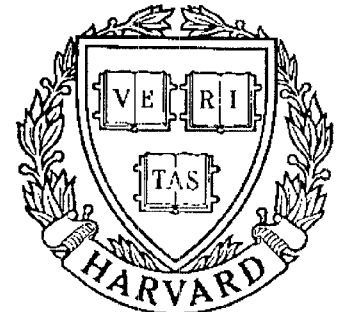


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**Worst-Case H_∞ Performance Under
Structured Perturbations with Known Bounds**

by M.K.H. Fan and A.L. Tits

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Abstract

The structured singular value (SSV or μ) is known to be an effective tool for assessing robust performance of linear time-invariant models subject to structured uncertainty. Yet all a single μ analysis provides is a bound β on the uncertainty under which stability as well as H_∞ performance level of k/β are guaranteed, where k is preselectable. In this paper, we introduce a related quantity, denoted by ν which, for a given β , provides a value α such that for any uncertainty bounded by β an H_∞ performance level of α (but none better than α) is guaranteed.

1. Introduction

Consider a linear time-invariant model affected by uncertainty. It is by now well known (see, e.g., [1]) that in many cases of interest such a system can be represented in "feedback" form as in Fig. 1. Here Δ represents the uncertainty and is typically block diagonal, each block corresponding to uncertainty affecting a specific subsystem. Both parametric and dynamic uncertainties can be accounted for. While the former appear as real scalar blocks in Δ , the latter are often represented by H_∞ -norm bounded linear time-invariant transfer functions. Under the assumption that the nominal model is internally stable, the overall system will be internally stable for all Δ of size (H_∞ -norm) no more than 1, Δ having the specified structure, if and only if

$$\sup_{\omega} \mu(P_{11}(j\omega)) < 1$$

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where μ is Doyle's structured singular value (SSV) for the given structure [2]. The SSV framework also permits to assess robust performance [1]. Namely, referring again to Fig. 1, suppose one desires to know whether the worst-case H_∞ performance is satisfactory, i.e., whether, for any structured Δ of size no more than 1, the H_∞ -norm of the transfer function $F_u(P, \Delta)$ from exogenous (e.g., disturbance) signal u to error signal e is small, say, no larger than 1. It turns out that this will be the case if and only if the system of Fig. 2 is internally stable for all Δ of size no more than 1, Δ having the specified structure, and for all δ of size no more than 1; equivalently, if and only if

$$\sup_{\omega} \tilde{\mu}(P(j\omega)) < 1$$

where $\tilde{\mu}$ is the structured singular value corresponding to the "augmented structure". Straightforward scaling then yields the following, given any $\beta > 0$: The system of Fig. 1 is stable for all structured Δ of size β or less, and the worst-case performance under such uncertainty is no more than $1/\beta$, if and only if

$$\sup_{\omega} \tilde{\mu}(P(j\omega)) < 1/\beta . \quad (1)$$

Thus an SSV analysis can answer the question.

- (Q1) what is the largest β such that, whenever the uncertainty has size β or less,
- (i) the system is stable and
 - (ii) the worst-case H_∞ performance is better than $1/\beta$?

While this does provide some kind of "stability-and-performance margin", it may well happen that a good estimate of the actual uncertainty bound is available. In this case, assuming that the uncertainty bound has been normalized, a question of interest is whether (i) the system is stable whenever the uncertainty has size less than 1, and (ii) if yes, what is the worst case performance for this same uncertainty size. In other words:

- (Q2) what is the smallest α such that, whenever the uncertainty has size 1 or less,
- (i) the system is stable and
 - (ii) the worst-case H_∞ performance is better than α ?

It is possible to answer (Q2) via an (infinite) sequence of SSV analyses as follows. Note that for any $\alpha > 0$, stability of the system in Fig. 1 is equivalent to stability of the system in Fig. 3, with $P_\alpha(s)$ given by

$$P_\alpha(s) = \begin{bmatrix} \alpha P_{11}(s) & \alpha P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}.$$

We conclude that the system of Fig. 1 is stable whenever $\|\Delta\|_\infty \leq 1$ (i.e., whenever $\|(1/\alpha)\Delta\|_\infty \leq 1/\alpha$), with worst-case performance better than α , if and only if

$$\sup_{\omega} \tilde{\mu}(P_\alpha(j\omega)) < \alpha . \quad (2)$$

Thus the answer to (Q2) is given by $\hat{\alpha}$, the infimum of those α satisfying (2). It can be shown the $\hat{\alpha}$ is the only root (except for, possibly, 0) of

$$\sup_{\omega} \tilde{\mu}(P_\alpha(j\omega)) = \alpha$$

and that it can be computed via the fixed point iteration

$$\alpha_{i+1} = \sup_{\omega} \tilde{\mu}(P_{\alpha_i}(j\omega)), \quad \alpha_0 > 0.$$

This iteration can be proven to converge (also see Section 4 below).

The purpose of this paper is to introduce a quantity closely related to the structured singular value, but yielding an answer to (Q2) in a *single analysis*. For simplicity of exposition, we consider the case of two-block structures (performance block and single uncertainty block). In Section 2 below we define the new function ν of a matrix and state a theorem (related to the Small μ Theorem) that shows its relation to (Q2). In Section 3, we discuss elementary properties of ν and in Section 4 we elucidate its correspondence with μ . Finally, in Section 5, we indicate how, for a complex matrix M , an upper bound to $\nu(M)$ ($\nu(M)$ itself in the special case emphasized in the remainder of this paper) can be efficiently computed. Throughout this paper, scalar functions, including value functions of optimization problems, take values in the extended real line $\mathbb{R} \cup \{\infty\}$.

2. A Measure of Robust Performance

Thus, for a complex $n \times n$ matrix M , consider the structure $\mathcal{K} = (k_1, k_2)$, k_1 and k_2 positive integers, with $k_1 + k_2 = n$. This corresponds to an uncertainty of dimension $k_1 \times k_1$ and an exogenous input and error signals of dimension k_2 . Below, we make use of the notation

$$\mathcal{D} = \{\text{block diag } (dI_{k_1}, I_{k_2}) : d > 0\},$$

$$\mathcal{U} = \{\text{block diag } (U_1, U_2), U_i : k_i \times k_i, \text{ unitary}\},$$

$$P_1 = \text{block diag } (I_{k_1}, O_{k_2}), \quad P_2 = \text{block diag } (O_{k_1}, I_{k_2}),$$

and M is partitioned according to

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

with $M_{ij} : k_i \times k_j$. Also, the unit sphere in \mathbb{C}^n is denoted by ∂B .

Recall [2] that, for the given structure, the structured singular value $\mu(M)$ of a complex matrix M is equal to zero if there is no $\Delta \in \mathcal{X}$ such that $\det(I + \Delta M) = 0$ and

$$\mu(M) = \left(\min_{\Delta \in \mathcal{X}} \{ \bar{\sigma}(\Delta) : \det(I + \Delta M) = 0 \} \right)^{-1}$$

otherwise, where \mathcal{X} is a subspace of $\mathbb{C}^{n \times n}$ given by

$$\mathcal{X} = \{ \text{block diag}(\Delta_1, \Delta_2) : \Delta_i \in \mathbb{C}^{k_i \times k_i}, i = 1, 2 \}.$$

Consider now the related quantity $\nu(M)$, equal to zero if there is no $\Delta \in \mathcal{Y}$ such that $\det(I + \Delta M) = 0$ and given by

$$\nu(M) = \left(\min_{\Delta \in \mathcal{Y}} \{ \bar{\sigma}(\Delta_2) : \det(I + \Delta M) = 0 \} \right)^{-1}$$

(possibly ∞) otherwise, where \mathcal{Y} is given by

$$\mathcal{Y} = \{ \text{block diag}(\Delta_1, \Delta_2) : \Delta_i \in \mathbb{C}^{k_i \times k_i}, i = 1, 2, \bar{\sigma}(\Delta_1) \leq 1 \}.$$

Note that, in the formula for $\nu(M)$, the size of Δ_1 is not minimized but merely kept below 1, reflecting the fact that the uncertainty has a known bound of 1. The following result, to be compared to the Small μ Theorem [1], can be proven.

Theorem 2.1. Suppose $P \in H_\infty$ is internally stable and let $\alpha > 0$. Then the system depicted in Fig. 1 is well formed and internally stable for all $\Delta \in H_\infty$, $\|\Delta\|_\infty \leq 1$, and $\|F_u(P, \Delta)\|_\infty < \alpha$ for all such Δ , if and only if

$$\sup_{\omega} \nu(P(j\omega)) < \alpha.$$

□

Thus (Q2) can be answered by means of a single “ ν ” analysis.

3. Properties of ν

The following properties of ν are to be compared to similar properties of μ given in [2]. (Recall that k_1 is strictly positive.)

Proposition 3.1.

- (a) $\nu(M) \geq \bar{\sigma}(M_{22})$.
- (b) $\nu(M) < \infty$ iff $\bar{\sigma}(M_{11}) < 1$.
- (c) Suppose $\bar{\sigma}(M_{11}) < 1$ and either $M_{12} = 0$ or $M_{21} = 0$. Then $\nu(M) = \bar{\sigma}(M_{22})$.
- (d) $\nu(\gamma M) \geq |\gamma|\nu(M)$ for any $|\gamma| \geq 1$.
- (e) $\nu(\gamma M) \leq |\gamma|\nu(M)$ for any $|\gamma| \leq 1$.
- (f) $\nu(DMD^{-1}) = \nu(M)$, for any $D \in \mathcal{D}$.
- (g) $\nu(UM) = \nu(MU) = \nu(M)$, for any $U \in \mathcal{U}$.
- (h) Whenever $\nu(M) < \infty$,

$$\nu(M) = \sup_{U \in \mathcal{U}} \rho(UM - P_1, P_2),$$

where, given two square matrices A and B of identical dimension, $\rho(A, B)$ is the largest (finite) zero of $\chi(\lambda) = \det(A + \lambda B)$.

- (i) $\nu(\cdot)$ is continuous at any M for which $\nu(M) < \infty$.

□

The discussion of Section 2 also suggests that ν may be related to μ in some recursive way. This is indeed the case as stated next.

Proposition 3.2. Let M be such that $\nu(M) < \infty$.

- (a) $\mu(M) \geq 1$ iff $\nu(M) \geq 1$ iff $\nu(M) \geq \mu(M)$. Furthermore, if $\mu(M) > \bar{\sigma}(M_{22})$, then $\mu(M) > 1$ iff $\nu(M) > 1$ iff $\nu(M) > \mu(M)$.
- (b) Suppose $\nu(M) < \infty$. Then

$$\mu \left(\begin{bmatrix} \nu(M)I_{k_1} & 0 \\ 0 & I_{k_2} \end{bmatrix} M \right) = \nu(M).$$

- (c) Suppose $\mu(M) > 0$. Then

$$\nu \left(\begin{bmatrix} \mu^{-1}(M)I_{k_1} & 0 \\ 0 & I_{k_2} \end{bmatrix} M \right) = \mu(M).$$

□

4. Computing ν via μ

Proposition 4.1. Let M be such that $\nu(M) < \infty$. Define the function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$

$$f(\alpha) = \mu \left(\begin{bmatrix} \alpha I_{k_1} & 0 \\ 0 & I_{k_2} \end{bmatrix} M \right).$$

Then

- (a) $f(\alpha)$ is continuous nondecreasing.
- (b) $f(\alpha) = \alpha$ has at least one solution at $\alpha = \nu(M)$ and has at most two solutions at $\alpha = \nu(M)$ and $\alpha = 0$. Furthermore, $f(\alpha) = \alpha$ has two solutions iff $\bar{\sigma}(M_{12})\bar{\sigma}(M_{21}) \neq 0$ and $M_{22} = 0$.
- (c) $\beta > \nu(M)$ implies that $f(\beta) < \beta$.
- (d) $0 < \beta < \nu(M)$ implies that $f(\beta) > \beta$. □

It follows that both the fixed point iteration

$$\alpha_{i+1} = f(\alpha_i), \alpha_0 > 0,$$

and the obvious bisection iteration can be used to compute $\nu(M)$.

5. Direct Computation of ν

The key question is now whether $\nu(M)$ can be easily computed. For the case under consideration, efficient algorithms are known for the computation of the structure singular value $\mu(M)$, based on the formulas ([1,3])

$$\mu(M) = \inf_{D \in \mathcal{D}} \bar{\sigma}(DM D^{-1}) \quad (3)$$

$$\mu(M) = \max_{\substack{x \in \partial B \\ \theta \geq 0}} \{\theta : \|P_i M x\| = \theta \|P_i x\|, i = 1, 2\}.$$

In particular, the optimization problem in (3) has no local minima that are not global and robust algorithm are available for its solution. Practical value of $\nu(M)$ is obviously contingent on the availability of a comparably efficient computational algorithm. Fortunately, it appears that such an algorithm can be constructed, based on the following results.

Theorem 5.1.

$$\nu(M) = \sup_{\substack{x \in \partial B \\ \theta \geq 0}} \{\theta : \|P_1 M x\| = \|P_1 x\|, \|P_2 M x\| = \theta \|P_2 x\|\}$$

$$\nu(M) = \begin{cases} 0 & \text{if } \psi(0) < 0 \\ \sup_{\gamma \geq 0} \{\sqrt{\gamma} : \psi(\gamma) \geq 0\} & \text{otherwise} \end{cases} \quad (4)$$

where

$$\psi(\gamma) = \min_{\substack{d_1, d_2 \geq 0 \\ d_1 + d_2 = 1}} \bar{\lambda} \left(M^H \begin{bmatrix} d_1 I_{k_1} & 0 \\ 0 & d_2 I_{k_2} \end{bmatrix} M - \begin{bmatrix} d_1 I_{k_1} & 0 \\ 0 & \gamma d_2 I_{k_2} \end{bmatrix} \right)$$

□

Formula (4) is closely related to a result recently obtained in the context of mixed parametric and dynamic uncertainty [4,5]. The optimization problem defining ψ is convex, and thus ψ can be evaluated efficiently. Further, ψ is monotone decreasing so that, e.g., a fast secant method can be used to compute ν .

6. Discussion

The results of Section 2, 3 and 4 can be extended in a straightforward manner to the case of more complex uncertainty structures, at the expense of more involved notation. The results of Section 5 can also be generalized. However, the generalization of (4) typically becomes a “ \leq ” inequality (but it is still always an equality in the case of two complex uncertainty blocks). This is to be expected since (3) also becomes a “ \leq ” inequality in that case.

□

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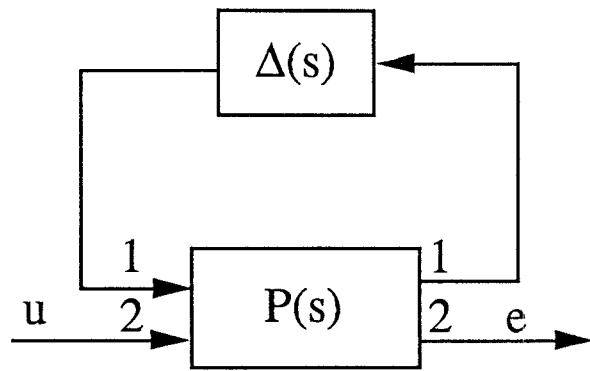


Fig. 1

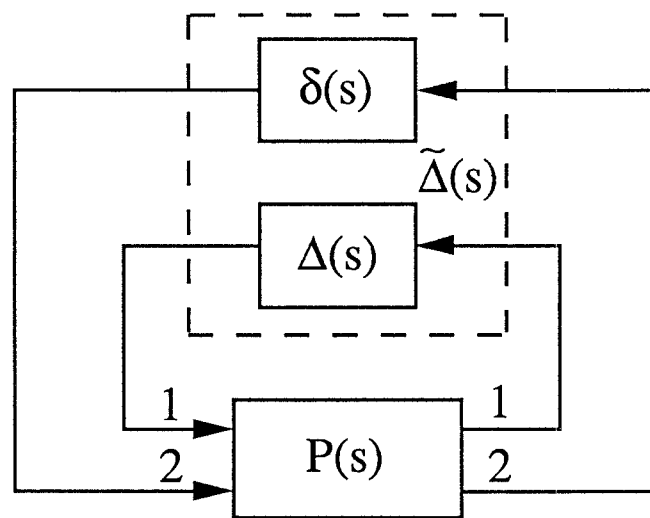


Fig. 2

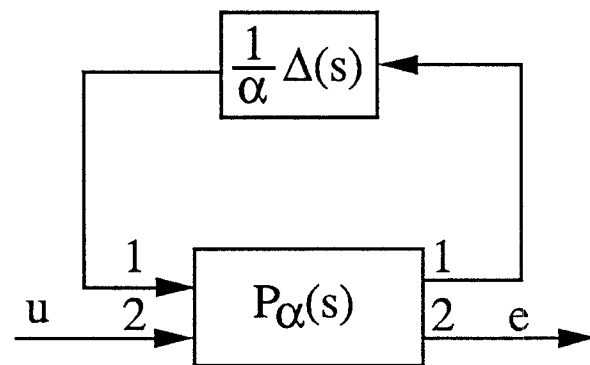


Fig. 3