The Structure of Divisible Discrete Random Sets and Their Randomized Superpositions

by N. Sidiropoulos, J. Baras and C. Berenstein

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The Structure of Divisible Discrete Random Sets and their Randomized Superpositions *

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Abstract

In this paper, we present an axiomatic formulation of Discrete Random Sets, and extend Choquet's uniqueness result to obtain a recursive procedure for the computation of the underlying event-space probability law, given a consistent Discrete Random Set specification via its generating functional. Based on this extension, we investigate the structure of Discrete Random Set models that enjoy the properties of independent decomposition / superposition, and present a design methodology for deriving models that are guaranteed to be consistent with some underlying event-space probability law. These results pave the way for the construction of various interesting models, and the solution of statistical inference problems for Discrete Random Sets.

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1 Introduction

Statistical inference techniques similar to maximum likelihood, or maximum a posteriori, are practically non-existent in Random Set theory [6]. This is due to inherent analytical difficulties, and the fact that, despite its usefulness, continous domain Random Set theory falls short of providing the necessary tools needed to construct efficient inference procedures. The richness of the event space poses another fundamental constraint: it is very difficult (if not impossible) to derive the measure on the event space, given a constructive Random Set specification.

The transition from continous domain Random Sets to discrete domain random sets is a troublesome one [20, 5]. For this reason, it is preferable to define discrete domain Random Sets (or Discrete Random Sets, for brevity) directly on the appropriate spaces, and base subsequent developments on this axiomatic definition. Discrete Random Sets have a very strong potential for applications in the areas of Machine Vision and Image Modeling and Understanding [20, 5, 7, 17, 24, 18, 3, 4, 1, 21, 22].

In this paper, we consider Discrete Random Sets, and present a result that establishes a vital link between Discrete Random Set theory and standard existing Statistical Inference techniques. Using this result as a starting point, we examine the structure of divisible Discrete Random Set models, which are discrete domain analogs of the popular continous domain Boolean and Germ-Grain Random Set models [19, 20, 5, 1, 4, 3]. Finally, we propose a procedure that yields new Discrete Random Set models, based on existing models.

The rest of this paper is organized as follows. Section (2) presents the axiomatic definition of a Discrete Random Set, along with a special case of a very important result due to Choquet [2]. Section (3) extends Choquet's result, and shows that, for Discrete Random Sets, the measure on the event space *can actually be recovered* from its "projections" (the collection of hit (or miss) probabilities over the sample space, which, in the case of Discrete Random Sets, is a subspace of the event space). A simple, yet important, example of a Discrete Random Set is considered in section (4), and its capacity functional is computed in section (5). The general structure of Boolean Discrete Random Set models is investigated in section (6). Section (7) presents a simple randomization process, which we call randomized superposition, and discusses its utility in deriving new Discrete Random Set models. Finally, section (8) contains some concluding remarks, and guidelines for future research.

2 Discrete Random Sets

Definition 1 *Let B be a bounded subset of $\mathcal{Z}^2$. Assume that B contains the origin. Let $\Sigma(\Omega)$ denote the σ-algebra on Ω. Let $\Sigma(B)$ denote the power set (i.e. the set of all subsets) of B, and let $\Sigma(\Sigma(B))$ denote the power set of $\Sigma(B)$. A* **Discrete Random Set (DRS)**, *X, on $B \subset \mathcal{Z}^2$, is a measurable mapping of a probability space $(\Omega, \Sigma(\Omega), P)$ into the measurable space $(\Sigma(B), \Sigma(\Sigma(B)))$.*

Definition 2 *The* **capacity functional**, *$T_X(K)$, of a DRS X, on $B \subset \mathcal{Z}^2$, is defined by*

$$T_X(K) = P_X(X \cap K \neq \emptyset), \quad K \in \Sigma(B)$$

The capacity functional $T_X(K)$, for all $K \in \Sigma(B)$, contains all the information about the DRS X.

Theorem 1 *[2, Choquet] Given $T_X(K)$, $\forall K \in \Sigma(B)$, there exists a unique probability measure, P_X, on $\Sigma(\Sigma(B))$, such that*

$$P_X(X \cap K \neq \emptyset) = T_X(K), \quad \forall K \in \Sigma(B)$$

Proof: The theorem is valid for Random Sets defined on $\mathcal{R}^2$, and for all $K \in \mathcal{K}$, the collection of all compact subsets of $\mathcal{R}^2$. Since $\mathcal{Z}^2 \subset \mathcal{R}^2$, the validity of the theorem for the discrete case follows. Briefly, Discrete Random Sets are a special case of Random Sets, defined on the integer coordinate grid. Therefore, by Choquet's theorem, the probability measure on $\Sigma(\Sigma(B))$ is uniquely determined by the knowledge of hit probabilities over the entire collection of compact subsets of $\mathcal{R}^2$. The hit probability of any particular compact subset, $K \subset \mathcal{R}^2$ is uniquely determined by the hit probability of $K \cap B \subseteq B$. Therefore, given the hit probabilities of all subsets of B, the probability measure on $\Sigma(\Sigma(B))$ is uniquely determined.

3 Probability functionals

Let $K, K_1, \cdots, K_n$ be $n+1$ elements of $\Sigma(B)$, and define the following probability functionals

$$\Gamma_{X,n}(K, K_1, K_2, \cdots, K_n) = P_X(X \cap K = \emptyset \ AND \ X \cap K_1 \neq \emptyset \ AND \ \cdots$$

$$\cdots\ AND\ X \cap K_n \neq \emptyset)$$

i.e.

$$\Gamma_{X,n}(K, K_1, K_2, \cdots, K_n) = P_X(X\ misses\ K\ AND\ X\ hits\ K_1\ AND\ \cdots$$
$$\cdots\ AND\ X\ hits\ K_n)$$

Using Bayes rule we have

$$P_X(X\ misses\ K\ AND\ X\ misses\ K_n\ AND\ X\ hits\ K_1\ through\ K_{n-1}) =$$
$$P_X(X\ misses\ K_n \mid X\ misses\ K\ AND\ X\ hits\ K_1\ through\ K_{n-1}) \cdot$$
$$\cdot P_X(X\ misses\ K\ AND\ X\ hits\ K_1\ through\ K_{n-1}) =$$
$$= [1 - P_X(X\ hits\ K_n \mid X\ misses\ K\ AND\ X\ hits\ K_1\ through\ K_{n-1})] \cdot$$
$$\cdot P_X(X\ misses\ K\ AND\ X\ hits\ K_1\ through\ K_{n-1}) =$$
$$= P_X(X\ misses\ K\ AND\ X\ hits\ K_1\ through\ K_{n-1}) -$$
$$P_X(X\ misses\ K\ AND\ X\ hits\ K_1\ through\ K_n)$$

Therefore

$$\Gamma_{X,n-1}(K \cup K_n, K_1, \cdots, K_{n-1}) = \Gamma_{X,n-1}(K, K_1, \cdots, K_{n-1}) -$$
$$\Gamma_{X,n}(K, K_1, \cdots, K_n)$$

and reorganizing

$$\Gamma_{X,n}(K, K_1, \cdots, K_n) = \Gamma_{X,n-1}(K, K_1, \cdots, K_{n-1}) -$$
$$\Gamma_{X,n-1}(K \cup K_n, K_1, \cdots, K_{n-1})$$

with

$$\Gamma_{X,0}(K) = 1 - T_X(K) = Q_X(K) \geq 0$$

The functional

$$Q_X(K) = 1 - T_X(K) = P_X(X \cap K = \emptyset)$$

is known as the **generating functional** of the DRS X. We remark that for all $n \geq 0$, and any collection, $K, K_1, \cdots, K_n$, of $n+1$ elements of $\Sigma(B)$, $\Gamma_{X,n}(K, K_1, \cdots, K_n) \geq 0$, because it is a probability which can be recursively computed using Bayes rule.

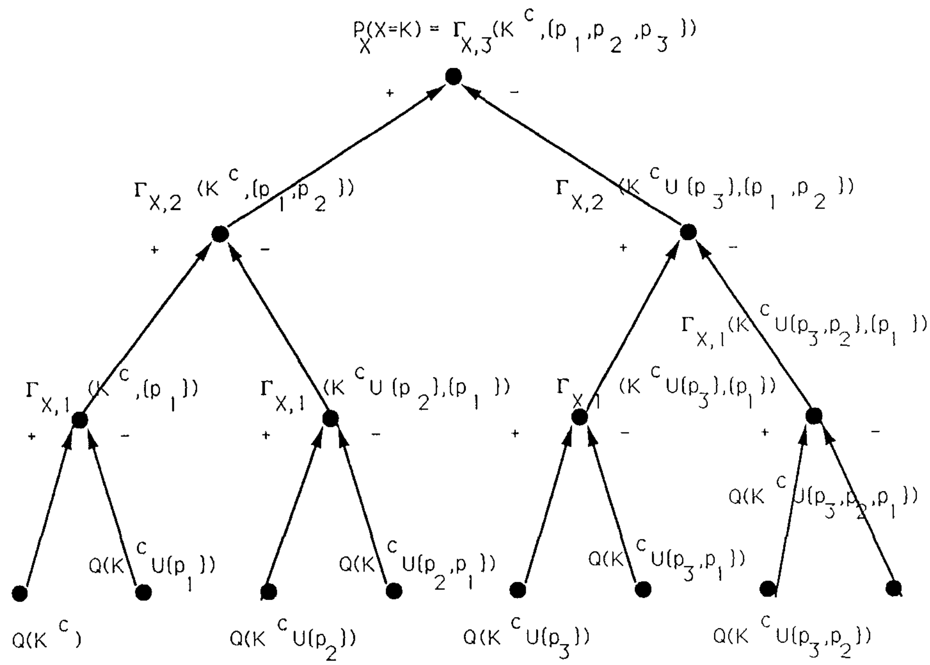

Figure 1: Example of binary tree generated for a three point observation set

For reasons that will soon become apparent, the recursion obtained above is a very interesting result. It implies that, given $T_X(K)$, for all $K \in \Sigma(B)$, we can recursively compute various important probability functionals. In particular, let

$$K = \cup_{i=1}^{n} \{p_i\}, \qquad K^c = B - K$$

then

$$\begin{aligned} P_X(X = K) &= \Gamma_{X,n}(K^c, \{p_1\}, \{p_2\}, \cdots, \{p_n\}) \\ &= \Gamma_{X,n-1}(K^c, \{p_1\}, \{p_2\}, \cdots, \{p_{n-1}\}) - \\ &\quad \Gamma_{X,n-1}(K^c \cup \{p_n\}, \{p_1\}, \{p_2\}, \cdots, \{p_{n-1}\}) \end{aligned}$$

Therefore, given any observation $K \in \Sigma(B)$, we can recursively compute the probability of this observation. An example of a binary tree generated by the above recursion is given in figure 1, for the case of a three point observation set, $K = \{p_1, p_2, p_3\}$. A closed-form expression can also be obtained. It is easy to see that

$$P_X(X = K) = \sum_{i=0}^{|K|}(-1)^i \sum_{K_i \subseteq K,\ |K_i|=i} Q_X(K^c \cup K_i)$$

It is interesting to note that, in order to compute $P_X(X = K)$, we only need the miss probabilities of all possible unions of K^c with subsets of K. The probability of any event in $\Sigma(\Sigma(B))$ can then be found by summing up the probabilities of its constituent elementary outcomes, all of which are in $\Sigma(B)$. Theorem 1 guarantees that, given $T_X(K)$ for all $K \in \Sigma(B)$ there exists a unique probability law P_X on $\Sigma(\Sigma(B))$, but here we claim even more: **the unique probability law can be recursively obtained, i.e. we can infer the underlying distribution on $\Sigma(\Sigma(B))$** . This is a striking result, especially because it is valid for *any* Discrete Random Set model. Its implications are significant, because, for example, all statistical inference problems involving Discrete Random Sets whose capacity functional can be specified are (in principle) solved.

The cost of deriving the underlying probability law on $\Sigma(\Sigma(B))$ is high. The number of operations needed to determine the probability of any $K \in \Sigma(B)$ is $O(2^{|K|})$. To see this, let $n = |K|$. Let $T(n)$ denote execution time, and let $W(n)$ denote workload (overall number of operations involved) associated with the computation of $P(X = K)$. From the recursion for $\Gamma_{X,n}$ we have

$$T(n) = T(n-1) + 1 \mapsto T(n) = O(n)$$

$$W(n) = 2W(n-1) + 1 \mapsto W(n) = O(2^n) \quad (\textit{exponential})$$

Therefore, the computation is very costly when implemented on a sequential machine, because in this case the run time is given by $W(n)$, and, hence, it is exponential. On the other hand, if one fully exploits the inherent parallelism of the recursion (i.e. recursively, simultaneously compute both subexpressions) then one can come up with a run time which is linear in $|K|$ (this *requires* the use of a massively parallel machine). The power set itself, $\Sigma(B)$, has $O(2^{|B|})$ elements. Therefore, the overall number of operations needed to find $P_X(X = K)$, $\forall K \in \Sigma(B)$, is $O(2^{2|B|})$.

4 The Boolean RS with Radial Convex Primary Grains model

The *Boolean random set* is an important and relatively simple example of a random set. Its importance stems from two principal considerations: its analytical tractability and its power in modeling many real-life applications. Despite its simplicity, it has many interesting properties, and, in fact, there are many unanswered questions [24, p65]. A Boolean model is a basic model in stereology and stochastic geometry [24, 10, 20, 5]. Typical applications include: random clumping of dust, or powder particles, or blood cells; modeling of geological structures, patterns in photographic emulsion, colloids in gel form, and structural inhomogeneities in amorphous matter [19, 20, 5, 1, 3, 4] and [24, p68, and references therein].

Here, we consider a restricted version of the discrete case analog of the Boolean random set, the *discrete Boolean random set with radial convex primary grains*. This version is still powerfull enough to model many real-life applications, yet restricted enough to allow for the analysis and design of efficient algorithms for the statistical inference of various model parameters. Recently, this version has been successfully used to model the degradation process in an attempt to derive the "optimal" reconstruction filters for communication of morphologically coded images [18]. An efficient procedure for almost-Bayesian binary hypothesis testing for this special case of discrete Boolean models has been devised [21]. It uses the discrete Morphological Skeletonization algorithm and some of its variants [5, 9, 26, 12, 11, 7, 17, 18]. An exact Bayesian solution to a more general M-ary testing problem can be found in [22].

The discrete "analog" of a planar Poisson Point Process (PPP), Ψ, observed through a bounded Borel window, B, is defined in what follows.

Definition 3 *Let B be a bounded subset of $\mathcal{Z}^2$. Assume that B contains the origin. Let $\Sigma(\Omega)$ denote the σ-algebra on Ω. Let $\Sigma(B)$ denote the power set (i.e. the set of all subsets) of B, and let $\Sigma(\Sigma(B))$ denote the power set of $\Sigma(B)$. A* **Discrete Point Process (DPP)**, *Ψ, on B, is a measurable mapping of a probability space $(\Omega, \Sigma(\Omega), P)$ into the measurable space $(\Sigma(B), \Sigma(\Sigma(B)))$.*

Observe that the DPP definition is identical to the DRS definition. In the discrete case (at least in principle) there is no distinction between random point processes and random sets; in fact discrete random sets *are* discrete random point processes, because the regularity conditions are automatically satisfied here (discrete random sets *are* locally finite and simple). Informally, Ψ can be thought of as a random pattern of points scattered over B. The continous case PPP can be derived from a generalization of the continous case Binomial Point Process [24, pp36-38], by using a limiting argument. Alternatively, the continous case PPP can also be derived from a generalization of the *Bernoulli Lattice Process (BLP)* [24, pp40-42], again by using a limiting argument. This, then, is the correct discrete-case analog of the PPP.

Definition 4 *A* **Generalized Bernoulli Lattice Process (GBLP)**, Ψ, *on* B, *is a discrete point process on* B *which is constructively defined in the following manner. Each point* $x \in B$ *is contained in* Ψ *with probability* $p\lambda_s(x)$, *independently of all others. Here,* $p \in (0,1]$ *and* $\lambda_s(x) \in [0,1]$, $\forall x \in B$.

Clearly, the GBLP enjoys the independent scattering property, i.e. the number of points that fall on n disjoint subsets of B form n independent random variables.

Definition 5 *The* **dilation**, $X \oplus H^s$, *of a set* $X \subset \mathcal{Z}^2$, *by a structuring element* $H \subset \mathcal{Z}^2$, *is defined as*

$$X \oplus H^s = \bigcup_{h \in H} X_{-h} = \{z \in \mathcal{Z}^2 \ni H_z \cap X \neq \emptyset\}$$

In the discrete case the notion of size is formalized via the operation of set dilation

$$rH = \begin{cases} \{\bar{0}\} \oplus H \oplus H \oplus \cdots \oplus H, \quad (r \text{ dilations}) & , r = 1, 2, \ldots \\ \{\bar{0}\} & , r = 0 \end{cases} \tag{1}$$

Let H be a nonempty, bounded, and convex subset of B, which contains the origin, and $H \subset B' \subset B$, $|H| << |B'| << |B|$. We have the following definition.

Definition 6 *Let* Ψ *be a GBLP on* B *with parameters* (p, λ_s). *Let* $\{G_1, G_2, \cdots\}$ *be a set of nonempty, bounded and convex i.i.d. discrete RS's, on* $B' \subset B$, $|B'| << |B|$, *each given by* $G_i = R_i H$, *where* $\{R_1, R_2, \cdots\}$ *form an i.i.d. sequence of* $\mathcal{Z}_+$*-valued r.v.'s which is independent of* Ψ, $R_i < \bar{R}$, $\forall i$, *and each*

R_i is distributed according to the pmf $f_R(r)$, which is compactly supported on $\{0, 1, \ldots, \bar{R}\}$. Define

$$X = \bigcup_{i=1,2,\ldots} G_i \oplus \{y_i\}$$

where $\Psi = \{y_1, y_2, \cdots\}$. Then X will be called a **Discrete Radial Boolean RS (DRBRS)** *, with parameters (p, λ_s, H, f_R), and will be denoted by (p, λ_s, H, f_R)-DRBRS. The points $\{y_1, y_2, \cdots\}$ are called* the germs, *and the RS's $\{G_1, G_2, \cdots\}$ are called* the primary grains *of the RS X.*

Strictly speaking, X *is not* a radial random set (for example, it does not necessarily contain the origin). Nevertheless, with some abuse of terminology, we shall call the resulting discrete Boolean RS a **Discrete Radial Boolean RS (DRBRS)**, to emphasize the nature of the underlying grain process. A typical realization of a DRBRS is given in figure 2.

Given the result of the previous section, one would be interested in obtaining the capacity functional for the model at hand. We proceed to do this in the section that follows.

5 Capacity functional for the DRBRS model

Let K be any subset of $B \subset \mathcal{Z}^2$. We want to compute $T_X(K) = P_X(X \cap K \neq \emptyset)$. Observe that the only points (germ locations) that can give rise to a primary grain that hits K are the ones in the set $K \oplus \bar{R}H^s$, as depicted in figure 4. Consider

$$Q_X(K) = 1 - T_X(K) = P_X(X \cap K = \emptyset)$$

and define

$$d^H(z, K) = min_{k \in K} ||z - k||_H$$

where

$$||z - k||_H = min\{n \geq 0 \mid (\{z\} \oplus nH) \cap \{k\} \neq \emptyset\}$$

Observe that for $z \in K$, $d^H(z, K) = 0$, since H contains the origin. We remark that $d^H(z, K)$, as defined above, is a digital uniform step metric, which is a generalization of the digital Housdorff metric. For a proof of the fact that $d^H(z, K)$ is indeed a metric, refer to [26]. With this notation in place, we now have

$$Q_X(K) = \prod_{z \in K \oplus \bar{R}H^s} \left[(1 - p\lambda_s(z)) + p\lambda_s(z) F_R(d^H(z, K) - 1)\right] \qquad (2)$$

Figure 2: Realization of a DRBRS. Here, the primary grain is a discrete CIRCLE (see figure 3)

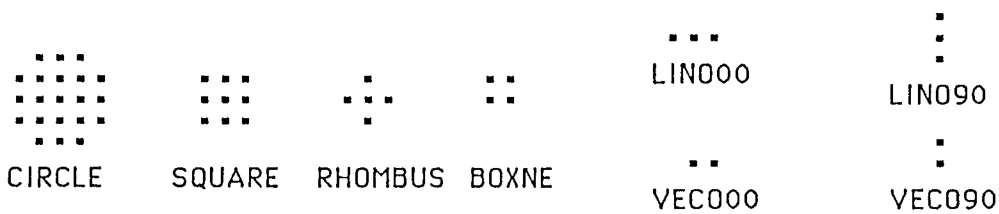

Figure 3: Some commonly used discrete structuring elements

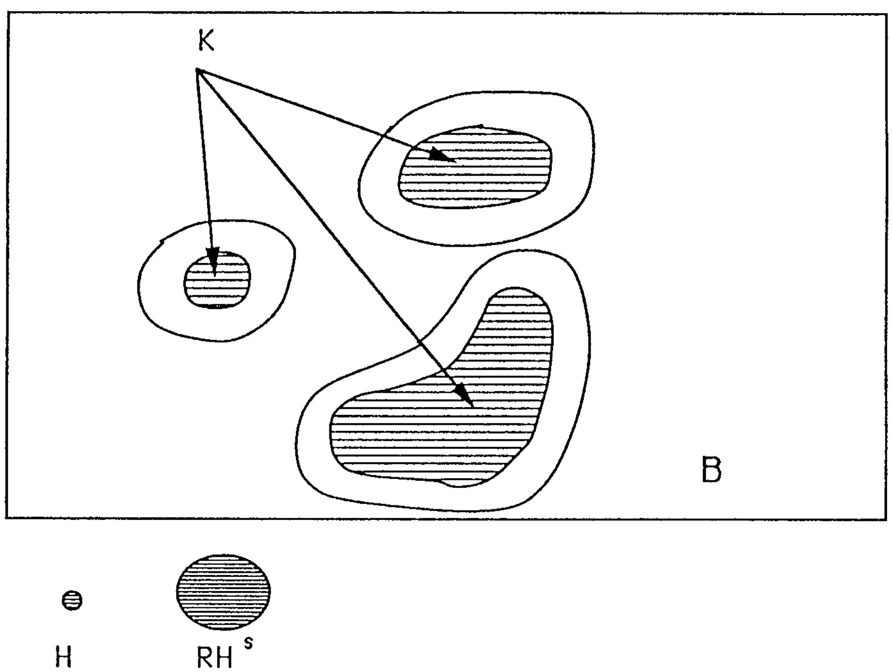

Figure 4: A set $K \subset B$, and its dilation $K \oplus \bar{R}H^s$

where

$$F_R(m) = \sum_{l=0}^{m} f_R(l)$$

and $F_R(-1) = 0$, by convention. Therefore, for a (p, λ_s, H, f_R)-DRBRS X, on B, and any $K \in \Sigma(B)$

$$T_X(K) = 1 - \prod_{z \in K \oplus \check{R}H^s} \left[(1 - p\lambda_s(z)) + p\lambda_s(z) F_R(d^H(z, K) - 1)\right] \qquad (3)$$

By using the closed-form solution of the recursion for computing $P_X(X = K)$ in terms of Q_X, we obtain

$$P_X(X = K) = \sum_{i=0}^{|K|} (-1)^i \sum_{K_i \subseteq K,\ |K_i| = i}$$

$$\prod_{z \in K^c \cup K_i \oplus \check{R}H^s} \left[(1 - p\lambda_s(z)) + p\lambda_s(z) F_R(d^H(z, K^c \cup K_i) - 1)\right]$$

6 The Structure of Boolean DRS models

We have seen that if $\Phi(K)$, $K \in \Sigma(B)$ are indeed the miss probabilities corresponding to some probability law P_X, on $\Sigma(\Sigma(B))$ (i.e. if $\Phi(K) = P_X(X \cap K = \emptyset)$, $\forall K \in \Sigma(B)$), then $P_X(X = K)$ can be recovered for all $K \in \Sigma(B)$, by using the recursion

$$P_X(X = K) = \Gamma_{X,n}(K^c, \{p_1\}, \{p_2\}, \cdots, \{p_n\})$$

$$= \Gamma_{X,n-1}(K^c, \{p_1\}, \{p_2\}, \cdots, \{p_{n-1}\}) -$$

$$\Gamma_{X,n-1}(K^c \cup \{p_n\}, \{p_1\}, \{p_2\}, \cdots, \{p_{n-1}\})$$

with

$$\Gamma_{X,0}(K) = \Phi(K)$$

$$K = \cup_{i=1}^{n} \{p_i\}, \qquad K^c = B - K$$

Alternatively, $P_X(X = K)$ can be recovered using the closed-form expression

$$P_X(X = K) = \sum_{i=0}^{|K|} (-1)^i \sum_{K_i \subseteq K,\ |K_i| = i} \Phi(K^c \cup K_i)$$

In general, given a probability functional, $\Phi(K)$, $K \in \Sigma(B)$, it may be the case that there exists no probability law on $\Sigma(\Sigma(B))$ such that the given probability functional gives the miss probabilities corresponding to this law. In this case, trying to recover a probability law which is consistent with the given miss probabilities is a hopeless pursuit. Therefore, we need to characterize miss probabilities which are consistent in the sense defined below.

Definition 7 *A probability functional, $\Phi(K)$, $K \in \Sigma(B)$, is* **consistent** *if and only if there exists some probability law, P_X, on $\Sigma(\Sigma(B))$, such that the given probability functional gives the miss probabilities corresponding to this law, i.e. if $\Phi(K) = P_X(X \cap K = \emptyset)$, $\forall K \in \Sigma(B)$.*

Therefore, if $\Phi(K)$, $K \in \Sigma(B)$ is consistent, the underlying probability law can be recovered. Conversely, if we blindly use the recursion (or the closed-form expression) and come up with a valid probability law, then $\Phi(K)$, $K \in \Sigma(B)$ is consistent, because the construction of the recursion tree is based on Bayes rule, and, therefore, traversing the tree upwards preserves the miss probabilities. Hence, $\Phi(K)$, $K \in \Sigma(B)$ is consistent if and only if the recursion (or closed-form expression) results in a valid probability law. Finally, by the definition of a DRS as a measurable mapping, $\Phi(K)$, $K \in \Sigma(B)$ is consistent if and only if there exists a DRS X with miss probabilities $Q_X(K) = P_X(X \cap K = \emptyset) = \Phi(K)$, $\forall K \in \Sigma(B)$ (because one such DRS is given by the canonical space construction, i.e. the identity mapping from $(\Sigma(B), \Sigma(\Sigma(B)), P_X)$ to $(\Sigma(B), \Sigma(\Sigma(B)), P_X)$).

Proposition 1 *Let $\{X_i\}_{i=0}^{\infty}$ be a sequence of (not necessarily independent) DRS's, on a bounded subset $B \subset \mathcal{Z}^2$. Then, for any finite N, $\cup_{i=0}^{N} X_i$ is a DRS on B, and, furthermore, the limit $\cup_{i=0}^{\infty} X_i$ always exists, and is a DRS on B.*

Proof: By definition, each X_i is a measurable mapping

$$(\Omega, \Sigma(\Omega), P) \overset{X_i}{\mapsto} (\Sigma(B), \Sigma(\Sigma(B)))$$

Let us consider all possible sample paths. For each $\omega \in \Omega$, $X_i(\omega)$ is some element of $\Sigma(B)$ (i.e. some subset of B). Since $\Sigma(B)$ is a σ-algebra on B, by definition of a σ-algebra, it must contain all finite and countably infinite unions of elements of $\Sigma(B)$. Therefore,

$$\bigcup_{i=0}^{N} X_i(\omega) \in \Sigma(B), \ \forall N < \infty$$

and

$$\bigcup_{i=0}^{\infty} X_i(\omega) \in \Sigma(B)$$

and this is true pointwise for all $\omega \in \Omega$. Therefore, we have established that $\bigcup_{i=0}^{N} X_i$, and $\bigcup_{i=0}^{\infty} X_i$ are mappings from Ω to $\Sigma(B)$. Since the sample space, $\Sigma(B)$, is discrete (countable), we can assume (without loss of generality) that Ω is a discrete space, and that $\Sigma(\Omega)$ is the power set of Ω. In this case *any* mapping from Ω to $\Sigma(B)$ is measurable. Hence, $\bigcup_{i=0}^{N} X_i$, and $\bigcup_{i=0}^{\infty} X_i$ are *measurable mappings* from $(\Omega, \Sigma(\Omega), P)$ to $(\Sigma(B), \Sigma(\Sigma(B)))$.

Proposition 2 *If $\Phi_i(K)$, $K \in \Sigma(B)$ is consistent, for all $i = 1, 2, \cdots, N$, then $\Phi_1(K)\Phi_2(K)\cdots\Phi_N(K)$, $K \in \Sigma(B)$ is also consistent, for any (finite or infinite) $N \in \mathcal{Z}_+^*$.*

Proof: It suffices to show that there exists some DRS Y, with the given miss probabilities, i.e. $Q_Y(K) = P_Y(Y \cap K = \emptyset) = \Phi_1(K)\Phi_2(K)\cdots\Phi_N(K)$, $\forall K \in \Sigma(B)$. Since $\Phi_i(K)$, $K \in \Sigma(B)$ is consistent, there exists DRS X_i, with $Q_{X_i}(K) = P_{X_i}(X_i \cap K = \emptyset) = \Phi_i(K)$, $\forall K \in \Sigma(B)$. Let $X_1, X_2, \cdots, X_N$ be N independent. DRS's, with generating functionals $\Phi_1(K), \Phi_2(K), \cdots$, $\Phi_N(K)$ respectively. Let

$$Y = X_1 \cup X_2 \cup \cdots \cup X_N$$

According to proposition 1, Y is a DRS.

$$\begin{aligned} Q_Y(K) = Pr(Y \cap K = \emptyset) &= Pr(X_1 \cap K = \emptyset, X_2 \cap K = \emptyset, \ldots, X_N \cap K = \emptyset) \\ &= Pr(X_1 \cap K = \emptyset) Pr(X_2 \cap K = \emptyset) \cdots Pr(X_N \cap K = \emptyset) \\ &= \Phi_1(K)\Phi_2(K)\cdots\Phi_N(K) \end{aligned}$$

Therefore, the proposition is proved.

Remark: By Choquet's uniqueness result, all DRS's with generating functional $\Phi_1(K)\Phi_2(K)\cdots\Phi_N(K)$, induce the same measure on the event space, $\Sigma(\Sigma(B))$. Therefore, they are all *equivalent.* According to the proof above, one such DRS, Y, is directly given as the union of N independent DRS's, $X_1, X_2, \cdots, X_N$, with generating functionals $\Phi_1(K), \Phi_2(K), \cdots$, $\Phi_N(K)$ respectively. Modulo this equivalence, Y is unique. From now on, whenever we write DRS $X \sim$ DRS Y, or DRS $X =$ DRS Y, or say that a DRS X is a specific member, Y, of a class of DRS's, we mean that X and Y induce the same measure on $\Sigma(\Sigma(B))$, and, therefore, from a probabilistic viewpoint they are equivalent.

Corollary 1 *If* $\Phi(K)$, $K \in \Sigma(B)$ *is consistent, then* $[\Phi(K)]^N$, $K \in \Sigma(B)$ *is also consistent, for any (finite or infinite)* $N \in \mathcal{Z}_+^*$.

The next proposition will facilitate the exposition of subsequent results. Its proof is simple, and will be ommited.

Proposition 3 *Let* $\{G^{(z)},\ z \in B\}$ *be a sequence of bounded and convex independent DRS's on* $B' \subset B$, $|B'| << |B|$, *given by* $G^{(z)} = R^{(z)}H$, $\forall z \in B$, *where* $\{R^{(z)},\ z \in B\}$ *form an independent sequence of* $\{-1, 0, 1, \ldots, \bar{R}\}$*-valued r.v.'s, with* $R^{(z)}$ *distributed according to the pmf* $\widetilde{f}_{R^{(z)}}(r)$, *and it is understood that* $(-1)H = \emptyset$. *Define*

$$X = \bigcup_{z \in B} G^{(z)} \oplus z \tag{4}$$

Then X *is a* (p, λ_s, H, f_R)*-DRBRS, with*

$$\widetilde{f}_{R^{(z)}}(r) = \begin{cases} (1 - p\lambda_s(z)) & , r = -1 \\ p\lambda_s(z) f_R(r) & , r \in \{0, 1, \ldots, \bar{R}\} \end{cases}$$

Define

$$\widetilde{F}_{R^{(z)}}(r) = \sum_{l=-1}^{r} \widetilde{f}_{R^{(z)}}(l)$$

Then

$$\widetilde{F}_{R^{(z)}}(r) = \begin{cases} (1 - p\lambda_s(z)) & , r = -1 \\ (1 - p\lambda_s(z)) + p\lambda_s(z) F_R(r) & , r \in \{0, 1, \ldots, \bar{R}\} \end{cases}$$

Therefore, the generating functional of a (p, λ_s, H, f_R)-DRBRS X, can be written as

$$Q_X(K) = \prod_{z \in B} \widetilde{F}_{R^{(z)}}(d^H(z, K) - 1) \tag{5}$$

The same result can also be obtained using propositions 2, 3, and the fact that the generating functional of the DRS $G^{(z)} \oplus z$ is

$$Q_{G^{(z)} \oplus z}(K) = \widetilde{F}_{R^{(z)}}(d^H(z, K) - 1) \tag{6}$$

The original constructive definition of a DRBRS X is intuitive, and parallels that of the continous analog of the DRBRS, namely the Boolean RS with radial convex primary grains. On the other hand, it complicates its specification, and makes notation cumbersome. For these reasons, we shall henceforth adopt the alternative specification of proposition 3, and use the notation $(H, \widetilde{f}_R)$-DRBRS.

The Poisson Point Process, and the continous Boolean RS model, share the important property of independent decomposition/superposition. This means that a continous Boolean RS can be decomposed into the union of (possibly infinitely many) independent Boolean RS's, of the same type, and the union of N independent Boolean RS's is a Boolean RS. This infinite decomposition property of the continous Boolean RS model is known as the property of *infinite divisibility.* The class of RS models that posses this property is known as the class of *Infinitely Divisible Random Sets.* The importance of this class stems from practical considerations: Infinitely Divisible Random Sets are good models for many applications, and they exhibit a certain degree of analytical tractability. In a sense, the reasons for adopting the class of Infinitely Divisible Random Sets to model applications, are very much the same as the reasons that led to the proliferation of the use of Poisson Point Processes in queueing theory and communication networks. The key properties are independent decomposition/superposition, memoryless behaviour, and the ability to state and prove asymptotic results.

In light of these observations, it is clear that we should be after some analogous properties for Discrete Boolean Random Set models.

Definition 8 *Let $\{X_1, X_2, \cdots, X_N\}$ be a sequence of N nontrivial independent DRS's. The* **divisibility degree** *of a DRS X is defined as*

$$dg(X) = sup\left\{N \mid X = \bigcup_{i=1}^{N} X_i\right\}$$

Here, nontrivial DRS means a DRS whose generating functional is not identically equal to 1 for all $K \in \Sigma(B)$. Equivalently, by proposition 2, $dg(X)$ can be defined as follows. Let $\Phi_i(K)$, $K \in \Sigma(B)$, $i = 1, 2, \cdots, N$, be a sequence of consistent probability functionals, none of which is identically equal to 1 for all $K \in \Sigma(B)$. Then

$$dg(X) = sup\left\{N \mid Q_X(K) = \prod_{i=1}^{N} \Phi_i(K),\ \forall K \in \Sigma(B)\right\}$$

It is clear that for any DRS X, $dg(X) \geq 1$.

Definition 9 *A DRS X is* **divisible** *if $dg(X) > 1$, and it is* **indivisible** *if $dg(X) = 1$.*

From equations 4, 5, 6, and propositions 2, 3, it is clear that a DRBRS X, on B, is divisible into the union of any possible collection of disjoint and collectively exhausting unions of elementary DRBRS's of the form $G^{(z)} \oplus z$, and that, if for all $z \in B$, $G^{(z)} \oplus z$ is indivisible, the divisibility degree of X is bounded above by $|B|$. We have the following important proposition.

Proposition 4 *Let $\{X_1, X_2, \cdots, X_N\}$ be a sequence of N independent DRBRS's, with*

$$X_i \sim (H, \tilde{f}_R^{(i)}) - DRBRS, \quad i = 1, 2, \cdots, N$$

Then,

$$X = \bigcup_{i=1}^{N} X_i$$

is a $(H, \tilde{f}_R^{max})$-DRBRS, with

$$\tilde{F}_{R^{(z)}}^{max}(r) = \prod_{i=1}^{N} \tilde{F}_{R^{(z)}}^{(i)}(r) \quad \forall z \in B, \ \forall r \in \{-1, 0, 1, \cdots, \bar{R}\}$$

where $\forall z \in B, \ \forall r \in \{-1, 0, 1, \cdots, \bar{R}\}$

$$\tilde{F}_{R^{(z)}}^{max}(r) = \sum_{l=-1}^{r} \tilde{f}_{R^{(z)}}^{max}(l)$$

and

$$\tilde{F}_{R^{(z)}}^{(i)}(r) = \sum_{l=-1}^{r} \tilde{f}_{R^{(z)}}^{(i)}(l), \quad i = 1, 2, \cdots, N$$

Proof: The validity of the proposition can be proven in two ways. The simplest way is to look at the union of the N independent elementary DRS's $R_{(i)}^{(z)} H \oplus \{z\}, \ i = 1, 2, \cdots, N$, corresponding to the pointwise (in z) contribution of the DRBRS's $\{X_i, \ i = 1, 2, \cdots, N\}$, and observe that the resulting DRS, $R_{max}^{(z)} H \oplus \{z\}$, will have radius equal to the maximum of the radii of the contributing DRS's. It is a standard exercise to show that the cdf of the maximum of N r.v.'s is equal to the product of the cdf's of the N r.v.'s. Therefore, the correctness of the proposition follows. Alternatively, by proposition 2, and equation 5, the generating functional of X is given by

$$Q_X(K) = \prod_{z \in B} \tilde{F}_{R^{(z)}}^{(1)}(d^H(z, K) - 1) \cdots \tilde{F}_{R^{(z)}}^{(N)}(d^H(z, K) - 1)$$

for all $K \in \Sigma(B)$. The validity of the proposition is now a direct consequence of Choquet's uniqueness theorem, the above result, and equation 5.

According to the above results, the DRBRS model posseses the desirable independent decomposition/superposition properties. A more general model, the Discrete Boolean RS, enjoys the same properties.

Definition 10 *Let $\{G^{(z)},\ z \in B\}$ be a sequence of bounded and independent DRS's on $B' \subset B$, $|B'| << |B|$, with $G^{(z)}$ characterized by the generating functional $Q_{G^{(z)}}(K)$. Define*

$$X = \bigcup_{z \in B} G^{(z)} \oplus z \tag{7}$$

Then X will be called a **Discrete Boolean RS (DBRS)**, *and it will be denoted by $\{Q_{G^{(z)}}(K),\ z \in B,\ K \in \Sigma(B')\}$-DBRS.*

Based on proposition 2, the generating functional of a $\{Q_{G^{(z)}}(K),\ z \in B,\ K \in \Sigma(B')\}$-DBRS X, on B, can be shown to be

$$Q_X(K) = \prod_{z \in B} Q_{G^{(z)} \oplus z}(K) = \prod_{z \in B} Q_{G^{(z)}}(K \oplus \{-z\}) \tag{8}$$

for all $K \in \Sigma(B)$. It is clear that a DBRS X, on B, is divisible into the union of any possible collection of disjoint and collectively exhausting unions of elementary DBRS's of the form $G^{(z)} \oplus z$, and that, if for all $z \in B$, $G^{(z)} \oplus z$ is indivisible, the divisibility degree of X is bounded above by $|B|$. This is in contrast with the continous domain case, where it is known that a Boolean RS is infinitely divisible [6]. The following is a generalization of proposition 4.

Proposition 5 *Let $\{X_1, X_2, \cdots, X_N\}$ be a sequence of N independent DBRS's, with*

$$X_i \sim \{Q^{(i)}_{G^{(z)}}(K),\ z \in B,\ K \in \Sigma(B')\} - DBRS, \quad i = 1, 2, \cdots, N$$

Then,

$$X = \bigcup_{i=1}^{N} X_i$$

is a $\{Q_{G^{(z)}}(K),\ z \in B,\ K \in \Sigma(B')\}$-DBRS, with

$$Q_{G^{(z)}}(K) = \prod_{i=1}^{N} Q^{(i)}_{G^{(z)}}(K), \quad \forall z \in B,\ \forall K \in \Sigma(B')$$

Proof: By proposition 2, and equation 8, the generating functional of X is given by

$$Q_X(K) = \prod_{z \in B} \prod_{i=1}^{N} Q^{(i)}_{G(z)}(K), \quad \forall K \in \Sigma(B)$$

The validity of the proposition is now a direct consequence of Choquet's uniqueness theorem, the above result, proposition 2, and equation 8.

7 Randomized Superpositions of DRS's

We have seen that the generating functional plays an important role in Random Set theory, and an even more important role in Discrete Random Set theory. In effect, the generating functional can be used as a modeling tool, which allows for the design of models which are consistent with some underlying event-space probability law. In this section we elaborate on this approach, and show that one can specify complex DRS models, by using a simple randomization process, and the (known) generating functional of the Boolean DRS model, or other simple models. Furthermore, some of the resulting DRS models possess the desirable properties of independent decomposition (divisibility) and superposition.

Proposition 6 *If $\Phi(K)$, $K \in \Sigma(B)$ is consistent, then $(1-p+p\Phi(K))^N$, $K \in \Sigma(B)$ is also consistent, for all $p \in (0,1)$ and all finite $N \in \mathcal{Z}^*_+$.*

Proof: Since $\Phi(K)$, $K \in \Sigma(B)$ is consistent, there exists a DRS, X, with generating functional $Q_X(K) = P_X(X \cap K = \emptyset) = \Phi(K)$, $\forall K \in \Sigma(B)$. Let M be a Binomial r.v. with parameters $p \in (0,1)$, $N \in \mathcal{Z}^*_+$, $N < \infty$, i.e.

$$Pr(M = m) = \binom{N}{m} p^m (1-p)^{N-m}, \quad m = 0, 1, 2, \ldots, N$$

Let $X_1, X_2, \cdots, X_M$ be M i.i.d. and independent of M DRS's, each with generating functional $Q_X(K) = \Phi(K)$, $\forall K \in \Sigma(B)$. Consider the DRS

$$Y = X_1 \cup X_2 \cup \cdots \cup X_M$$

$$Q_Y(K) = P_Y(Y \cap K = \emptyset) = Pr(X_1 \cap K = \emptyset, \ldots, X_M \cap K = \emptyset)$$
$$= \sum_{m=0}^{N} Pr(X_1 \cap K = \emptyset, \ldots, X_M \cap K = \emptyset,\ M = m)$$

$$= \sum_{m=0}^{N} Pr(X_1 \cap K = \emptyset, \ldots, X_M \cap K = \emptyset \mid M = m) Pr(M = m)$$

$$= \sum_{m=0}^{N} Pr(X_1 \cap K = \emptyset, \ldots, X_m \cap K = \emptyset) Pr(M = m)$$

$$= \sum_{m=0}^{N} [Pr(X_1 \cap K = \emptyset)]^m \, Pr(M = m)$$

$$= \sum_{m=0}^{N} [Q_X(K)]^m \, Pr(M = m)$$

$$= \sum_{m=0}^{N} [Q_X(K)]^m \binom{N}{m} p^m (1-p)^{N-m}$$

$$= (1-p)^N \sum_{m=0}^{N} \binom{N}{m} \left[\frac{p}{1-p} Q_X(K) \right]^m$$

From the Binomial summation formula

$$\sum_{m=0}^{N} \binom{N}{m} p^m (1-p)^{N-m} = 1$$

it is easy to show that, for any $\lambda \in (0, \infty)$, the following holds

$$\sum_{m=0}^{N} \binom{N}{m} \lambda^m = (1 + \lambda)^N$$

Therefore,

$$Q_Y(K) = (1-p)^N \left(1 + \frac{p}{1-p} Q_X(K) \right)^N$$

$$= (1 - p + pQ_X(K))^N = (1 - p + p\Phi(K))^N$$

Therefore, there exists a DRS with the given miss probabilities, and thus $(1 - p + p\Phi(K))^N$ is consistent.

The construction above corresponds to the following experiment. Each independent DRS component is independently included in the union with probability p. Therefore, the construction involves a sequence of independent Bernoulli trials that determine the inclusion of the component DRS's.

We will call this sequence the *inclusion sequence.* Since the inclusion sequence is a stochastic sequence, we will call this and subsequent constructions *randomized superpositions.* It is important to note that finite-length randomized superpositions are realizable, given that we have the means to realize the component DRS's.

A DRS, Y, with generating functional $(1-p+p\Phi(K))^N$, $K \in \Sigma(B)$, where $\Phi(K)$, $K \in \Sigma(B)$ is the generating functional of a DRS X, $p \in (0,1)$, and $N \in \mathcal{Z}_+^*$, is clearly divisible into the union of $m \leq N$ independent DRS's, of the same type as Y, but with $N = N_i$, $i = 1,2,\ldots,m$, such that $\sum_{i=1}^m N_i = N$. If X is indivisible, then $dg(Y) = N$. Furthermore (see proposition 2), if $\{Y_i\}_{i=1}^m$ is a sequence of independent DRS's, with generating functionals $(1-p+p\Phi(K))^{N_i}$, $K \in \Sigma(B)$, where $\Phi(K)$, $K \in \Sigma(B)$ is the generating functional of a DRS X, $p \in (0,1)$, and $N_i \in \mathcal{Z}_+^*$, $i = 1,2,\ldots,m$, then the DRS $Y = \cup_{i=1}^m Y_i$, has generating functional $(1-p+p\Phi(K))^N$, $K \in \Sigma(B)$, where $N = \sum_{i=1}^m N_i$, and, therefore, it is of the same type as its component DRS's. Hence, the class of DRS's with generating functional of the above type is closed under the operation of independent superposition.

It is interesting to consider what happens to the above construction when the length of the inclusion sequence goes to infinity, i.e. when $N \to \infty$. Suppose that $N \to \infty$ and $p \to 0$, in such a way that $Np \to \lambda$, where $\lambda \in (0,\infty)$. Then, in the limit, the Binomial pmf becomes the Poisson pmf, with parameter λ. We have the following proposition.

Proposition 7 *If $\Phi(K)$, $K \in \Sigma(B)$ is consistent, then $e^{-\lambda(1-\Phi(K))}$, $K \in \Sigma(B)$ is also consistent, for all $0 < \lambda < \infty$.*

Proof: Since $\Phi(K)$, $K \in \Sigma(B)$ is consistent, there exists a DRS, X, with generating functional $Q_X(K) = P_X(X \cap K = \emptyset) = \Phi(K)$, $\forall K \in \Sigma(B)$. Let M be a Poisson r.v. with parameter λ. Let $X_1, X_2, \cdots, X_M$ be M i.i.d. and independent of M DRS's, each with generating functional $Q_X(K) = \Phi(K)$, $\forall K \in \Sigma(B)$. Consider the DRS

$$Y = X_1 \cup X_2 \cup \cdots \cup X_M$$

$$Q_Y(K) = P_Y(Y \cap K = \emptyset) = Pr(X_1 \cap K = \emptyset, \ldots, X_M \cap K = \emptyset)$$

$$= \sum_{m=0}^{\infty} Pr(X_1 \cap K = \emptyset, \ldots, X_M \cap K = \emptyset,\ M = m)$$

$$= \sum_{m=0}^{\infty} Pr(X_1 \cap K = \emptyset, \ldots, X_M \cap K = \emptyset \mid M = m) Pr(M = m)$$

$$= \sum_{m=0}^{\infty} Pr(X_1 \cap K = \emptyset, \ldots, X_m \cap K = \emptyset) Pr(M = m)$$

$$= \sum_{m=0}^{\infty} [Pr(X_1 \cap K = \emptyset)]^m \, Pr(M = m)$$

$$= \sum_{m=0}^{\infty} [Q_X(K)]^m \frac{\lambda^m}{m!} e^{-\lambda} = e^{-\lambda} \sum_{m=0}^{\infty} \frac{[\lambda Q_X(K)]^m}{m!}$$

$$= e^{-\lambda} e^{\lambda Q_X(K)} = e^{-\lambda(1-Q_X(K))} = e^{-\lambda(1-\Phi(K))}$$

Therefore, there exists a DRS with the given miss probabilities, and thus $e^{-\lambda(1-\Phi(K))}$ is consistent.

If a DRS, Y, has generating functional $e^{-\lambda(1-\Phi(K))}$, $K \in \Sigma(B)$, where $\Phi(K)$, $K \in \Sigma(B)$ is consistent, and $0 < \lambda < \infty$, then, for *any* positive integer N, Y is equivalent to the union of N i.i.d. DRS's, of the same type as Y, but with parameter $\lambda(N) = \lambda/N$. Therefore, $dg(Y) = \infty$, i.e. Y is infinitely divisible. Furthermore (see proposition 2), if $\{Y_i\}_{i=1}^{m}$ is a sequence of independent DRS's, with generating functionals $e^{-\lambda_i(1-\Phi(K))}$, $K \in \Sigma(B)$, where $\Phi(K)$, $K \in \Sigma(B)$ is consistent, $0 < \lambda_i < \infty$, $i = 1, 2, \ldots, m$, then the DRS $Y = \cup_{i=1}^{m} Y_i$, has generating functional $e^{-\lambda(1-\Phi(K))}$, $K \in \Sigma(B)$, where $\lambda = \sum_{i=1}^{m} \lambda_i$, and, therefore, it is of the same type as its component DRS's. Hence, the class of DRS's with generating functional of the above type is closed under the operation of independent superposition.

By the same token, if the number of inclusions (units), M, in the inclusion process is modeled by the state of a continous parameter birth-death process, with constant birth and death rates, evolving since the beginning of time ($t = -\infty$), then M is distributed according to the modified geometric pmf, and the following result is obtained.

Proposition 8 *If* $\Phi(K)$, $K \in \Sigma(B)$ *is consistent, then*

$$\frac{1-p}{1-p\Phi(K)}, \quad K \in \Sigma(B)$$

is also consistent, for all $p \in (0,1)$.

Proof: Again, since $\Phi(K)$, $K \in \Sigma(B)$ is consistent, there exists a DRS, X, with generating functional $Q_X(K) = P_X(X \cap K = \emptyset) = \Phi(K)$, $\forall K \in \Sigma(B)$. Let M be a r.v. distributed according to the modified geometric pmf, with parameter $p \in (0,1)$, i.e.

$$Pr(M = m) = (1-p)p^m, \quad m = 0, 1, 2, \ldots$$

Let $X_1, X_2, \cdots, X_M$ be M i.i.d. and independent of M DRS's, each with generating functional $Q_X(K) = \Phi(K)$, $\forall K \in \Sigma(B)$. Consider the DRS

$$Y = X_1 \cup X_2 \cup \cdots \cup X_M$$

As before,

$$\begin{aligned} Q_Y(K) &= \sum_{m=0}^{\infty} [\Phi(K)]^m \, Pr(M = m) \\ &= \sum_{m=0}^{\infty} [\Phi(K)]^m \, (1-p)p^m \\ &= (1-p) \sum_{m=0}^{\infty} [p\Phi(K)]^m \\ &= \frac{1-p}{1-p\Phi(K)} \end{aligned}$$

Therefore, there exists a DRS with the given miss probabilities, and thus

$$\frac{1-p}{1-p\Phi(K)}, \quad K \in \Sigma(B)$$

is consistent.

Generalizing (see proposition 1), randomized superposition *always* leads to consistent probability functionals, regardless of the specific pmf of M. At this point, some remarks on its utility are in place.

In applications, a systems designer who attempts to model the discretized observations of a physical two-dimensional process as a DRS, has to cope with a severe handicap: only a limited number of DRS models are completely specified, and even less are well understood. One of the few exceptions is the DBRS model. It is interesting to note that if $\Phi(K)$ is the generating functional of the DBRS model, then none of the randomized superpositions above results in a generating functional which can be written in the product form that characterizes DBRS models. Therefore, no DBRS exists which is equivalent to the DRS constructed using any of the above randomized superpositions. Hence, randomized superposition can be used to generate new classes of DRS models, based on simple models such as the DBRS model. Furthermore, the resulting DRS's are completely specified and amenable to analysis, since their generating functional is known in advance, and, therefore, the induced event-space probability law corresponding to any one of them can be found.

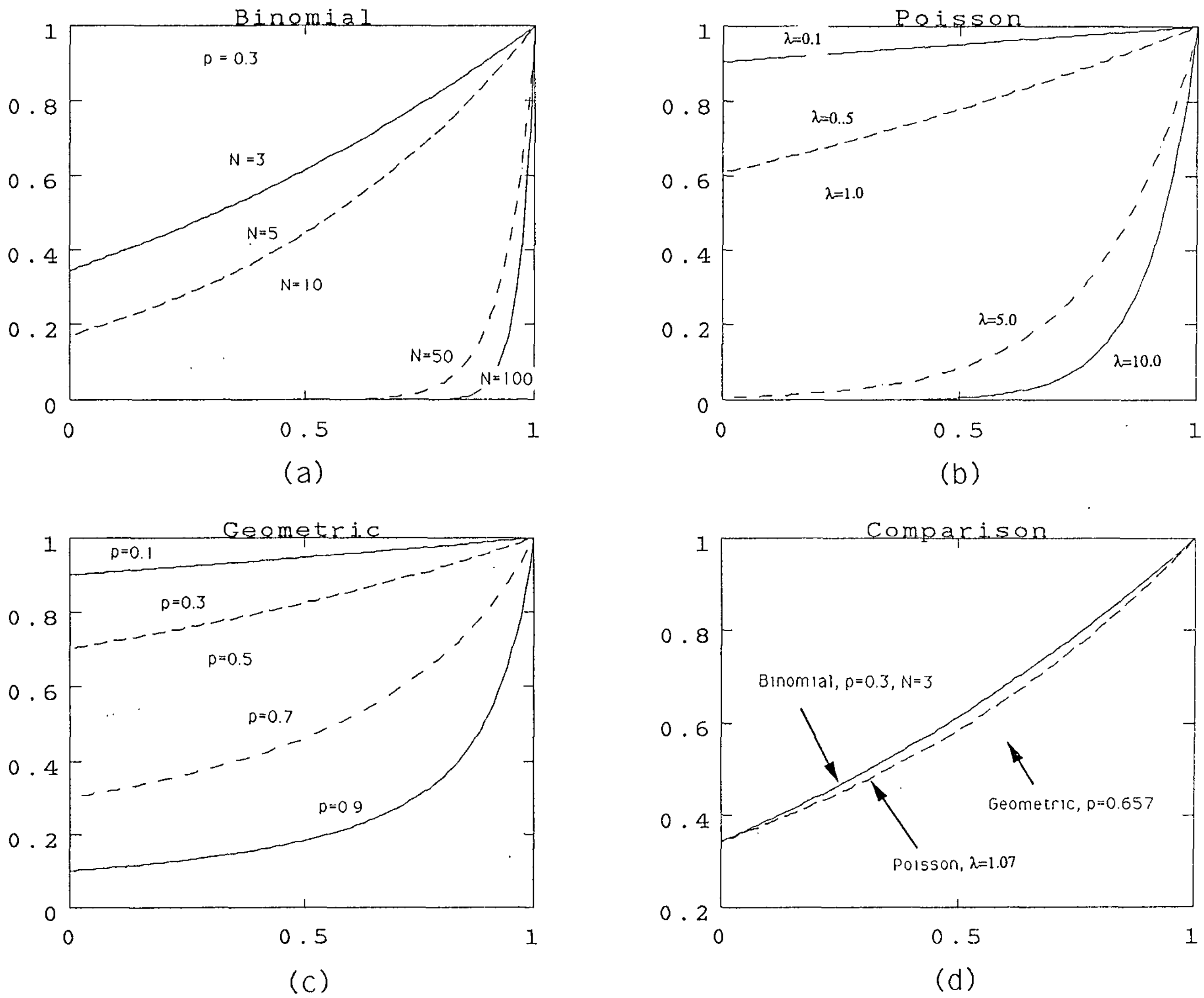

Figure 5: Randomized superposition corresponds to a nonlinear deformation of the generating functional of the component DRS's

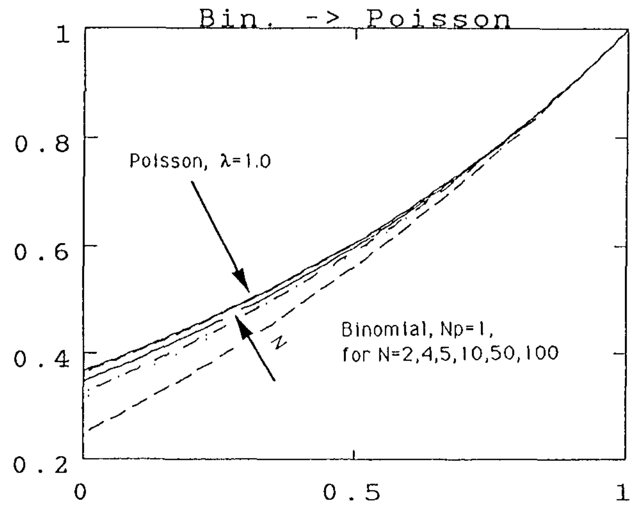

Figure 6: Convergence of Binomial deformation

Randomized superposition corresponds to a nonlinear deformation of the generating functional of the component DRS's, as it can be clearly seen in figure 5. Figure 5 (a) presents a plot of the values of the generating functional of the DRS result of randomized superposition, when M is Binomialy distributed with parameters N, p, versus the values of the generating functional of the component DRS's, for $p = 0.3$, and for various values of N. Figure 5 (b) presents the same plot, when M is distributed according to the Poisson pmf with parameter λ, for various values of λ. Similarly, figure 5 (c) corresponds to the case where M is distributed according to the geometric pmf with parameter p, for various possible values of p. Figure 5 (d) compares the three deformations, for one particular choice of the parameters which forces the left endpoints to coincide. Figure 6 presents a plot of the values of the generating functional of the DRS result of randomized superposition, when M is Binomialy distributed with parameters N, p, versus the values of the generating functional of the component DRS's, when the product Np is fixed to unity, and for various values of N. The generating functional of the DRS result of randomized superposition, when M is distributed according to the Poisson pmf with parameter $\lambda = 1$, is also plotted in the same figure. Clearly, when N goes to infinity, the family of generating functionals corresponding to the Binomial pmf tends to the generating functional corresponding to the Poisson pmf. In fact, convergence seems to be relatively fast. We have the following proposition.

Proposition 9 *Let Y_N be the DRS result of randomized superposition, with Binomialy distributed number of components, M, with parameters N, p such that $Np = \lambda$, $0 < \lambda < \infty$, λ = fixed, $\forall N \in \mathcal{Z}_+^*$. Then Y_N converges in* *distribution to Y, where Y is the DRS result of randomized superposition, with Poisson distributed number of components, M, with parameter λ.*

Proof: As $N \longrightarrow \infty$, the Binomial pmf with parameters N, p such that $Np = \lambda$, converges to the Poisson pmf with parameter λ. Therefore (see the proofs of propositions 7, 6), the generating functional of Y_N converges to the generating functional of Y. Since, by Choquet's result, the generating functional uniquely determines the probability law on $\Sigma(\Sigma(B))$, the result follows.

Proposition 9 provides us with a means to approximate infinitely divisible DRS's by using finitely divisible DRS's, *which can be realized* using randomized superpositions of finite extent.

8 Conclusions and future work

We have attempted to demonstrate the power of an axiomatic approach to Discrete Random Set theory. This power stems from the ability to recover the measure on the appropriate subset of the event space, given knowledge of the generating functional, and the possibility of estimating the generating functional from real life observations (although such an estimation procedure seems plausible, no techniques have been developed so far towards this end; therefore, the utility of this approach remains to be proven). Taken together, these imply the applicability of standard nonparametric techniques to statistical inference problems for the case of Discrete Random Sets [13].

One way to break the complexity barrier would be to extend the range and diversity of available Discrete Random Set models. This would give a systems designer additional freedom and flexibility in choosing a model for a specific application. This, in effect, should make parametric techniques much more attractive [21, 22]

It would be interesting to relax the independence requirements imposed on the randomized superposition construction, and consider what happens when the component sequence is Markovian, and/or the components depend on the length of the sequence. This seems difficult at this point. Research on these and other ideas is currently underway.

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