigma}(\Delta_1) = 1$, $\overline{\sigma}(\Delta_2) = \theta^{-1}$, and $\det(I + \Delta M) = 0$. The claim then follows from the definition of $\nu(M)$.

(c) $(\theta, x) \in \mathbb{R}^+ \times \mathbb{C}^n$ is feasible in (3.10) if and only if, for any $\phi \in \mathbb{R}$, there exist unitary matrices $U_1 \in \mathbb{C}^{k_1 \times k_1}$ and $U_2 \in \mathbb{C}^{k_2 \times k_2}$ such that

$$U_1 Q_1 M x = Q_1 x$$

$$U_2 Q_2 M x = \theta e^{j\phi} Q_2 x$$

which happens if and only if, for any $\phi \in \mathbb{R}$, there exists $U \in \mathcal{U}$ such that

$$(UM - P_1)x = \theta e^{j\phi} P_2 x.$$

Therefore, (c) follows directly from (3.10) and the definition of $\rho(\cdot, \cdot)$.

(d) Follows directly from (c). □

4. Computing ν via μ

In Section 1 above it was suggested that (Q2) and (Q3) could be tackled via an infinite sequence of μ analyses. How ν can indeed be obtained from μ is made precise here. Define the function $f : \mathbb{R} \to \mathbb{R}$

$$f(\alpha) = \mu\left(\begin{bmatrix} \alpha I_{k_1} & 0 \\ 0 & I_{k_2} \end{bmatrix} M\right) .$$

It can be easily checked (see, e.g., [9]) that the function $f(\alpha)$ is continuous and nondecreasing.

Proposition 4.1.

(a) $\beta > \nu(M)$ implies that $f(\beta) < \beta$.

(b) $0 < \beta < \nu(M) < \infty$ implies that $f(\beta) > \beta$.

□

Theorem 4.1. Let $\{\alpha_k\}$ be the sequence generated by the fixed point iteration

$$\alpha_{k+1} = f(\alpha_k), \quad k = 1, 2, \ldots$$

with α_0 any positive number. Then $\lim_{k\to\infty} \alpha_k = \nu(M)$.

Proof. Follows directly from Proposition 4.1 and the continuity of f. □

5. Direct Computation of ν

The key question is now whether $\nu(M)$ can be easily computed. For the case under consideration, efficient algorithms are known for the computation of the structured singular value $\mu(M)$, based on the formulas ([4,5])

$$\mu(M) = \inf_{D\in\mathcal{D}} \overline{\sigma}(DMD^{-1}) \tag{5.1}$$

$$\mu(M) = \max_{\substack{x\in\partial B \\ \theta\geq 0}} \{\theta : \quad \|P_i M x\| = \theta \|P_i x\|, \; i = 1, 2\}.$$

In particular, the optimization problem in (5.1) has no local minima that are not global and robust algorithms are available for its solution. Practical value of $\nu(M)$ is obviously contingent on the availability of a comparably efficient computational algorithm. We show below that $\nu(M)$ is the optimal value of certain quasi-convex optimization problem.[2] For A, B Hermitian, with $\overline{\lambda}(A) \geq 0$ and $B \geq 0$, define

$$\eta(A, B) = \sup\{\gamma \in \mathbb{R} : \overline{\lambda}(A - \gamma B) \geq 0\}.$$

[2] Theorem 5.2 as well as the present formulation of Theorem 5.1 were prompted by S.P. Boyd's observation that the result in [7] can be expressed in terms of a quasi-convex optimization problem involving a certain generalized eigenvalue problem.

The following is easily proven.

Proposition 5.1. Given A, B Hermitian, with $\overline{\lambda}(A) \geq 0$ and $B \geq 0$, the following statements hold.

(a) $\eta(A, B) \geq 0$.

(b) For any $t \in \mathbb{R}^+$, $\eta(A, B) < t$ if and only if $\overline{\lambda}(A - tB) < 0$.

(c) If $\overline{\lambda}(A, B)$ is finite and $\overline{\lambda}(A - \overline{\lambda}(A, B)B) = 0$, then $\eta(A, B) = \overline{\lambda}(A, B)$; otherwise, $\eta(A, B) = \infty$.

□

Thus $\eta(A, B)$ can be computed by solving a generalized eigenvalue problem. Now, let $G(d) = M^H(dP_1 + P_2)M - dP_1$. For any $d \geq 0$, $x^H P_2 G(d) P_2 x \geq 0$ for any x, and thus $\overline{\lambda}(G(d)) \geq 0$. Thus, in view of Proposition 5.1 (a), $\eta(G(d), P_2) \geq 0$ for any $d \geq 0$.

Theorem 5.1. $\nu(M) = \inf\limits_{d > 0} \sqrt{\eta(G(d), P_2)}$.

Proof. From (3.10), we have, for any $d \geq 0$,

$$\begin{aligned}
\nu(M) &= \sup_{\substack{x \in \partial B \\ \theta \geq 0}} \left\{\theta : x^H(M^H P_1 M - P_1)x \geq 0,\ x^H(M^H P_2 M - \theta^2 P_2)x \geq 0\right\} \qquad (5.2)\\
&\leq \sup_{\substack{x \in \partial B \\ \theta \geq 0}} \left\{\theta : x^H(G(d) - \theta^2 P_2)x \geq 0\right\}\\
&= \sup_{\theta \geq 0} \left\{\theta : \overline{\lambda}(G(d) - \theta^2 P_2) \geq 0\right\}\\
&= \sqrt{\eta(G(d), P_2)}
\end{aligned}$$

Therefore, $\nu(M) \leq \inf\limits_{d \geq 0} \sqrt{\eta(G(d), P_2)}$. If $\nu(M) = \infty$, the proof is complete. Suppose now that $\nu(M) < \infty$. For $\theta \geq 0$, consider the numerical range

$$W(\theta) = \left\{ \begin{bmatrix} x^H(M^H P_1 M - P_1)x \\ x^H(M^H P_2 M - \theta^2 P_2)x \end{bmatrix} : x \in \partial B \right\}.$$

Let $t > \nu(M)$. Then, in view of (5.2), the intersection of $W(t)$ and the (closed) first quadrant in $\mathbb{R}^2$ must be empty. Since the numerical range is convex [12], there exists $d_1 > 0$ such that, for any $z \in W(t)$, $[d_1\ 1]z < 0$, i.e.,

$$[d_1 \quad 1] \begin{bmatrix} x^H(M^H P_1 M - P_1)x \\ x^H(M^H P_2 M - t^2 P_2)x \end{bmatrix} = x^H(G(d_1) - t^2 P_2)x < 0$$

for all $x \in \partial B$. Therefore, $\overline{\lambda}(G(d_1) - t^2 P_2) < 0$, i.e., in view of Proposition 5.1(b), $\eta(G(d_1), P_2) < t^2$. Thus $\inf\limits_{d \geq 0} \sqrt{\eta(G(d), P_2)} < t$. Hence $\inf\limits_{d \geq 0} \sqrt{\eta(G(d), P_2)} \leq \nu(M)$, and the proof is complete. □

Theorem 5.2. The function $\eta(G(d), P_2)$ is quasi-convex in $d \in \mathbb{R}^+$.

Proof. For $d \geq 0$, let $g(d) = \eta(G(d), P_2)$. We show the claim by contradiction. Suppose $g(\alpha d_1 + (1-\alpha)d_2) > \max\{g(d_1), g(d_2)\}$ for some $d_1, d_2 > 0$ and $\alpha \in (0,1)$. Let t be such that $g(\alpha d_1 + (1-\alpha)d_2) > t > \max\{g(d_1), g(d_2)\}$. Let $g_t(d) := \overline{\lambda}(G(d) - tP_2)$. In view of Proposition 5.1 (b), we have $g_t(d_1) < 0$, $g_t(d_2) < 0$ and $g_t(\alpha d_1 + (1-\alpha)d_2) \geq 0$. This contradicts the fact that g_t is convex (composition of an affine function and a convex function). □

Quasi-convex optimization problems can be solved, e.g., by a cutting-plane method (see, e.g., [1] and the references therein). It was also recently shown that optimization problems such as that of Theorem 5.1 can be solved in polynomial time by an interior path method [10].

6. Structured Uncertainty

The results presented in Sections 2 to 5 can be extended to the case, considered in Section 1, where the uncertainty is structured, i.e., $\Delta(s)$ in Fig. 1 or Fig. 6 is constrained to be block-diagonal with blocks of specified dimensions. It may happen that for some uncertainty blocks, the true bound on uncertainty size is known, while for other, one seeks to determine how large a bound would be acceptable. Similarly, there can be a fixed desired performance level, or one may seek as good a performance as possible. Without loss of generality, assume that, for some ℓ, blocks number 1 thus ℓ have size bounded by 1 (this includes the performance block if the performance level is prescribed) and it is desired to determined the maximum acceptable size for blocks $\ell+1$ and above (this includes the performance block if as good as possible performance is desired). Thus consider the pair of multiindexes $\mathcal{K} = (k_1, \ldots, k_\ell;\ k_{\ell+1}, \ldots, k_m)$ and the corresponding families of matrices

$$\begin{aligned}
\mathcal{D} &= \{\text{block diag}\ (d_1 I_{k_1}, \ldots, d_{m-1} I_{k_{m-1}}, I_{k_m}) : d_i > 0\}, \\
\mathcal{U} &= \{\text{block diag}\ (U_1, \ldots, U_m), U_i : k_i \times k_i,\ \text{unitary}\}, \\
P_i &= \text{block diag}\ (O_{k_1}, \ldots, O_{k_{i-1}}, I_{k_i}, O_{k_{i+1}}, \ldots, O_{k_m}),\ i = 1, \ldots, m;
\end{aligned}$$

let M be partitioned as

$$M = \begin{bmatrix} M_{11} & \ldots & M_{1m} \\ \vdots & \ddots & \vdots \\ M_{m1} & \ldots & M_{mm} \end{bmatrix} = \begin{bmatrix} \mathbf{M_{11}} & \mathbf{M_{12}} \\ \mathbf{M_{21}} & \mathbf{M_{22}} \end{bmatrix},$$

with $M_{ij} : k_i \times k_j$, $\mathbf{M_{ij}} : \mathbf{k_1} \times \mathbf{k_2}$, where $\mathbf{k_1} = \sum_{i=1}^{\ell} k_i$, $\mathbf{k_2} = \sum_{i=\ell+1}^{m} k_i$; and let $\mathbf{P_1} =$ block diag $(I_{\mathbf{k_1}}, O_{\mathbf{k_2}})$, $\mathbf{P_2} =$ block diag $(O_{\mathbf{k_1}}, I_{\mathbf{k_2}})$. In the multiblock case, we define $\nu(M)$ to be zero if there is no $\Delta \in \mathcal{Y}$ such that $\det(I + \Delta M) = 0$ and to be given by

$$\nu(M) = \left(\min_{\Delta \in \mathcal{Y}} \left\{ \overline{\sigma} \left(\begin{bmatrix} \Delta_{\ell+1} & & \\ & \ddots & \\ & & \Delta_m \end{bmatrix} \right) : \det(I + \Delta M) = 0 \right\} \right)^{-1}$$

(possibly ∞) otherwise, where $\mathcal{Y}$ is given by

$$\mathcal{Y} = \left\{\text{block diag } (\Delta_1, \ldots, \Delta_m) : \Delta_i \in \mathbb{C}^{k_i \times k_i},\ i = 1, \ldots,\ m, \bar{\sigma}(\Delta_i) \leq 1,\ i = 1, \ldots,\ \ell\right\}.$$

Theorems 2.1 and 2.2 are easily extended to this case (Fig. 6 is used if the performance level is prescribed, and Fig. 1 is used otherwise). Propositions 3.1 and 3.2 hold with all subscripts replaced by their boldface counterparts and $\bar{\sigma}(M_{11})$ replaced by $\mu(M_{\mathbf{11}})$, where μ is taken with respect to the obvious structure. Propositions 3.3 and 3.4 are replaced by the following. In Proposition 6.1, for any $D \in \mathcal{D}$, we let $M^D = DMD^{-1}$ and, for $\mathbf{i}, \mathbf{j} = 1, 2$, we let $M^D_{\mathbf{ij}} = (M^D)_{\mathbf{ij}}$.

Proposition 6.1. $\nu(M) \geq \mu(M_{\mathbf{22}})$. Suppose that there exists $D \in \mathcal{D}$ such that $\bar{\sigma}(M^D_{\mathbf{11}}) < 1$. Then

$$\nu(M) \leq \bar{\sigma}(M^D_{\mathbf{22}}) + \frac{\bar{\sigma}(M^D_{\mathbf{12}})\bar{\sigma}(M^D_{\mathbf{21}})}{1 - \bar{\sigma}(M^D_{\mathbf{11}})}$$

□

Proposition 6.2. Suppose that $\nu(M) < \infty$. Then the following statements are true.

(a) Suppose that M has rank one. Write $M = uv^H$ for some $u, v \in \mathbb{C}^n$. Then

$$\nu(M) = \frac{\sum_{i=\ell+1}^{m} \|P_i u\| \|P_i v\|}{1 - \sum_{i=1}^{\ell} \|P_i u\| \|P_i v\|}.$$

(b)

$$\begin{aligned}\nu(M) &= \sup_{\substack{x \in \partial B \\ \theta \geq 0}} \{\theta : \|P_i M x\| = \|P_i x\|\ , i = 1, \ldots, \ell, \|P_j M x\| = \theta \|P_j x\|, j = \ell+1, \ldots, m\} \\ &= \sup_{\substack{x \in \partial B \\ \theta \geq 0}} \{\theta : \|P_i M x\| \geq \|P_i x\|\ , i = 1, \ldots, \ell, \|P_j M x\| \geq \theta \|P_j x\|, j = \ell+1, \ldots, m\}\end{aligned}$$

(c) $\nu(M) = \max_{U \in \mathcal{U}} \rho(UM - P_1, P_2)$.

(d) $\nu(\cdot)$ is continuous at M.

□

Proposition 4.1 holds in the structured uncertainty case and so does Theorem 4.1 with the definition of f suitably adapted. Finally, Theorems 5.1 and 5.2 are generalized as follows.

Theorem 6.1. Define

$$h(d_1, \ldots, d_{m-1}) = \eta\left(M^H\left(\sum_{i=1}^{m-1} d_i P_i + P_m\right) M - \sum_{i=1}^{\ell} d_i P_i, \sum_{i=\ell+1}^{m-1} d_i P_i + P_m\right)$$

Then $\nu(M) \leq \inf_{d_i \geq 0, i=1,\ldots,m-1} \sqrt{h(d_1,\ldots,d_{m-1})}$. Furthermore, the equality holds if $m \leq 3$. □

Theorem 6.2. The function $h(d_1,\ldots,d_{m-1})$ defined in Theorem 6.1 is (jointly) quasi-convex in d_i, $i = 1,\ldots,m-1$ over the first orthant. □

Finally, extension of the results of this paper to the case of mixed parametric uncertainty and unmodeled dynamics (as is done in [7] in the context of the structured singular value) does not present any conceptual difficulties.

Appendix. Proofs of Propositions 3.1, 3.2 and 4.1

The following lemma is used in proving Propositions 3.1 and 4.1.

Lemma A. Suppose $\mu(M) > 0$. Let $\Delta = \text{block diag}\{\Delta_1, \Delta_2\} \in \mathcal{X}$ be such that $\overline{\sigma}(\Delta) = \mu^{-1}(M)$ and $\det(I + \Delta M) = 0$. Suppose that $\overline{\sigma}(\Delta_2) < \mu^{-1}(M)$. Then $\det(I + \text{block diag}\{\Delta_1, O_{k_2}\}M) = 0$.

Proof. Consider the polynomial $p(s) = \det(I + \text{block diag}\{\Delta_1, s\Delta_2\}M)$. We have $p(1) = 0$ by assumptions. Now suppose $p(s) \not\equiv 0$. Since the roots of a polynomial are continuous with respect to its coefficients, it holds that there exist $\alpha < 1$ and $z \in \mathbb{C}$ such that $\overline{\sigma}(z\Delta_2) < \mu^{-1}(M)$ and $\det(I + \text{block diag}\{\alpha\Delta_1, z\Delta_2\}M) = 0$. In view of the definition of $\mu(\cdot)$, this contradicts the fact that $\overline{\sigma}(\Delta) = \mu^{-1}(M)$. Therefore, $p(s) \equiv 0$ and, in particular, $p(0) = 0$. □

Proof of Proposition 3.1.

(a) Let $\nu = \nu(M)$ and let μ_ν be the left hand side of (3.1). We show that $\mu_\nu \geq \nu$ and $\nu \geq \mu_\nu$. The former holds trivially if $\nu = 0$. Thus, assume $\nu > 0$. In view of the definition of $\nu(\cdot)$, there exists $\Delta = \text{diag}\{\Delta_1, \Delta_2\} \in \mathcal{Y}$ such that $\overline{\sigma}(\Delta_2) = \nu^{-1}$ and

$$\det(I + \Delta M) = \det\left(I + \begin{bmatrix} \nu^{-1}\Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} \nu I_{k_1} & 0 \\ 0 & I_{k_2} \end{bmatrix} M\right) = 0 \,.$$

Since $\overline{\sigma}\left(\begin{bmatrix} \nu^{-1}\Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}\right) = \nu^{-1}$, in view of the definition of $\mu(\cdot)$, we have $\mu_\nu \geq \nu$. Again, $\nu \geq \mu_\nu$ holds trivially if $\mu_\nu = 0$. Thus, assume $\mu_\nu > 0$. Let $\hat{\Delta} = \text{diag}\{\hat{\Delta}_1, \hat{\Delta}_2\} \in \mathcal{X}$ be such that $\overline{\sigma}(\hat{\Delta}_1) = \overline{\sigma}(\hat{\Delta}_2) = \mu_\nu^{-1}$ and

$$\det\left(I + \hat{\Delta}\begin{bmatrix} \nu I_{k_1} & 0 \\ 0 & I_{k_2} \end{bmatrix} M\right) = \det\left(I + \begin{bmatrix} \nu\hat{\Delta}_1 & 0 \\ 0 & \hat{\Delta}_2 \end{bmatrix} M\right) = 0 \,.$$

Since $\mu_\nu \geq \nu$, we have $\text{diag}\{\nu\hat{\Delta}_1, \hat{\Delta}_2\} \in \mathcal{Y}$ and, in view of the definition of $\nu(\cdot)$,

$$\nu \geq (\overline{\sigma}(\hat{\Delta}_2))^{-1} = \mu_\nu \,.$$

(b) Let $\mu = \mu(M)$ and ν_μ be the left hand side of (3.2). We show that $\nu_\mu \geq \mu$ and $\mu \geq \nu_\mu$. Let $\hat{\Delta} = \text{diag}\{\hat{\Delta}_1, \hat{\Delta}_2\} \in \mathcal{X}$ be such that $\overline{\sigma}(\hat{\Delta}_1) = \overline{\sigma}(\hat{\Delta}_2) = \mu^{-1}$ and

$$\det(I + \hat{\Delta}M) = \det\left(I + \begin{bmatrix} \mu\hat{\Delta}_1 & 0 \\ 0 & \hat{\Delta}_2 \end{bmatrix} \begin{bmatrix} \mu^{-1} I_{k_1} & 0 \\ 0 & I_{k_2} \end{bmatrix} M\right) = 0 \,.$$

Since $\overline{\sigma}(\mu\hat{\Delta}_1) = 1$, in view of the definition of $\nu(\cdot)$, we have $\nu_\mu \geq \mu$. Again, $\mu \geq \nu_\mu$ holds trivially if $\nu_\mu = 0$. Thus, assume $\nu_\mu > 0$. Let $\Delta = \text{diag}\{\Delta_1, \Delta_2\} \in \mathcal{Y}$ be such that $\overline{\sigma}(\Delta_2) = \nu_\mu^{-1}$ and

$$\det\left(I + \Delta \begin{bmatrix} \mu^{-1} I_{k_1} & 0 \\ 0 & I_{k_2} \end{bmatrix} M\right) = \det\left(I + \begin{bmatrix} \mu^{-1}\Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} M\right) = 0 \ .$$

Since $\nu_\mu \geq \mu$ and $\overline{\sigma}(\Delta_1) \leq 1$, in view of the definition of $\mu(\cdot)$, we have $\overline{\sigma}\left(\begin{bmatrix} \mu^{-1}\Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}\right) = \mu^{-1}$. Therefore, either $\overline{\sigma}(\Delta_1) = 1$ or $\nu_\mu = \mu$. Suppose $\overline{\sigma}(\Delta_1) = 1$ and $\nu_\mu > \mu$. Then, in view of Lemma A, we have

$$\det(I + \mu^{-1}\Delta_1 M_{11}) = \det\left(I + \begin{bmatrix} \mu^{-1}\Delta_1 & 0 \\ 0 & 0 \end{bmatrix} M\right) = 0$$

which implies $\overline{\sigma}(M_{11}) \geq \mu$, a contradiction. □

Proof of Proposition 3.2

(a) $\nu(M) = \infty$ if and only if there exists $\Delta = \text{diag}\{\Delta_1, 0\} \in \mathcal{Y}$ such that

$$\det(I + \Delta M) = \det(I + \Delta_1 M_{11}) = 0 \ .$$

Since Δ_1 is arbitrary with $\overline{\sigma}(\Delta_1) \leq 1$, this happens if and only if $\overline{\sigma}(M_{11}) \geq 1$.

(b) From (a), $\nu(M) = \infty$ implies $\nu(\alpha M) = \infty$ for any $|\alpha| \geq 1$. Therefore, the inequality in (3.3) holds if $\nu(M) = \infty$. Thus, assume $\nu(M) < \infty$. Let $\Delta = \text{diag}\{\Delta_1, \Delta_2\} \in \mathcal{Y}$ be such that $\overline{\sigma}(\Delta_2) = \nu^{-1}(M)$ and

$$\det(I + \Delta M) = \det(I + (\alpha^{-1}\Delta)(\alpha M)) = 0 \ .$$

Since $\alpha^{-1}\Delta \in \mathcal{Y}$ and $\overline{\sigma}(\alpha^{-1}\Delta_2) = (|\alpha|\nu(M))^{-1}$, in view of the definition of $\nu(\cdot)$,

$$\nu(\alpha M) \geq (\overline{\sigma}(\alpha^{-1}\Delta_2))^{-1} = |\alpha|\nu(M) \ .$$

(c) It is obvious that $\nu(0) = 0$ and, therefore, the inequality in (3.4) holds if $\alpha = 0$. Thus, assume $\alpha \neq 0$. From (b), $\nu(M) = \nu(\alpha^{-1}\alpha M) \geq |\alpha|^{-1}\nu(\alpha M)$, i.e., $\nu(\alpha M) \leq |\alpha|\nu(M)$.

(d) The equality in (3.5) holds since, for any $\Delta \in \mathcal{Y}$, $D \in \mathcal{D}$, the following holds

$$\det(I + \Delta D M D^{-1}) = \det(I + D^{-1}\Delta D M) = \det(I + \Delta M) \ .$$

(e) The equality in (3.6) holds since, for any $\Delta = \text{diag}\{\Delta_1, \Delta_2\} \in \mathcal{Y}$, $U = \text{diag}\{U_1, U_2\} \in \mathcal{U}$, we have $\overline{\sigma}(\Delta_i U_i) = \overline{\sigma}(U_i \Delta_i) = \overline{\sigma}(\Delta_i)$, $i = 1, 2$, and

$$\det(I + \Delta(UM)) = \det(I + (\Delta U)M) = \det(I + \Delta(MU)) = \det(I + (U\Delta)M) \ .$$

□

Proof of Proposition 4.1

(a) We show that the condition $f(\beta) \geq \beta$ implies $\nu(M) \geq \beta$. The claims follows trivially if $\beta = 0$. Thus assume $f(\beta) \geq \beta > 0$. In view of the definition of $\mu(\cdot)$, there exists $\hat{\Delta} = \text{diag}\{\hat{\Delta}_1, \hat{\Delta}_2\} \in \mathcal{X}$ such that $\overline{\sigma}(\hat{\Delta}_1) = \overline{\sigma}(\hat{\Delta}_2) = f^{-1}(\beta)$ and

$$\det(I + \hat{\Delta}\text{diag}\{\beta I_{k_1}, I_{k_2}\}M) = \det(I + \text{diag}\{\beta\hat{\Delta}_1, \hat{\Delta}_2\}M) = 0 \ .$$

Since $\overline{\sigma}(\beta\hat{\Delta}_1) \leq 1$, in view of the definition of $\nu(\cdot)$, we have $\nu(M) \geq (\overline{\sigma}(\hat{\Delta}_2))^{-1} = f(\beta) \geq \beta$.

(b) Let $0 < \beta < \nu(M) < \infty$. In view of the definition of $\nu(\cdot)$, there exists $\Delta = \text{diag}\{\Delta_1, \Delta_2\} \in \mathcal{Y}$ such that $\overline{\sigma}(\Delta_2) = (\nu(M))^{-1}$ and

$$\det(I + \Delta M) = \det(I + \text{diag}\{\beta^{-1}\Delta_1, \Delta_2\}\text{diag}\{\beta I_{k_1}, I_{k_2}\}M) = 0 \ .$$

Since $\overline{\sigma}(\beta^{-1}\Delta_1) \leq \beta^{-1}$ and $\overline{\sigma}(\Delta_2) = (\nu(M))^{-1} < \beta^{-1}$, we have $f(\beta) \geq \beta$. However, in view of Lemma A, $f(\beta) = \beta$ $(= (1/\overline{\sigma}(\beta^{-1}\Delta_1)))$ would imply that $\nu(M) = \infty$, which is a contradiction. □

Acknowledgements. The authors wish to thank Dr. C.N. Nett, Ms. Batool Nekooie and Dr. J. Chen for helpful discussions.

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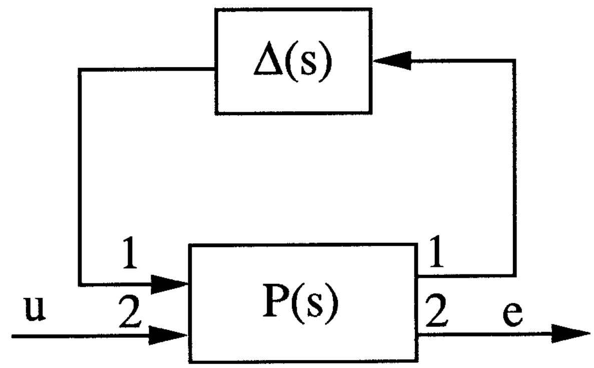


Fig. 1

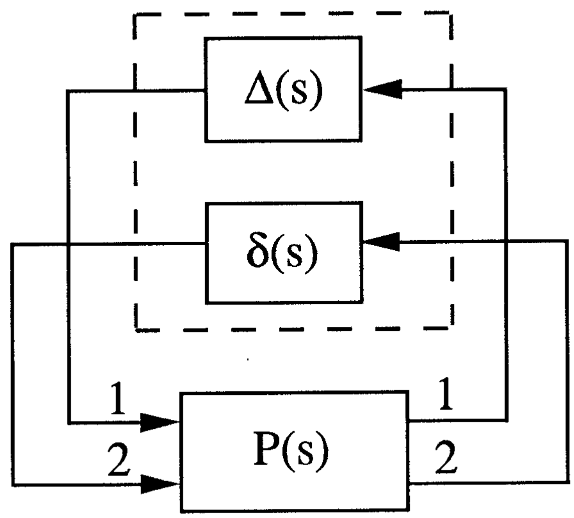
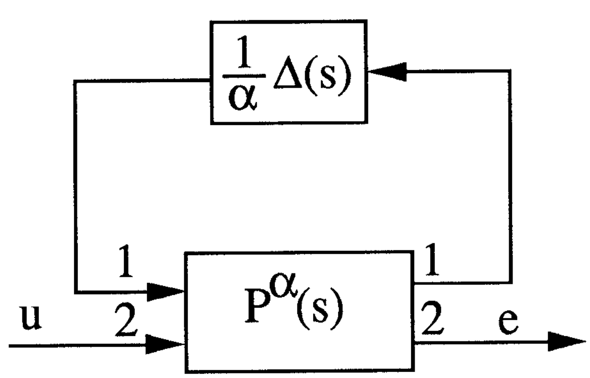


Fig. 2



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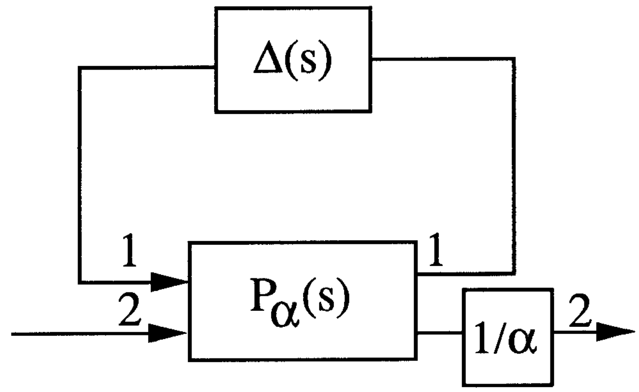

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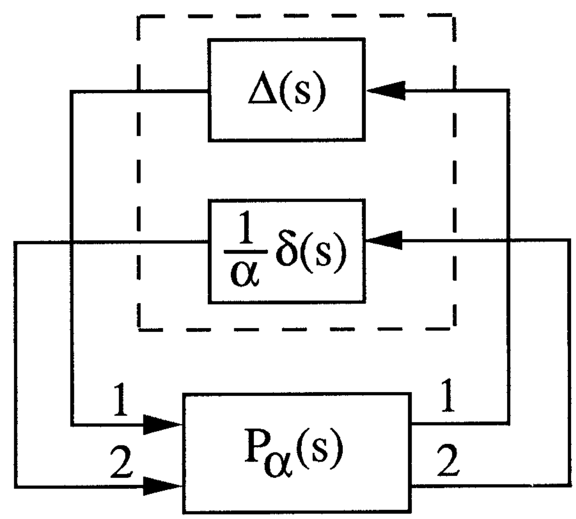

Fig. 5

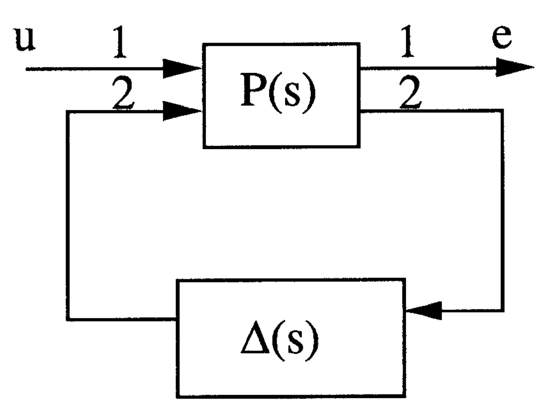

Fig. 6