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Asymptotic Stability of Nonlinear Systems with Holomorphic Structure

By

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Abstract

We consider the local asymptotic stability of a system $dx/dt = F(z)$, $z \in C^n$, $F : C^n \to C^n$ is holomorphic, $t \in R$, and show that if the system is locally asymptotically stable at some equilibrium point in the N^{th} approximation for some N, then necessarily its linear part is asymptotically stable also.

1 Introduction

Consider the system of ordinary differential equations

$$dz/dt = F(z), \tag{1.1}$$

where $z \in C^n$, $F : C^n \to C^n$ is holomorphic, $F(0) = 0$, and t is a real variable. The question we raise here is to what extent does the local asymptotic

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stability of system (1.1) at the origin relate to the asymptotic stability of its linear part. In this note we prove the following result.

Theorem 1.1 *Suppose that (1.1) is locally asymptotically stable at the origin in the N^{th} approximation for some N, i.e., the terms of degree higher than N do not play a role in determining the local asymptotic stability. Then the linear part is asymptotically stable also.*

The proof is given in the next section and it is along the following lines. The local asymptotic stability of (1.1) implies that the degree of the vector field F at the origin (as a real vector field) is equal to one (see [Br,KZ], etc.). This and the holomorphy now imply that the linear part of F is nonsingular (see, e.g., page 19 of [GH]). It is now left to show that the linear part of F does not have any imaginary eigenvalues. This is done by using the Poincaré normal forms and writing F as the sum of a linear vector field and a nonlinear vector field such that the two summands commute, the linear one is diagonal and has purely imaginary elements. Furthermore the construction is such that the linear part of the nonlinear summand is singular. Now we conclude that the nonlinear summand has to be asymptotically stable as well. But this contradicts the nonsingularity requirement for the linear part.

2 Proof of Theorem 1.1

Write $F = \sum_{i=1}^{\infty} F_i$ where F_i is a homogeneous polynomial vector field of degree i. It is well known (see [Br,KZ]) that the local asymptotic stability implies that the degree of the vector field F at the origin (as a real vector field of R^{2n}) is equal to one. Now the holomorphy implies that F_1 has a nonsingular linear part. Let us now assume that at least one of the eigenvalues of the linear part of F_1 is purely imaginary and we will reach a contradiction. The construction given below is the one carried out to obtain Poincaré normal forms (see [AR]). We carry it out for the sake of completeness.

Without any loss of generality we may assume that F_1 is in its complex Jordan form. Now let $F_1 = A + N$ where A is diagonal and N is nilpotent. We will denote the space of vector fields in C^n which are homogeneous of some degree k by S^k. If B is some linear vector field, then $/mboxad(B)$: $S^k \to S^k$ is a linear transformation. Since $\mathrm{ad}(A)$ and $\mathrm{ad}(N)$ commute and $\mathrm{ad}(N)$ is nilpotent it follows that for each k

$$\text{range of } (\mathrm{ad}(A+N)) + \text{null space of } (\mathrm{ad}(N)) = S^k. \tag{2.2}$$

We now use this fact to find a holomorphic coordinate transformation near the origin of C^n of the form $z = \phi(w) = w + \sum_{i=2}^N f_i(w)$, which transforms the vector field F (more precisely, a modification of F which agrees with F up to and including terms of order N. We will tacitly assume the necessity to modify F in what follows and refer to the new vector field as F also) into a vector field $H(w) = Aw + Nw + \sum_{i=2}^N X_i(w)$ where $X_i(w)$ is homogeneous of degree i and the vector fields Aw and $Nw + \sum_{i=2}^N X_i(w)$ commute. Now the transformation of F gives

$$\begin{aligned} H(w) &+ \sum_{i=2}^{N}(Df_i(w))(H(w)) \\ &= F_1\left(w + \sum_{i=2}^{\infty} f_i(w)\right) + \sum_{i=2}^{\infty} F_i\left(w + \sum_{i=2}^{N} f_i(w)\right). \end{aligned} \tag{2.3}$$

The linear parts in w in (2.3) are already equal. Now equating the quadratic terms we obtain

$$\mathrm{ad}(F_1)(f_2) = X_2 - F_2. \tag{2.4}$$

By using (2.3) we find f_2 such that $X_2 \in$ null space $(\mathrm{ad}(A))$. We now continue to find $f_3, \ldots, f_N$ and $X_3, \ldots, X_N$ such that $\mathrm{ad}(A)(X_k) = 0$, $k = 2, \ldots, N$. Now the local coordinate transformation $z = \phi(w) = w + \sum_{i=2}^N f_i(w)$ transforms the vector field $H(w) = F_1(w) + \sum_{i=2}^N X_i(w)$ into a vector field in the z coordinates which agrees with F up to and including terms of degree N. We will call this new vector field F also. By our hypothesis in Theorem 1.1 it is locally asymptotically stable at the origin also and hence so is H.

Now we focus on H. By construction, $\mathrm{ad}(A)X_i = 0$, $i = 2, \ldots, N$ and $\mathrm{ad}(A)N = 0$. For a fixed $k \in \{2, \ldots, N\}$ let us fix a basis of S^k in the following way. Let e_j denote the j^{th} standard basis vector of C^n and for each multiindex $m = (m_1, \ldots, m_n)$ and $w \in C^n$ we denote $w_1^{m_1} \ldots w_n^{m_n}$ by w^m. Now

$$\{w^m e_j\}_{\sum_{i=1}^n m_i = k,\ j \in \{1, \ldots, n\}}$$

is a basis of S^k. Let $A = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$. Then

$$\text{ad}(A)(w^m e_j) = (< \lambda, m > -\lambda_j) w^m e_j,$$

where $< \lambda, m >= \sum_{i=1}^n \lambda_i m_i$. Thus $\{w^m e_j\}$ is the set of eigenvectors of $\text{ad}(A) : S^k \to S^k$ with corresponding eigenvalues $\{< \lambda, m > -\lambda_j\}$. In particular, it follows that the real and the imaginary parts of A commute with N and each X_i as well. Let A_r and A_i denote the real and the imaginary parts of A respectively and denote $A_r + N + \sum_{i=2}^N X_i$ by Y. Then

$$H = A_i + Y \tag{2.5}$$

and

$$[H, A_i] = 0. \tag{2.6}$$

Since all eigenvalues of A_i are imaginary and A_i is diagonal it follows that $-A_i$ is a stable vector field. By (2.6) the flow of Y in positive time can be obtained by concatenating the flows of H and $-A_i$ and thus Y is a locally asymptotically stable holomorphic vector field. However, our hypothesis that A has at least one imaginary eigenvalue now implies that Y has singular linear part which was shown to be impossible before. This contradiction concludes the proof of Theorem 1.1.

Q.E.D.

3 Concluding Remarks

An interesting open question now is to decide whether the hypothesis on the asymptotic stability in the N^{th} approximation can be relaxed to just local asymptotic stability.

References

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