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Robust Stability via the Guardian Map Approach: A Perspective*

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Abstract

Guardian and semiguardian maps were recently introduced as tools for investigating generalized stability of parametrized families of matrices and of polynomials. Here we give a brief survey of the motivation and scope of application of the guardian and semiguardian map concepts.

1. Background

In the last few years, several authors have obtained algebraic necessary and sufficient conditions for the generalized stability of parametrized families of matrices or of polynomials. First Białas [1] showed that in the case of the convex hull of two real matrices or two real polynomials, Hurwitz stability can be assessed by checking whether or not all real eigenvalues of a test matrix are negative. This matrix is of the form $M_0^{-1}M_1$, where M_0 and M_1 have size n (case of polynomials) or $n(n+1)/2$ (case of matrices). Similar results were obtained independently by Fu and Barmish [2]. Later, Ackermann and Barmish [3] extended these results to the case of Schur stability of polytopes of real polynomials, and Fu and Barmish [4] pointed out that the key to all the previous results was the transformation of stability problems into nonsingularity problems. Tesi and Vicino [5] showed that results similar to those of Białas follow for families of matrices depending polynomially (rather than affine linearly) on an uncertain parameter, and obtained results for generalized stability with respect to some other domains of interest of the complex plane, among which is the unit circle (Schur stability). The case of the convex hull of two *complex* polynomials has been addressed by Bose [6], who introduced tests for both Hurwitz and Schur stability based on the resultant of certain auxiliary polynomials. Bose's tests amount to checking that these resultants, as polynomials in the uncertain parameter, have no zeros in the interval of interest; they do not involve inverting a matrix. Simplified tests are provided in [6] for the case of real polynomials.

2. Guardian and Semiguardian Maps

Introduction of the concepts of guardian and semiguardian maps [7,8] was motivated by the belief that the essential element in the work of Białas [1] and Fu and Barmish [2] could be employed in broader contexts. The approach taken independently by Fu and Barmish [4] is similar in that respect.

Definition 1. ([8]) Let $\mathcal{X}$ denote the set of all $n \times n$ complex matrices, or the set of all monic polynomials of degree n with complex coefficients, and let $\mathcal{S}$ be an open subset of $\mathcal{X}$. Let ν map $\mathcal{X}$ into $\mathbf{C}$. We say that ν is *semiguarding* for $\mathcal{S}$ if the implication

$$x \in \partial\mathcal{S} \quad \Longrightarrow \quad \nu(x) = 0$$

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holds. The map ν *guards* $\mathcal{S}$ if the converse implication also holds whenever $x \in \overline{\mathcal{S}}$. The map ν is said to be *polynomic* if it is a polynomial function of the entries (matrix case) or coefficients (polynomial case) of its argument, and of their complex conjugates. □

Suppose that we wish to determine whether or not a pathwise connected family $\Phi = \{x(r),\ r \in U\}$ of matrices or polynomials lies entirely in an open set $\mathcal{S}$ of interest (e.g., the set of Hurwitz stable matrices or polynomials). Assume "nominal stability", i.e., $x(\hat{r}) \in \mathcal{S}$ for some $\hat{r} \in U$. Then, clearly, (i) if ν is a guardian map for $\mathcal{S}$, then $\Phi \subset \mathcal{S}$ if and only if $\nu(x(r)) \neq 0$ for all $r \in U$, and (ii) if ν is (merely) a semiguardian map for $\mathcal{S}$, then $\Phi \subset \mathcal{S}$ if and only if $x(r) \in \mathcal{S}$ for all $r \in U$ such that $\nu(x(r)) = 0$. If ν is polynomic and $x(r)$ depends polynomially (e.g., affine linearly) on r, these are algebraic tests (involving multivariate polynomials if r is not scalar). If moreover ν is a *guardian* map, these tests are nothing but polynomial positivity tests.

The stability tests in [1] and [2] can be directly formulated in terms of specific guardian maps for Hurwitz stable matrices and polynomials. In fact, each of the results mentioned above can be interpreted in the guardian map framework, and their validity can be proved through a systematic approach which entails verifying that a certain map meets the guardian map criterion for the set $\mathcal{S}$ of interest.

Polynomic semiguardian maps can be constructed in a systematic manner for sets $\mathcal{S}(\Omega)$ characterized by confinement of eigenvalues (zeros) in any given algebraic domain Ω of the complex plane for both real [8] and complex [9] matrices (polynomials). A sufficient condition for these to be guardian is derived in [8]. It holds in many cases of interest. A point worth noting is that, for the set $\mathcal{S}$ of Hurwitz stable matrices, one such guardian map is given by

$$\nu(A) = \det(A \cdot I) \det A \qquad (1)$$

where $A \cdot I$ is the bialternate product of A and the identity (see, e.g., [8,10]), a matrix of size $n(n-1)/2$; this indicates that the test proposed for the matrix case by Białas, which involves matrices of size $n(n+1)/2$, is equivalent to two similar tests involving matrices of size $n(n-1)/2$ and n respectively, and the latter entails slightly less computation.

The guardian map framework, by its simplicity, suggests connections between seemingly distinct results. Such a connection is made in [9] in the case of families of complex polynomials. In the case of real polynomials, it is pointed out in [9] that a test proposed by Bose [6] is computationally equivalent to Białas' test (both can be expressed as generalized eigenvalue tests). It turns out moreover that the guardian map that yields Białas' test, namely

$$\nu(p) = \det H(p),$$

where $H(p)$ is the Hurwitz matrix associated with polynomial p, is identical (up to a constant factor) to the map (1), with A a companion matrix associated to p. Thus Białas' polynomial test and matrix test (applied to a corresponding companion matrix) are essentially equivalent.

Finally, the geometric intuition underlying the idea of guardian and semiguardian maps leads naturally to extensions of its application. For instance, a necessary and sufficient condition for stability of LTI singularly perturbed systems, for all sufficiently small values of the small parameter, can be derived, and parametric families of such systems can be dealt with [11]; a test for dominant poles location can be obtained by constructing semiguardian maps corresponding to certain disconnected domains of the complex plane [12]; and guardian maps for sets of matrices not characterized by a domain of the complex plane, e.g., the strictly aperiodic matrices, can be constructed [8].

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