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A Unified Approach for Dyadic Shift Invariance and Cyclic Shift Invariance

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ABSTRACT

In this paper, the concept of the dyadic shift invariance (DSI) and cyclic shift invariant (CSI) functions are proposed. First, basic properties of the DSI and CSI functions are presented. Then, we can show that the Walsh-Hadamard transform (WHT) and discrete Fourier transform (DFT) functions are, in fact, special cases of the DSI and CSI functions, respectively. Many properties of the WHT and DFT can then be obtained easily from DSI and CSI points of view. The proposed unified approach is simple and rigorous. We will show that the properties of the WHT and DFT are the consequence of the basic principles of the DSI and CSI functions.

1 Introduction

There are many signal processing applications where the use of effective transformation such as the Walsh-Hadamard transform (WHT) and discrete Fourier transform (DFT) is essential [1, 2, 3, 4, 5]. Basically, there are three different kinds of orderings for the WHT, specifically, the Walsh ordering, the Hadamard ordering, and the Dyadic/Paley rodering [1, 2]. It is well known that the power spectrum of the WHT's are invariant to a dyadic shift of the data sequence. Unfortunately, up to now, there is still no unified proof of the invariance for all of the orderings. The only proof available is of show-by-example illustration [1, 2, 4, 5] which, though demonstrates the invariant property, did not provide a rigorous treatment of the dyadic shift invariance. Here, we not only propose a simple but rigorous treatment of the subject, but also provide a more general way to look into various properties of the WHT. We first propose the concept of dyadic decomposable and present some properties of the dyadic shift invariant function. Then, based on these, we will show that all the WHT's are dyadic shift invariant functions and present unified treatments for various known properties of the WHT. This unified approach provides a deep insight into various properties of the WHT. The results can be easily extended to other transform function such as DFT which, on the other hand, is a cyclic shift invariant function. This extension is also considered in this paper. We will see that all the well-known properties of the WHT and DFT are the consequence of the basic principles of the special functions. In fact, the WHT is a special case of the dyadic shift invariant function and the DFT is a special case of a cyclic shift invariant function.

The dyadic Shift invariance is presented in Section 2 followed by the cyclic shift invariance in Section 3.

2 Dyadic Shift Invariance

A function $h(m,n)$ is said to be *dyadic decomposable* if it satisfies

$$h(m \oplus k, n) = h(m,n) \cdot h(k,n), \tag{1}$$

where $\oplus$ is the moduls 2 addition. A transformation is said to be *dyadic shift invariant* (DSI) if the transform function is dyadic decomposible with unity norm. That is

1. $\|h(m,n)\|^2 = h(m,n) \cdot h^*(m,n) = 1,$

2. $h(m \oplus k, n) = h(m,n) \cdot h(k,n),$

where * is the complex conjugate operation. Let $\{x(m)\}$ be a real-valued N-periodic sequence and $\{X(n)\}$ be the DSI transformation of $\{x(m)\}$. We have

$$X(n) = \frac{1}{N} \sum_{m=0}^{N-1} x(m) h(m,n). \tag{2}$$

Theorem 1 *The power spectrum of a DSI transform function is dyadic shift invariant.*

Proof: Let $\{X_k(n)\}$ be the transformation of $\{x(m \oplus k)\}$, where $\{x(m \oplus k)\}$ is the sequence obtained by subjecting $\{x(m)\}$ to a dyadic shift of size k. Since the modulo 2 addition is the same operation as the modulo 2 substraction, one obtains

$$\begin{aligned} X_k(n) &= \frac{1}{N} \sum_{m=0}^{N-1} x(m \oplus k) h(m,n) \\ &= \frac{1}{N} \sum_{\hat{m}=0}^{N-1} x(\hat{m}) h(\hat{m} \oplus k, n). \end{aligned}$$

Since $h(m,n)$ is DSI, we have

$$X_k(n) = \frac{1}{N} \sum_{\hat{m}=0}^{N-1} x(\hat{m}) h(\hat{m}, n) h(k,n) = X(n) h(k,n),$$

and the power spectrum of $\{X_k(n)\}$ is

$$\|X_k(n)\|^2 = \|X(n)\|^2 \cdot \|h(k,n)\|^2 = \|X(n)\|^2$$

The power spectrum is dyadic shift invariant. □

Let m, n be any two real-valued parameters which have the binary representation

$$\begin{aligned} m &= m_{N-1}2^{N-1} + m_{N-2}2^{N-2} + \cdots + m_1 2^1 + m_0 2^0, \\ n &= n_{N-1}2^{N-1} + n_{N-2}2^{N-2} + \cdots + n_1 2^1 + n_0 2^0 \end{aligned} \tag{3}$$

The bit-valued inner product $< m, n >$ is defined as

$$< m, n >= \sum_{s=0}^{N-1} m_s n_s. \tag{4}$$

Lemma 1 *The WHT function* $(-1)^{<m,n>}$ *is DSI.*

Proof: Since $(-1)^{<m\oplus k,n>} = (-1)^{\sum_{s=0}^{N-1}(m_s \oplus k_s)n_s}$, and from Table 1, the Boolean function $(m_s \oplus k_s)n_s$ is equivalent to the function $m_s n_s \oplus k_s n_s$. We have

$$(-1)^{<m\oplus k,n>} = (-1)^{\sum_{s=0}^{N-1} m_s n_s \oplus k_s n_s} = (-1)^{<m,n>\oplus<k,n>}. \tag{5}$$

From Table 2, it can be found

$$\begin{aligned} (-1)^{<m,n>\oplus<k,n>} &= (-1)^{m_1 n_1 \oplus k_1 n_1} \cdot (-1)^{m_2 n_2 \oplus k_2 n_2} \cdot (-1)^{n_{N-1} n_{N-1} \oplus k_{N-1} n_{N-1}} \\ &= (-1)^{m_1 n_1 + k_1 n_1} \cdot (-1)^{m_2 n_2 + k_2 n_2} \cdot (-1)^{n_{N-1} n_{N-1} + k_{N-1} n_{N-1}} \\ &= (-1)^{<m,n>} \cdot (-1)^{<k,n>}. \end{aligned} \tag{6}$$

As $\|(-1)^{<m,n>}\|^2 = 1$, one concludes that WHT function is a DSI function. □

In general, the WHT is defined as

$$X(n) = \frac{1}{N} \sum_{m=0}^{N-1} x(m)(-1)^{<m,r(n)>}, \tag{7}$$

where $r(n)$ is a function which depends on the ordering of WHT. Let

$$< m, r(n) >= \sum_{s=0}^{N-1} m_s r_s(n),$$

where $r_s(n), s = 0, 1, \cdots, N-1$ is the binary representation of $r(n)$ as in (3). For the Hadamard oedering, $r_s(n) = n_s$. For the Walsh ordering,

$$r_s(n) = \begin{cases} n_{N-s} + n_{N-s-1} & \text{for } s \neq 0 \\ n_{N-1} & \text{for } s = 0. \end{cases}$$

For the Dyadic/Paley oedering, $r_s(n) = n_{N-1-s}$. We have the following theorem.

Theorem 2 *The power spectrum of all rderings of the WHT are dyadic shift invariant.*

Proof: By *Lemma* 1, we know the function $(-1)^{<m,r(n)>}$ is DSI. From *Theorem* 1, the power spectrum of WHT is dyadic shift invariant. □

Lemma 2 *Let $\{X\}$ be the WHT transform of $\{x\}$ and $\{X_k\}$ be the transform of $\{x\}$ subjected to a dyadic of size k, where $\{x\}$ is an N-periodic sequence. Then the relationship between X_k and X is*

$$X_k(n) = X(n) \cdot (-1)^{<k,n>}. \tag{8}$$

Proof: Since $X_k(n) = \frac{1}{N} \sum_{m=0}^{N-1} x(m \oplus k)(-1)^{<m,n>}$, by *Lemma* 1, we have

$$X_k(n) = \frac{1}{N} \sum_{\hat{m}=0}^{N-1} x(\hat{m})(-1)^{<\hat{m},n>}(-1)^{<k,n>} = X(n) \cdot (-1)^{<k,n>} \square$$

With *Lemma* 2, we can interpret that for the WHT, the dyadic shift in the time domain results in a "phase shift" of either 0 or π in the frequency domain.

The dyadic cross-correlation function of two real-valued N-periodic sequences $\{x(m)\}$ and $\{y(m)\}$ is defined as

$$z(m) = \frac{1}{N} \sum_{h=0}^{N-1} x(h) y(m \oplus h). \tag{9}$$

Let the DSI transform of $z(m)$ be $Z(n)$ and the DSI transform of sequences $x(m)$ and $y(m)$ be $X(n)$ and $Y(n)$ respectively. We have

$$\begin{aligned} Z(n) &= DSIT(z(m)) = \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{l=0}^{N-1} x(l) y(m \oplus l) h(m,n) \\ &= \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{l=0}^{N-1} x(l) y(m \oplus l) h(m,n) h(l,n) h^*(l,n), \end{aligned} \tag{10}$$

$$Z(n) = \frac{1}{N^2} \sum_{l=0}^{N-1} x(l) h^*(l,n) \sum_{m=0}^{N-1} y(m \oplus l) h(m \oplus l, n) = \bar{X}^*(n) \cdot Y(n), \tag{11}$$

where $\bar{X}^* \triangleq (DSI(x^*))^*$. If it is a real DSI function such as WHT, then $Z(n) = X(n) \cdot Y(n)$. Obvious, the power spectrum is dyadic shift invariant in the way that

$$\|Z(n)\|^2 = \|X(n)\|^2 \cdot \|Y(n)\|^2. \tag{12}$$

A two-dimensional transformation which is said to be DSI if the transform function satisfies

1. $\|h(m_1, m_2, n_1, n_2)\|^2 = 1$,

2. $h(m_1 \oplus k, m_2 \oplus l, n_1, n_2) = h(m_1, m_2, n_1, n_2) \cdot h(k, l, n_1, n_2)$.

Analogous to 1-D case, it can be shown that the power spectrum of a two-dimensional (2-D) DSI transform function is dyadic shift invariant. The 2-D WHT function is $(-1)^{<m_1,n_1>+<m_2,n_2>}$. Since

$$\begin{aligned} (-1)^{<m_1 \oplus k, n_1>+<m_2 \oplus l, n_2>} &= (-1)^{<m_1,n_1>}(-1)^{<k,n_1>}(-1)^{<m_2,n_2>}(-1)^{<l,n_2>} \\ &= (-1)^{<m_1,n_1>+<m_2,n_2>} \cdot (-1)^{<k,n_1>+<l,n_2>}, \end{aligned} \tag{13}$$

the 2-D WHT function is a 2-D DSI function. Therefore, all the properties of the 2-D WHT follow as discussed in the 1-D case. For instance, denote $X_{k,l}(n_1, n_2)$ as the transformation of $\{x(m_1 \oplus k, m_2 \oplus l)\}$ which is the sequence obtained by subjecting $\{x(m_1, m_2)\}$ to a dyadic shift of size k in m_1 direction and of size l in m_2 direction. It can be easily obtained that

$$X_{k,l}(n_1, n_2) = X(n_1, n_2)(-1)^{<n_1,k>+<n_2,l>}. \tag{14}$$

From the analogy in the 1-D case, the 2-D cross-correlation function can be defined as

$$z(m_1, m_2) = \frac{1}{N_1 N_2} \sum_{h_2=0}^{N_2-1} \sum_{h_1=0}^{N_1-1} x(m_1 \oplus h_1, m_2 \oplus h_2) y(h_1, h_2). \tag{15}$$

Let the 2-D DSI transform of z, x, and y be Z, X, and Y respectively. With the same derivation as before, we can show that

$$Z(n_1, n_2) = \bar{X}^*(n_1, n_2) \cdot Y(n_1, n_2). \tag{16}$$

Again, if the 2-D DSI transform function is real-valued such as 2-D WHT, then $Z(n_1, n_2) = X(n_1, n_2) \cdot Y(n_1, n_2)$.

3 Cyclic Shift Invariance

The above results can be extended to cyclic shift invariance. A function $g(m, n)$ is *cyclic decomposable* if it satisfies

$$g(m + k, n) = g(m, n) \cdot g(k, n). \tag{17}$$

A transformation is said to be cyclic shift invariant (CSI) if the transform function is cyclic decomposable with unity norm. That is

1. $||g(m, n)||^2 = 1$,

2. $g(m+k,n) = g(m,n)\cdot g(k,n)$.

Theorem 3 *The power spectrum of a CSI transform function is cyclic shift invariant.*

Proof: Let $\{x(m)\}$ be a real-valued N-periodic sequence and $\{X(n)\}$ be transformation of $\{x(m)\}$. We have

$$X(n) = \frac{1}{N}\sum_{m=0}^{N-1} x(m)h(m,n). \tag{18}$$

Let $\{X_k(n)\}$ be the transformation of $\{x(m+k)\}$, where $\{x(m+k)\}$ is the sequence obtained by subjecting $\{x(m)\}$ to a cyclic shift of size k. One obtain

$$\begin{aligned} X_k(n) &= \frac{1}{N}\sum_{m=0}^{N-1} x(m+k)h(m,n) \\ &= \frac{1}{N}\sum_{\hat{m}=0}^{N-1} x(\hat{m})h(\hat{m}-k,n). \end{aligned} \tag{19}$$

Since $h(m,n)$ is cyclic decomposible, we have

$$X_k(n) = \frac{1}{N}\sum_{\hat{m}=0}^{N-1} x(\hat{m})h(\hat{m},n)h(-k,n) = X(n)h(-k,n).$$

By the property that the norm of $h(m,n)$ equals unity,

$$\|X_k(n)\|^2 = \|X(n)\|^2 \cdot \|h(-k,n)\|^2 = \|X(n)\|^2. \square$$

The cyclic cross-correlation (or convolution) function of two real-valued N-periodic sequences $\{x(m)\}$ and $\{y(m)\}$ is defined as

$$z(m) = \frac{1}{N}\sum_{h=0}^{N-1} x(h)y(m+h). \tag{20}$$

Let the CSI transform of $z(m)$ be $Z(n)$ and the CSI transform of sequences $x(m)$ and $y(m)$ be $X(n)$ and $Y(n)$ respectively. We have

$$\begin{aligned} Z(n) &= CSIT(z(m)) = \frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{l=0}^{N-1} x(l)y(m+l)h(m,n) \\ &= \frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{l=0}^{N-1} x(l)y(m+l)h(m,n)h(l,n)h^*(l,n), \end{aligned} \tag{21}$$

$$Z(n) = \frac{1}{N^2} \sum_{l=0}^{N-1} x(l)h^*(l,n) \sum_{m=0}^{N-1} y(m+l)h(m+l,n) = \bar{X}^*(n) \cdot Y(n), \tag{22}$$

where $\bar{X}^* \triangleq (CSI(x^*))^*$. It can be easily shown that the discrete Fourier transform (DFT) function $\exp(-j2\pi kn/N)$ is a CSI function. Therefore, the power spectrum of DFT is cyclic shift invariant. Many well-know properties of the DFT can then be easily obtained by the same derivations as in Section 2.

4 Conclusions

Basic properties of the dyadic shift invariance and the cyclic shift invariance are presented in this paper. As we have shown, the WHT and DFT are the special cases of the DSI and CSI functions, respectively. Therefore, all the properties of the DSI and CSI functions are preserved in the WHT and DFT, respectively. Many properties are then easily derived by using this proposed approach. This approach also provides a deep insight into those well-known properties. In conclusion, the properties of the WHT and DFT are the consequence of the basic principles of the DSI and CSI functions.

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m_s	l_s	n_s	$m_s \oplus l_s$	$m_s n_s$	$l_s n_s$	$(m_s \oplus l_s) n_s$	$m_s n_s \oplus l_s n_s$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	1	1	1
1	0	0	1	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	0	0	0	0	0
1	1	1	0	1	1	0	0

Table 1: Truth table for $(m_s \oplus k_s) n_s$ and $m_s n_s \oplus k_s n_s$

m_s	n_s	$m_s \oplus n_s$	$m_s + n_s$	$(-1)^{m_s \oplus n_s}$	$(-1)^{m_s + n_s}$
0	0	0	0	1	1
0	1	1	1	-1	-1
1	0	1	1	-1	-1
1	1	0	2	1	1

Table 2: Truth table for $(-1)^{m_s \oplus n_s}$ and $(-1)^{m_s + n_s}$