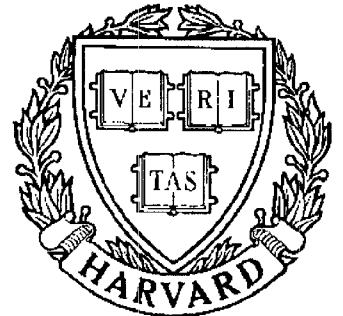


TECHNICAL RESEARCH REPORT



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*Supported by the
National Science Foundation
Engineering Research Center
Program (NSFD CD 8803012),
the University of Maryland,
Harvard University,
and Industry*

Research support for this
report has been provided
in part by the
Aerospace Corp.

Computation of the Circular Error Probability Integral

by J.T. Gillis

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March 10, 1991

Abstract

This note describes a simplified derivation of the representation of the circular error probability (CEP) integral, which is the integral over a disk centered at the origin of a zero mean two dimensional Gaussian random variable, as a one-dimensional integral. In addition, a rapidly converging series expression is derived for the CEP.

The integral occurs in the evaluation of communication and radar signals, and other statistical applications.

1 Introduction

The circular error probability (CEP) integral occurs in a variety of physical situations, including the evaluation of communication and radar systems as well as wind shear. The CEP is given by:

$$P_\rho = \frac{1}{2\pi\sqrt{|\Sigma|}} \int \int_{x^2+y^2 < \rho^2} e^{-\frac{1}{2}(x,y)\Sigma^{-1}(x,y)^t} dx dy.$$

The quantity P_ρ is the probability that a realization of an $N(0, \Sigma)$ random vector lies inside a circle of radius ρ . A derivation of the CEP integral is given in a NASA Technical Memorandum [5]. That derivation proceeds by expansion of the CEP integral as a series and regrouping of terms. The derivation is considerably longer than the one presented here. This paper also derives an efficient series for evaluation of the CEP integral.

The CEP integral has a long history in applications, one application is the assessment of when to fire artillery. The shells have a circular “kill-zone” and the target locations are specified by probability ellipses. The eccentricity of the ellipse is significant as it is easy to collect information

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along front (you hold these positions) and hard to collect information perpendicular to the front. This problem typifies the applications in which the CEP integral occurs: that is that the natural axis of the law (the major and minor axis in the quadratic form in the exponential) do not coincide with the area in which the probability is desired. More realistic models might have different natural axis for the location of the target and the accuracy of the ordinance delivery. The CEP integral can be used to compute integrals of the form:

$$\int \int_{(x,y)\Theta(x,y)^t < \rho^2} e^{-\frac{1}{2}(x,y)\Sigma^{-1}(x,y)^t} dx dy$$

by a change of variables that sends $(x, y)\Theta(x, y)^t$ into $x'^2 + y'^2$, clearly Θ must be positive definite. The application that motivated the author was the evaluation of cumulative errors caused by portions of a spacecraft which had different (in fact varying) natural axis for the error sources. In addition, circular equivalent errors are unchanged by coordinate transformations and hence are easy to deal with visavis coordinate transformations.

2 The CEP Calculation

2.1 The CEP Integral

For a given radius ρ , the probability that the Gaussian random vector $(X = (x, y) \sim N(0, \Sigma))$ has magnitude less than ρ is:

$$P_\rho = P(X^t X < \rho^2) = \frac{1}{2\pi\sqrt{|\Sigma|}} \int_{X^t X < \rho^2} e^{-\frac{1}{2}(X^t \Sigma^{-1} X)} dX,$$

where Σ is the covariance matrix (positive definite). Since the area of integration is a disk which is invariant under rotations, we can rotate the covariance matrix to be diagonal without loss of generality. We shall assume that the covariance matrix is non-degenerate. The diagonal elements of Σ are denoted as a^2 and b^2 . We then have:

$$P_\rho = \frac{1}{2\pi ab} \int \int_{x^2 + y^2 < \rho^2} e^{-\frac{1}{2}(\frac{x^2}{a^2} + \frac{y^2}{b^2})} dx dy.$$

Rewriting in polar coordinates, this expression becomes:

$$P_\rho = \frac{1}{2\pi ab} \int_0^\rho \int_0^{2\pi} e^{-\frac{r^2}{2}(\frac{\cos^2(\theta)}{a^2} + \frac{\sin^2(\theta)}{b^2})} r d\theta dr.$$

Substituting $\cos^2 = 1 - \sin^2$ and simplifying the exponential the expression is:

$$P_\rho = \frac{1}{2\pi ab} \int_0^\rho \int_0^{2\pi} e^{-\frac{r^2}{2a^2}} \cdot e^{-\frac{1}{2}(\frac{r}{b})^2(1 - (\frac{b}{a})^2)\sin^2(\theta)} r d\theta dr.$$

Letting $2\xi = \frac{1}{2} \left(\frac{r}{b}\right)^2 \left(1 - \left(\frac{b}{a}\right)^2\right)$, the expression becomes:

$$P_\rho = \frac{1}{2\pi ab} \int_0^\rho \int_0^{2\pi} e^{-\frac{r^2}{2a^2}} \cdot e^{-2\xi \sin^2(\theta)} r d\theta dr.$$

Upon the substitution $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta$ or $-idz/z = d\theta$; hence the integration is transformed to a line integral along the curve $|z| = 1$. Using Euler's formula: $\sin(\theta) = (z - z^{-1})/(2i)$, the integrand can be rewritten as:

$$\begin{aligned} P_\rho &= \frac{-i}{2\pi ab} \int_0^\rho \oint_{|z|=1} e^{-\frac{r^2}{2a^2}} \cdot e^{-2\xi \left(\frac{z-z^{-1}}{2i}\right)^2} \frac{r}{z} dz dr \\ &= \frac{-i}{2\pi ab} \int_0^\rho \oint_{|z|=1} e^{-\frac{r^2}{2a^2}} \cdot e^{\frac{\xi}{2}(z^2 - 2 + z^{-2})} \frac{r}{z} dz dr. \end{aligned}$$

The appropriate change of variable here is to let $w = z^2$, hence $dw/2w = dz/z$. The path of integration is now traversed two times, so the integral is doubled — this cancels the factor of a half in the differential. After an easy simplification, the expression is:

$$\frac{-i}{2\pi ab} \int_0^\rho \oint_{|w|=1} e^{-\frac{r^2}{2a^2} - \xi} \cdot e^{\left(\frac{\xi}{2}\right)\left(w + \frac{1}{w}\right)} \frac{r}{w} dw dr.$$

The contour integral is easily related to I_0 , the modified Bessel function of order zero (c.f. Arfken[1], p416):

$$I_0(\xi) = \frac{-i}{2\pi} \oint_{|w|=1} e^{\left(\frac{\xi}{2}\right)\left(w + \frac{1}{w}\right)} \frac{dw}{w}.$$

The expression for the CEP is then:

$$P_\rho = \frac{1}{ab} \int_0^\rho I_0(\xi) e^{-\frac{r^2}{2a^2} - \xi} r dr.$$

Re-expanding ξ :

$$\frac{1}{ab} \int_0^\rho I_0 \left(-\frac{1}{4} \left(\frac{r}{b}\right)^2 \left(1 - \left(\frac{b}{a}\right)^2\right) \right) e^{-\frac{r^2}{2a^2} - \frac{1}{4} \left(\frac{r}{b}\right)^2 \left(1 - \left(\frac{b}{a}\right)^2\right)} r dr.$$

The argument of the exponent can be simplified as follows:

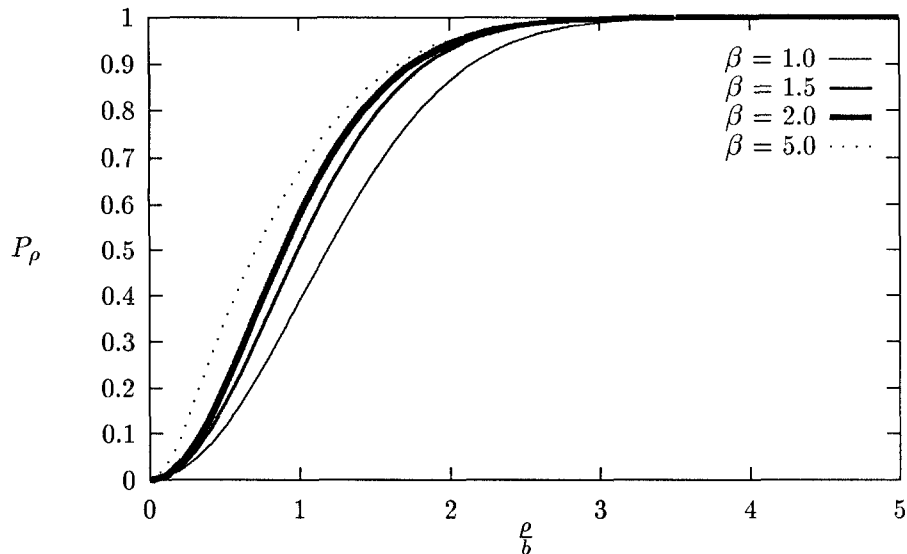
$$\begin{aligned} \frac{r^2}{2a^2} + \frac{1}{4} \left(\frac{r}{b}\right)^2 \left(1 - \left(\frac{b}{a}\right)^2\right) &= \frac{r^2}{2a^2} + \frac{1}{4} \left(\frac{r}{b}\right)^2 - \frac{1}{4} \left(\frac{r}{b}\right)^2 \left(\frac{b}{a}\right)^2 \\ &= \frac{r^2}{4b^2} \left(\frac{b^2}{a^2} + 1\right). \end{aligned}$$

Upon substitution of $t = \frac{r}{b}$ and $\beta = \frac{b}{a}$ in the integral, the expression is seen in its final form:

$$P_\rho = \beta \int_0^{\frac{\rho}{b}} I_0 \left(\frac{1}{4} t^2 (\beta^2 - 1) \right) e^{-\frac{1}{4} t^2 (1 + \beta^2)} t dt \quad (1)$$

This is precisely the expression tabulated in the NASA technical memorandum[5].

Figure 1: P_ρ for various values of β .



3 Computational Techniques

The actual value for the CEP integral was computed by integration of Eq. 1 using Romberg integration (the specific integration routine was taken from Numerical Recipes [4] and converted to double precision). This method was cross checked with the NASA table [5] for a variety of values and was verified to the table accuracy. Figure 1 shows the CEP integral for various values of β .

If $\beta = 1$ (that is $a = b$), the integration is easily performed and yields (n.b. $I_0(0) = 1$):

$$P_\rho = \int_0^{\frac{\rho}{b}} t e^{-\frac{1}{2}t^2} dt \quad (2)$$

$$= 1 - e^{-\frac{\rho^2}{2b^2}} \quad (3)$$

The CEP integral is more sensitive to variations at $\beta \sim 1$, than it is to β large as can be seen in Figure 1. Because of this Eq. 3 is not very useful for approximation of the CEP integral.

3.1 A Series Expression For P_ρ

Our goal in this section is to derive a simple series expression for the CEP integral that converges quickly.

We begin by two changes of variables, the first is $s = t^2$ (in Eq. 4), and the second is $\xi = \frac{\beta^2+1}{4}s$ (in Eq. 5). Hence:

$$\begin{aligned}
P_\rho &= \beta \int_0^{\frac{\rho}{b}} I_0 \left(\frac{1}{4} t^2 (\beta^2 - 1) \right) e^{-\frac{1}{4} t^2 (1 + \beta^2)} t dt \\
&= \frac{\beta}{2} \int_0^{(\frac{\rho}{b})^2} I_0 \left(\frac{\beta^2 - 1}{4} s \right) e^{-\frac{\beta^2 + 1}{4} s} ds
\end{aligned} \tag{4}$$

$$= \frac{2\beta}{\beta^2 + 1} \int_0^{\frac{\beta^2 + 1}{4} (\frac{\rho}{b})^2} I_0 \left(\frac{\beta^2 - 1}{\beta^2 + 1} \xi \right) e^{-\xi} d\xi \tag{5}$$

The Frobenius series (see Lebedev[3], p. 108) for I_0 is given by:

$$I_0 \left(\frac{\beta^2 - 1}{\beta^2 + 1} \xi \right) = \sum_{k=0}^{\infty} \frac{(\beta^2 - 1)^{2k} \xi^{2k}}{(k!)^2 4^k (\beta^2 + 1)^{2k}}$$

We can substitute this expression into Eq. 5 and formally interchange summation and integration to get:

$$P_\rho = \frac{2\beta}{\beta^2 + 1} \int_0^{\frac{\beta^2 + 1}{4} (\frac{\rho}{b})^2} \sum_{k=0}^{\infty} \frac{(\beta^2 - 1)^{2k} \xi^{2k}}{(k!)^2 4^k (\beta^2 + 1)^{2k}} e^{-\xi} d\xi \tag{6}$$

$$\begin{aligned}
&= \frac{2\beta}{\beta^2 + 1} \sum_{k=0}^{\infty} \frac{(\beta^2 - 1)^{2k}}{(k!)^2 4^k (\beta^2 + 1)^{2k}} \int_0^{\frac{\beta^2 + 1}{4} (\frac{\rho}{b})^2} \xi^{2k} e^{-\xi} d\xi \\
&= \frac{2\beta}{\beta^2 + 1} \sum_{k=0}^{\infty} \frac{(\beta^2 - 1)^{2k}}{(k!)^2 4^k (\beta^2 + 1)^{2k}} \gamma(2k + 1, \frac{\beta^2 + 1}{4} (\frac{\rho}{b})^2)
\end{aligned} \tag{7}$$

Here

$$\gamma(k, x) = \int_0^x \xi^{k-1} e^{-\xi} d\xi$$

is the incomplete gamma function [3]. The interchange of integration and summation is justified as the summation is uniformly convergent, as can be seen by the Weierstrass M-test. Alternately one could use the Lebesgue bounded convergence theorem. The series converges for all $\beta \geq 1$, and this can be shown using the ratio test. The asymptotic ratio of the terms is:

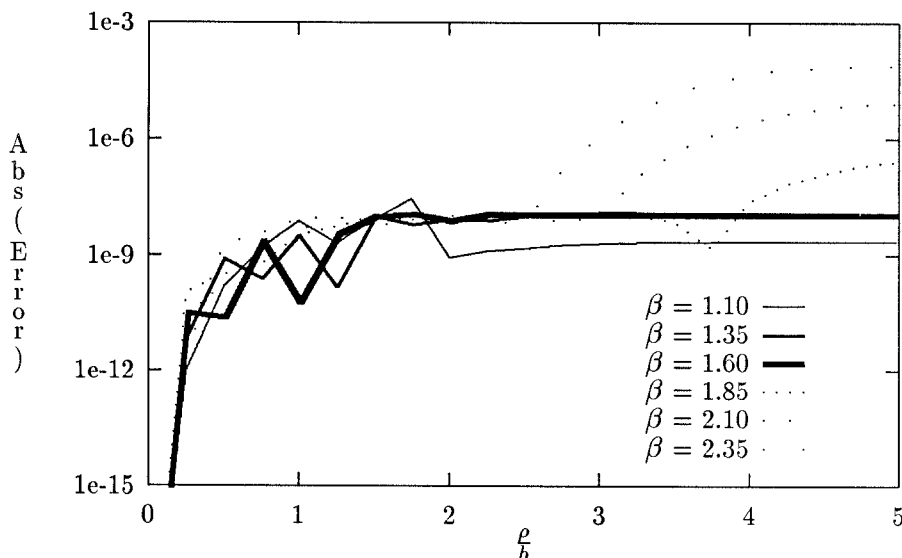
$$\frac{a_k}{a_{k+1}} \sim \frac{1}{4} + \frac{1}{4k + 1}. \tag{8}$$

By the analogous arguments we can show

$$P_\rho = 1 - \frac{2\beta}{\beta^2 + 1} \sum_{k=0}^{\infty} \frac{(\beta^2 - 1)^{2k}}{(k!)^2 4^k (\beta^2 + 1)^{2k}} \Gamma(2k + 1, \frac{\beta^2 + 1}{4} (\frac{\rho}{b})^2) \tag{9}$$

where $\Gamma(k, x) = \int_x^\infty \xi^{k-1} e^{-\xi} d\xi$ is the (other) incomplete gamma function. Figures 2 and 3 show typical interpolation results using these series to show that these methods are computationally effective; in all cases less than ten terms of the (appropriate) series were used. The ‘truth’ model, integration of Eq. 1, was accurate to eight decimal places – hence the floor on the approximation

Figure 2: The error between the γ -series and direct integration of the integral for $\beta > 1$.



error occurs at that value. The γ series seems to be at least as accurate as the Γ series, and is often better; we see no particular reason for this and presume that it reflects the accuracy of the routines use to compute the incomplete gamma functions or imperfections in the ‘truth’ model ([4]).

3.2 Shanks Transformation

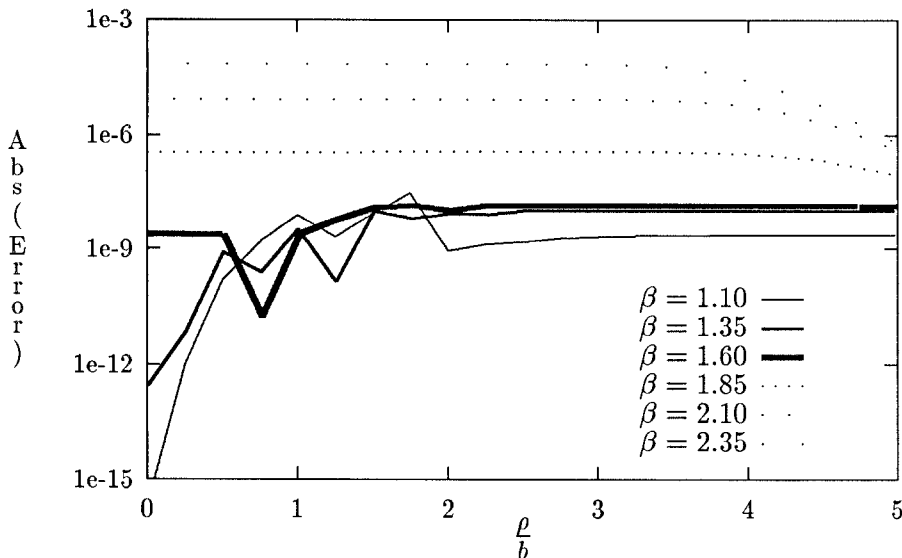
The asymptotic ratio of the terms of the summation (Eq. 8) shows that the series is a good candidate for using Shanks transformation [2]. This technique applies a transformation on the sequence of partial sums in order to accelerate convergence. Specifically let A_n be the n^{th} partial sum of $\sum_j a_j$ (we are thing of Eq. 7). Then the Shanks transformation of the sequence of partial sums $\{A_n \rightarrow A\}$ is given by

$$S(A_n) = S_n = \frac{A_{n+1}A_{n-1} - A_n^2}{A_{n+1} + A_{n-1} - 2A_n}.$$

This transformation is most effective if the partial sum has the general form $A_n = A + \alpha q^n$ (with $|q| < 1$). In this case one can show that $S(A_n) = A$. Thus if the partial sums are dominated by a single ‘transient’ one expects that Shanks transformation can be used to accelerate the convergence of the series. It can be shown [2] that if $A_n = \sum_{k=0}^n a_k$ is the series and $\frac{a_{n+1}}{a_n} \sim c_0 + \frac{c_1}{n} + \dots$ then

$$S_n - S_{n-1} \sim -a_n \frac{c_1}{(1 - c_0)^2 n^2}$$

Figure 3: The error between the Γ -series and direct integration of the integral for $\beta > 1$.



and hence the sequence S_n has an accelerated convergence rate (as compared to the original series A_n). Figures 4 and 5 show typical interpolation results using the accelerated series. Repeated application of Shanks transformations are possible to further accelerate convergence; however this was not investigated. Figures 6 and 7 gives typical convergence behavior of the partial sum as a function of the number of terms (with and without Shanks transformation). Clearly the Shanks transformation is useful for the γ -series for large β . The Shanks transformation is very helpful for the Γ -series for β in general (compare Figures 3 and 5), and especially helpful near one. This is not surprising as the regions where the Shanks transformation is effective are precisely the regimes in which the respective series are closest to the bounding series $a_n = (2n)!/(n!)^2$ and hence have the prescribed form for the ratio of terms (i.e. Eq. 8). Fortunately for both series these are the regions in which convergence is the slowest – so some acceleration is useful.

4 Conclusions

We have presented a simplified derivation of the CEP integral in its usual form. In addition, two series have been presented which can be used to compute the CEP integral. The domain of applicability of these series and methods for acceleration of the convergence of these series have been discussed. The series method provides a computationally efficient method for the calculation of the CEP integral.

Figure 4: The error between the Shanks' accelerated γ -series and direct integration of the integral for $\beta > 1$.

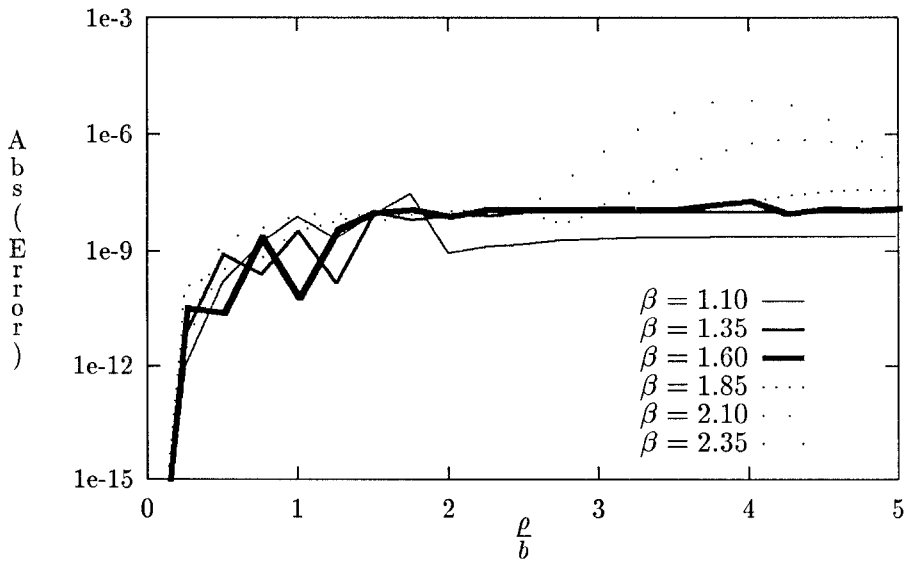


Figure 5: The error between the Shanks' accelerated Γ -series and direct integration of the integral for $\beta > 1$.

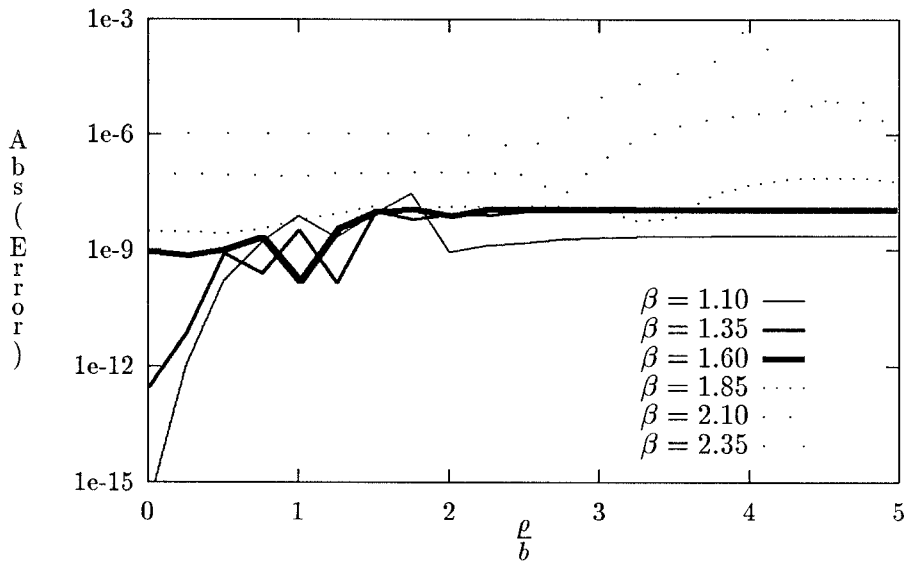


Figure 6: The γ -series approximation error, as a function of the number of terms in the partial sum, for several points ($\beta = 1.5$), with and without the Shanks transformation.

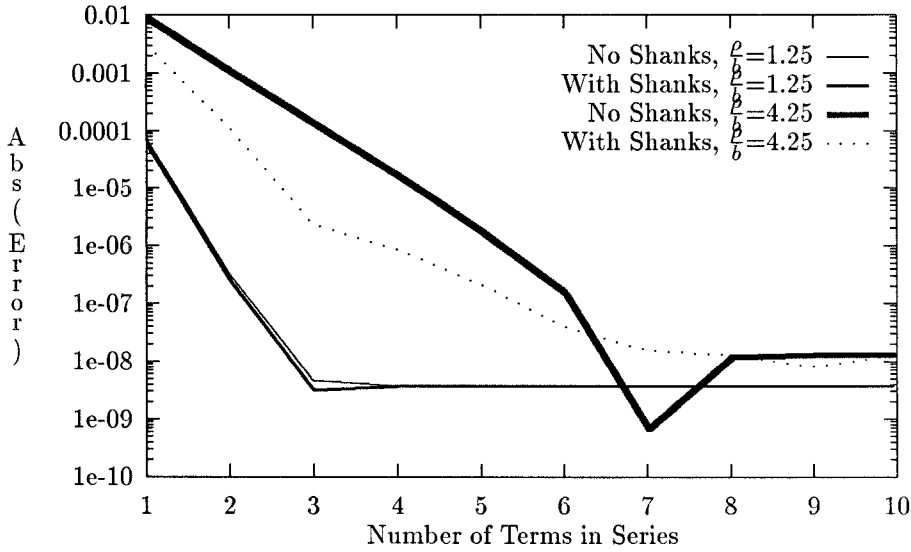
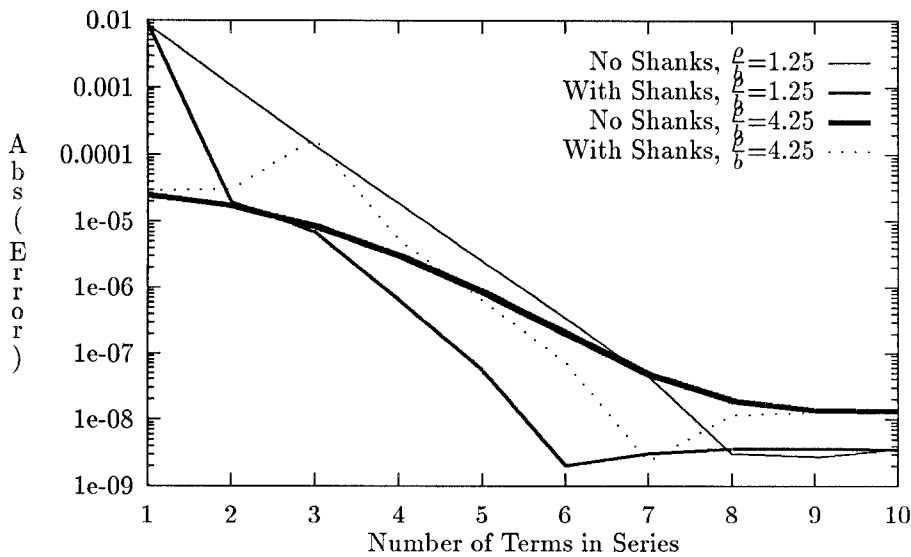


Figure 7: The Γ -series approximation error, as a function of the number of terms in the partial sum, for several points ($\beta = 1.5$), with and without the Shanks transformation.



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