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An Analog Tracking System

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AN ANALOG TRACKING SYSTEM

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ABSTRACT

"An Analog Tracking System"

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This paper investigates the application of classical PID control laws to the special case of an analog tracking system. Through a step-by-step development of this tracking system, several fundamental properties of general PID control are highlighted. A broad class of circuits is shown to satisfy the PID transfer function, which leads to a discussion of performance objectives and error considerations. Once the design has been completed, the performance of the tracking circuit is analyzed in a variety of scenarios. Three examples verify the concepts presented and illustrate the advantages of PID control.

AN ANALOG TRACKING SYSTEM

Outline of Contents

INTRODUCTION

The following discussion is an introduction to Proportional Integral Derivative (PID) control as used in tracking systems. The forthcoming development of a tracking computer should lead to an intuitive grasp of general PID control. The reader should be familiar with basic systems terminology and methods.

This report was produced through a combination of three microcomputer programs. Framework II handled all word processing. ProDesign II generated all diagrams and equations. Numerical computations for the examples were performed on MathCad version 1.1.

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A GENERAL TRACKING SYSTEM: DESCRIPTION AND CONSTRUCTION

The primary aim of this paper is to develop and investigate an analog tracking system which implements the Proportional Integral Derivative (PID) method of process control. In developing such a system, the characteristics and properties of general PID control should become clear. Simply stated, tracking is a specific control application which involves matching the output of a given system to its input. Tracking systems are useful wherever a process or plant must adjust to changing external conditions. These external changes may be intentional or unintentional and may be either abrupt or slowly time varying.

A good example of a tracking system in just this setting may be found in the navigation system of a large ship. The Captain wishes to change the ship's velocity from A to B. The tracking system initiates and monitors this intentional change until the ship's actual velocity matches the Captain's desired velocity, B. The ship is also subject to various unintentional external influences such as ocean currents, winds and similar unpredictable conditions. The tracking system must take these perturbations into account and make adjustments accordingly. Clearly intentional and unintentional inputs are fundamentally different both in source and effect; intentional inputs must be tracked as closely as possible while unintentional inputs robustly overcome. This paper will consider only applied inputs in designing the tracking circuit.

Elements of a Tracking System

No matter how simple or complex, any tracking system (less external perturbations) can be modeled in the following general manner:

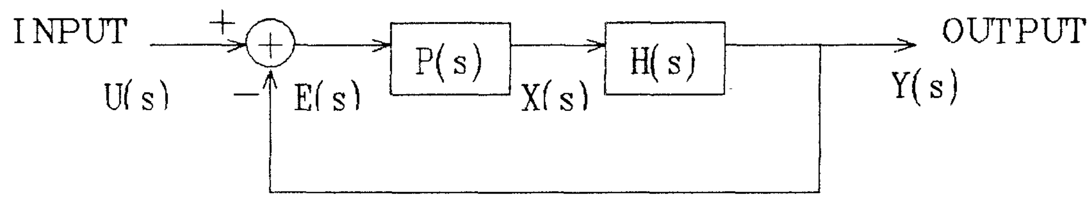


Figure 1: General Tracking

With system transfer function:

$$Q(s) = \frac{Y(s)}{U(s)} = \frac{PH}{1+PH} \tag{1}$$

The system input function, U(s), represents any arbitrary transformed input function, while Y(s) is the transformed output function, also called the system output.

The "system function," H(s), is a mathematical model of the process to be controlled. Though assumed given, H(s) may be very difficult if not impossible to find. For the large ship, even a modest approximation to H(s) would be of high order. For ease of calculation, H(s) will remain manageably small in the following discussion. This is not a serious restriction as a large number of processes can be modeled adequately through a lower order system function.

The heart of any tracking system lies in its "compensator function," P(s) (Oppenheim, 696). Operating on the error signal, E(s), P(s) governs how fast and in what fashion the output tracks the applied input. P(s) generates a compensated input, X(s). X(s) may justly be considered the controller's "improved" version of the input to the process. The success or failure of the entire system rests in the design of P(s). A suitable definition for a successful controller will be considered later.

Note that tracking as defined need not necessarily be one to one; tracking to within a multiplicative constant is completely acceptable. Given steady-state output of 1/K in response to a unit step input, simply inserting an amplifier of appropriate gain, K, at the output of the compensator returns one to one tracking to the entire system:

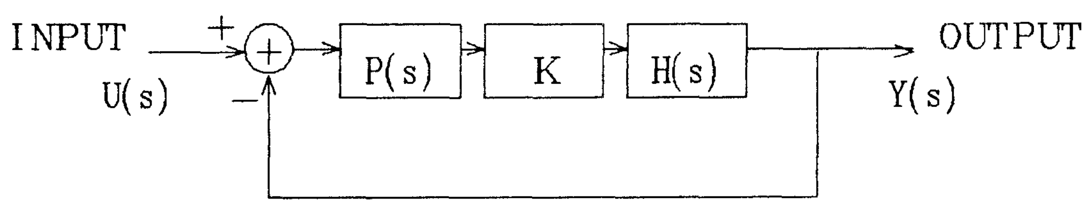


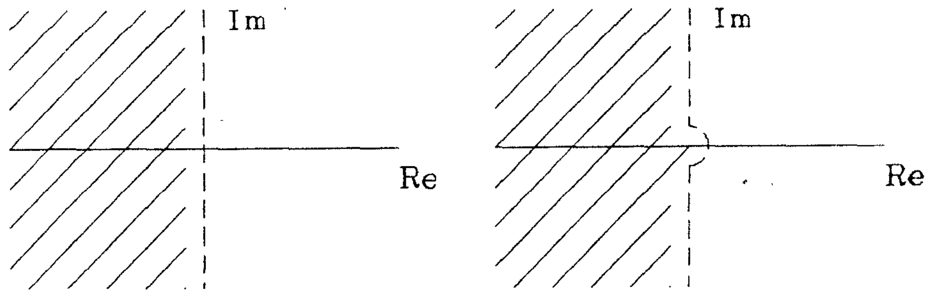
Figure 2: Tracking with Amplifier

The system transfer function now becomes:

$$(2) \qquad Q(s) = \frac{Y(s)}{U(s)} = \frac{PKH}{1+PKH}$$

Both transfer functions, (1) and (2), are of the same functional form, hence both are trackers, the latter being one to one.

The slightly generalized definition of tracking described above also generalizes the region of stability in the complex plane to include the origin.



Region of Stability for One to One Tracking

Region of Stability for Tracking Within a Multiplicative Constant

PID Control Description

For PID control, the compensator function, P(s), is also called the PID control equation and represents a transfer function (Refer to Figure 1):

$$(3) \quad P(s) = \frac{X(s)}{E(s)}$$

As this name suggests, P(s) is of second order and superficially resembles the differential equation describing the classical RLC circuit. For this reason, the PID method is often referred to as the "classical" approach.

In the time domain,

$$(4) \quad x(t) = P\,e(t) + I\int_{t_0}^{t} e(\tau)\,d\tau + D\frac{d}{dt}e(t)$$

Which transforms to

(5) $$X(s) = PE(s) + \frac{I}{s}E(s) + Ds\,E(s)$$

in the frequency domain.

Therefore equation (3) may be written,

(6) $$P(s) = \frac{X(s)}{E(s)} = P + \frac{I}{s} + sD$$

In the above equations, P, I, and D are control constants associated with proportion, integration, and differentiation respectively. These constants control various aspects of system output and may be tuned to produce a wide variety of desired response curves.

Notice that P(s) comprises terms which are linearly independent; this has significant physical implications. A PID controller may be considered a linear combination of simpler controllers. For example, a full PID controller may be considered a "P + I + D", "PI + D", or "P + ID." Thus, any controller (and associated characteristics, good or bad) may be completely removed simply by setting its control constant to zero.

Logic Underlying a PID Controller

The reasoning behind the PID controller is intuitive and easily understood. The general PID and all related controllers operate on an error signal generated from the output. Each element in the controller reacts to a different characteristic of this signal. In a well-tuned circuit, each linearly independent component complements the others, contributing uniquely to improve system tracking.

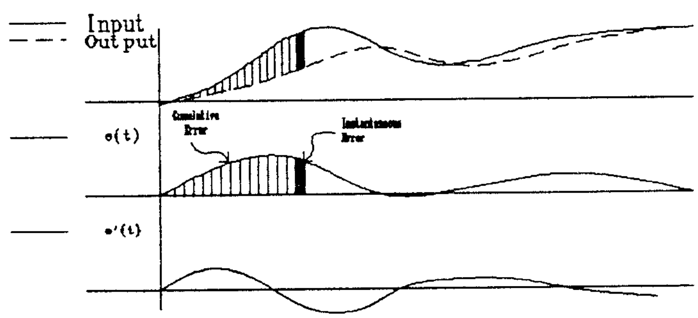


Figure 3: Error and its Derivative

The proportional portion of a PID controller uses the instantaneous value of error shown in Figure 3. In the proposed circuit of Figure 5, this error becomes a voltage source driving the controller. Since negative feedback is used, the P controller opposes any difference between input and output (i.e. error) and works to minimize this difference. The result is eventual tracking; when the error becomes zero (or even merely constant) the P controller is "satisfied" and no longer strives to correct the output.

The integral part of the controller works very much like its proportional counterpart except that the entire history of the error function is used to track the input function as opposed to merely the instantaneous error (Figure 4). Operating on this weighted error signal, the I controller reacts more slowly to higher-frequency inputs than does the P controller. In this sense, an I controller might well serve to eliminate the undesirable external perturbations affecting the large ship mentioned earlier.

The derivative controller, modeled by an inductor, functions quite differently than its cousin controllers. Unlike the capacitor found in the I-controller, the inductor acts strongly to control abrupt, high-frequency inputs. This will be evident in the D-controller's reaction to the unit step input.

In short, the PID is a combination of quite different controllers. The integrator performs best in a slowly time varying setting as in the slow wearing of a gear with age. The D-controller finds use in controlling abrupt events such as governing a fracture of the same gear. The proportional element, in situations it can govern, generally serves equally well for all frequencies.

Construction of a PID Controller

Having identified P(s), the next step is to devise a circuit which generates P(s), the PID controller's transfer function. All of the preceding discussion would be meaningless if such a device could not be built to suit.

As a first attempt at constructing a PID control block, recall the similarity of equation (4) to the series RLC differential equation. Possibly an RLC circuit might generate P(s) as defined above.

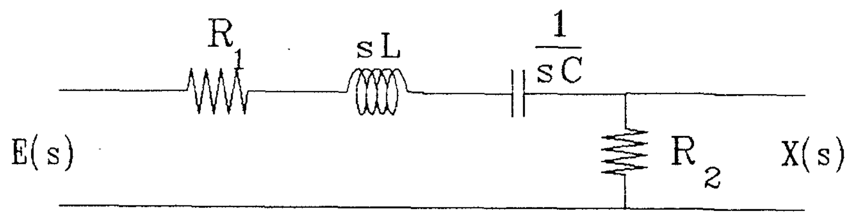


Figure 4: Series RLC Circuit

In this case:

$$(7)\quad \frac{1}{P(s)} = \frac{E(s)}{X(s)} = \frac{R_1 + R_2}{R_2} + s\frac{L}{R_2} + \frac{1}{CR_2}\frac{1}{s}$$

Where,

$$(8)\quad P = \frac{R_1 + R_2}{R_2}\ ,\ I = \frac{1}{CR_2}\ ,\ D = \frac{L}{R_2}$$

One sees immediately that this circuit won't work. Input, E(s), and output, X(s), are reversed, and there the similarity of P(s) to an RLC circuit ends. As it turns out, no combination consisting solely of RLC elements (i.e. passive elements) will work. A different kind of circuit altogether is needed.

Recall the basic transfer function of an inverting operational amplifier:

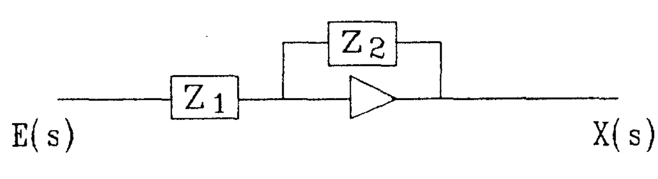


Figure 5: Inverting Amplifier

$$(9) \quad P(s) = \frac{X(s)}{E(s)} = -\frac{Z_2}{Z_1}$$

This is very promising. Taking this idea a step further, a series of n operational amplifiers in series might appear so:

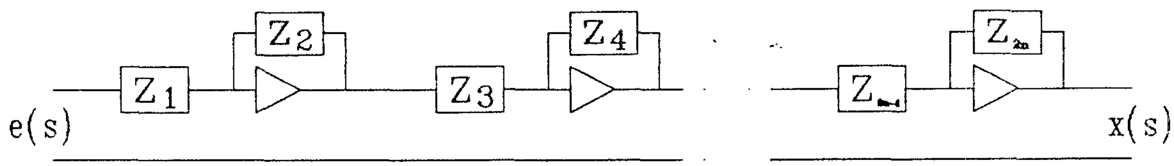


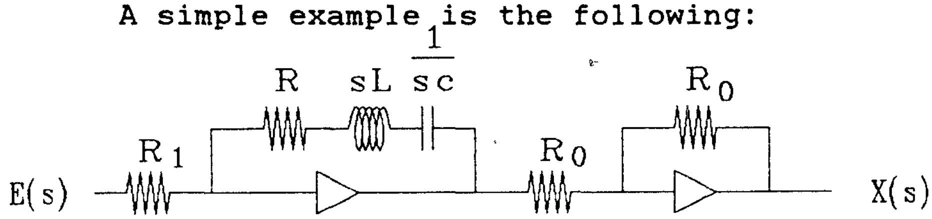
Figure 6: n Inverting Operational Amplifiers in Series

This construction generates the transfer function:

$$(10) \quad P(s) = \frac{X(s)}{E(s)} = (-1)^n \prod_{k=1}^{n} \frac{Z_{2k}}{Z_{2k-1}}$$

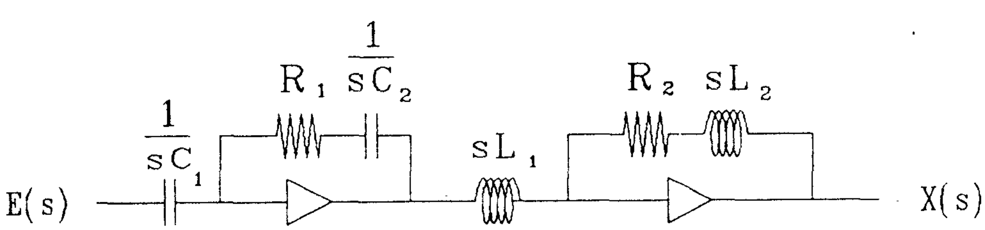
From here, any number of configurations may be assembled which satisfy the compensator function, P(s). Any P, PI, PD, ID, or higher order controller may similarly be built.

A simple example is the following:



Where: $P = \frac{R}{R_1}$, $I = \frac{L}{R_1}$, $D = \frac{1}{R_1 C}$

A more complicated circuit might likewise appear:



Note that the control constants P, I, and D can never be negative as this would require negative-valued circuit elements. This is not inconvenient; for tracking applications considered below, only positive constants are useful.

It is worth mentioning that the above and all similar considerations which involve securing "real" components do not exist in the digital/discrete domain. Negative and purely imaginary components are available in digital form. There are many advantages in using a computer to solve control problems if one is willing to work in discrete time.

An Analog Computer

Circuits like those described above now make possible a programmable analog computer implementing P(s), the sole function of which is to track input functions.

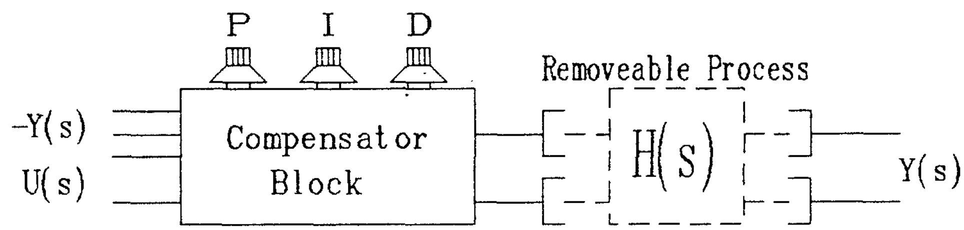


Figure 7: Analog Computer

This computer is programmable in the sense that the values of P, I, and D are at any instant the "program" contained in the computer's memory. The values of these control constants may be changed depending on the particular process attached to the compensator block.

Testing Considerations and Objectives

Because the elements comprising a PID controller are linearly independent, it make sense to test and compare the responses of each constituent controller building up to and including a full PID controller (that is: P, I, D, PD, PI, ID, PID). In this way, higher order controllers may be quantitatively compared to lower order controllers.

To evaluate each controller, a unit step input will be applied. Two important reasons make the unit step function a sensible choice. First, the unit step response completely defines any linear system's response to an arbitrary input. Secondly, the resulting system output immediately reveals a particular controller's ability (or inability) to track a given input, something its unit impulse response does not provide quite so readily. On the flip side, the unit step is an idealization and not physically attainable. The unit step input will produce some unexpected effects in the output, especially at time t=0. (See Appendix 2.)

In order to rate and compare the success of different controllers as well as to compare the same controller operating under varying controller settings, a "good" response must be defined. Such a definition would ideally be applicable to any controller governing any process.

To begin, one broad requirement for a good response is that it converge to within a multiplicative constant of the applied input. This precludes any possibility of instability. As shown earlier, an amplifier could be attached to the output of the compensator to provide one to one convergence, if needed.

There are many ways by which to evaluate the system output of a given controller. A straightforward approach is simply to evaluate the absolute value of the error in system response. Earlier the error function, E(s), was defined as the difference between the applied input, U(s), and the resulting output, Y(s). A definition for a good response may be based on this error. Let an <u>Error Criterion</u>, E, be defined,

$$(11) \qquad E = \int_0^\infty \left| e(t) \right| dt$$

Intuitively, a value of zero for E signifies a perfect response. Conversely, if E tends toward infinity, the response is not stable. Evidently this definition incorporates the aforementioned convergence requirement. The lower the value of E, the better the response.

There are undoubtedly many, more sophisticated possibilities for rating a given response curve, however, for the purposes of this discussion, the above definition will suffice.

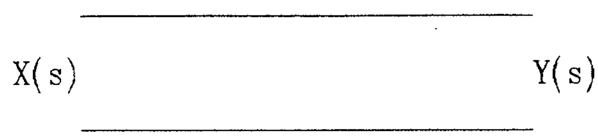
As a last general restriction, assume all controller "knobs" to be set at time t=0 and not allowed to vary thereafter. Here simplification is the main motive, though one could easily imagine intelligently time-varying control constants as a tremendous aid in controlling a process (i.e. $P = P(t)$, $I = I(t)$, etc.).

Controller Analysis

Having arrived at an appropriate input function, the unit step, and having defined a rating system by which to compare various response curves, E, the time has arrived to thoughtfully investigate a series of related controllers, ranging from no control to a full PID controller. For each controller, several important results will be tabulated. These results include the frequency and time responses as well as general system response characteristics. In this way one might quickly note similarities and differences between the successive controllers.

The three examples which follow use fairly representative lower-order processes and should give a good indication of the typical response curves associated with each controller. Solutions for the first two examples are given in general form, that is, no specific values are assigned to the control constants, P, I, and D. In the third example, the control constants are set equal to one and then solutions are obtained. In general, the error function is a function of the control constants and may well be written: $e(t) = e(t,P,I,D)$. For this reason, the specific values for the error criteria will be given only for the third example where P, I, and D are assigned actual values ($P=I=D=1$).

Example 1: Zeroth order process.



$H(s) = 1$

Input: Unit Step Function.

Steady State Response

Controller Type	Stable	Final Value	Amplifier Required
Open Loop	Yes	1	No
P	Yes	$\frac{P}{P + 1}$	Yes
I	Yes	1	No
D	No	0	N/A
PD	Yes	$\frac{P}{P + 1}$	Yes
ID - Over Damped	Yes	1	No
ID - Critically Damped	Yes	1	No
ID - Under Damped	Yes	1	No
PID - Over Damped	Yes	1	No
PID - Critically Damped	Yes	1	No
PID - Under Damped	Yes	1	No

Frequency and time responses for this example follow.

Example 1 - Frequency and Time Responses

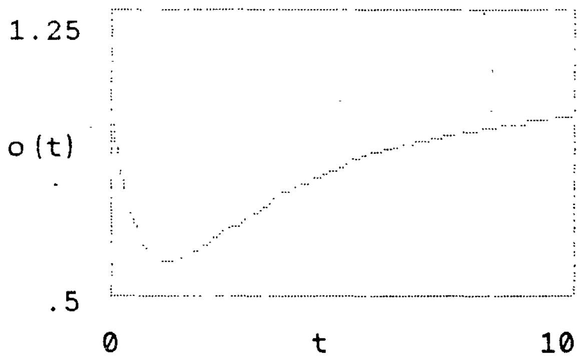
Ctrl	Y(s)	y(t)
Open	$\frac{1}{s}$	$u(t)$
P	$\frac{P}{P+1}\frac{1}{s}$	$u(t)\frac{P}{P+1}$
I	$\frac{I}{s\,(s+I)}$	$u(t)\cdot\left[1-e^{-(I\,t)}\right]$
D	$\frac{1}{s+\frac{1}{D}}$	$u(t)\cdot e^{\frac{-1}{D}t}$
PD	$\frac{P+sD}{s\cdot(sD+(P+1))}$	$u(t)\cdot\frac{1}{P+1}\cdot\left[P+e^{\frac{-(P+1)}{D}t}\right]$
PI	$\frac{P+\frac{I}{s}}{(P+1)\cdot s+I}$	$u(t)\cdot\left[1-\frac{1}{P+1}e^{\frac{-I}{P+1}\cdot t}\right]$

Ctrl	Y(s)	y(t)
ID	$\dfrac{\frac{I}{s} + D s}{D \cdot s^2 + s + I}$	Case: ID < .25, Over Damped. $u(t) \cdot \left[1 + \frac{1}{D(a - b)} \left[e^{a t} - e^{b \cdot t}\right]\right]$ where: $a = \frac{-1}{2 \cdot D} + \frac{1}{2} \sqrt{\frac{1}{D^2} - \frac{4 \cdot I}{D}}$ $b = \frac{-1}{2 \cdot D} - \frac{1}{2} \sqrt{\frac{1}{D^2} - \frac{4 \cdot I}{D}}$
ID	"	Case: ID = .25, Critically Damped. $u(t) \left[1 - \frac{1}{D} \cdot t \cdot e^{a \cdot t}\right]$ where: $a = -\left[\frac{1}{2 D}\right]$
ID	"	Case: ID > .25, Under Damped. $u(t) \cdot \left[1 - \frac{1}{Db} \cdot e^{a \cdot t} \cdot \sin(b \cdot t)\right]$ where: $a = -\left[\frac{1}{2 D}\right]$ $b = \sqrt{\frac{I}{D} - \frac{1}{4 D^2}}$

Ctrl	Y(s)	y(t)
PID	$\dfrac{P + \frac{I}{s} + D s}{D s^2 + s + I}$	Case: $ID < \dfrac{(P + 1)^2}{4}$, Over Damped. $u(t) \left[1 - \dfrac{1}{D\,(a - b)} \left(e^{a t} - e^{b t} \right) \right]$ where: $a = -\left[\dfrac{P + 1}{2 \cdot D} \right] + \dfrac{1}{2} \sqrt{\dfrac{1}{D^2} - \dfrac{4 I}{D}}$ $b = -\left[\dfrac{P + 1}{2 \cdot D} \right] - \dfrac{1}{2} \sqrt{\dfrac{1}{D^2} - \dfrac{4 I}{D}}$
PID	"	Case: $ID = \dfrac{(P + 1)^2}{4}$, Critically Damped. $u(t) \left[1 - \dfrac{1}{D} t\, e^{a t} \right]$ where: $a = -\left[\dfrac{P + 1}{2 \cdot D} \right]$
PID	"	Case: $ID > \dfrac{(P + 1)^2}{4}$, Under Damped. $u(t) \left[1 - \dfrac{1}{Db} e^{a t} \sin(b\, t) \right]$ where: $a = -\left[\dfrac{1}{2 \cdot D} \right]$ $b = \sqrt{\dfrac{I}{D} - \dfrac{1}{4\, D^2}}$

Some Typical PID Response Curves For Example 1

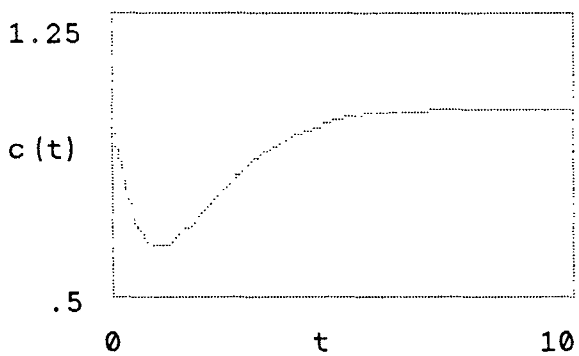
Over damped: P=1, D=1, I=.5



Error:

$$\int_0^{15} |1 - o(t)| \, dt = 1.933$$

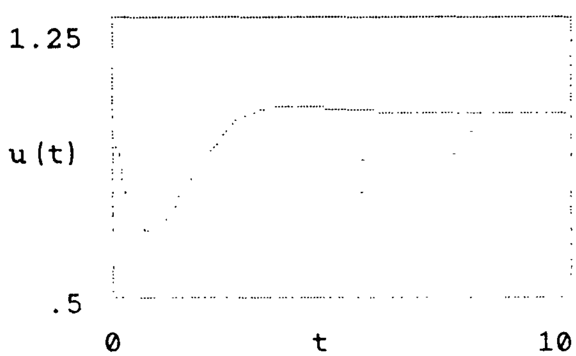
Critically damped: P=1, D=1, I=1



Error:

$$\int_0^{15} |1 - c(t)| \, dt = 1$$

Under damped: P=1, D=1, I=2



Error:

$$\int_0^{15} |1 - u(t)| \, dt = 0.545$$

Example 1 - Summary

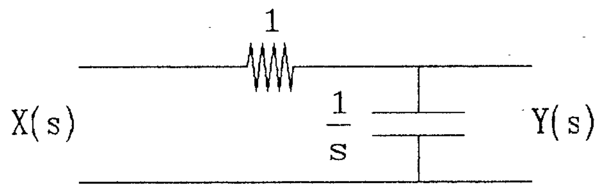
1) The most obvious result from this simple example is that the open loop response is perfectly matched to the input. At once this leads one to conclude that there are at least some processes better left alone. Simply setting P, I, and D equal to zero accomplishes this. The P controller can also produce perfect tracking if an amplifier of gain $\frac{P+1}{P}$ is present in the feed forward path.

2) The D controller can not stand alone as a tracker. Only in tandem with a P or I controller or in combination with both the P and I can the D controller prove useful as a tracking controller.

3) The PD and PI controllers are both first order and produce exponentially rising response curves which converge uniformly to 1. The time constant in each case is a function of the control constants, P, I, and D.

4) The ID and PID controllers are both second order and contain three types of response curves depending on the settings of the control constants: over damped, critically damped, and under damped. The error criterion, as defined, does not penalize a response curve for alternating sign as do some other methods for evaluating a given response. One consequence of this is that of the three types, the under damped response usually (though not always) rates the highest.

Example 2: First order process.



$$H(s) = \frac{1}{s+1}$$

Input: Unit Step Function.

Steady State Response

Controller Type	Stable	Final Value	Amplifier Required
Open Loop	Yes	1	No
P	Yes	$\frac{P}{P+1}$	Yes
I - Over Damped	Yes	1	No
I - Critically Damped	Yes	1	No
I - Under Damped	Yes	1	No
D	No	0	N/A
PD	Yes	$\frac{P}{P+1}$	Yes
PI - Over Damped	Yes	1	No
PI - Critically Damped	Yes	1	No
PI - Under Damped	Yes	1	No
ID - Over Damped	Yes	1	No
ID - Critically Damped	Yes	1	No
ID - Under Damped	Yes	1	No
PID - Over Damped	Yes	1	No
PID - Critically Damped	Yes	1	No
PID - Under Damped	Yes	1	No

Frequency and time responses for this example follow.

Ctrl	Y(s)	y(t)
Open	$\frac{1}{s\,(s+1)}$	$u(t)\left[1 - e^{-t}\right]$
P	$\frac{P}{s\,(s + (P+1))}$	$u(t)\,\frac{P}{P+1}\left[1 - e^{-(P+1)\,t}\right]$
D	$\frac{D}{(D+1)\,s + 1}$	$u(t)\left[\frac{D}{D+1}\right] e^{-(D+1)\,t}$
I	$\frac{I}{s\left[s^{2} + s + I\right]}$	Case: I < .25 , Over Damped. $u(t)\left[1 - \frac{1}{a-b}\left[(a+1)\,e^{a\,t} - (b+1)\,e^{b\,t}\right]\right]$ where: $a = \frac{-1}{2} + \frac{1}{2}\sqrt{1 - (4\,I)}$ $b = \frac{-1}{2} - \frac{1}{2}\sqrt{1 - (4\,I)}$
I	"	Case: I = .25 , Critically Damped. $u(t)\left[1 - e^{-.5\,t} - .5\,e^{-.5\,t}\right]$
I	"	Case: I < .25 , Under Damped. $u(t)\left[1 - e^{-.5\,t}\left[\cos(b\,t) + \left[\frac{1}{2b}\right]\sin(b\,t)\right]\right]$ where: $b = \sqrt{I - .25}$
PD	$\frac{P + sD}{s\,((D+1)\,s + (P+1))}$	$u(t)\left[\frac{P}{P+1} + \left[\frac{D}{D+1} - \frac{P}{P+1}\right] e^{-\left[\frac{P+1}{D+1}\right] t}\right]$

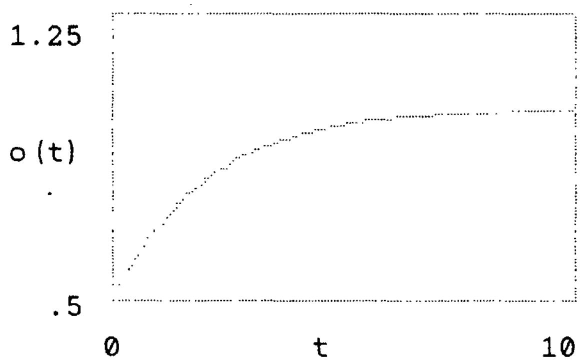
Ctrl	Y(s)	y(t)
PI	$\dfrac{P + s D}{s((D + 1)s + (P + 1))}$	Case: $(P + 1)^2 > 4 I$, Over Damped. $u(t)\left[1 + \dfrac{1}{D(a - b)}\left[e^{a t} - e^{b t}\right]\right]$ where: $a = -\left[\dfrac{P + 1}{2}\right] + \dfrac{1}{2}\sqrt{(P + 1) - 4 I}$ $b = -\left[\dfrac{P + 1}{2}\right] - \dfrac{1}{2}\sqrt{(P + 1) - 4 I}$
PI	"	Case: $(P + 1)^2 = 4 I$, Critically Damped. $u(t)\left[1 - e^{a t} - (a + 1)\, t\, e^{a t}\right]$ where $a = -\left[\dfrac{P + 1}{2}\right]$
PI	"	Case: $(P + 1)^2 < 4 I$, Under Damped. $u(t)\left[1 - e^{a t}\left[\cos(b\,t) + \dfrac{a + 1}{b}\sin(b\,t)\right]\right]$ $a = -\left[\dfrac{P + 1}{2}\right]$ $b = \sqrt{I - \dfrac{(P + 1)^2}{4}}$

Ctrl	Y(s)	y(t)
ID	$\dfrac{\left[\dfrac{I}{s}\right] + s \cdot D}{(D + 1)s^2 + s + I}$	Case: $\dfrac{1}{D + 1} > 4\ I$, Over Damped. $u(t)\left[1 - \dfrac{1}{(D + 1)(a - b)}\left[(a + 1)e^{at} - (b + 1)e^{bt}\right]\right]$ where: $a = -\left[\dfrac{1}{2(D + 1)}\right] + \dfrac{1}{2}\sqrt{\left[\dfrac{1}{(D + 1)^2}\right] - 4\dfrac{I}{D + 1}}$ $b = -\left[\dfrac{1}{2(D + 1)}\right] - \dfrac{1}{2}\sqrt{\left[\dfrac{1}{(D + 1)^2}\right] - 4\dfrac{I}{D + 1}}$
ID	"	Case: $\dfrac{1}{D + 1} = 4 \cdot I$, Critically Damped. $u(t) \cdot \left[1 - \dfrac{1}{D + 1}\left[e^{a \cdot t} + (a + 1) \cdot t\ e^{a \cdot t}\right]\right]$ where: $a = -\left[\dfrac{1}{2\ (D + 1)}\right]$
ID	"	Case: $\dfrac{1}{D + 1} < 4\ I$, Under Damped. $u(t) \cdot \left[1 - \dfrac{e^{a\ t}}{D + 1}\left[\cos(b\ t) + \dfrac{1}{b}\sin(b\ t)\right]\right]$ where: $a = -\left[\dfrac{1}{2\ (D + 1)}\right]$ $b = \sqrt{\left[\dfrac{I}{D + 1}\right] - \dfrac{1}{4\ (D + 1)^2}}$

Ctrl	Y(s)	y(t)
PID	$\dfrac{P + \frac{I}{s} + s D}{(D + 1)s^2}$ $(P + 1)\, s + I$	Case: $\dfrac{(P + 1)^2}{D + 1} > 4\, I$, Over Damped. $u(t) \left[1 - \dfrac{1}{(D + 1)(a - b)} \left[(a + 1)e^{at} - (b + 1)e^{bt} \right. \right.$ where: $a = -\left[\dfrac{P + 1}{2(D + 1)}\right] + \dfrac{1}{2}\sqrt{\left[\dfrac{P + 1}{(D + 1)}\right]^2 - 4\, \dfrac{I}{D + 1}}$ $b = -\left[\dfrac{P + 1}{2(D + 1)}\right] - \dfrac{1}{2}\sqrt{\left[\dfrac{P + 1}{(D + 1)}\right]^2 - 4\, \dfrac{I}{D + 1}}$
PID	"	Case: $\dfrac{(P + 1)^2}{D + 1} = 4\, I$, Critically Damped. $u(t) \left[1 - \dfrac{1}{D + 1} \left[e^{a t} + (a + 1)\, t\, e^{a t} \right] \right]$ where: $a = -\left[\dfrac{P + 1}{2\,(D + 1)}\right]$
PID	"	Case: $\dfrac{(P + 1)^2}{D + 1} < 4\, I$, Under Damped. $u(t) \left[1 - \dfrac{e^{a t}}{D + 1} \left[\cos(b\, t) + \dfrac{1}{b} \sin(b\, t) \right] \right]$ where: $a = -\left[\dfrac{P + 1}{2\,(D + 1)}\right]$ $b = \sqrt{\left[\dfrac{I}{D + 1}\right] - \left[\dfrac{1}{4}\right]\left[\dfrac{P + 1}{D + 1}\right]^2}$

Some Typical PID Response Curves For Example 2

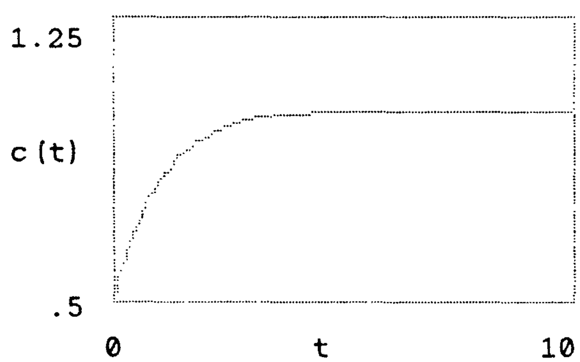
Over damped: P=2, I=1, D=1



Error:

$$\int_0^{15} |1 - o(t)|\ dt = 1$$

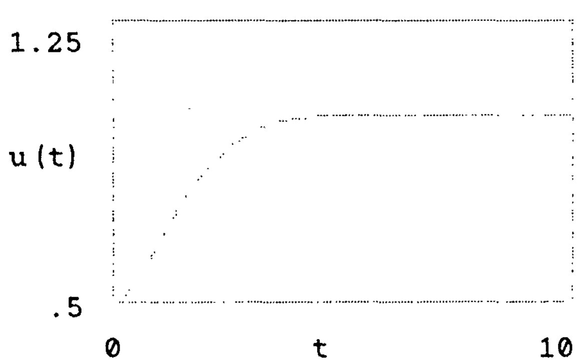
Critically damped: P=3, I=2, D=1



Error:

$$\int_0^{15} |1 - c(t)|\ dt = 0.5$$

Under damped: P=1, I=1, D=1



Error:

$$\int_0^{15} |1 - u(t)|\ dt = 0.804$$

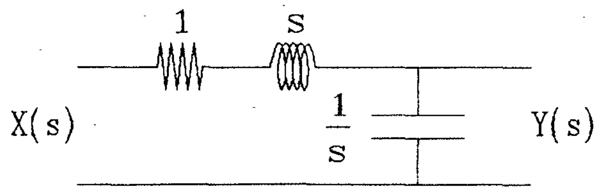
Example 2 - Summary

1) This example serves to reinforce the fact that the best response curve (minimum E) is strongly dependent on the values given the control constants.

2) The error criterion for the open loop response is 1. One would expect there to be some values of P, I, and D for which E would be less than 1. These values do exist and are illustrated below.

3) An interesting observation is that the second order controllers, the ID and PID, generate response curves which are discontinuous at time $t = 0$. In the real world, this is not possible, but then the unit step is not a physically realizable input function. A continuous input function will always produce a feasible output function (see Appendix 2).

Example 3: Second order process.



$$H(s) = \frac{1}{s^2 + s + 1}$$

Steady State Response

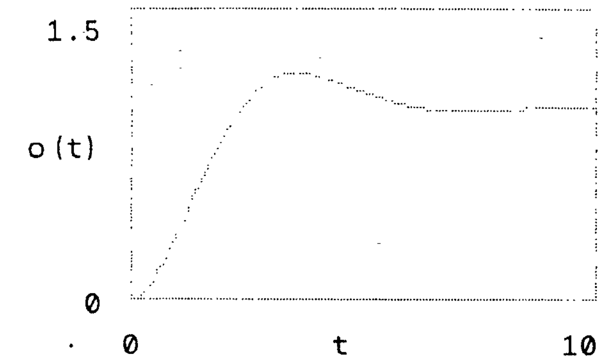
Controller Type	Stable	Final Value	Amplifier Required	Error Criterion
Open Loop	Yes	1	No	1.713
P	Yes	$\frac{1}{2}$	Yes	1.481
I	No	- -	N/A	- -
PD	Yes	$\frac{1}{2}$	Yes	1.140
PI	Yes	1	No	2.479
ID	Yes	1	No	3.004
PID	Yes	1	No	1.000

Frequency and time responses for this example follow.

Example 3 - Frequency and Time Responses

Ctrl	Y(s)	y(t)
Open	$\frac{1}{s\left(s^2+s+1\right)}$	$u(t)\left[1-e^{-.5t}\left[\cos\left(\frac{\sqrt{3}}{2}t\right)+\frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]\right]$
P	$\frac{1}{s\left(s^2+s+2\right)}$	$u(t)\frac{1}{2}\left[1-e^{-.5t}\left[\cos\left(\frac{\sqrt{7}}{2}t\right)+\frac{\sqrt{7}}{7}\sin\left(\frac{\sqrt{7}}{2}t\right)\right]\right]$
I	$\frac{1}{s\left(s^3+s^2+s+1\right)}$	$u(t)\left[1-\frac{1}{2}\left[e^{-t}+\cos(t)+\sin(t)\right]\right]$
PD	$\frac{1+s}{s\left(s^2+2s+2\right)}$	$u(t)\frac{1}{2}\left[1-e^{-t}(\cos(t)+\sin(t))\right]$
PI	$\frac{1+\frac{1}{s}}{s^3+s^2+2s+1}$	$u(t)\left[1-.4e^{-.57t}-e^{-.22t}(.6\cos(1.3t)+.37\sin(1.3t))\right.$
ID	$\frac{\frac{1}{s}+s}{s^3+2s^2+s+1}$	$u(t)\left[1-.7e^{-1.8t}-e^{-.13t}(.3\cos(.74t)+.4\sin(.74t))\right.$
PID	$\frac{1+\frac{1}{s}+s}{s^3+2s^2+2s+1}$	$u(t)\left[1-e^{-t}\right]$

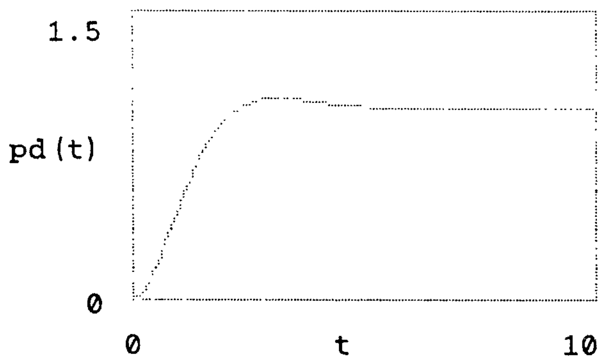
Some Typical Response Curves For Example 3



Open loop

Error:

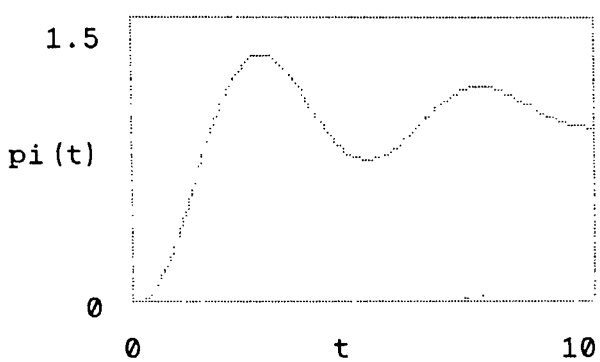
$$\int_0^{20} |1 - o(t)| \, dt = 1.715$$



PD response
(with amplifier of gain 2)

Error:

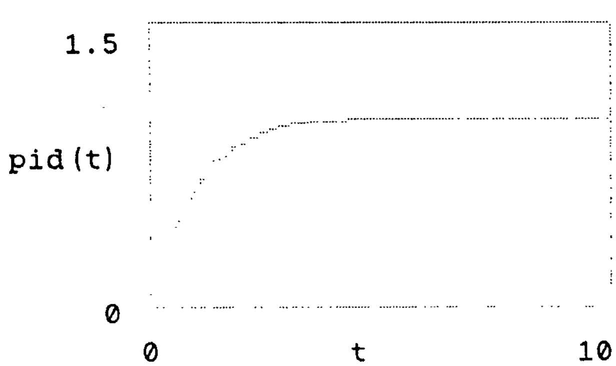
$$\int_0^{20} |1 - pd(t)| \, dt = 1.14$$



PI response

Error:

$$\int_0^{20} |1 - pi(t)| \, dt = 2.457$$



PID response

Error:

$$\int_0^{20} |1 - pid(t)| \, dt = 1$$

Example 3 - Summary

1) Unlike the previous two examples, the control constants were given. Numerical values for E can be obtained and compared. The full PID controller generated the lowest value for E, followed in turn by the PD and P controllers.

2) A surprising result is the instability of the I controller. Apparently the inductor in the compensator block and the capacitor in the process send energy back and forth with no attenuation to slow the exchange. The effect is also evident in the large oscillations and the resulting high value of E in the PI controller's response.

3) One final point of interest, the open loop response of Example 2 and the PID response of Example 3 are identical. *In effect, the introduction of PID control allows the process to assume an additional order (i.e. include an inductor) with no loss of response quality.*

CONCLUSIONS

The preceding work hopefully shed some light on the many properties and characteristics of a simple class of analog tracking systems. The design and testing of PID and related controlling structures proved useful in illustrating a few of the advantages found in employing a programmable tracking computer to open loop processes.

As well as answering basic questions concerning tracking and control, the foregoing discussion revealed many areas deserving of further consideration:

1) The introduction of intelligently time-varying control "constants" to further improve system performance.

2) The analysis of controllers of higher order than the PID. For example, using conventional notation, a $PIDD_2$ would include the second derivative of e(t) and thus could be considered a third order controller.

3) The analysis of the control of nth order processes as well as the important consideration of time varying processes.

4) The investigation of discrete time tracking systems, particularly computer-aided control.

5) The use of a microcomputer to develop an "expert" control system which could recognize often encountered processes and instantly change control settings to generate the best possible response curves from memory.

Each of the above topics is a logical extension of the concepts and results presented in this paper. The pursuit of any one is certain to lead to many interesting and novel results.

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Gabel, Robert A. Signals and Linear Systems, New York, NY: John Wiley & Sons, 1973.

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APPENDIX 1
A METHOD FOR EVALUATING OSCILLATING ERROR FUNCTIONS

Often in calculating the value of E for a particular controller's response, the error function, e(t), is identical to the absolute value of the error function. In these cases, the determination of E reduces to the evaluation of elementary integrals. Unfortunately, not all error functions have this property; the underdamped responses in particular contain error functions which vary in sign. Using the properties of the exponentially-decaying sinusoid, a simple algorithm facilitates the process of determining the Error Criterion, E.

In each example, all oscillating error functions assumed the form

(1) $$e(t) = \left[e^{ct}(A\cos(bt) + B\sin(bt))\right] u(t) \quad .$$

Or equivalently,

(2) $$e(t) = \left[Fe^{ct}(\cos(bt + a))\right] u(t)$$

where $F = \sqrt{A^2 + B^2}$ and $a = \mathrm{Tan}^{-1}\left[\frac{-B}{A}\right]$.

If one considers the integral of e(t) to be a series of areas of alternating sign, the integral of the absolute value of e(t) is simply the sum of the absolute values of all areas.

Denote each successive area A_n, whereby:

(3) $$\int |e(t)|\, dt = A_0 + (-1)^n \sum_{1} A_n .$$

The zeroes of equation (2) represent the end-points of each area and are located at $\dfrac{(n - .5)\pi - a}{b}$.

In this light, the integral of equation (1) is found to be:

(4) $$\frac{e^{ct}}{c^2 + b^2} (A(c \cos(bt) + b \sin(bt)) + B(c \sin(bt) - b \cos(bt))) .$$

Area A_0 is equation (4) evaluated from $t_1 = 0$ to $t_2 = \dfrac{\pi}{2b} - \dfrac{a}{b}$.

Area A_n is similarly determined by evaluating (4) from

$$t_1 = \frac{(n - .5)\pi}{2b} - \frac{a}{b} \quad \text{to} \quad t_2 = \frac{(n + .5)\pi}{2b} - \frac{a}{b} .$$

Using this method, the error in the P-controller's response of Example 3 is found to be:

n	t_1	t_2	Eqn (4)	$(-1)^n$ Error
0	0.00	1.46	.839	.839
1	1.46	3.84	-.444	.444
2	3.84	6.22	.175	.135
3	6.22	8.60	-.041	.041
4	8.60	10.98	.012	.012
			Error for n = 4:	1.471

APPENDIX 2
REAL VS. IDEAL INPUT FUNCTIONS

As noted earlier, the unit step function is an idealized input. While simple and quite revealing as a test input, the ideal nature of this function leads to some unexpected system responses. An illustration of this may be found in Example 2. In the real world, the voltage across a capacitor must be continuous; however, in the cases of the PD, ID, and PID controllers, there exists a discontinuity at time $t = 0$. This discontinuity is due to the unit step input. If a physically realizable input function is applied, the system output will certainly be well behaved.

For instance, if a continuous input function, U(s), is applied to the ID-controller of Example 2, the voltage output across the capacitor will also be continuous.

$$\text{Let } U(s) = \frac{1}{s} - \frac{1}{s+1} .$$

The system response, y(t), will be:

$$y(t) = u(t) \cdot \left[1 - .5 \cdot e^{-t} - e^{-.75t} \left(.5 \cdot \cos(t) - .125 \sin(t) \right) \right]$$

From a graph of input and output, both are seen to be continuous:

U(t) - Input

y(t) - Output

Error:

$$\int_0^{15} \left| U(t) - y(t) \right| dt = 0.243$$