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Robust Stability of Complex Families of Matrices and Polynomials

L. SAYDY, A.L. TITS AND E.H. ABED

Abstract

Recently, the authors introduced the "guardian map" approach as a unifying tool in the study of robust generalized stability questions for parametrized families of matrices and polynomials. Real matrices and polynomials have been emphasized in previous reports on this approach. In the present note, the approach is discussed in the context of complex matrices and polynomials. In the case of polynomials, some algebraic connections with other recent work are uncovered.

1 Introduction

In a recent paper [1], the authors developed the "guardian map" approach to the study of generalized stability of families of matrices and polynomials. The case of real matrices and polynomials was emphasized in [1]. For some applications however, such as network realizability and filter design, the stability tests involve polynomials and matrices with complex coefficients (see [2–4] and the references therein). The aim of this note is to explicate the results of [1] for problems involving complex matrices and polynomials. A systematic procedure for finding guardian and semiguardian maps for the complex case is given. We specialize the results to the Hurwitz and Schur stability of families of polynomials, and proceed to uncover algebraic connections between constructions of Bose [2] and the guardian maps employed here.

2 Guardian Maps and Robust Stability

The guardian map approach was introduced in [1,5] as a unifying tool for the study of generalized stability of parametrized families of matrices and polynomials. In this section we summarize the elements of this approach.

2.1 Notation

s^*: Conjugate of the complex number s
$\mathring{\mathbb{C}}_-, \overline{\mathbb{C}}_-$: Open, closed left-half complex plane
$\overline{\mathcal{D}}, \partial\mathcal{D}$: Closure, boundary of set $\mathcal{D}$
A^H: Conjugate transpose of matrix A
$\sigma(A)$: Eigenvalues of matrix A (counting multiplicities)
$\mathcal{Z}(p)$: Zeros of polynomial p (counting multiplicities)
$\mathcal{P}_n$: Set of all monic complex polynomials of degree n.
$\mathcal{C}(p)$: Companion matrix associated with polynomial p
$\otimes, \oplus$: Kronecker product, Kronecker sum ($A \oplus B = A \otimes I + I \otimes B$)
$\mathcal{S}(\Omega)$: Set of all $n \times n$ complex matrices with spectrum inside Ω. Also denotes the set of all polynomials in $\mathcal{P}_n$ with zeros inside Ω.

2.2 Guardian and Semiguardian Maps

The notions of guardian, semiguardian and polynomic map are recalled in the next definition [1].

Definition 1. Let $\mathcal{X}$ be either the set of $n \times n$ complex matrices, or the set of all monic polynomials of degree n with complex coefficients , and let $\mathcal{S}$ be an open subset of $\mathcal{X}$. Let ν map $\mathcal{X}$ into $\mathbb{C}$. We say that ν is *semiguarding* for $\mathcal{S}$ if for all $x \in \overline{\mathcal{S}}$, the implication

$$x \in \partial\mathcal{S} \quad \Longrightarrow \quad \nu(x) = 0 \tag{1}$$

holds. Moreover, ν *guards* $\mathcal{S}$ if the converse implication also holds. The map ν is said to be ***polynomic*** if it is a polynomial function of the entries (matrix case) or coefficients (polynomial case) of its argument and of their complex conjugates.[1]

In [1], guardian and semiguardian maps are exhibited for a large variety of sets of interest, the focus of the representation being the case of real matrices and polynomials. Analogous maps for corresponding sets of *complex* matrices and polynomials are constructed in a straightforward manner. For

[1]Complex conjugates are not explicit in the definition given in [1], as that paper focuses on the case of *real* matrices and polynomials.

example, the map $\nu : A \mapsto \det(A \oplus A^H)$ guards the set of $n \times n$ Hurwitz stable complex matrices. This map is obtained by noting that the spectrum of the Kronecker sum of two square matrices A and B consists of all pairwise sums of eigenvalues of A and B (see, e.g., [6]). Note that this same result also yields a guardian map for the Hurwitz stable complex polynomials, namely $\nu(p) = \det(\mathcal{C}(p) \oplus \mathcal{C}^H(p))$.

Let Ω be an open subset of the complex plane. Of particular importance are sets $\mathcal{S}$ of the form $\mathcal{S}(\Omega)$, where $\mathcal{S}(\Omega)$ is given, for matrix stability problems, by

$$\mathcal{S}(\Omega) = \{A \in \mathbb{C}^{n\times n} : \sigma(A) \subset \Omega\}, \tag{2}$$

and, for polynomial stability problems, by

$$\mathcal{S}(\Omega) = \{p \in \mathcal{P}_n : \mathcal{Z}(p) \subset \Omega\}. \tag{3}$$

Such sets $\mathcal{S}(\Omega)$ are referred to as *(generalized) stability sets.*

The example above dealing with Hurwitz stable families is a particular case of a more general result given next. This result gives a systematic procedure for constructing semiguardian and guardian maps for a large class of stability sets, corresponding to domains of the complex plane with polynomial boundaries.

Let $p(x, y)$ be a real bivariate polynomial of the form

$$p(x, y) = \sum_{k,\ell} p_{k\ell} x^k y^\ell. \tag{4}$$

Denote by Ω the subset of the complex plane

$$\Omega = \{s = x + jy : \ p(x, y) < 0\}, \tag{5}$$

and associate with p the complex polynomial

$$q(\lambda, \mu) := p\left(\frac{\lambda+\mu}{2}, \frac{\lambda-\mu}{2j}\right) \tag{6}$$

$$= \sum_{k,\ell} c_{k\ell}(\lambda+\mu)^k(\lambda-\mu)^\ell. \tag{7}$$

Here

$$c_{k\ell} := (-j)^\ell \left(\frac{1}{2}\right)^{k+\ell} p_{k\ell}. \tag{8}$$

Rewrite (6) as

$$q(\lambda, \mu) = \sum_{k,\ell} q_{k\ell} \lambda^k \mu^\ell. \tag{9}$$

With this notation, Ω and $\partial\Omega$ have the alternative expressions

$$\Omega = \{\lambda \in \mathbb{C} : \quad q(\lambda, \lambda^*) < 0\}, \tag{10}$$
$$\partial\Omega = \{\lambda \in \mathbb{C} : \quad q(\lambda, \lambda^*) = 0\}. \tag{11}$$

Throughout the remainder of this section we focus on the matrix case with the understanding that similar considerations apply in the polynomial case. For example, we may specialize any general matrix result to companion matrices, as in the example above.

Consider the mapping $\mathcal{F} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n^2\times n^2}$ given by

$$\mathcal{F}(A) := \sum_{k,\ell} q_{k\ell} A^k \otimes (A^H)^\ell. \tag{12}$$

Let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$. Then Stéphanos' Theorem [1,6,7] implies that

$$\sigma(\mathcal{F}(A)) = \{q(\lambda_i, \lambda_j^*) : \quad i, j = 1, \ldots, n\}. \tag{13}$$

As direct consequences of this fact, we have the following two results.

Theorem 1. ([1]) *The map ν given by*

$$\nu : \quad A \mapsto \det \mathcal{F}(A) \tag{14}$$

is polynomic semiguarding for $\mathcal{S}(\Omega)$.

Proposition 2. ([1]) *The map ν of Eq. (14) guards $\mathcal{S}(\Omega)$ if and only if q satisfies the condition*

$$q(\lambda, \lambda^*) < 0 \ \ \textit{and} \ \ q(\mu, \mu^*) < 0 \ \Rightarrow \ q(\lambda, \mu^*) \neq 0. \qquad (\textit{Condition } C)$$

Condition C is in general difficult to check. A sufficient condition for it to hold is given by the next proposition.

Proposition 3. ([1]) *Assume that*

$$q_{kk} \geq 0, \quad \text{for all } k \geq 1, \tag{15}$$

$$q_{k\ell} = -q_{\ell k}, \quad \text{for all } k \neq \ell, \ \ k\ell \neq 0. \tag{16}$$

Then q satisfies Condition C.

2.3 Robust Stability

The robust stability problem for parametrized families of matrices or polynomials may be stated as follows. Let $r = (r_1, \ldots, r_k) \in U$, where U is a pathwise connected subset of $\mathbb{R}^k$, and let $x(r)$ be an element of $\mathcal{X}$ which depends continuously on the parameter vector r. Given an open subset $\mathcal{S}$ of $\mathcal{X}$, we seek basic conditions for $x(r)$ to lie within $\mathcal{S}$ for all values of r in U. The next theorem gives a basic necessary and sufficient condition for this problem both for guarded and semiguarded sets $\mathcal{S}$. Typically, $\mathcal{S}$ is a stability set of the form $\mathcal{S}(\Omega)$ where Ω is a given subset of the complex plane.

Given a continuous map $\nu : \mathcal{X} \longrightarrow \mathbb{C}$, define the *critical set*

$$U_{\mathrm{cr}} := \{r \in U : \nu(x(r)) = 0\}. \tag{17}$$

Theorem 4. ([1]) *Assume that the family* $\Phi := \{x(r) : r \in U\}$ *is nominally stable relative to* Ω, *i.e., assume that* $x(r^0) \in \mathcal{S}(\Omega)$ *for some* $r^0 \in U$. *Then: (i) if* $\mathcal{S}(\Omega)$ *is guarded by* ν, *then* Φ *is stable relative to* Ω *if and only if* $U_{\mathrm{cr}} = \emptyset$, *(ii) if* $\mathcal{S}(\Omega)$ *is semiguarded by* ν, *then* Φ *is stable relative to* Ω *if and only if* $x(r) \in \mathcal{S}(\Omega)$ *for all* $r \in U_{\mathrm{cr}}$.

In the case of polynomic guardian or semiguardian maps, unless U_{cr} is all of U, it is finite and the theorem above yields computable conditions for robust stability of parametrized families of matrices or polynomials [1]. The guardian or semiguardian maps obtained in Section 2.2 are all of the polynomic type. It is not surprising that results analogous to those obtained for one- and two-parameter families of *real* matrices or polynomials in [1] hold for *complex* families . We give special attention to the cases of Hurwitz and Schur stability of families of complex polynomials, uncovering algebraic connections between recent results of Bose [2] and those which follow from the present approach.

3 Families of Polynomials

Let $\{p(r) : r \in U\}$ be a polynomic family of polynomials in $\mathcal{P}_n$ and let Ω be a given open subset of the complex plane. We seek conditions under which this family is stable relative to Ω.

Theorem 4 states that if ν is a polynomic guardian map for $\mathcal{S}(\Omega)$ and at least one polynomial $p(\hat{r})$, $\hat{r} \in U$, is stable relative to Ω, then a necessary and sufficient condition for the entire family to be stable relative to Ω is that the polynomial $\nu(p(r))$ have no zeros in U. In the cases of Hurwitz and Schur stability, this translates into the polynomials

$$\det\left(\mathcal{C}(p(r)) \oplus \mathcal{C}^H(p(r))\right) \tag{18}$$

$$\det\left(\mathcal{C}(p(r)) \otimes \mathcal{C}^H(p(r)) - I\right) \tag{19}$$

having no zeros in U.

We now proceed to show that (i) the results of [2] are particular cases of Theorem 4, and (ii) the corresponding "resultant-based" guardian maps are in fact identical to (18) and (19). The Hurwitz case is treated first.

Following [2], we associate with any complex polynomial

$$p(s) \quad := \quad p_0 + p_1 s + p_2 s^2 + \ldots + p_{n-1} s^{n-1} + s^n \tag{20}$$

the pair of polynomials

$$\alpha_p(s) := \frac{1}{2}\left(p(s) + (p(-s^*))^*\right), \tag{21}$$

$$\beta_p(s) := \frac{1}{2}\left(p(s) - (p(-s^*))^*\right). \tag{22}$$

Also, given two polynomials x and y with degrees n and m respectively, the *resultant* of x and y is the determinant of the $(m+n) \times (m+n)$ matrix $R(x,y)$ given by[2,3]

$$\begin{bmatrix}
x_n & x_{n-1} & \cdot & \cdot & \cdot & x_0 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & x_n & x_{n-1} & \cdot & \cdot & \cdot & x_0 & 0 & \cdot & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x_0 & 0 \\
\cdot & \cdot & \cdot & \cdot & 0 & x_n & x_{n-1} & \cdot & \cdot & \cdot & x_0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & y_m & y_{m-1} & \cdot & \cdot & y_0 \\
\cdot & \cdot & \cdot & \cdot & 0 & y_m & y_{m-1} & \cdot & \cdot & y_0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & y_m & y_{m-1} & \cdot & \cdot & y_0 & 0 & \cdot & \cdot & \cdot & 0 \\
y_m & y_{m-1} & \cdot & \cdot & y_0 & 0 & \cdot & \cdot & \cdot & \cdot & 0
\end{bmatrix}. \tag{23}$$

Proposition 5. *The set of Hurwitz stable complex monic polynomials is guarded by the (polynomic) map δ given by*

$$\delta(p) = \det R(\alpha_p, \beta_p). \tag{24}$$

Proof. Let $p \in \overline{\mathcal{S}(\mathring{\mathbb{C}}_-)}$, i.e., assume p has all its zeros in $\overline{\mathbb{C}}_-$. Then we have

$$\begin{aligned}
\delta(p) = 0 \quad &\Leftrightarrow \quad \alpha_p \text{ and } \beta_p \text{ have a common zero} \\
&\Leftrightarrow \quad p(s) = (p(-s^*))^* = 0.
\end{aligned}$$

2 A vanishing resultant signals that x and y have a common zero.

3 Eq. (23) corresponds to the case $n = m + 1$.

Since p has all its zeros in $\overline{\mathbb{C}}_-$, it follows that

$$\begin{aligned}\delta(p)=0 \quad &\Leftrightarrow \quad p(s)=0 \text{ for some } s\in\partial\overset{\circ}{\mathbb{C}}_- \\ &\Leftrightarrow \quad p\in\partial\mathcal{S}(\overset{\circ}{\mathbb{C}}_-).\end{aligned}$$

Q.E.D.

In the light of this proposition, an alternate necessary and sufficient condition for the convex hull of two polynomials p^0 and p^1 to be Hurwitz stable, given that p^0 is Hurwitz stable, is that the polynomial $\delta((1-r)p^0+rp^1)$ in the indeterminate r be nonzero for all $r\in(0,1]$ This appears as Theorem 1 of [2].[4]

The question now arises as to the relationship between guardian maps (18) and (24). It turns out that, essentially, these maps are identical, as is shown next.

Proposition 6. *Let $p\in\mathcal{P}_n$. Then the following identity holds:*

$$\det\left(\mathcal{C}(p)\oplus\mathcal{C}^H(p)\right)=(-1)^{\frac{n(n-1)}{2}}2^nR\left(\alpha_p,\beta_p\right). \tag{25}$$

Proof. Recall [2] that if x and y are polynomials with zeros given by $v_i,\ i=1,\ldots,n$ and $w_k,\ k=1,\ldots,m$, respectively, then

$$\det R(x,y)=(-1)^{\frac{n(n-1)}{2}}x_n^m y_m^n\prod_{i=1}^{n}\prod_{k=1}^{m}(v_i-w_k). \tag{26}$$

Here, the x_i's and y_i's denote the coefficients of x and y respectively. Note that this can also be written as

$$\det R(x,y)=(-1)^{\frac{n(n-1)}{2}}x_n^m\prod_{i=1}^{n}y(v_i). \tag{27}$$

Since $\alpha_p(s)+\beta_p(s)=p(s)$, it follows from (21)-(22) that at least one of $\alpha_p(s)$ and $\beta_p(s)$ has degree n. Without loss of generality, assume $\alpha_p(s)$ has degree n and denote by a_n the coefficient of the highest power. Let $\mathcal{Z}(p)=:\{z_1,\ldots,z_n\}$, $\mathcal{Z}(\alpha_p)=:\{\alpha_1,\ldots,\alpha_n\}$ and $\mathcal{Z}(\beta_p)=:\{\beta_1,\ldots,\beta_{n_2}\}$, where n_2 denotes the degree of p. Now let $\nu(p)=\det(C(p)\oplus C^H(p))$. Since

$$\sigma\left(\mathcal{C}(p)\oplus\mathcal{C}^H(p)\right)=\{z_i+z_k^*:\ i,k=1,\ldots,n\}, \tag{28}$$

[4] The additional condition given in [2] on the polynomial coefficients is apparently superfluous.

we can write

$$\nu(p) = \prod_{i=1}^{n} \prod_{k=1}^{n} (z_i + z_k^*). \tag{29}$$

It also follows that

$$\nu(p) = (-1)^{n^2} \prod_{i=1}^{n} \widehat{p}(z_i), \tag{30}$$

where

$$\widehat{p}(s) := (p(-s^*))^*. \tag{31}$$

With this notation, Eqs. (21), (22) may be rewritten as

$$\alpha_p(s) = \frac{1}{2}\left(p(s) + \widehat{p}(s)\right), \tag{32}$$

$$\beta_p(s) = \frac{1}{2}\left(p(s) - \widehat{p}(s)\right). \tag{33}$$

Eq. (33) implies that $\widehat{p}(z_i) = -2\beta_p(z_i),\ i = 1, \ldots, n$, whence

$$\nu(p) = (-1)^{n^2} \prod_{n=1}^{n} \widehat{p}(z_i) = 2^n \prod_{i=1}^{n} \beta_p(z_i). \tag{34}$$

From the fact that $\alpha_p(s) + \beta_p(s) = p(s)$, we have $\alpha_p(z_i) = -\beta_p(z_i)$, $i = 1, \ldots, n$. Hence

$$\nu(p) = (-1)^n 2^n \prod_{i=1}^{n} \alpha_p(z_i). \tag{35}$$

We now use the following easily proved fact: Given monic polynomials $A(s)$ and $B(s)$ with zeros $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$, respectively,

$$\prod_{i=1}^{m} A(y_i) = (-1)^{mn} \prod_{i=1}^{n} B(x_i). \tag{36}$$

It follows from (35) and (36) that

$$\nu(p) = 2^n a_n^n \prod_{i=1}^{n} p(\alpha_i) \tag{37}$$

$$= 2^n a_n^n \prod_{i=1}^{n} \beta_p(\alpha_i). \tag{38}$$

On the other hand, Eq. (27) implies

$$\delta(p) = \det R(\alpha_p, \beta_p) = (-1)^{\frac{n(n-1)}{2}} a_n^{n^2} \prod_{i=1}^{n} \beta_p(\alpha_i). \tag{39}$$

Comparison of Eqs. (38) and (39) yields

$$\nu(p) = (-1)^{\frac{n(n-1)}{2}} 2^n a_n^{n-n_2} \delta(p). \tag{40}$$

Note that, if $n_2 < n$, then $a_n = p_n = 1$. Thus, $a_n^{n-n_2} = 1$ both for $n_2 = n$ and for $n_2 < n$. Q.E.D.

Remark 1. As a reviewer has pointed out to the authors, the guardian map $\delta(p)$ may alternatively be written as the determinant of the *Bezoutian* $B(p, \widehat{p})$, modulo a constant factor. The Bezoutian of two n^{th} degree polynomials is an $n \times n$ matrix [8]. The reduction in dimensionality as compared to the resultant formulation used here may lead to a savings in computation.

Finally, it can be shown that similar results hold for the case of Schur stability where ν and δ now take the form

$$\nu(p) = \det \left(\mathcal{C}(p) \otimes \mathcal{C}^H(p) - I \right) \tag{41}$$

and

$$\delta(p) = \det R(\alpha_p, \beta_p). \tag{42}$$

Here, polynomials α_p and β_p are defined by

$$\alpha_p(s) := \frac{1}{2} \left(p(s) + s^n (p(1/s^*))^* \right), \tag{43}$$

$$\beta_p(s) := \frac{1}{2} \left(p(s) - s^n (p(1/s^*))^* \right). \tag{44}$$

The case of polynomials with real coefficients is considered in the following remark.

Remark 2. Bose [2] shows that if p^0 and p^1 have real coefficients and are both Hurwitz stable, then the convex hull $\{p(r) = (1-r)p^0 + rp^1, r \in [0,1]\}$ is Hurwitz stable if and only if $\widehat{\delta}(p(r))$ has no zeros in $[0,1]$, where $\widehat{\delta}(p(r))$ is the determinant of a resultant matrix of size $(n-1) \times (n-1)$, linear in the coefficients of p. It can be easily verified that $p_0 \widehat{\delta}(p)$, with p_0 as in (20), is in fact a guardian map for the set of Hurwitz stable real monic polynomials. Białas' test [9] on the other hand, corresponds to the guardian map $\det H(p)$, where

$$H(p) = \begin{bmatrix} p_{n-1} & p_{n-3} & p_{n-5} & \cdot & \cdot & \cdot & \cdot & 0 \\ 1 & p_{n-2} & p_{n-4} & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & p_{n-1} & p_{n-3} & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & p_n & p_{n-2} & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & p_3 & p_1 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & p_2 & p_0 \end{bmatrix}$$

is the $n \times n$ Hurwitz test matrix associated with p, which too is linear in the coefficients of p. Since clearly $\det H(p) = p_0 \cdot \det \tilde{H}(p)$, with $\tilde{H}(p)$ an $(n-1) \times (n-1)$ matrix, it is clear that Bose's test and Bialas' test are identical from a computational point of view. A similar comment holds in the Schur case regarding Bose's test and that proposed by Ackermann and Barmish [10].

We conclude with an example illustrating the use of Theorem 4 and Proposition 5.

Example 3. We test the one-parameter family of polynomials

$$p(r)(s) = s^3 + 3(1-r^2)s^2 + 3s + 1 - (1-j)r$$

for Hurwitz stability. Clearly $p(0)$ is stable. Using, e.g., the symbolic manipulation system MACSYMA,[5] one obtains

$$(p(r)(-s^*))^* = -\frac{1}{8}(s^3 + 3(r^2-1)s^2 + 3s + (1+j)r - 1),$$

$$\alpha_p(r)(s) = \frac{1}{16}(7s^3 + 27(1-r^2)s^2 + 21s + 9 - (9-7j)r),$$

and

$$\beta_p(r)(s) = \frac{1}{16}(9s^3 + 21(1-r^2)s^2 + 27s + 7 - (7-9j)r)$$

yielding

$$\begin{aligned}\delta(p(r)) = \; & -884736r^8 + 2654208r^6 + 2654208r^5 - 5898240r^4 \\ & -4096000r^3 + 6094848r^2 + 1572864r - 2097152.\end{aligned}$$

The real roots of $\delta(p(r))$ are given approximately by

$$-0.904, -0.864, 1.0, 1.126$$

Since $p(0)$ is stable, we now have that $p(r)$ is Hurwitz stable for any $r \in [-0.864, 1.0]$. It can be checked that $p(-0.864)$ and $p(1.0)$ do indeed have zeros on the imaginary axis.

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[5]MACSYMA is a registered trademark of Symbolics, Inc., Cambridge, MA.

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