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Application of Center Manifold Reduction to System Stabilization

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APPLICATION OF CENTER MANIFOLD REDUCTION TO SYSTEM STABILIZATION

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ABSTRACT

The Center Manifold Theorem is applied to the local feedback stabilization of nonlinear systems in critical cases. The paper addresses two particular critical cases, for which the system linearization at the equilibrium point of interest is assumed to possess either a simple zero eigenvalue or a complex conjugate pair of simple, pure imaginary eigenvalues. In either case, the noncritical eigenvalues are taken to be stable. The results on stabilizability and stabilization are given explicitly in terms of the nonlinear model of interest in its original form, i.e., before reduction to the center manifold. Moreover, the formulation given in this paper uncovers connections between results obtained using the center manifold reduction and those of an alternative approach.

Keywords

Nonlinear systems, stabilization, center manifold reduction.

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1. INTRODUCTION

Recently, center manifold reduction has been employed in nonlinear stabilization, resulting in stabilizing control law designs for various classes of nonlinear systems in the so-called "critical cases." Critical cases occur when the linearized system at an equilibrium point has at least one eigenvalue on the imaginary axis, with the remaining eigenvalues in the open left half of the complex plane.

Aeyels [1], who initiated application of the center manifold reduction in nonlinear stabilization, investigated the existence of smooth stabilizing feedback control laws for a class of third-order nonlinear systems for which the linearized model possesses an uncontrollable pair of pure imaginary eigenvalues. Behtash and Sastry [10] used the same approach to study stabilization for nonlinear systems whose linearized model has two distinct pairs of complex conjugate pure imaginary eigenvalues, or a double pole at the origin, or a pole at the origin and a complex conjugate pair of pure imaginary eigenvalues. In [10], the design was undertaken for the reduced system on the center manifold using normal form calculations, and for simplicity, a scalar stable mode was assumed. Generally, there is a need for considering cases with any finite number of stable modes. Moreover, it is desirable to express the control laws directly in terms of the original model rather than in terms of transformed versions.

A main goal of this paper is to derive such stabilizing control algorithms for general nonlinear systems in critical cases. The development focuses on general nonlinear systems in two specific critical cases. In the first critical case of interest here, a simple zero eigenvalue occurs, while in the second case a pair of pure imaginary eigenvalues occurs. In either case, the critical eigenvalues of the linearized model need not be controllable. The feedback laws obtained include purely linear state feedbacks, purely nonlinear state feedbacks and feedback control laws containing both linear and nonlinear terms in the state. Results of this paper are compared with those of [6], [7].

2. PRELIMINARIES

Consider a class of nonlinear autonomous systems

$$\dot{\eta} = A_{11}\eta + A_{12}\xi + F(\eta, \xi) \qquad (1a)$$

$$\dot{\xi} = A_{21}\eta + A_{22}\xi + G(\eta, \xi), \tag{1b}$$

where $\eta \in I\!R^n$, $\xi \in I\!R^m$. In (1), A_{ij} for $i, j = 1, 2$ are constant matrices, and the functions F, G are sufficiently smooth, with their values and first derivatives vanishing at the origin. If A_{12} and A_{21} vanish, the matrix A_{11} has all its eigenvalues on the imaginary axis, and A_{22} is Hurwitz, then the Center Manifold Theorem asserts the existence of a locally invariant manifold for (1) near the origin. This manifold is given by the graph of a function $\xi = h(\eta)$.

In applying the Center Manifold Theorem to feedback stabilization problems, it is convenient to give a restatement of the theorem in a way that does not require vanishing of the "linear coupling" matrices A_{12} and A_{21}. This is especially true when the feedback is allowed to possess linear terms. For the purposes of this paper, a restatement allowing nonzero A_{21} but with $A_{12} = 0$ suffices. A linear transformation of variables is now employed to achieve this. Consider the equation

$$AM + MB = C, \tag{2}$$

where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ and $M, C \in \mathbb{C}^{m\times n}$. For $n = m$ and $B = A^T$, Eq. (2) is a Liapunov matrix equation [5]. Let $\mathcal{F}$ denote the linear operator

$$\mathcal{F} : M \mapsto AM + MB \tag{3}$$

for $M \in \mathbb{C}^{m\times n}$.

The following result is a direct generalization of [5, Theorem F-1 and Corollary F-1a].

Theorem 1. Let n, m be positive integers. If the sum of any eigenvalue of A and any eigenvalue of B is nonzero, then the linear matrix equation (2) has a unique solution for matrix M.

We now apply the Center Manifold Theorem to the stability analysis of (1) for the case in which $A_{12} = 0$, with A_{21} not necessarily zero. Let A_{22} be Hurwitz and A_{11} have all its eigenvalues on the imaginary axis. By Theorem 1, the equation

$$A_{22}E - EA_{11} + A_{21} = 0 \tag{4}$$

has a unique solution for the $m \times n$ matrix E. Letting $\nu := \xi - E\eta$, we can rewrite system (1) as

$$\dot{\eta} = A_{11}\eta + F(\eta, \nu + E\eta) \tag{5a}$$

$$\dot{\nu} = A_{22}\nu + G(\eta, \nu + E\eta) - E \cdot F(\eta, \nu + E\eta). \tag{5b}$$

The Center Manifold Theorem for (1) can now be restated as follows:

Lemma 1. Assume $A_{12} = 0$, A_{22} is Hurwitz, and all eigenvalues of A_{11} have zero real parts. Then the origin of (1) is asymptotically stable (unstable) if the origin is asymptotically stable (unstable) for the reduced model

$$\dot{\eta} = A_{11}\eta + F(\eta, h(\eta) + E\eta), \tag{6}$$

where h satisfies the partial differential equation

$$\begin{aligned} &Dh(\eta)\{A_{11}\eta + F(\eta, h(\eta) + E\eta)\} \\ &\quad = A_{22}h(\eta) + G(\eta, h(\eta) + E\eta) - E \cdot F(\eta, h(\eta) + E\eta) \end{aligned} \tag{7}$$

with E the solution of Eq. (4) and boundary conditions: $h(0) = 0$ and $Dh(0) = 0$.

We employ Taylor series expansions in the development below, using multilinear function notation for the terms in these expansions. The definition of multilinear function is recalled as follows.

Definition 1. (e.g., [9]) Let $V_1, V_2, \ldots, V_k$ and W be vector spaces over the same field. A map $\psi : V_1 \times V_2 \times \ldots \times V_k \rightarrow W$ is multilinear (or k-linear) if it is linear in each of its arguments. That is, for any $v_i, \tilde{v}_i \in V_i$, $i = 1, \ldots, k$, and for any scalars $a, \tilde{a}$, we have

$$\begin{aligned} \psi(v_1, \ldots, av_i + \tilde{a}\tilde{v}_i, \ldots, v_k) &= a\psi(v_1, \ldots, v_i, \ldots, v_k) \\ &\quad + \tilde{a}\psi(v_i, \ldots, \tilde{v}_i, \ldots, v_k). \end{aligned} \tag{8}$$

The integer k is the degree of the multilinear function ψ.

The next definition deals with the special case in which $V_1 = V_2 = \ldots = V_k = V$.

Definition 2. [9] A k-linear function $\psi : V \times V \times \ldots \times V \rightarrow W$ is symmetric if the vector $\psi(v_1, v_2, \ldots, v_k)$ is invariant under arbitrary permutations of the argument vectors v_i. A function $\phi : I\!R^n \rightarrow I\!R^m$ is homogeneous of degree k (k an integer), if for each scalar α, $\phi(\alpha\eta) = \alpha^k \phi(\eta)$ for all $\eta \in I\!R^n$.

Note that, in the sequel prime denotes the transpose of both vector and matrix and I denotes the identity matrix.

3. GENERAL FRAMEWORK

Consider a nonlinear control system

$$\dot{\eta} = A_{11}\eta + b_1 u + F(\eta, \xi), \tag{9a}$$

$$\dot{\xi} = A_{22}\xi + b_2 u + G(\eta, \xi), \tag{9b}$$

where η, ξ are real vectors, and a preliminary block diagonalization has been applied to remove any linear coupling term in the dynamics between η and ξ. For simplicity, u is supposed to be a scalar control. It is not difficult to extend the study to the case in which the input is a vector control. In the following, we apply the center manifold result in Lemma 1 to design stabilizing control laws for (9) for which all eigenvalues of A_{11} lie on the imaginary axis.

Let us first consider the case in which b_1 is nonzero. In the simple critical cases, where A_{11} is the scalar 0 or is a 2×2 matrix with a pair of pure imaginary eigenvalues, linear theory will imply the existence of a linear stabilizing feedback control for (9). Consider next the existence of a purely nonlinear smooth feedback (i.e., one with vanishing linear part).

Since now we focus on purely nonlinear stabilizing controllers, system (9) retains the linear decoupling property upon control. Thus, if A_{22} is stable, then according to center manifold theorem (e.g., [3], [8]) there is a locally invariant manifold $\xi = h(\eta)$ for (9). Furthermore, h satisfies

$$\begin{aligned} &Dh(\eta)\{A_{11}\eta + b_1 u(\eta, h(\eta)) + F(\eta, h(\eta))\} \\ &= A_{22}h(\eta) + b_2 u(\eta, h(\eta)) + G(\eta, h(\eta)) \end{aligned} \tag{10}$$

with boundary conditions $h(0) = 0$ and $Dh(0) = 0$. Then, we seek a purely nonlinear stabilizing feedback control law by using stability conditions for the reduced model

$$\dot{\eta} = A_{11}\eta + b_1 u(\eta, h(\eta)) + F(\eta, h(\eta)). \tag{11}$$

Note that, for the case in which A_{22} is not stable, a linear state feedback $K_2\xi$ is needed to first stabilize $A_{22} + b_2 K_2$.

Next, consider the case of $b_1 = 0$ and assume the feedback control to be of the form

$$u(\eta,\xi) = K_1\eta + K_2\xi + U(\eta,\xi), \tag{12}$$

where $U(\cdot,\cdot)$ is a smooth, purely nonlinear function whose first derivatives vanish at the origin. Rewrite the system dynamics (9) as

$$\dot{\eta} = A_{11}\eta + F(\eta,\xi), \tag{13}$$

$$\dot{\xi} = b_2K_1\eta + (A_{22} + b_2K_2)\xi + b_2U(\eta,\xi) + G(\eta,\xi). \tag{14}$$

From Eq. (14), the feedback has given rise to a linear coupling term between η and ξ in the dynamics. As discussed in preceding section, there is a constant matrix E such that, with $\nu := \xi - E\eta$, the transformed version of the control system (13)-(14) is in block diagonal form. Here, E is the (unique) solution of the Liapunov-like equation

$$b_2K_1 + (A_{22} + b_2K_2)E - EA_{11} = 0. \tag{15}$$

We assume that $A_{22} + b_2K_2$ is stable. Moreover, since all the eigenvalues of A_{11} lie on the imaginary axis, then Theorem 1 guarantees existence of a solution E to Eq. (15). The transformed dynamics in the states η and ξ is then

$$\dot{\eta} = A_{11}\eta + F(\eta,\nu + E\eta), \tag{16a}$$

$$\dot{\nu} = (A_{22} + b_2K_2)\nu + b_2U(\eta,\nu + E\eta) + G(\eta,\nu + E\eta). \tag{16b}$$

Eq. (16) has a center manifold given by the graph of a function $\nu = h(\eta)$, where h satisfies

$$\begin{aligned} Dh(\eta)\{A_{11}\eta + F(\eta,h(\eta) + E\eta)\} &= (A_{22} + b_2K_2)h(\eta) \\ &\quad + b_2U(\eta,h(\eta) + E\eta) + G(\eta,h(\eta) + E\eta) \end{aligned} \tag{17}$$

with boundary conditions $h(0) = 0$ and $Dh(0) = 0$.

Lemma 1 implies asymptotic stability of the origin for (16) if the control gains K_1, K_2 and the nonlinear function U are chosen such that (i) $A_{22} + b_2K_2$ is Hurwitz, and (ii) the origin of reduced model ($16a$) with $\nu = h(\eta)$ is asymptotically stable.

We now proceed to consider two special cases in which the system has only simple critical modes (i.e., one zero eigenvalue or a pair of pure imaginary eigenvalues) and the rest of the eigenvalues are stabilizable.

4. ONE ZERO EIGENVALUE

In this section, we first consider stability conditions for scalar systems with a zero eigenvalue. These conditions are then employed in the design of stabilizing control laws for higher order systems with a simple zero eigenvalue.

Consider a scalar real nonlinear system

$$\dot{x} = dx^2 + ex^3 + \cdots. \tag{18}$$

Stability conditions for system (18) are given next.

Lemma 2. The origin is asymptotically stable for system (18) if $d = 0$ and $e < 0$. The origin is unstable for (18) if $d \neq 0$.

Now consider Eq. (9), with the scalar x replacing the critical state η, and with

$$\begin{aligned} f(x,\xi) :=& F(x,\xi) \\ =& f_{xx}x^2 + xf_{x\xi}\xi + \xi' f_{\xi\xi}\xi + f_{xxx}x^3 + x^2 f_{xx\xi}\xi \\ & + x\xi' f_{x\xi\xi}\xi + f_{\xi\xi\xi}(\xi,\xi,\xi) + O(||(x,\xi)||^4), \end{aligned} \tag{19}$$

$$\begin{aligned} G(x,\xi) =& x^2 G_{xx} + xG_{x\xi}\xi + G_{\xi\xi}(\xi,\xi) + x^3 G_{xxx} \\ & + x^2 G_{xx\xi}\xi + xG_{x\xi\xi}(\xi,\xi) + G_{\xi\xi\xi}(\xi,\xi,\xi) \\ & + O(||(x,\xi)||^4). \end{aligned} \tag{20}$$

The coefficients in the Taylor series expansions (19)-(20) are either constants or symmetric multilinear functions of their arguments. For instance, $f_{\xi\xi\xi}(\xi,\xi,\xi)$ and $G_{\xi\xi}(\xi,\xi)$ denote a symmetric trilinear scalar function and a bilinear vector function of ξ, respectively.

In the remainder of this section, stabilizing control laws will be obtained for system (9) under one or the other of the following two hypotheses.

Hypothesis 1A. The matrix $A_{11} = 0$ is a scalar and $b_1 \neq 0$.

Hypothesis 1B. The matrix $A_{11} = 0$ is a scalar and $b_1 = 0$.

4.1. The case $b_1 \neq 0$

In this subsection, we consider the case in which Hypothesis 1A holds. The control law is taken to be purely nonlinear. Existence of a linear stabilizing feedback for this case is evident. Nonlinear feedback controllers are none the less desirable in certain applications. We assume A_{22} is stable and the scalar control input is of the form

$$\begin{aligned} u(x,\xi) =& U(x,\xi) \\ :=& u_{xx}x^2 + xu_{x\xi}\xi + \xi' u_{\xi\xi}\xi + u_{xxx}x^3 \\ &+ x^2 u_{xx\xi}\xi + x\xi' u_{x\xi\xi}\xi + u_{\xi\xi\xi}(\xi,\xi,\xi). \end{aligned} \tag{21}$$

According to center manifold theorem, the stability of the origin for (9) coincides with the stability of the origin for the reduced model

$$\dot{x} = b_1 u(x, h(x)) + F(x, h(x)). \tag{22}$$

Here, h solves Eq. (10) with η replaced by x and with boundary conditions $h(0) = 0$ and $Dh(0) = 0$. Indeed, solving (10) we have

$$h(x) = x^2 h_{xx} + O(|x|^3), \tag{23}$$

where

$$h_{xx} = -A_{22}^{-1}(b_2 u_{xx} + G_{xx}). \tag{24}$$

From Lemma 2, we now have

Lemma 3. Let A_{22} be stable. Under Hypothesis 1A, the origin is asymptotically stable for (9) if $f_{xx} + b_1 u_{xx} = 0$ and $f_{xxx} + b_1 u_{xxx} - (f_{x\xi} + b_1 u_{x\xi})A_{22}^{-1}(G_{xx} + b_2 u_{xx}) < 0$.

It is obvious from Lemma 3 that a purely quadratic stabilizing control law exists.

Corollary 1. Assume that A_{22} is stable. Under Hypothesis 1A, the origin of (9) is asymptotically stabilizable by a *purely quadratic* feedback of the form $u = u_{xx}x^2 + xu_{x\xi}\xi$ if $A_{22}^{-1}G_{xx} \neq 0$.

Furthermore, below we have a purely cubic stabilizing controller for system (9) when $f_{xx} = 0$.

Corollary 2. Assume that A_{22} is stable and $f_{xx} = 0$. Under Hypothesis 1A, the origin of (9) is asymptotically stabilizable by a *purely cubic* feedback of the form $u = u_{xxx}x^3$.

For the case in which A_{22} is not stable, a linear feedback $K_2\xi$ is needed to guarantee the existence of a locally invariant manifold. Then the design of stabilizing control laws proposed in Lemma 3 and Corollaries 1 and 2 can be applied directly.

4.2. The case $b_1 = 0$

Next, we consider the case in which Hypothesis 1B holds and consider feedback control has the form as

$$u(x, \xi) = k_1 x + K_2 \xi + U(x, \xi) \tag{25}$$

with k_1 a scalar control gain and the nonlinear control function U as in (21).

From Section 3, the stability of control system (9) in this critical case coincides with the stability of the reduced model

$$\dot{x} = f(x, h(x) + Ex), \tag{26}$$

where E and $h(\cdot)$ solve Eqs. (15) and (17), respectively, under the conditions: $(A_{22} + b_2K_2)$ is stable and η and K_1 are substituted by x and k_1, respectively.

As above, h is as in (23). We assume that $A_{22} + b_2K_2$ is stable. Solving Eqs. (15) and (17), we have

$$E = -(A_{22} + b_2K_2)^{-1} b_2 k_1, \quad \text{and} \tag{27}$$

$$\begin{aligned} h_{xx} = (A_{22} + b_2K_2)^{-1}\{[f_{xx} + f_{x\xi}E + E'f_{\xi\xi}E]E - [b_2 u_{xx} \\ + G_{xx} + (b_2 u_{x\xi} + G_{x\xi})E + b_2 E' u_{\xi\xi} E + G_{\xi\xi}(E, E)]\}. \end{aligned} \tag{28}$$

The reduced model (26) is then given by

$$\begin{aligned} \dot{x} = \{f_{xx} + f_{x\xi}E + E'f_{\xi\xi}E\}x^2 + \{f_{x\xi}h_{xx} + 2E'f_{\xi\xi}h_{xx} + f_{xxx} \\ + f_{xx\xi}E + E'f_{x\xi\xi}E + f_{\xi\xi\xi}(E, E, E)\}x^3 + O(|x|^4). \end{aligned} \tag{29}$$

Note that E and h as given in (27) and (28) depend on the control u. Using Lemma 2, we have the following stabilization result for control system (9).

Lemma 4. Let the control input u be of the form as in (25). Then under Hypothesis 1B, the origin of the closed-loop system (9) is asymptotically stable if $A_{22} + b_2 K_2$ is stable and following two conditions hold:

$$f_{xx} + f_{x\xi} E + E' f_{\xi\xi} E = 0, \quad \text{and} \tag{30}$$

$$f_{x\xi} h_{xx} + 2E' f_{\xi\xi} h_{xx} + f_{xxx} + f_{xx\xi} E + E' f_{x\xi\xi} E + f_{\xi\xi\xi}(E, E, E) < 0. \tag{31}$$

where E is as in (27) and

$$h_{xx} = -(A_{22} + b_2 K_2)^{-1} \{ b_2 u_{xx} + G_{xx} + (b_2 u_{x\xi} + G_{x\xi}) E + b_2 E' u_{\xi\xi} E + G_{\xi\xi}(E, E) \}. \tag{32}$$

From Eqs. (27) and (32), and the fact that A_{22} is invertible, we have $E = 0$ and $h_{xx} = -A_{22}^{-1} G_{xx}$ for the uncontrolled system. The next stability criterion for the uncontrolled version of system (9) follows readily from Lemma 4.

Corollary 3. Suppose Hypothesis 1B holds. Then the origin is asymptotically stable for (9) (with $u = 0$) if A_{22} is stable, $f_{xx} = 0$ and $f_{xxx} - f_{x\xi} A_{22}^{-1} G_{xx} < 0$.

In the rest of this subsection, we assume that the stability conditions given in Corollary 3 do not hold, and seek stabilizing control laws for system (9).

Linear stabilizing control laws follow readily from Lemma 4, and are as given next.

Proposition 1. Suppose Hypothesis 1B holds and let $M := (A_{22} + b_2 K_2)^{-1}$. Then there is a purely linear feedback which asymptotically stabilizes the origin of (9) if there exist feedback gains k_1 and K_2 for which $(A_{22} + b_2 K_2)$ is stable,

$$f_{xx} - k_1 f_{x\xi} M b_2 + k_1^2 b_2' M' f_{\xi\xi} M b_2 = 0, \quad \text{and} \tag{33}$$

$$\begin{aligned} f_{xxx} &- f_{x\xi} M G_{xx} + k_1 \{ f_{x\xi} M G_{x\xi} + 2G_{xx}' M' f_{\xi\xi} - f_{xx\xi} \} M b_2 \\ &+ k_1^2 \{ b_2' M' f_{x\xi\xi} M b_2 - f_{x\xi} M G_{\xi\xi}(M b_2, M b_2) \\ &- 2 b_2' M' f_{\xi\xi} M G_{x\xi} M b_2 \} - k_1^3 \{ f_{\xi\xi\xi}(M b_2, M b_2, M b_2) \\ &- 2 b_2' M' f_{\xi\xi} M G_{\xi\xi}(M b_2, M b_2) \} < 0. \end{aligned} \tag{34}$$

Remark 1. The linear stabilizing control rule proposed in Proposition 1 is a composite-type controller design. First, the feedback gain K_2 is chosen to stabilize state ξ. Then the remaining feedback gain k_1 is selected to satisfy the conditions (32) and (33) based on the chosen gain K_2.

Since k_1 is a scalar, conditions (32) and (33) do not necessarily hold for any given K_2. Thus, a stabilizing linear feedback does not always follow from Corollary 2. A special result, for the case in which the non-critical state ξ is a scalar, is given below to demonstrate such a design is not vacuous. Note that $G_{\xi\xi}(\xi,\xi) := G_{\xi\xi}\xi^2$ in the next corollary.

Corollary 4. Suppose the non-critical state ξ is a scalar and Hypothesis 1B holds. Then there is a *purely linear* feedback which asymptotically stabilizes the origin of (9) if either of the following conditions holds:

(i) $f_{\xi\xi} = 0, f_{x\xi} \neq 0$ and $f_{x\xi}G_{xx} - f_{xx}G_{x\xi} + \frac{1}{f_{x\xi}}G_{\xi\xi}f_{xx}^2 < 0$.

(ii) $f_{\xi\xi} \neq 0, f_{x\xi}^2 - 4f_{xx}f_{\xi\xi} > 0$ and either $G_{xx} + G_{x\xi}E^+ + G_{\xi\xi}(E^+)^2 < 0$ or $G_{xx} + G_{x\xi}E^- + G_{\xi\xi}(E^-)^2 > 0$, where

$$E^{\pm} = \frac{1}{2f_{\xi\xi}}\{-f_{x\xi} \pm \sqrt{f_{x\xi}^2 - 4f_{xx}f_{\xi\xi}}\}. \tag{35}$$

According to the stability conditions given in Lemma 4, the cubic terms of both the function G and the control input u do not contribute to the stability criteria of system (9). A general linear-plus-quadratic feedback control law can then be abstracted as

$$u(x,\xi) = k_1x + K_2\xi + u_{xx}x^2 + xu_{x\xi}\xi + \xi' u_{\xi\xi}\xi \tag{36}$$

while the control gains satisfying the conditions of Lemma 4.

As implied by Lemmas 1 and 2 and the discussions above, we have the next result.

Lemma 5. Suppose A_{22} is stable and Hypothesis 1B holds. Then there exists no *purely quadratic* feedback stabilizer for the origin of system (9) if $f_{xx} \neq 0$. However, the origin of (9) is asymptotically stabilizable by a *purely quadratic* feedback of the form $u = u_{xx}x^2$ if $f_{xx} = 0$ and $f_{x\xi}A_{22}^{-1}b_2 \neq 0$.

Note that the stabilization results given in Corollaries 1 and 2 and Lemma 5 agree with those obtained in [7].

5. PAIR OF PURE IMAGINARY EIGENVALUES

In this section, we consider system (9) in which A_{11} has a pair of pure imaginary eigenvalues. Specifically, we take A_{11} to be of the form (38) below.

First, however, consider the stability of a planar system

$$\dot{\eta} = A_{11}\eta + Q(\eta,\eta) + C(\eta,\eta,\eta) + \cdots, \tag{37}$$

where $\eta = (x,y)'$, and

$$A_{11} = \begin{pmatrix} 0 & \Omega_1 \\ -\Omega_2 & 0 \end{pmatrix} \tag{38}$$

with $\Omega_1\Omega_2 > 0$. Without loss of generality, we may express the quadratic and cubic terms in Eq. (37) in the form

$$Q(\eta,\eta) = \begin{pmatrix} q_{11}x^2 + q_{12}xy + q_{13}y^2 \\ q_{21}x^2 + q_{22}xy + q_{23}y^2 \end{pmatrix}, \tag{39}$$

$$C(\eta,\eta,\eta) = \begin{pmatrix} c_{11}x^3 + c_{12}x^2y + c_{13}xy^2 + c_{14}y^3 \\ c_{21}x^3 + c_{22}x^2y + c_{23}xy^2 + c_{24}y^3 \end{pmatrix}, \tag{40}$$

respectively. Note the linearization of (37) at the origin has the pair of pure imaginary eigenvalues $\pm i\sqrt{\Omega_1\Omega_2}$, where $i = \sqrt{-1}$.

Applying a general stability criterion for planar systems undergoing Hopf bifurcation (see, e.g., [8]), we find that a sufficient condition for the stability of the origin for (37) is:

$$\begin{aligned} &\frac{1}{8}\{q_{22}(\frac{1}{\Omega_2}q_{21} + \frac{1}{\Omega_1}q_{23}) - q_{12}(\frac{1}{\Omega_1}q_{11} + \frac{\Omega_2}{\Omega_1^2}q_{13}) + \frac{2}{\Omega_2}q_{11}q_{21} \\ &\quad - \frac{2\Omega_2}{\Omega_1^2}q_{13}q_{23} + 3(c_{11} + \frac{\Omega_2}{3\Omega_1}c_{13} + \frac{1}{3}c_{22} + \frac{\Omega_2}{\Omega_1}c_{24})\} < 0. \end{aligned} \tag{41}$$

In the following, we apply the stability criterion (41) to the design of stabilizing control laws for the more general (nonplanar) system (9) in which both $\eta = (x,y)'$ and $b_1 := (b_{11}, b_{12})'$ are two-dimensional vectors, and $F(\eta,\xi) = (f(x,y,\xi), g(x,y,\xi))'$.

Results obtained in this section will apply under one or the other of the following two hypotheses.

Hypothesis 2A. The matrix A_{11} (appearing in (9)) is a 2×2 matrix of the form (38) above, and $b_1 \neq 0$.

Hypothesis 2B. The matrix A_{11} (appearing in (9)) is a 2×2 matrix of the form (38) above, and $b_1 = 0$.

5.1. The case $b_1 \neq 0$

First, we consider the case in which at least one of b_{11} and b_{12} is nonzero. Although this assumption guarantees the controllability of the subsystem (9), here we consider only purely nonlinear control laws. Assume that A_{22} is stable and the control input $u = U(x, y, \xi)$ is a smooth, purely nonlinear function. From Section 3, the stability of the origin of (9) now coincides with the stability of the origin of the reduced model

$$\dot{x} = \Omega_1 y + b_{11} U(x, y, h(x, y)) + f(x, y, h(x, y)) \tag{42}$$

$$\dot{y} = -\Omega_2 x + b_{12} U(x, y, h(x, y)) + g(x, y, h(x, y)). \tag{43}$$

Here, h solves Eq. (10) with η replaced by $(x, y)'$ and with boundary conditions $h(0) = 0$ and $Dh(0) = 0$. Indeed, h takes the form

$$h(x, y) = x^2 h_{xx} + xy h_{xy} + y^2 h_{yy} + O(||(x, y)||^3), \tag{44}$$

where h_{xx}, h_{xy}, h_{yy} are constant vectors.

In the following, we restrict the nonlinear control function U to be a function of x and y only, as follows:

$$\begin{aligned} U(x, y, \xi) = &u_{xxx}x^2 + u_{xy}xy + u_{yy}y^2 + u_{xxx}x^3 \\ &+ u_{xxy}x^2y + u_{xyy}xy^2 + u_{yyy}y^3. \end{aligned} \tag{45}$$

A stability criterion for the control system (9) in this case is given next.

Lemma 6. Suppose A_{22} is stable and Hypothesis 2A holds. Then the origin is asymptotically stable for (9) if

$$(b_{12}u_{xy} + g_{xy})\{\frac{1}{\Omega_2}(b_{12}u_{xx} + g_{xx}) + \frac{1}{\Omega_1}(b_{12}u_{yy} + g_{yy})\}$$

$$
\begin{aligned}
&-(b_{12}u_{xy}+f_{xy})\{\frac{1}{\Omega_1}(b_{11}u_{xx}+f_{xx})+\frac{\Omega_2}{\Omega_1^2}(b_{11}u_{yy}+f_{yy})\} \\
&+\frac{2}{\Omega_2}(b_{11}u_{xx}+f_{xx})(b_{12}u_{xx}+g_{xx})-\frac{2\Omega_2}{\Omega_1^2}(b_{11}u_{yy}+f_{yy})(b_{12}u_{yy}+g_{yy}) \\
&+3\{b_{11}u_{xxx}+f_{xxx}+f_{x\xi}h_{xx}+\frac{\Omega_2}{3\Omega_1}(b_{11}u_{xyy}+f_{xyy}+f_{x\xi}h_{yy}+f_{y\xi}h_{xy}) \\
&+\frac{1}{3}(b_{12}u_{xxy}+g_{xxy}+g_{x\xi}h_{xy}+g_{y\xi}h_{xx}) \\
&+\frac{\Omega_2}{\Omega_1}(b_{12}u_{yyy}+g_{yyy}+g_{y\xi}h_{yy})\}<0, \qquad (46)
\end{aligned}
$$

where

$$
\begin{aligned}
h_{xy}=&\{A_{22}^2+4\Omega_1\Omega_2 I\}^{-1}\{2\Omega_2(u_{yy}b_2+G_{yy}) \\
&-2\Omega_1(u_{xx}b_2+G_{xx})-A_{22}(u_{xy}b_2+G_{xy})\}, \qquad (47)\\
h_{xx}=&-A_{22}^{-1}(u_{xx}b_2+G_{xx}+\Omega_2 h_{xy}), \qquad (48)\\
h_{yy}=&-A_{22}^{-1}(u_{yy}b_2+G_{yy}-\Omega_1 h_{xy}). \qquad (49)
\end{aligned}
$$

It is observed from Lemma 6, generically there exists a quadratic-plus-cubic feedback stabilizer for system (9). In addition, a purely quadratic state feedback stabilizing control law and a purely cubic state feedback stabilizing control law follow readily from Lemma 6 as given in the next two corollaries.

Corollary 5. Let A_{22} be stable and Hypothesis 2A hold. Then the origin of system (9) is stabilizable by a *purely quadratic* state feedback of the form $u=u_{xy}xy$ if

$$
\begin{aligned}
&b_{12}\{\frac{1}{\Omega_1}(g_{yy}-f_{xx})+\frac{1}{\Omega_2}g_{xx}-\frac{\Omega_2}{\Omega_1^2}f_{yy}\} \\
&\quad-\frac{1}{3}\{\Omega_2(2g_{y\xi}-8f_{x\xi})A_{22}^{-1}+\frac{\Omega_2}{\Omega_1}f_{y\xi}+g_{x\xi}\}(A_{22}^2+4\Omega_1\Omega_2 I)^{-1}A_{22}b_2\neq 0. \qquad (50)
\end{aligned}
$$

Corollary 6. Let A_{22} be stable and Hypothesis 2A hold. Then the origin of system (9) is stabilizable by a *purely cubic* state feedback of the form $u=u_{xxx}x^3+u_{xxy}x^2y+u_{xyy}xy^2+u_{yyy}y^3$.

5.2. The case $b_1 = 0$

Next, we consider the case in which Hypothesis 2B holds, i.e., $b_1 = 0$ and A_{11} is as in (38). Let the control input be of the form

$$u = k_{11}x + k_{12}y + K_2\xi + U(x, y, \xi), \tag{51}$$

where U is defined in (45).

Assume that $A_{22} + b_2K_2$ is stable. From Section 3, the stability of the origin of (9) agrees with the stability of the origin of the reduced model

$$\dot{x} = \Omega_1 y + f(x, y, E_1x + E_2y + h(x, y)) \tag{52}$$

$$\dot{y} = -\Omega_2 x + g(x, y, E_1x + E_2y + h(x, y)). \tag{53}$$

Here, $E = (E_1, E_2)$ and $h(x, y)$ are the solutions of Eqs. (15) and (17), respectively, with $K_1 = (k_{11}, k_{12})$.

Since $(A_{22}+b_2K_2)$ is stable, matrices $(A_{22}+b_2K_2)^2+\Omega_1\Omega_2 I$ and $(A_{22}+b_2K_2)^2+4\Omega_1\Omega_2 I$ are then both invertible.

Let

$$\begin{aligned} H(x, y) :=& b_2U(x, y, E_1x + E_2y) + G(x, y, E_1x + E_2y) \\ & - f(x, y, E_1x + E_2y)E_1 - g(x, y, E_1x + E_2y)E_2 \\ =& x^2H_{xx} + xyH_{xy} + y^2H_{yy} + O(\|(x, y)\|^3). \end{aligned} \tag{54}$$

As before, we take h to be of the form (44). Solving Eqs. (15) and (17), we have

$$E_1 = -\{(A_{22} + b_2K_2)^2 + \Omega_1\Omega_2 I\}^{-1}\{k_{11}(A_{22} + b_2K_2) - \Omega_2k_{12}I\}b_2 \tag{55}$$

$$E_2 = -\{(A_{22} + b_2K_2)^2 + \Omega_1\Omega_2 I\}^{-1}\{k_{12}(A_{22} + b_2K_2) + \Omega_1k_{11}I\}b_2 \tag{56}$$

and

$$\begin{aligned} h_{xy} =& \{(A_{22} + b_2K_2)^2 + 4\Omega_1\Omega_2 I\}^{-1}\{2\Omega_2H_{yy} - 2\Omega_1H_{xx} \\ & - (A_{22} + b_2K_2)H_{xy}\}, \end{aligned} \tag{57}$$

$$h_{xx} = -(A_{22} + b_2K_2)^{-1}(H_{xx} + \Omega_2h_{xy}), \tag{58}$$

$$h_{yy} = -(A_{22} + b_2K_2)^{-1}(H_{yy} - \Omega_1h_{xy}). \tag{59}$$

The reduced model (52)-(53) is obtained as

$$\begin{aligned}\dot{x} =&\Omega_1 y + \hat{f}_{xx}x^2 + \hat{f}_{xy}xy + \hat{f}_{yy}y^2 + \hat{f}_{xxx}x^3 \\ &+ \hat{f}_{xxy}x^2y + \hat{f}_{xyy}xy^2 + \hat{f}_{yyy}y^3 + O(||(x,y)||^4) \qquad (60)\\ \dot{y} =& -\Omega_2 x + \hat{g}_{xx}x^2 + \hat{g}_{xy}xy + \hat{g}_{yy}y^2 + \hat{g}_{xxx}x^3 \\ &+ \hat{g}_{xxy}x^2y + \hat{g}_{xyy}xy^2 + \hat{g}_{yyy}y^3 + O(||(x,y)||^4). \qquad (61)\end{aligned}$$

Here, $\hat{f}_{ij}, \hat{g}_{ij}, \hat{f}_{ijk}$ and $\hat{g}_{ijk}$, $i,j,k \in \{x,y,z\}$, denote the new versions of the quadratic terms and cubic terms, the values of which are given in Appendix A.

Referring to the stability criterion (41) and the preceding discussions, we obtain stability conditions for the control system (9) as summarized in the following lemma.

Lemma 7. Suppose Hypothesis 2B holds and that the control input is of the form (51) with nonlinear function U as in (45). Then the origin of Eq. (9) is asymptotically stable if $A_{22} + b_2 K_2$ is stable and

$$\begin{aligned}&\hat{g}_{xy}\left(\frac{1}{\Omega_2}\hat{g}_{xx} + \frac{1}{\Omega_1}\hat{g}_{yy}\right) - \hat{f}_{xy}\left(\frac{1}{\Omega_1}\hat{f}_{xx} + \frac{\Omega_2}{\Omega_1^2}\hat{f}_{yy}\right) + \frac{2}{\Omega_2}\hat{f}_{xx}\hat{g}_{xx} \\ &- \frac{2\Omega_2}{\Omega_1^2}\hat{f}_{yy}\hat{g}_{yy} + 3\left(\hat{f}_{xxx} + \frac{\Omega_2}{3\Omega_1}\hat{f}_{xyy} + \frac{1}{3}\hat{g}_{xxy} + \frac{\Omega_2}{\Omega_1}\hat{g}_{yyy}\right) < 0. \qquad (62)\end{aligned}$$

Remark 2. From (62) and Appendix A, we observe that only quadratic terms of the function G, and the linear and quadratic terms of the control input u contribute to the stability conditions. A linear and/or quadratic feedback stabilizing control law readily follows from Lemma 7. Moreover, a stability criterion for the uncontrolled version of system (9) is also implied by Lemma 7 by letting $u = 0$.

Similarly, although Lemma 7 addresses the design of a linear feedback stabilizing control law, such a linear stabilizing control law need not exist. In the next result, we consider a special case of which the non-critical state ξ of system (9) is a scalar. Since ξ is a scalar, as observed from Eqs. (55)-(56), we always have solutions for the control gains k_{11} and k_{12} for arbitrary given values of E_1, E_2 and K_2. According to the formulations as in Appendix A, we can select $E_1 = 0$ and E_2 large enough (or $E_2 = 0$ and E_1 large enough) such that the condition (62) in Lemma 7 holds while $g_{\xi\xi\xi} < 0$ ($f_{\xi\xi\xi} < 0$). The following result hence readily implied by Lemma 7.

Corollary 7. Suppose the non-critical state ξ is a scalar and Hypothesis 2B holds. Then there is a *purely linear* feedback which asymptotically stabilizes the origin of (9) if either $f_{\xi\xi\xi} < 0$ or $g_{\xi\xi\xi} < 0$.

Referring to Eqs. (54)-(56), for the general case of which the state ξ of system (9) may not be a scalar, we have $H(x,y) = b_2 U(x,y,0) + G(x,y,0)$ while $k_{11} = k_{12} = 0$ and $K_2 = 0$. A purely quadratic stabilizing control law can then be obtained as follows.

Corollary 8. Assume that Hypothesis 2B holds, A_{22} is stable and the origin of system (9) is unstable. Then a *purely quadratic* stabilizing feedback in the form $u = u_{xx}x^2 + u_{xy}xy + u_{yy}y^2$ exists for the origin of (9) if one of the following three conditions holds:

(i) $M_0 A_{22} b_2 \neq 0$, or

(ii) $\{(3f_{x\xi} + \frac{1}{3}g_{y\xi})A_{22}^{-1} + 2\Omega_1 M_0\}b_2 \neq 0$, or

(iii) $\{2\Omega_2 M_0 - \frac{\Omega_2}{\Omega_1}(\frac{1}{3}f_{x\xi} + g_{y\xi})A_{22}^{-1}\}b_2 \neq 0$, where

$$M_0 = \frac{1}{3}\{\Omega_2(2g_{y\xi} - 8f_{x\xi})A_{22}^{-1} + \frac{\Omega_2}{\Omega_1}f_{y\xi} + g_{x\xi}\}(A_{22}^2 + 4\Omega_1\Omega_2 I)^{-1}. \tag{63}$$

We note that Aeyels' stabilization conditions for a third-order system [1] are special cases of those given in Corollary 8. Moreover, Corollary 8 easily extends to the case in which A_{22} is not stable but the pair $(A_{22},\ b_2)$ is stabilizable. This involves use of an additional linear feedback $K_2\xi$ to ensure the existence of a locally invariant manifold and the stability of the Jacobian matrix of Eq. (9*b*). Furthermore, we note that the stabilizing control laws obtained in Corollaries 6 and 8 agree with those obtained by Abed and Fu [6], where an asymptotic expansion method based on bifurcation analysis is used for controller design.

6. CONCLUDING REMARKS

In this paper, the center manifold reduction technique has been applied to the smooth feedback stabilization of nonlinear systems in two critical cases. The stabilizing control laws were obtained in a two step composite-type design. Linear stability for the noncritical state ξ is first ensured, then the remaining control gains are chosen to stabilize the origin of the reduced model whose eigenvalues all lie on the imaginary axis. Stabilizing control laws have been designed in linear and/or nonlinear feedback form.

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APPENDIX A

The coefficients in the Taylor expansions of $\hat{f}, \hat{g}$ are given below in terms of those of f, g. Here, ρ denotes either f or g, and $i \neq j$ for $i, j \in \{x, y\}$ with $E_{[x]} = E_1$, and $E_{[y]} = E_2$.

$$
\begin{aligned}
\hat{\rho}_{ii} &= \rho_{ii} + \rho_{i\xi} E_{[i]} + E'_{[i]} \rho_{\xi\xi} E_{[i]} \\
\hat{\rho}_{ij} &= \rho_{ij} + \rho_{i\xi} E_{[j]} + \rho_{j\xi} E_{[i]} + 2E'_{[i]} \rho_{\xi\xi} E_{[j]} \\
\hat{\rho}_{iii} &= \rho_{iii} + \rho_{ii\xi} E_{[i]} + E'_{[i]} \rho_{i\xi\xi} E_{[i]} + \rho_{\xi\xi\xi}(E_{[i]}, E_{[i]}, E_{[i]}) \\
&\quad + \rho_{i\xi} h_{ii} + 2E'_{[i]} \rho_{\xi\xi} h_{ii} \\
\hat{\rho}_{iij} &= \rho_{j\xi} h_{ii} + \rho_{i\xi} h_{ij} + 2E'_{[j]} \rho_{\xi\xi} h_{ii} + 2E'_{[i]} \rho_{\xi\xi} h_{ij} \\
&\quad + \rho_{iij} + \rho_{ij\xi} E_{[i]} + \rho_{ii\xi} E_{[j]} + E'_{[i]} \rho_{j\xi\xi} E_{[i]} \\
&\quad + 2E'_{[i]} \rho_{i\xi\xi} E_{[j]} + 3\rho_{\xi\xi\xi}(E_{[i]}, E_{[i]}, E_{[j]}).
\end{aligned}
$$