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Stochastic Convexity of Sums of I.I.D. Non-Negative Random Variables with Applications

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STOCHASTIC CONVEXITY OF SUMS

OF I.I.D. NON–NEGATIVE RANDOM VARIABLES

WITH APPLICATIONS

by

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ABSTRACT

We present some monotonicity and convexity properties for the sequence of partial sums associated with a sequence of non–negative independent identically distributed random variables. These results are applied to a system of parallel queues with Bernoulli routing, and are useful in establishing a performance comparison between two scheduling strategies in multi–processor systems.

Key words: stochastic convexity, i.i.d. non–negative random variables, forward recurrence times, multi–processor systems, Fork–Join, random routing.

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I. INTRODUCTION

Let $\{X_k,\ k = 1,2,\ldots\}$ be a sequence of non-negative independent and identically distributed (i.i.d.) random variables (rvs) with probability distribution function B. We define the partial sums $\{S_K,\ K = 1,2,\ldots\}$ by

$$S_K := \sum_{k=1}^{K} X_k \qquad K = 1,2,\ldots \quad (1.1)$$

with the usual convention $S_0 = 0$. With any Borel mapping $\phi : \mathbb{R}_+ \to \mathbb{R}$ which is locally integrable, we associate the mapping $\Phi : \mathbb{R}_+ \to \mathbb{R}$ by setting

$$\Phi(t) = \int_0^t \phi(\tau)d\tau, \quad t \geq 0. \tag{1.2}$$

In this paper, we discuss, among other things, the monotonicity and convexity properties of the sequences $\{\frac{1}{K}E[\phi(S_K)],\ K = 1,2,\ldots\}$ and $\{\frac{1}{K}E[\Phi(S_K)],\ K = 1,2,\ldots\}$ for various choices of the mapping ϕ, specifically ϕ increasing and ϕ convex.

These properties, which we develop in Sections II and III, have probabilistic interpretations and prove crucial in [3] when establishing a performance comparison between two scheduling strategies in multi-processor systems. We discuss this issue in some detail in Section IV.

The remainder of this paper is organized as follows. In Section II we prove certain convexity properties of the sequence of partial sums $\{S_K,\ K = 1,2,\ldots\}$. In Section III we prove some related results for the sequence of forward recurrence times (or residual lifetimes) $\{\tilde{S}_K,\ K = 1,2,\ldots\}$ of the partial sums $\{S_K,\ K = 1,2,\ldots\}$. In Section IV, we present an application of these results to queues with Bernoulli routing and some related results on the comparison of the Fork-Join queue and a system of queues with Bernoulli routing. Finally, in Section V we draw conclusions and present some open problems. Appendix A contains sufficient material on stochastic convexity to make the paper self contained.

Our notation follows that used in [1,5–7]; in particular, we denote the set of real (resp. non-negative real) numbers by $\mathbb{R}$ (resp. $\mathbb{R}_+$). We assume the reader to be familiar with notions of stochastic orderings; the main references are the monographs by Ross [4] and Stoyan [8] which contain additional information and properties on the orderings $\leq_{st}$, $\leq_{cx}$ and $\leq_{icx}$.

II. MONOTONICITY AND CONVEXITY OF $\{S_K,\ K = 1,2,\ldots\}$

For every Borel mapping $\phi : \mathbb{R}_+ \to \mathbb{R}$, we set

$$\hat{\phi}(K) := E[\phi(S_K)] \qquad K = 1,2,\ldots \quad (2.1)$$

provided the expectation exists. Note that $\hat{\phi}(K)$ is always well defined, though possibly infinite with $\hat{\phi}(K) > -\infty$, either if ϕ is increasing or if ϕ is convex and the i.i.d rvs $\{X_k,\ k = 1,2,\ldots\}$ are integrable. The monotonicity and convexity properties for the sequence of partial sums $\{S_K,\ K = 1,2,\ldots\}$ are contained in the next proposition.

Theorem 1. *The following facts hold true for the partial sums $\{S_K,\ K = 1,2,\ldots\}$: If the mapping $\phi : \mathbb{R}_+ \to \mathbb{R}$ is*

1. *increasing, then $K \to \hat{\phi}(K)$ is increasing;*
2. *convex, then $K \to \hat{\phi}(K)$ is integer convex;*
3. *convex with $\phi(0) = 0$, then $K \to \frac{1}{K}\hat{\phi}(K)$ is increasing.*

Claims 1 and 2 are well known [4, Lemma 8.6.7, p. 278], and are given here for the sake of completeness, so that only Claim 3 needs to be established. Before doing so, we present in probabilistic terms some simple consequences of Theorem 1. To that end, let $\{b_k,\ k = 1,2,\ldots\}$ be a sequence of $\{0,1\}$–valued rvs such that

$$P[b_k = 1] = \frac{1}{k} = 1 - P[b_k = 0]. \qquad k = 1,2,\ldots \quad (2.2)$$

Theorem 2. *Assume the sequences $\{X_k,\ k = 1,2,\ldots\}$ and $\{b_k,\ k = 1,2,\ldots\}$ to be mutually independent. The following facts hold true for the partial sums $\{S_K,\ K = 1,2,\ldots\}$:*

1. *The collection of rvs $\{S_K,\ K = 1,2,\ldots\}$ is SICX, in fact SICX(sp); and*
2. *The sequence $\{b_K S_K,\ K = 1,2,\ldots\}$ is increasing in the sense of the ordering $\leq_{cx}$, i.e.,*

$$b_K S_K \leq_{cx} b_{K+1} S_{K+1}. \qquad K = 1,2,\ldots \quad (2.3)$$

Proof. Claim 1 follows readily by combining Claims 1 and 2 of Theorem 1; we refer the reader to recent work by Shaked and Shanthikumar [5,6] for a more complete description of the property SICX(sp).

Finally, for every convex mapping $\phi : \mathbb{R}_+ \to \mathbb{R}$, we get

$$E[\phi(b_K S_K)] = \frac{1}{K} E[\phi(S_K)] + (1 - \frac{1}{K})\phi(0)$$
$$= \frac{1}{K} E[\phi_0(S_K)] + \phi(0) \qquad K = 1, 2, \ldots \quad (2.4)$$

where the mapping $\phi_0 : \mathbb{R}_+ \to \mathbb{R} : x \to \phi(x) - \phi(0)$ is also convex with $\phi_0(0) = 0$. From Claim 3 of Theorem 1, we conclude that the mapping $K \to E[\phi(b_K S_K)]$ is increasing, thus establishing Claim 2.

□

The next two lemmas prove useful in the proof of Theorem 1. The first lemma is a well–known property of convex functions; its proof is elementary and is omitted in the interest of brevity.

Lemma 3. *For any convex mapping $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi(0) = 0$, the inequalities*

$$\sum_{k=1}^{K} \phi(x_k) \le \phi\left(\sum_{k=1}^{K} x_k\right) \qquad K = 1, 2, \ldots \quad (2.5)$$

hold true for arbitrary $x_k \ge 0$, $1 \le k \le K$.

Lemma 4. *For any convex mapping $\phi : \mathbb{R}_+ \to \mathbb{R}$ with $\phi(0) = 0$, the inequality*

$$E[\phi(X_K)] \le \frac{1}{K} E[\phi(S_K)] \qquad K = 1, 2, \ldots \quad (2.6)$$

holds true, whence

$$X_K \le_{cx} b_K S_K. \qquad K = 1, 2, \ldots \quad (2.7)$$

Proof. Since the non–negative rvs $\{X_k,\ k = 1, 2, \ldots\}$ are i.i.d., we observe that $KE[\phi(X_K)] = E[\sum_{k=1}^{K} \phi(X_k)]$ for all $K = 1, 2, \ldots$, and the inequality (2.6) follows immediately from Lemma 3. Finally, in view of (2.4), we see that (2.7) holds true provided

$$E[\phi(X_K)] \le \frac{1}{K} E[\phi(S_K)] + (1 - \frac{1}{K})\phi(0) \qquad K = 1, 2, \ldots \quad (2.8)$$

for every convex mapping $\phi : \mathbb{R}_+ \to \mathbb{R}$. The proof is now completed by observing that this last inequality is simply (2.6) applied to the convex mapping $\phi_0 : \mathbb{R}_+ \to \mathbb{R} : x \to \phi(x) - \phi(0)$ (for which $\phi_0(0) = 0$).

□

Proof of Theorem 1. As pointed out earlier, only Claim 3 needs to be established, namely that for any convex mapping $\phi : \mathbb{R}_+ \to \mathbb{R}$ with $\phi(0) = 0$, the inequalities

$$\frac{1}{K}E[\phi(S_K)] \leq \frac{1}{K+1}E[\phi(S_{K+1})] \qquad K = 1, 2, \ldots \quad (2.9)$$

hold true. This is now shown by induction on K.

- The basis step: For $K = 1$, (2.9) reduces to the inequality

$$E[\phi(X_1)] \leq \frac{1}{2}E[\phi(S_2)] \tag{2.10}$$

which is exactly (2.6) (with $K = 1$ since $X_1 =_{st} X_2$).

- The induction step: Assuming now that (2.9) indeed holds for some $K \geq 1$, we want to show that (2.9) also holds for $K + 1$. With this in mind, we define the mapping $\phi_K : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\phi_K(x) := E[\phi(S_{K-1} + x)], \quad x \geq 0. \tag{2.11}$$

Clearly the mapping ϕ_K is convex whenever ϕ is convex.

Under our assumptions, the rv S_{K-1} is independent of the rv $X_K + X_{K+1}$, so that

$$\begin{aligned} E[\phi(S_{K+1})] &= E[\phi_K(X_K + X_{K+1})] \\ &= \phi_K(0) + E[\phi_K(X_K + X_{K+1}) - \phi_K(0)]. \end{aligned} \tag{2.12}$$

From the basis step (2.10), we then conclude that

$$\begin{aligned} E[\phi(S_{K+1})] &\geq \phi_K(0) + 2E[\phi_K(X_{K+1}) - \phi_K(0)] \\ &= 2E[\phi_K(X_K)] - \phi_K(0) \\ &= 2E[\phi(S_K)] - E[\phi(S_{K-1})]. \end{aligned} \tag{2.13}$$

where the first equality follows from the fact that $X_K =_{st} X_{K+1}$. Therefore, (2.9) will hold for $K+1$ provided we can show that

$$\frac{1}{K}E[\phi(S_K)] \leq \frac{1}{K+1}\left(2E[\phi(S_K)] - E[\phi(S_{K-1})]\right). \tag{2.14}$$

By simple arithmetic we see that (2.14) is equivalent to (2.9) (for K) and this establishes the induction step.

□

Before closing this section, it is worth pointing out several facts concerning the results of this section: First, Claim 3 of Theorem 1 appears to be the best result possible under the enforced assumptions on ϕ. For instance, the condition $\phi(0) = 0$ cannot be dispensed with. Indeed, for the convex mapping $\phi : x \to x+1$, the mapping $K \to \frac{1}{K}\hat{\phi}(K)$ is in fact decreasing (assuming of course that the i.i.d. rvs $\{X_k,\ k = 1,2,\ldots\}$ are integrable). Moreover, the convexity of $K \to \frac{1}{K}\hat{\phi}(K)$ should not be expected from the convexity of ϕ alone. To see this, take $X_k \equiv 1$ for all $k = 1,2,\ldots$, and $\phi(x) = x(\log x)^+$ for all $x \geq 0$ (with $\phi(0) = 0$). It is plain that while ϕ is convex, $K \to \frac{1}{K}\hat{\phi}(K) = \log(K)$ is not integer convex; it is , in fact, integer concave. Finally, the inequalities (2.3) and (2.7) cannot hold in the sense of the ordering $\leq_{st}$ (except in degenerate cases) since all involved rvs have the same mean yet different probability distrbutions [8, p. 5].

III. MONOTONICITY AND CONVEXITY OF $\{\tilde{S}_K,\ K = 1,2,\ldots\}$.

In this section, we assume the i.i.d. $\mathbb{R}_+$–valued rvs $\{X_k,\ k = 1,2,\ldots\}$ to be integrable, with common mean m. For each $K = 1,2,\ldots$, we define the $\mathbb{R}_+$-valued rv $\tilde{S}_K$ as the forward recurrence time associated with S_K [2]; its probability distribution is given by

$$P[\tilde{S}_K > t] = \frac{\int_t^\infty P[S_K > \tau]d\tau}{Km}, \quad t \geq 0. \qquad K = 1,2,\ldots \tag{3.1}$$

Also, for every mapping $\phi : \mathbb{R}_+ \to \mathbb{R}$, we define

$$\tilde{\phi}(K) := E[\phi(\tilde{S}_K)] \qquad K = 1,2,\ldots \tag{3.2}$$

provided the expectation exists. We readily see that $\tilde{\phi}(K)$ is always well defined, though possibly infinite with $\hat{\phi}(K) > -\infty$, either if ϕ is increasing or if ϕ is convex and the i.i.d rvs $\{X_k,\ k = 1,2,\ldots\}$ have finite second moments (in which case the forward recurrence times $\{\tilde{S}_K,\ K = 1,2,\ldots\}$ all have first finite moments [2, p. 173].

We are now in position to state monotonicity and convexity results for the sequence $\{\tilde{S}_K,\ K = 1,2,\ldots\}$ which parallel those obtained for the partial sums $\{S_K,\ K = 1,2,\ldots\}$ in Theorem 1.

Theorem 5. *The following facts hold true for the sequence $\{\tilde{S}_K,\ K = 1,2,\ldots\}$: If the mapping $\phi : I\!R_+ \to I\!R$ is*

1. *increasing, then $K \to \tilde{\phi}(K)$ is increasing;*
2. *convex, then $K \to \tilde{\phi}(K)$ is integer convex.*

On combining Claims 1 and 2 of Theorem 5, we already get the following property.

Corollary 6. *The collection of rvs $\{\tilde{S}_K,\ K = 1,2,\ldots\}$ is SICX.*

The proof of Theorem 5 will be given in two steps. To prepare for it, consider a Borel mapping $\phi : I\!R_+ \to I\!R$ which is locally integrable so that the corresponding mapping Φ given by (1.2) is indeed well defined; note that any convex or increasing mapping ϕ is locally integrable. Clearly, we have

$$E[\phi(\tilde{S}_K)] = \int_0^\infty \phi(t)\frac{P[S_K > t]}{Km}dt$$

$$= \frac{1}{Km}\int_0^\infty \Phi(t)' P[S_K > t]dt \qquad K = 1,2,\ldots \ (3.3)$$

where $'$ denotes differentiation. Integrating (3.3) by parts and using the fact that $\Phi(0) = 0$, we see that

$$E[\phi(\tilde{S}_K)] = \frac{E[\Phi(S_K)]}{Km}, \qquad K = 1,2,\ldots \ (3.4)$$

with both expectations in (3.4) either finite or infinite simultaneously. With our earlier notation, we can write (3.4) as

$$\tilde{\phi}(K) = \frac{\hat{\Phi}(K)}{Km}. \qquad K = 1,2,\ldots \ (3.5)$$

In addition to being useful in the discussion, the expressions (3.4)–(3.5) also show in what sense Claim 2 of Theorem 5 is an improvement to Claim 3 of Theorem 1. Although $K \to \frac{1}{K}\hat{\phi}(K)$ is not necessarily convex if ϕ is a convex mapping with $\phi(0) = 0$, this will however be the case if ϕ is the anti–derivative Φ of a convex function.

Proof of Claim 1. If the mapping $\phi : \mathbb{R}_+ \to \mathbb{R}$ is increasing, then the corresponding mapping Φ is convex (with $\Phi(0) = 0$). Therefore Claim 1 follows immediately from Claim 3 of Theorem 1 and the relations (3.4)–(3.5).

□

The proof of Claim 2 of Theorem 5 is much more involved. The next lemma identifies the crucial property that underlies its proof.

Lemma 7. *For every convex mapping $\phi : \mathbb{R}_+ \to \mathbb{R}$, the inequalities*

$$2E\left[\int_{S_K}^{S_{K+1}} \phi(t)dt\right] \le E\left[\int_{S_{K+1}}^{S_{K+2}} \phi(t)dt\right] + E\left[\int_{S_{K-1}}^{S_K} \phi(t)dt\right] \qquad K = 1,2,\ldots \quad (3.6)$$

hold true.

Proof. For any convex mapping $\phi : \mathbb{R}_+ \to \mathbb{R}$, set

$$\Delta\Phi(K) = \int_{S_K}^{S_{K+1}} \phi(t)dt, \qquad K = 0,1,\ldots \quad (3.7)$$

and observe that (3.6) is equivalent to

$$E[\Delta\Phi(K)] - E[\Delta\Phi(K-1)] \le E[\Delta\Phi(K+1)] - E[\Delta\Phi(K)]. \qquad K = 1,2,\ldots \quad (3.8)$$

To proceed, we fix $K = 1,2,\ldots$ and introduce the mutually independent $\mathbb{R}_+$–valued rvs S, U, V and W such that

$$S =_{st} S_{K-1} \quad \text{and} \quad U =_{st} V =_{st} W =_{st} X_1. \quad (3.9)$$

Under the enforced assumptions on the rvs $\{X_k,\ k = 1,2,\ldots\}$, we see that

$$\Delta\Phi(K+1) =_{st} \int_{S+V+W}^{S+V+W+U} \phi(t)dt \quad (3.10)$$

$$\Delta\Phi(K) =_{st} \int_{S+V}^{S+V+U} \phi(t)dt =_{st} \int_{S+W}^{S+W+U} \phi(t)dt \tag{3.11}$$

and

$$\Delta\Phi(K-1) =_{st} \int_{S}^{S+V} \phi(t)dt. \tag{3.12}$$

Consequently, using (3.10)–(3.12), we get

$$\begin{aligned} E[\Delta\Phi(K+1)] - E[\Delta\Phi(K)] =& E\left[\int_{S+V+W}^{S+V+W+U} \phi(t)dt\right] - E\left[\int_{S+V}^{S+V+U} \phi(t)dt\right] \\ =& E\left[\int_0^U (\phi(S+V+W+\tau) - \phi(S+V+\tau))\, d\tau\right] \end{aligned} \tag{3.13}$$

where the last equality follows by a simple change of variables. Similarly, from (3.11)–(3.12) we obtain

$$\begin{aligned} E[\Delta\Phi(K)] - E[\Delta\Phi(K-1)] =& E\left[\int_{S+W}^{S+W+U} \phi(t)dt\right] - E\left[\int_{S}^{S+U} \phi(t)dt\right] \\ =& E\left[\int_0^U (\phi(S+W+\tau) - \phi(S+\tau))\, d\tau\right]. \end{aligned} \tag{3.14}$$

To conclude, we need only notice that the convexity of ϕ implies

$$\phi(S+W+\tau) - \phi(S+\tau) \leq \phi(S+W+V+\tau) - \phi(S+V+\tau), \quad \tau \geq 0 \tag{3.15}$$

and (3.6) follows upon combining (3.13)–(3.15).

□

Proof of Claim 2. Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a convex mapping. Using the fact that $\Phi(S_{K+1}) = \Phi(S_K) + \Delta\Phi(K)$, we readily obtain the relation

$$m\left(\tilde{\phi}(K+1) - \tilde{\phi}(K)\right) = \frac{KE[\Delta\Phi(K)] - E[\Phi(S_K)]}{K(K+1)} \qquad K = 1, 2, \ldots \tag{3.16}$$

so that the asserted integer convexity of $K \to \tilde{\phi}(K)$, i.e., $\tilde{\phi}(K)-\tilde{\phi}(K-1) \leq \tilde{\phi}(K+1)-\tilde{\phi}(K)$ for all $K = 2, 3 \ldots$, is equivalent to

$$\frac{(K-1)E[\Delta\Phi(K-1)] - E[\Phi(S_{K-1})]}{K(K-1)} \leq \frac{KE[\Delta\Phi(K)] - E[\Phi(S_K)]}{K(K+1)}.$$

$$K = 2, 3, \ldots \quad (3.17)$$

Upon writing $\Phi(S_K) = \Phi(S_{K-1}) + \Delta\Phi(K)$ and rearranging terms, we see after some algebra that (3.17) holds provided

$$(K^2 + K - 2)E[\Delta\Phi(K-1)] \leq 2E[\Phi(S_{K-1})] + K(K-1)E[\Delta\Phi(K)].$$

$$K = 2, 3, \ldots \quad (3.18)$$

To show (3.18) we proceed by induction on K.

- The basis step: For $K = 2$, (3.18) reduces to the inequality

$$2E\left[\int_{S_1}^{S_2} \phi(t)dt\right] \leq E\left[\int_0^{S_1} \phi(t)dt\right] + E\left[\int_{S_2}^{S_3} \phi(t)dt\right] \quad (3.19)$$

which is exactly (3.6) for $K = 1$.

- The induction step: Assuming now that (3.18) indeed holds for some $K \geq 2$, we want to show that (3.18) also holds for $K + 1$, namely

$$(K^2 + 3K)E[\Delta\Phi(K)] \leq 2E[\Phi(S_K)] + K(K+1)E[\Delta\Phi(K+1)]. \quad (3.20)$$

Since $\Phi(S_K) = \Phi(S_{K-1}) + \Delta\Phi(K-1)$, we get

$$\begin{aligned} &2E[\Phi(S_K)] + K(K+1)E[\Delta\Phi(K+1)] \\ &= 2E[\Phi(S_{K-1})] + 2E[\Delta\Phi(K-1)] + K(K+1)E[\Delta\Phi(K+1)] \quad (3.21) \\ &\geq K(K+1)E[\Delta\Phi(K-1)] - K(K-1)E[\Delta\Phi(K)] + K(K+1)E[\Delta\Phi(K+1)]. \end{aligned}$$

$$(3.22)$$

This last inequality is obtained by (lower) bounding $2E[\Phi(S_{K-1})]$ in (3.21) through a straightforward application of the induction hypothesis (3.18). Therefore, (3.20) holds true if we can show that

$$K(K+1)E[\Delta\Phi(K-1)] - K(K-1)E[\Delta\Phi(K)] + K(K+1)E[\Delta\Phi(K+1)] \geq K(K+3)E[\Delta\Phi(K)]. \tag{3.23}$$

This last inequality is readily seen to be equivalent to (3.8) (thus to (3.6) of Lemma 7) and the induction step is established.

□

IV. APPLICATIONS

In a companion paper [3], the monotonicity and convexity properties discussed earlier were crucial in establishing a performance comparison between two multi-processor queueing structures, namely the Fork Join queue and a system of parallel queues with Bernoulli routing. Both systems have K (≥ 2) identical servers operating in *parallel* with infinite waiting rooms. Jobs that arrive to these systems are assumed to consist of exactly K tasks, the service requirements of the tasks being i.i.d. Upon arrival into the Fork–Join system, a job is instantaneously decomposed into its K constituent tasks and the k^{th} task is routed to the k^{th} queue where it is served in FCFS order. As soon as a task completes service, it is put into a synchronization buffer, and a job leaves the system when all of its constituent tasks have completed service. In the system of parallel queues with Bernoulli routing, an arriving job is routed to the k^{th} queue with probability $\frac{1}{K}$, $1 \leq k \leq K$, with the routing decision being independent of any other event, past, present or future. In each queue, jobs are processed in a FCFS manner.

For $n = 0, 1, \ldots$, let $T_n^{(K)}$ and $S_n^{(K)}$ denote the response times of the n^{th} job in the Fork–Join queue system with K processors and in the system of K parallel queues with Bernoulli routing, respectively. In [3], we sought to compare $T_n^{(K)}$ and $S_n^{(K)}$ (or their steady–state versions) in the sense of the stochastic ordering $\leq_{icx}$ by using the stochastic monotonicity and convexity/concavity (in K) of the sequences of response times. More specifically, we first showed that the sequence of response times $\{T_n^{(K)},\ K = 1, 2, \ldots\}$ is

SICV(st). Therefore, since $T_n^{(K)} =_{st} S_n^{(K)}$ for $K = 1$, the stochastic montonicity and convexity of $\{S_n^{(K)},\ K = 1,2,\ldots\}$ would then imply that the desired comparison can reverse its direction only once. In fact, if it could be shown that the comparison $T_n^{(K)} \leq_{icx} S_n^{(K)}$ holds for $K = 2$, then the convexity/concavity properties would imply the comparison for *all* $K \geq 2$; the reader is referred to [3] for additional details.

In the remaining part of this section, we show how the results of Sections II and III can be used to obtain the desired monotonicty and convexity properties of the sequence of response times $\{S_n^{(K)},\ K = 1,2,\ldots\}$. To do this, we first point out, as in [3], that the system of parallel queues with Bernoulli routing can be adequately studied through a single server equivalent – sometimes referred to as $GI/GI/1$ queues with Bernoulli loading – which we now describe. For each $K = 1,2,\ldots$, we postulate $\mathbb{R}_+$–valued rvs $\{\tau_{n+1},\ n = 0,1,\ldots\}$ and $\{\sigma_n^k,\ k = 1,2,\ldots;\ n = 0,1,\ldots\}$, and $\{0,1\}$–valued rvs $\{\beta_n^{(K)},\ n = 0,1,\ldots\}$. We assume these three sequences of rvs to be mutually independent sequences of i.i.d. rvs with common probability distribution A, B and

$$P[\beta_n^{(K)} = 1] = \frac{1}{K} = 1 - P[\beta_n^{(K)} = 0], \qquad n = 0,1,\ldots \quad (4.1)$$

respectively. We also introduce the sequence of i.i.d. rvs $\{\sigma_n^{(K)},\ n = 0,1,\ldots\}$ by setting

$$\sigma_n^{(K)} = \sum_{k=1}^{K} \sigma_n^k. \qquad n = 0,1,\ldots \quad (4.2)$$

Next we consider the $\mathbb{R}_+$–valued rvs $\{U_n^{(K)},\ n = 0,1,\ldots\}$ which are generated through the Lindley recursion

$$\begin{aligned} U_{n+1}^{(K)} &= \left[U_n^{(K)} + \beta_n^{(K)}\sigma_n^{(K)} - \tau_{n+1}\right]^+, \\ U_0^{(K)} &= 0. \end{aligned} \qquad n = 0,1,\ldots \quad (4.3)$$

These rvs are the successive customer waiting times in a $GI/GI/1$ queue with interarrival times $\{\tau_{n+1},\ n = 0,1,\ldots\}$ and service times $\{\beta_n^{(K)}\sigma_n^{(K)},\ n = 0,1,\ldots\}$. It is easy to see that

$$S_n^{(K)} =_{st} U_n^{(K)} + \sigma_n^{(K)}. \qquad n = 0, 1, \ldots \quad (4.4)$$

From Claim 2 of Theorem 2 we conclude for all $K = 1, 2, \ldots$, that

$$\beta_n^{(K)} \sigma_n^{(K)} \leq_{icx} \beta_n^{(K+1)} \sigma_n^{(K+1)} \qquad n = 0, 1, \ldots \quad (4.5)$$

since a comparison in the ordering $\leq_{cx}$ implies a similar comparison in the ordering $\leq_{icx}$, whence

$$U_n^{(K)} \leq_{icx} U_n^{(K+1)} \qquad n = 0, 1, \ldots \quad (4.6)$$

by making use of [4, Thm. 8.6.2, p. 274]. That the sequence $\{S_n^{(K)},\ K = 1, 2, \ldots\}$ is increasing in the ordering $\leq_{icx}$ then follows from (4.4) and (4.6) upon observing that $\sigma_n^{(K)} \leq \sigma_n^{(K+1)}$.

It is worth pointing out that it does not seem possible to strengthen this result to hold in the ordering $\leq_{st}$ if only the basic monotonicity result [4, Thm. 8.6.2, p. 274] is used for the Lindley recursion (4.3). This follows from the fact that for each $n = 0, 1, \ldots$, the rvs $\{\beta_n^{(K)} \sigma_n^{(K)},\ K = 1, 2, \ldots\}$ have the same mean, and therefore cannot be increasing (in K) in the sense of the ordering $\leq_{st}$ [8, p. 5]. As a result, the sequence of waiting times $\{U_n^{(K)},\ K = 1, 2, \ldots\}$ cannot be expected either to be SICX(st) or SICX(sp) for each $n = 1, 2, \ldots$; similar comments apply for the response times. However, in the important special case when the arrival process is Poisson, i.e.,

$$P[\tau_{n+1} > t] = e^{-\lambda t}, \quad t \geq 0 \qquad n = 0, 1, \ldots \quad (4.7)$$

for some $\lambda > 0$, we have been able to show that the steady–state waiting and response times are indeed SICX (in K).

Before developing this result, we note that the single server equivalents determined by the Lindley recursion (4.3) are *all* stable if and only if

$$\rho := \lambda E[B] < 1 \quad (4.8)$$

with $E[B]$ denoting the mean of the service time distribution B (which we assume finite). In particular, under (4.8), for every $K = 1, 2, \ldots$, there exists an $\mathbb{R}_+$–valued rv $U_\infty^{(K)}$

such that as n goes to infinity, the convergence in distribution $U_n^{(K)} \Longrightarrow U_\infty^{(K)}$ takes place. Moreover, it is readily verified [2] that $U_\infty^{(K)}$ coincides with the steady–state waiting time in a $M/GI/1$ queue with interarrival times $\{K\tau_{n+1},\ n = 0, 1, \ldots\}$ and service times $\{\sigma_n^{(K)},\ n = 0, 1, \ldots\}$. This $M/GI/1$ system is characterized by the arrival rate $\frac{\lambda}{K}$ and the service time distribution $B^{*(K)}$, the K–fold convolution of B with itself; the server utilization is still ρ as given by (4.8).

We are now in a position to take advantage of a representation result for the steady–state waiting time in $M/GI/1$ queues [2, p. 201]: Fix $K = 1, 2, \ldots$, and let $\{\tilde{\sigma}_n^{(K)},\ n = 1, 2, \ldots\}$ be a sequence of i.i.d. rvs, each one distributed as the forward recurrence time associated with $B^{*(K)}$. It is a simple matter to see from (3.1) that

$$P[\tilde{\sigma}_n^{(K)} > t] = \frac{1}{KE[B]} \int_t^\infty \left(1 - B^{*(K)}(\tau)\right) d\tau, \quad t \geq 0. \qquad n = 1, 2, \ldots \ (4.9)$$

Let $\nu(\rho)$ be a $\{0, 1, \ldots\}$–valued rv which is geometrically distributed with parameter ρ, i.e.,

$$P[\nu(\rho) = n] = (1 - \rho)\rho^n, \qquad n = 0, 1, \ldots \ (4.10)$$

and which is independent of the sequence $\{\tilde{\sigma}_n^{(K)},\ n = 1, 2, \ldots\}$. We also define the partial sums $\{\tilde{S}_n^{(K)},\ n = 1, 2, \ldots\}$ by

$$\tilde{S}_n^{(K)} = \sum_{m=1}^{n} \tilde{\sigma}_m^{(K)}, \qquad n = 1, 2, \ldots \ (4.11)$$

with the usual convention $\tilde{S}_0^{(K)} = 0$. Since $U_\infty^{(K)}$ can be viewed as the equilibrium waiting time of an $M/GI/1$ queue with arrival rate $\frac{\lambda}{K}$ and service time distribution $B^{*(K)}$, we readily conclude from [2, Eqn. (5.111), p. 201] that the rv $U_\infty^{(K)}$ has the representation

$$U_\infty^{(K)} =_{st} \tilde{S}_{\nu(\rho)}^{(K)}. \qquad (4.12)$$

To obtain the desired convexity result, we proceed as follows: By Corollary 6, for each $n = 0, 1, \ldots$, the collection of rvs $\{\tilde{\sigma}_n^{(K)},\ K = 1, 2, \ldots\}$ is SICX. Under the enforced independence assumptions, we can invoke the closure property of SICX under convolution

[7, Thm. 5.6] to conclude that the collection of rvs $\{\tilde{S}_n^{(K)},\ K = 1, 2, \ldots\}$ is also SICX; in other words, for every increasing (resp. increasing convex) mapping $\phi : \mathbb{R}_+ \to \mathbb{R}$, the mapping $K \to \phi(\tilde{S}_n^{(K)})$ is increasing (resp. increasing integer–convex) for each $n = 1, 2, \ldots$. We now conclude that the collection of rvs $\{U_\infty^{(K)},\ K = 1, 2, \ldots\}$ is SICX as an immediate consequence of these remarks and of the relation

$$E[\phi(U_\infty^{(K)})] = \sum_{n=0}^{\infty}(1-\rho)\rho^n E[\phi(\tilde{S}_n^{(K)})] \tag{4.13}$$

implied by the representation (4.12).

Under the stability condition (4.8), we also get from (4.4) that $S_n^{(K)} \Longrightarrow U_\infty^{(K)} + \sigma^{(K)}$ where $\sigma^{(K)}$ is an $\mathbb{R}_+$–valued rv which is independent of $U_\infty^{(K)}$ and distributed according to $B^{*(K)}$. Using the closure property given in Theorem 5.6 of [7], we then see that $\{S_\infty^{(K)},\ K = 1, 2, \ldots\}$ is SICX since this property holds for the collections $\{U_\infty^{(K)},\ K = 1, 2, \ldots\}$ (as just shown) and $\{\sigma^{(K)},\ K = 1, 2, \ldots\}$ (as implied by Theorem 2).

REFERENCES

[1] F. Baccelli and A. M. Makowski, "Queueing models for systems with synchronization constraints," *Proceedings of the IEEE* **77** (1989), Invited paper, Special Issue on Dynamics of Discrete Event Systems, pp. 138–161.

[2] L. Kleinrock, *Queueing Systems I: Theory*, J. Wiley & Sons, New York (NY), 1976.

[3] A.M. Makowski, T.K. Philips and S. Varma, "Comparing scheduling strategies in multi–processors systems," IBM Research Report **RC** (1990).

[4] S. Ross, *Stochastic Processes*, J. Wiley & Sons, New York (NY), 1984.

[5] M. Shaked and J.G. Shanthikumar, "Stochastic convexity and its applications," *Advances in Applied Probability* **20** (1988), pp. 427–446.

[6] M. Shaked and J.G. Shanthikumar, "Convexity of a set of stochastically ordered random variables," *Advances in Applied Probability* **22** (1990), pp. 160–177.

[7] M. Shaked and J.G. Shanthikumar, "Parametric stochastic convexity and concavity of stochastic processes," Preprint (1990).

[8] D. Stoyan, *Comparison Methods for Queues and Other Stochastic Models*, English Translation (D.J. Daley, Editor), J. Wiley & Sons, New York (NY), 1984.

APPENDIX A
STOCHASTIC CONVEXITY

In this appendix, we briefly recall several notions of stochastic convexity which have been recently introduced by Shaked and Shantikumar [5,6]: Throughout, Θ is a convex subset of $\mathbb{R}$ and $\{X(\theta),\ \theta \in \Theta\}$ is a collection of $\mathbb{R}$-valued rvs. For any Borel mapping $\phi : \mathbb{R} \to \mathbb{R}$, we define the mapping $\hat{\phi} : \Theta \to \mathbb{R}$ by

$$\hat{\phi}(\theta) := E[\phi(X(\theta))], \quad \theta \in \Theta \tag{A.1}$$

whenever these expectations exist. The collection of rvs $\{X(\theta),\ \theta \in \Theta\}$ is then said to be

1. stochastically increasing (resp. decreasing) convex in the usual stochastic ordering – in short SICX(st) (resp. SDCX(st)) – $\hat{\phi}$ is increasing (resp. decreasing) convex whenever ϕ is increasing;

2. stochastically increasing (resp. decreasing) convex – in short SICX (resp. SDCX) – if $\hat{\phi}$ is increasing (resp. decreasing) whenever ϕ is increasing (resp. decreasing) *and* if $\hat{\phi}$ is increasing (resp. decreasing) convex whenever ϕ is increasing convex;

3. stochastically increasing convex in the sample path sense – in short SICX(sp) – if for any four points θ_i, $i = 1, \ldots, 4$, in Θ, such that $\theta_1 \le \theta_2 \le \theta_3 \le \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$, there exist four rvs $\tilde{X}_i$, $i = 1, \ldots, 4$, defined on a common probability space such that $\tilde{X}_i =_{st} X(\theta_i)$, $i = 1, \ldots, 4$, and

$$\tilde{X}_j \le \tilde{X}_4, \quad j = 1,2,3 \quad \text{and} \quad \tilde{X}_2 + \tilde{X}_3 \le \tilde{X}_1 + \tilde{X}_4 \quad a.s. \tag{A.2}$$

A few words on these definitions: When the rvs $\{X(\theta),\ \theta \in \Theta\}$ are non-negative rvs, we note in the definition of SICX(st) and SICX that we need only consider $\mathbb{R}_+$-valued mappings ϕ, in which case $\hat{\phi}$ is always well defined. Moreover, when Θ is a subset of $\{0, 1, \ldots\}$, convexity is understood as integer convexity.

The following implications were discussed in (Shaked and Shantikumar [11,12]):

$$\text{SICX(st)} \Longrightarrow \text{SICX(sp)} \Longrightarrow \text{SICX}. \tag{A.3}$$

In general, the implications SICX(sp) $\Longrightarrow$ SICX(st) and SICX $\Longrightarrow$ SICX(sp) are not true as can be seen on simple counterexamples.