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On the Closed-Loop Stability of Constrained QDMC

by E. Zafiriou

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Abstract

The presence of constraints in the on-line optimization problem solved by Model Predictive Control algorithms results in a nonlinear control system, even if the plant and model dynamics are linear. This is the case both for physical constraints, like saturation constraints, as well for performance or safety constraints on outputs or other variables of the process. This paper discusses how constraints affect the stability properties of the closed-loop nonlinear system. In particular we concentrate on presenting a formulation that allows one to relate hard as well as soft constraints to stability. The degree of softening can be determined to guarantee stability.

1 Introduction

Model Predictive Control (MPC) encopasses a large class of process control algorithms sharing the common characteristic of explicitly using a model of the process to predict future behavior and take control action by optimizing some performance objective. A performance measure made popular because of its simplicity and its successful use in industrial applications is a quadratic objective function that includes the predicted deviation from desired setpoint values over a future horizon. In the Quadratic Dynamic Matrix Control (QDMC) formulation (Garcia and Morshedi, 1986), the objective function also includes a penalty term on excessive control moves and its minimization is carried out on-line at each sampling point, subject to satisfaction of hard constraints on several process variables.

The great attraction of QDMC is that the straightforward formulation of an optimization problem will result in satisfaction of the control specifications. Saturation constraints on the manipulated variables, as well as performance and safety constraints on outputs and other state variables can be taken care of by simply listing them and minimizing the quadratic objective function subject to their satisfaction. When the model used for prediction is linear, the on-line optimization is a Quadratic Program (QP), for which efficient algorithms exist, especially if the similarity of the optimization problems that are solved at successive sampling points is taken into account (Ricker, 1985). Formulations that use nonlinear models for prediction have also been developed. In this case, the on-line optimization is a Nonlinear Program, which with appropriate mathematical techniques and / or approximations can be transformed into a series of QPs (Li and Biegler, 1989; Peterson et al., 1990). Eaton and Rawlings (1990) also consider the parametric sensitivity of the optimal solution. An industrial application of QDMC that utilizes a nonlinear model is described in Garcia (1984).

There are, however, certain issues that make the use of QDMC more complex than it is apparent. The on-line optimization solves an open-loop control problem, given the information available up to that point. The control action that is calculated at a sampling point is optimal only if the sequence of control moves found by the optimization is implemented uninterrupted. This will not happen, though, because a new optimization problem will be solved at the next sampling point utilizing in the prediction the newly acquired information from the measurements. The fact that QDMC is implemented as a closed-loop control system is not incorporated in the on-line optimization. Closed-loop stability cannot be assumed simply because the on-line optimization finds a solution. This issue of closed-loop stability is complicated by two facts: first, there is always uncertainty associated with the model used in the prediction; second, the presence of constraints in the optimization problem results in a nonlinear closed-loop system even if the

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model and plant dynamics are assumed linear. In the unconstrained case, robust linear control theory can be used to study robustness with respect to modeling error (see, e.g., Prett and Garcia, 1988). For the constrained case, Zafiriou (1989) suggested a framework that allows the translation of the robust stability of the constrained, and therefore nonlinear, closed-loop system into robustness conditions for a set of linear systems.

This paper mainly looks at the effect of output constraints on the closed-loop stability of QDMC. The ability to include output constraints in the on-line optimization distinguishes QDMC from other efficient methods that deal with constraints on the manipulated variables only (e.g., Campo and Morari, 1990). Zafiriou and Marchal (1991) showed in detail how output constraints can result in very aggressive controllers. Ricker *et al.*(1989) suggested that softening such constraints may help avoid these problems. Since not all constraints can be softened, as is the case, e.g., for saturation constraints, one needs a framework that can deal with a mix of hard and soft constraints. This is accomplished in this paper by extending the framework discussed in Zafiriou (1990) to include the effect of softening on closed-loop stability.

2 Closed-Loop Stability

We focus on Quadratic Dynamic Matrix Control, which is a popular MPC algorithm, extensively used in the industry (Prett and Garcia, 1988). An impulse response model is used (see, e.g., Garcia and Morari, 1982):

$$\tilde{y}(k) = H_1 u(k-1) + H_2 u(k-2) + ... + H_N u(k-N) \quad (1)$$

where $\tilde{y}$ is the model output vector, u is the input vector and N the truncation number, i.e., it is assumed that $H_i = 0$ for $i > N$. The plant is assumed to be open-loop stable, but it may be non-square. Other types of models can also be used, e.g., step response models (Garcia and Morshedi, 1986) or state space descriptions (Li *et al.*, 1989; Ricker, 1990). The z-transfer function, $\tilde{p}^*(z)$, describing the process model is related to (1) through

$$\tilde{p}^*(z) = \sum_{i=1}^{N} H_i z^{-i} \quad (2)$$

At sampling point k, QDMC minimizes:

$$\min_{\Delta u(k),...,\Delta u(k+M-1)} \sum_{l=1}^{P} [e(k+l)^T \Gamma^2 e(k+l)$$

$$+ u(k+l-1)^T B^2 u(k+l-1)$$

$$+ \Delta u(k+l-1)^T D^2 \Delta u(k+l-1)] \quad (3)$$

The minimization of the objective function is carried out over the values of $\Delta u(k)$, $\Delta u(k+1)$,..., $\Delta u(k+M-1)$, where M is a specified parameter. The minimization is subject to possible hard constraints on the inputs u, their rate of change Δu, the outputs y and other process variables usually referred to as associated variables. The details on the formulation of the optimization problem can be found in Prett and Garcia (1988). Note that the standard form of QDMC does not include the term corresponding to a penalty on u, i.e, $B = 0$. We have incuded the case $B \neq 0$ in this paper in order to also cover the extension studied in Garcia and Morari (1982). After the problem is solved on-line at k, only the optimal value for the first input vector $\Delta u(k)$ is implemented and the problem is solved again at $k+1$. The optimal $u(k)$ depends on the tuning parameters of the optimization problem, the current output measurement $y(k)$ and the past inputs $u(k-1)$,..., $u(k-N)$ that are involved in the model output prediction. Let f describe the $u(k)$ that is obtained by adding $u(k-1)$ to the $\Delta u(k)$ that is the result of the optimization solved at sampling point k :

$$u(k) = f(y(k), u(k-1), \ldots, u(k-N); r_P(k), d(k))$$
$$(4)$$

where $r_P(k)$ includes all the values of the reference signal (setpoint) during the prediction horizon from $k+1$ to $k+P$ and $d(k)$ is the disturbance effect at the output at k.

The optimization problem of the QDMC algorithm can be written as a standard Quadratic Programming problem:

$$\min_{v} q(v) = \frac{1}{2} v^T G v + g^T v \quad (5)$$

subject to

$$A^T v \geq b \quad (6)$$

where

$$v = [\ \Delta u(k) \quad \ldots \quad \Delta u(k+M-1) \ ]^T \quad (7)$$

and the matrices G, A, and vectors g, b are functions of the tuning parameters (weights, horizon P, M, some of the hard constraints). The vectors g, b are also linear functions of $y(k)$, $u(k-1)$,..., $u(k-N)$. Efficient algorithms exist for solving this optimization, especially if the similarity between the problems solved at successive sampling points is taken into account (Ricker, 1985). For the optimal solution v^* we have (Fletcher, 1981):

$$\begin{bmatrix} G & -\hat{A} \\ -\hat{A}^T & 0 \end{bmatrix} \begin{bmatrix} v^* \\ \lambda^* \end{bmatrix} = - \begin{bmatrix} g \\ \hat{b} \end{bmatrix} \quad (8)$$

where $\hat{A}^T$, $\hat{b}$ consist of the rows of A^T, b that correspond to the constraints that are active at the optimum and λ^* is the vector of the Lagrange multipliers corresponding to these constraints. The optimal $\Delta u(k)$ corresponds to the first m elements of the v^* that solves (8), where m is the dimension of u.

The special form of the LHS matrix in (8) allows the numerically efficient computation of its inverse in a partitioned form, as discussed in detail in Fletcher (1981):

$$\begin{bmatrix} G & -\hat{A} \\ -\hat{A}^T & 0 \end{bmatrix}^{-1} = \begin{bmatrix} H & -T \\ -T^T & U \end{bmatrix} \quad (9)$$

Then

$$v^* = -Hg + T\hat{b} \quad (10)$$

$$\lambda^* = T^T g - U\hat{b} \quad (11)$$

and

$$u(k) = u(k-1) + \underbrace{\begin{bmatrix} I & 0 & \ldots & 0 \end{bmatrix}}_{M} v^*$$

$$\triangleq f(y(k), u(k-1), \ldots, u(k-N), r_P(k)) \quad (12)$$

For linear model dynamics, Zafiriou (1990) showed that the *constrained* QDMC is *piece-wise linear*, meaning that the dynamics of QDMC for a certain constraint set J_i active, are those of a discrete linear controller. This linear controller, denoted $c_{J_i}(z)$, depends explicitly only on J_i; it depends only implicitly on the past and current values of the plant inputs and outputs. These values together with external inputs (setpoints, disturbances) determine the J_i that corresponds to a sampling point. However, if at different sampling points the QP solution results in the same J_i, the QDMC dynamics at those points are those of the *same* linear controller.

Define:

$$x_j(k) \triangleq u(k-j), \quad 1 \le j \le N \quad (13)$$

and let f describe the QP solution given by (12) for a particular J_i (which determines the rows of A^T, b that are included in $\hat{A}^T$, $\hat{b}$):

$$f(y(k), u(k-1), \ldots, u(k-N); r_P(k), d(k))_{J_i}$$

$$\triangleq [I \; 0 \; \ldots \; 0] v^*_{J_i} + u(k-1) \quad (14)$$

Zafiriou (1990) showed that the c_{J_i} controller is given by:

$$c_{J_i}(z) \triangleq$$

$$-[I - (\nabla_{x_1} f)_{J_i} z^{-1} - \ldots - (\nabla_{x_N} f)_{J_i} z^{-N}]^{-1} (\nabla_y f)_{J_i} \quad (15)$$

Let us define the "state" of the system as $x(k) \triangleq [x_1(k), \ldots, x_N(k)]^T$ where the x_js are those of (13). Knowledge of $x(k)$ and of the external inputs $r_P(k)$, $d(k)$, allows the computation of $x(k+1)$ by applying the plant and controller equations on it. Let us denote by F the operator that maps $x(k)$ to $x(k+1)$:

$$x(k+1) = F(x(k); r_P(k), d(k)) \quad (16)$$

Define the transfer functions:

$$Q_{J_i}(z) \triangleq -[I - (\psi_1)_{J_i} z^{-1} - \ldots - (\psi_N)_{J_i} z^{-N}]^{-1} (\nabla_y f)_{J_i} \quad (17)$$

where

$$(\psi_j)_{J_i} \triangleq (\nabla_{x_j} f)_{J_i} + (\nabla_y f)_{J_i} H_j, \quad 1 \le j \le N \quad (18)$$

Then the following theorem holds:

Theorem 1 (Zafiriou, 1990) *F can be a contraction for all plants in a set Π, only if all feedback controllers $c_{J_i}(z)$, $i \ni (\nabla_y f)_{J_i} \ne 0$, stabilize all unconstrained plants in the set Π and all transfer functions $Q_{J_i}(z)$, $i \ni (\nabla_y f)_{J_i} = 0$, are stable.*

Theorem 1 allows one to handle any set Π that robust linear control theory can (for a discussion of common types of Π see Morari and Zafiriou, 1989). One should note that F being a contraction implies stability of the closed-loop nonlinear system. However, a violation of the necessary contraction condition does not always imply instability, but it must be considered as a warning that the control parameters should be modified. A sufficient condition (Zafiriou, 1990) can be derived but it is often conservative.

3 Effect of Constraint Softening on Stability

The issue examined in this section is whether it is possible to stabilize a closed-loop unstable QDMC algorithm by softening some or all hard constraints. Let us consider the case where a particular c_{J_i} results in an unstable control system if applied to the process. Let us then proceed and soften the hard constraints included in this set J_i. How will that affect the parameters, and therefore the stability, of c_{J_i}?

The set J_i corresponds to a constraint matrix $\hat{A}$ in (6). These constraints are softened by changing them to

$$\hat{A}^T v + I\epsilon \ge \hat{b} \quad (19)$$

and adding a term $W^2 \epsilon^2$ in the objective function, where W is a weighting matrix and ϵ a vector of non-

negative variables corresponding to the constraint violations. Let us look at a situation where at the optimal of the on-line optimization all of the constraints in J_i had to be softened, i.e., all the elements of the optimal ϵ are nonzero and (19) satisfied as an equality. In this case, the system of equations (8) is expanded to:

$$\begin{bmatrix} G & 0 & -\hat{A} \\ 0 & W^2 & -I \\ -\hat{A}^T & -I & 0 \end{bmatrix} \begin{bmatrix} v^* \\ \epsilon^* \\ \lambda^* \end{bmatrix} = - \begin{bmatrix} g \\ 0 \\ \hat{b} \end{bmatrix} \quad (20)$$

Solving for ϵ^* results in

$$\begin{bmatrix} G & -\hat{A} \\ -\hat{A}^T & -W^{-2} \end{bmatrix} \begin{bmatrix} v^* \\ \lambda^* \end{bmatrix} = - \begin{bmatrix} g \\ \hat{b} \end{bmatrix} \quad (21)$$

The effect that softening has on the behavior of the control system can be examined by comparing the solution of (21) to that of (8) and through them the effect on the corresponding c_{J_i}, by using (14) and (15). We can write:

$$\begin{bmatrix} G & -\hat{A} \\ -\hat{A}^T & -W^{-2} \end{bmatrix} = \begin{bmatrix} G & -\hat{A} \\ -\hat{A}^T & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ I \end{bmatrix} (-W^{-2}) \begin{bmatrix} 0 & I \end{bmatrix} \quad (22)$$

The following Lemma (Horn and Johnson, 1988; p. 18) can be used to proceed:

Lemma 1 *Let A_1, A_2, R, X, Y, be matrices of dimensions $n \times n$, $n \times n$, $r \times r$, $n \times r$, $r \times n$, respectively, where $n \geq r$. If $A_2 = A_1 + XRY$, and A_1, A_2, R are nonsingular, then:*

$$A_2^{-1} = A_1^{-1} - A_1^{-1}X(R^{-1} + YA_1^{-1}X)^{-1}YA_1^{-1}$$

Use of Lem. 1 on (22) and some algebra result in:

$$\begin{bmatrix} G & -\hat{A} \\ -\hat{A}^T & -W^{-2} \end{bmatrix}^{-1} = \begin{bmatrix} H & -T \\ -T^T & U \end{bmatrix}$$
$$\begin{bmatrix} I & 0 \\ (-W^2 + U)^{-1}T^T & I - (-W^2 + U)^{-1}U \end{bmatrix} \quad (23)$$

where H, T, and U are defined in (9). From (21), (23) it follows that:

$$v_s^* = -H_s g + T_s b \quad (24)$$

where
$$H_s = H - T(-W^2 + U)^{-1}T^T \quad (25)$$
$$T_s = T - T(-W^2 + U)^{-1}U \quad (26)$$

and the subscript s indicates that the constraints have been softened.

The effect of the softening on closed-loop stability can be evaluated by using H_s and T_s in (12) instead of H, T. This affects the coefficients in (15) and stability through Thm. 1. The limiting situations are relatively simple to see. For $W \to \infty$, (25), (26) yield $H_s = H$ and $T_s = T$, which means that the constraints become hard. For $W = 0$ we easily get $T_s = 0$. By substitung in (25) the expressions for H, T, U, given in Fletcher (1981) and doing some algebra we obtain $H_s = G^{-1}$, which corresponds to the unconstrained case.

In the case, where at the optimum some of the constraints that are active at the optimum (set J_i) are hard and some constraints have been softened by allowing nonzero ϵ-violations, we can proceed in a similar manner. Assume, without loss of generality, that the rows of $\hat{A}$ have been reordered so that

$$\hat{A}^T = \begin{bmatrix} \hat{A}_1^T \\ \hat{A}_2^T \end{bmatrix}$$

where $\hat{A}_1^T$ corresponds to the hard constraints and $\hat{A}_2^T$ to the softened. Then, after solving for ϵ, we find that the optimal v satisfies:

$$\begin{bmatrix} G & -\hat{A}_1 & -\hat{A}_2 \\ -\hat{A}_1^T & 0 & 0 \\ -\hat{A}_2^T & 0 & -W^2 \end{bmatrix} \begin{bmatrix} v^* \\ \lambda_1^* \\ \lambda_2^* \end{bmatrix} = - \begin{bmatrix} g \\ \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} \quad (27)$$

We can then compute the effect of softening on closed-loop stability as in the previous case.

4   Special Case

In this section we use the simple case of a SISO process with an output constraint to demonstrate the effect of softening on closed-loop stability. The simplicity of this case allows us to obtain analytic expressions for the sufficient condition developed in Zafiriou (1989).

In the formulation here, the same violation variable $\epsilon \geq 0$ is used for all the points in the constraint window. Hence the output constraints are softened to be:

$$y_L - \epsilon \leq y(k + l) \leq y_U + \epsilon, \quad w_b \leq l \leq w_e \quad (28)$$

The term $W^2\epsilon^2$ is added to the objective function, where W is the weight that determines the extent of softening. For $W = \infty$ we get hard constraints. $W = 0$ corresponds to completely removing the constraints. For a nonzero finite W, and when the on-line QP results in a nonzero ϵ, then at the optimum for at least one of the points in the constraint window, say for $N_a \in [w_b, w_e]$, we will have $y(k + N_a) = y_U + \epsilon$ or

$y(k + N_a) = y_L - \epsilon$. Otherwise a smaller ϵ would reduce the objective function, while still satisfying the constraints. This point is the one for which satisfaction of the constraint presents the greatest difficulty.

We will consider the case $M = 1$, which in Zafiriou and Marchal (1991) was shown to be a risky one, when output constraints are used. Let the subscripts u and h correspond to the unconstrained and hard constrained cases, respectively, and f_u, f_h the result of the QDMC optimization for these cases as defined in (12). Then by using the results of section 3, it can be shown that when the constraint is softened, we have for the coefficients of the c_{J_i} (from (15)) that corresponds to the softened constraint at $k + N_a$:

$$\nabla_{x_j} f_s = \frac{1}{1 + G^{-1} S_{N_a}^2 W^2} \nabla_{x_j} f_u$$
$$+ \frac{G^{-1} S_{N_a}^2 W^2}{1 + G^{-1} S_{N_a}^2 W^2} \nabla_{x_j} f_h \qquad (29)$$

for $j = 1, \ldots, N$ and also for ∇_y, where the subscript s stands for soft. S_{N_a} is the value of the open-loop unit-step response of the process model at the N_a sampling point. From (29) and (18) it follows that

$$\psi_{j,s} = \frac{1}{1 + G^{-1} S_{N_a}^2 W^2} \psi_{j,u} + \frac{G^{-1} S_{N_a}^2 W^2}{1 + G^{-1} S_{N_a}^2 W^2} \psi_{j,h} \qquad (30)$$

From Zafiriou (1990) we know that a sufficient condition for closed-loop stability for a c_{J_i} is:

$$|\psi_1| + \cdots + |\psi_N| < 1 \qquad (31)$$

For the hard constraint case, we cannot influence the value of the ψs, since, as shown in Zafiriou and Marchal (1991) we have:

$$|\psi_{1,h}| + \cdots + |\psi_{N,h}|$$
$$= (|H_{N_a+1}| + \cdots + |H_N|)/|S_{N_a}| \overset{\Delta}{=} \alpha_h \qquad (32)$$

For a system with inverse response, α_h is greater than 1 for small N_a and therefore stability cannot be guaranteed.

For the unconstrained QDMC though, we have methods for obtaining values for the tuning parameters that result in a stable control system. Hence we can obtain $\alpha_u < 1$, where

$$\alpha_u \overset{\Delta}{=} |\psi_{1,u}| + \cdots + |\psi_{N,u}| \qquad (33)$$

Then it follows from (30) that for closed-loop stability after softening the output constraints, it is sufficient that

$$\frac{1}{1 + G^{-1} S_{N_a}^2 W^2} \alpha_u + \frac{G^{-1} S_{N_a}^2 W^2}{1 + G^{-1} S_{N_a}^2 W^2} \alpha_h < 1 \qquad (34)$$

Hence we can obtain a value for W:

$$W^2 < \frac{1 - \alpha_u}{G^{-1} S_{N_a}^2 (\alpha_h - 1)} \qquad (35)$$

Example 1

Consider the model for a multi-effect evaporator given in Ricker (1985), relating concentration to steam flow:

$$\tilde{p}(s) = \frac{2.69(-6s + 1)e^{-1.5s}}{(20s + 1)(5s + 1)} \qquad (36)$$

The sampling time is selected $T = 3$ and the truncation number $N = 25$. This system has an "unstable" zero and exhibits inverse response characteristics. Zafiriou and Marchal (1991) showed for this type of systems, hard output constraints at early future points result in instability. This is predicted by Thm. 1; a more detailed analysis of the theoretical reasons behind the instability can be found in the reference.

By using (35) we can find a value for W that defines a degree of softening sufficient to guarantee stability. The table below shows the relevant values for an output constraint window consisting of the first 5 future sampling points. We use $M = 1$, $\Gamma = 1$, $B = D = 0$. The horizon P is computed to make α_u small. It is possible to do so, by selecting a relatively large P. The algebraic expressions in the proof of that theorem in Garcia and Morari (1982) can be used to get a value for P. We use $P = 30 > N + M - 1$, which yields $\alpha_u = 0.136$.

N_a	1	2	3	4	5
α_h	16.29	12.04	35.29	13.73	4.69
S_{N_a}	-0.1744	-0.2312	-0.07444	0.1733	0.4485
W	14.86	13.19	23.25	16.39	11.76

From the table we see that by selecting $W = 11.76$ or smaller, closed-loop stability is guaranteed, regardless of which of the 5 points in the constraint window presents the greatest difficulty in satisfying the constraint during the on-line optimization. Simulations confirm this prediction. □

5   Concluding Remarks

This paper extends a framework that has been developed for the study of the nominal and robust stability of constrained MPC algorithms to the case where all or some of the constraints are softened. A major advantage of this approach is that the stability of the

constrained system has been translated into stability conditions for a set of linear controllers. Hence all the results of robust linear control theory can be used in the study of constrained MPC. The effect of softening the hard constraints was discussed. It was demonstrated how the degree of softening can be directly related to closed-loop stability. A sufficient condition for SISO processes with output constraints was given for illustration purposes.

Acknowledgements

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