

**A Systematic Methodology for the  
Dynamic Analysis of Articulated  
Gear-Mechanisms**

**By**

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Articulated Gear-Mechanism**

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**Abstract**

This paper describes a systematic methodology for the dynamic analysis of a general class of articulated gear-mechanisms. The approach is based on recently published results related to the kinematic and dynamic analyses of such gear systems. We have shown that the Lagrange's equations of motion can be derived in terms of the minimum number of generalized coordinates in a systematic manner. We have also shown that constraint forces and torques can be evaluated systematically and efficiently, once the equations of motion have been solved. The procedure can be automated in a computer program.



## Introduction

Since Hooker and Margulies(1965) started to investigate the dynamics of general n-body satellite, there have been numerous articles on the dynamics of multi-body systems. At first, only open tree-structured systems made up of rigid bodies connected by perfect revolute joints were considered. Gradually, systems involving closed loops and flexible bodies connected by joints other than revolute ones have also been studied not just for space application but also for ground-based application. One of more recent application areas is robot manipulators. Out of these efforts many general purpose programs capable of modeling certain classes of systems become available; some are government sponsored, such as NBOD2 and DISCOS(Bodley et al., 1978), some are commercially available, such as ADAMS(MDI, 1981), DADS(CADSI, 1988) and SD/FAST(SDI, 1988), and still some others have been developed by corporations for their internal uses. Although existing programs can be applied to many different mechanical systems, they are not suitable for the gear systems in some robot manipulators such as the one shown schematically in Fig. 1.

Gears have been used for centuries by humans to transmit mechanical power from one rotation axis to another. They are also used to achieve desired rotation speeds at the output axes. With the advent of robotic manipulators in our work places comes a complicated form of gear systems that provide rotation motions similar to those of a human wrist. These robot wrists are made to be compact and light weight, and some of them have all their motors mounted away from the wrist so that the complete robot manipulators have more favorable mass distributions. The system depicted in Fig. 1 is an example of such wrist mechanism. Motors can be attached to each of links 5, 6 and 8 as the mechanical power source. Through spur and bevel gear trains, the power is transmitted to the output link 4 on which payload can be attached.

In order for a designer to better design such a wrist mechanism, detailed dynamic analysis of such gear systems need to be accurately and efficiently performed. Extensive literature search does not result in any articles addressing this issue directly. However, some recent developments in the kinematics and the dynamics of articulated gear-mechanisms provide the necessary bases for achieving such analysis systematically. This paper describes this systematic method for performing the dynamic analysis of a general class of articulated gear-mechanisms.

### **Background**

The dynamic analysis of a multi-degree-of-freedom articulated gear-mechanism, such as the one shown schematically in Fig. 1, can be quite involved. This is especially true when the dynamic loads on the bearings and on the gear teeth are to be evaluated for engineering design. The complexity arises because of the large number of moving links and joints between the links, ten moving links and seventeen joints in this case. Conventional methods for formulating the dynamic equations will be discussed in the following to give an indication of the complexity. Here and throughout this paper, rigid links and perfect revolute joints and gear meshings are assumed.

The first method to be discussed is the use of Lagrange's equations and Lagrange multipliers which is employed in some commercially available multibody dynamic analysis programs such as DADS and ADAMS. In this method, six or more generalized coordinates are used to describe the motion of each rigid body. Constraints due to joints and gear meshings are then introduced to restrict the motion. Each constraint equation requires a Lagrange multiplier in the Lagrange's equations. With six coordinates for each rigid body, there will be totally sixty coordinates and thus

sixty Lagrange's equations for the system shown in Fig. 1. Since the system has only three degrees of freedom, there should be fifty-seven constraint equations and, thus, fifty-seven Lagrange's multipliers. Therefore, simulation of this system is accomplished by solving simultaneously the one hundred and seventeen variables, sixty coordinates and fifty-seven multipliers, in one hundred and seventeen differential algebraic equations, sixty Lagrange's equations and fifty-seven constraint equations. This approach demands significant computation efforts. Furthermore, although Lagrange multipliers are known to be related to the constraint forces and torques, they may not be exactly equal. If constraint forces and torques are of interest, additional analyses have to be performed. In addition, inverse dynamic analysis for evaluating the driving torques for a given motion of the system also requires significant amount of computation. Its advantage is the relative ease of implementation for general mechanisms.

Next, an alternative method will be discussed. It can be observed that kinematic analysis of the system can be performed to determine the motions of all the links using only three generalized coordinates. This task usually requires capable analysts. Then the kinetic energy of the system is formulated in terms of these three coordinates and three Lagrange's equations of motion can be derived. Alternatively, Kane's equations or virtual work principle can be applied to derive the same equations. These equations can be numerically solved to yield the coordinates as functions of time. As for the loads on the bearings and the gears, free-body diagram must be developed for each link, and force/moment balance equations can be written for determining the loads. It is often the case that all the constraint forces and torques applied on a particular link, usually the input or the output link, can be solved with the use of its free-body diagram when its motion is known. Once they are solved, the

constraint forces and torques on an adjacent link may become solvable with its free-body diagram also. This can continue throughout the system until all the loads are found. For each link, there are only six linear algebraic equations to be solved simultaneously. As for the inverse dynamics, the three Lagrange's equations can easily be rearranged for the analysis.

From the narratives above, it can be easily argued that the second method is much more efficient computationally. It, however, requires the analyst to apply skillful and significant amount of analysis on each individual system before computer can be used to perform the numerical computations. It is the objective of this paper to present a systematic and efficient approach for the dynamics analysis of a general class of articulated gear-mechanisms.

### **Systematic Procedure**

In the following, a systematic procedure for the dynamic analysis of multi-degree-of-freedom articulated gear-mechanisms will be given. The fundamental concepts behind each step will be briefly discussed and references will be given for their more detailed descriptions. The step will then be applied to the example shown in Fig. 1 to demonstrate the principle.

#### **1. Kinematic Analysis**

A systematic procedure for the kinematic analysis of spatial robotic bevel-gear trains is presented in Tsai(1988). The procedure involves the development of canonical graph representation for the system, the identification of equivalent open-loop chain, and the formulation of angular displacement equations.

Graph representation is used to organize the analysis. In a graph



representation, links are denoted by vertices, turning pairs by thin edges and gear meshings by heavy edges. The thin edges are labeled according to their axes locations in space. Fig. 2 shows the graph representation of the mechanism shown in Fig. 1, and Fig. 3 shows its canonical graph representation. A canonical graph is a uniquely defined representation of a gear train in which there is no repeated axis labels in a thin-edged path starting from the base link(Tsai, 1988).

The equivalent open-loop chain of a system is defined as the linkage made up of those links and joints along the thin-edged path from the base link to the output link in the canonical graph. Fig. 4 shows the equivalent open-loop chain of the example system. Each link in Fig. 4 is referred to as a major link, and the relative angles  $\theta_{2,1}$ ,  $\theta_{3,2}$  and  $\theta_{4,3}$  about  $z_1$ ,  $z_2$  and  $z_3$  axes, respectively, are referred to as the open-loop joint angles. All the links other than the major links will be called the carried links. In the canonical graph representation, all the carried links are connected directly to one of the major links by a revolute joint. Thus, in Fig. 3, links 5, 6, 7, 8 and 9 are carried by link 1, link 10 is carried by link 2, and link 11 is carried by link 3. Throughout this paper relative angle  $\theta_{i,j}$  will be used to denote the angular displacement of link  $i$  relative to link  $j$ . The relationship between the position and orientation of the output link and the open-loop joint angles can be found using the matrix method or the vector approach.

Coaxial and fundamental circuit equations(Tsai, 1988) are then used to find the angular displacements of all the carried links in terms of the open-loop joint angles. A fundamental circuit consists of a gear pair and a carrier, relative to which the gears can be considered to be performing simple fixed axis rotations. For example, in Fig. 3, gear pair 9 and 10 perform fixed axis rotation relative to link 2, and thus one can write

$$\theta_{9,2} = N_{10,9} \theta_{10,2} \quad (1)$$

Here  $N_{10,9}$  is a positive or a negative gear ratio with its sign determined by the convention adopted on the positive direction of relative rotations.

When two turning pairs share a common joint axis, they are coaxial and simple relationship exists between relative angular displacements. For example, in Fig. 3, pairs (9,1) and (1,2) are coaxial, and one can write

$$\theta_{9,2} = \theta_{9,1} + \theta_{1,2} = \theta_{9,1} - \theta_{2,1} \quad (2)$$

Using the fundamental circuit equations and the appropriate coaxiality conditions, the angular displacements of the carried links with respect to their carriers can be derived in terms of the open-loop joint angles. For the example system, the displacement equations can be arranged in a matrix form as

$$\begin{bmatrix} \theta_{5,1} \\ \theta_{6,1} \\ \theta_{7,1} \\ \theta_{8,1} \\ \theta_{9,1} \\ \theta_{10,2} \\ \theta_{11,3} \end{bmatrix} = \begin{bmatrix} N_{2,5} & 0 & & 0 & & & \\ N_{7,6} & N_{3,7}N_{7,6} & & 0 & & & \\ 1 & N_{3,7} & & 0 & & & \\ N_{9,8} & N_{10,9}N_{9,8} & N_{4,11}N_{11,10}N_{10,9}N_{9,8} & & & & \\ 1 & N_{10,9} & N_{4,11}N_{11,10}N_{10,9} & & & & \\ 0 & 1 & N_{4,11}N_{11,10} & & & & \\ 0 & 0 & N_{4,11} & & & & \end{bmatrix} \begin{bmatrix} \theta_{2,1} \\ \theta_{3,2} \\ \theta_{4,3} \end{bmatrix} \quad (3)$$

## 2. Dynamic Analysis

First, all the carried links are treated as being rigidly attached to their carriers according to the canonical graph representation and then the generalized inertia forces due to the resulted open-loop linkage are formulated. Next, the additional effects of the relative rotations of the carried links are derived and added to the generalized inertia forces mentioned above. Finally, the generalized active forces are formulated and combined with the generalized inertia forces to form the equations of

motion.

The basic concept for this approach is presented by Chen(1989). All the carried links contribute to the generalized inertia force of the dynamic equations in two parts. The first part is due to the motions of the carriers, which are the major links. The second part is due to their motions relative to the carriers. It should be computationally advantageous to incorporate the first part of the generalized inertia forces with that of the appropriate major links. This is done by treating the carried links as being fixed in the major links.

The second part can be derived by applying Lagrange's equation on the additional kinetic energy contributed by the carried links due to the rotations relative to their respective carriers. For link  $i$  performing fixed axis rotation relative to major link  $j$ , the additional kinetic energy  $K'_{i,j}$  due to link  $i$  is

$$K'_{i,j} = 1/2 J_i (\dot{\theta}_{i,j})^2 + J_i \dot{\theta}_{i,j} (\underline{w}_j \cdot \underline{e}_i) \quad (4)$$

where  $J_i$  is the moment of inertia of the carried link  $i$  about its rotation axis on link  $j$ ,  $\underline{e}_i$  is the "positive" unit vector along the axis of rotation of link  $i$ ,  $\underline{w}_j$  is the angular velocity of link  $j$  with respect to the inertial frame and  $\dot{\theta}_{i,j}$  is the rotation rate of link  $i$  relative to link  $j$ .

For our example, the three degrees of freedom linkage shown in Fig. 4 is the open-loop manipulator whose generalized inertia force will be formulated first. The mass and inertia properties of links 5 to 9 should be incorporated in link 1, which is fixed to the ground and thus has no effects. The mass and inertia properties of link 10 should be include in link 2 while those of link 11 in link 3. The formulation of dynamic equations for such open-loop chain has been presented in many articles

(Hollerbach, 1980; Lee, et al., 1985; Thomas and Tesar, 1982; Walker and Orin, 1982) and they can be readily established in a computationally efficient form. The generalized inertia forces  $[F_e^*]$  (Kane and Levinson, 1985) for the system shown in Fig. 4 can always be put in the following form.

$$[F_e^*] = H_e(q)[\ddot{q}] + C_e(q, \dot{q}) + G_e(q) \quad (5)$$

where

$$[q] = [q_1, q_2, q_3]^T = [\theta_{2,1}, \theta_{3,2}, \theta_{4,3}]^T,$$

$H_e(q)$  is a 3x3 inertia matrix,  $C_e(q, \dot{q})$  is a 3x1 matrix specifying the coriolis and centrifugal effects, and  $G_e(q)$  is a 3x1 matrix specifying the gravitational effects.

As for the additional terms due to relative rotations, contributions from links 5, 6, 7, 9, 8, 10 and 11 need to be evaluated. Those for the last three will be demonstrated. Substituting the appropriate equations and their derivatives of eq. (3) into eq. (4), we obtain for link 8

$$K'_{8,1} = 1/2 J_8 (\mu_1 \dot{q}_1 + \mu_2 \dot{q}_2 + \mu_3 \dot{q}_3)^2 \quad (6)$$

where

$$\mu_1 = N_{9,8} \quad (7)$$

$$\mu_2 = \mu_1 N_{10,9} \quad (8)$$

$$\mu_3 = \mu_2 N_{4,11} N_{11,10} \quad (9)$$

Applying Lagrange's equations, one can obtain the additional contributions to the generalized inertia forces of link 8 as

$-\mu_1 J_8 (\mu_1 \ddot{q}_1 + \mu_2 \ddot{q}_2 + \mu_3 \ddot{q}_3)$	associated with $q_1$ ,
$-\mu_2 J_8 (\mu_1 \ddot{q}_1 + \mu_2 \ddot{q}_2 + \mu_3 \ddot{q}_3)$	associated with $q_2$ , and
$-\mu_3 J_8 (\mu_1 \ddot{q}_1 + \mu_2 \ddot{q}_2 + \mu_3 \ddot{q}_3)$	associated with $q_3$ .

For link 10, the additional kinetic energy is

$$K'_{10,2} = 1/2 J_{10} (\dot{q}_2 + \mu_4 \dot{q}_3)^2 \quad (10)$$

where

$$\mu_4 = N_{4,11} N_{11,10} \quad (11)$$

and its contributions to the generalized inertia forces are

0	associated with $q_1$ ,
$-J_{10}(\ddot{q}_2 + \mu_4 \ddot{q}_3)$	associated with $q_2$ , and
$-\mu_4 J_{10}(\ddot{q}_2 + \mu_4 \ddot{q}_3)$	associated with $q_3$ .

For link 11, the additional kinetic energy is

$$K'_{11,3} = 1/2 J_{11} (N_{4,11} \dot{q}_3)^2 + J_{11} N_{4,11} \dot{q}_1 \dot{q}_3 \cos(q_2) \quad (12)$$

Here it is assumed that the axis of rotation of link 11 relative to link 3 is parallel to  $z_3$  axis, which is always perpendicular to  $z_2$  axis while  $z_2$  axis remains perpendicular to  $z_1$  axis. The additional contributions to the generalized inertia forces due to link 11 are

$-J_{11} N_{4,11} \{ \ddot{q}_3 \cos(q_2) - \dot{q}_2 \dot{q}_3 \sin(q_2) \}$	associated with $q_1$ ,
$-J_{11} N_{4,11} \dot{q}_1 \dot{q}_3 \sin(q_2)$	associated with $q_2$ , and
$-J_{11} (N_{4,11})^2 \ddot{q}_3 - J_{11} N_{4,11} \{ \ddot{q}_1 \cos(q_2) - \dot{q}_1 \dot{q}_2 \sin(q_2) \}$	associated with $q_3$ .

The additional contributions to the generalized inertia forces of the carried links can then be added to the right-hand side of Eq. (5) according to the generalized coordinates they are associated with.

The formulation of generalized active forces is rather straightforward; for our example, the generalized active forces  $F_i$  associated with  $q_i$  ( $i=1,2,3$ ) due to the motor torques  $T_5$ ,  $T_6$ , and  $T_8$  acted on links 5, 6, and 8, respectively, are



of one gear meshing. The gear surfaces are usually designed to have this normal force acting along a fixed direction, known as the line of action, relative to the carrier of the meshing gears. For more discussion on this subject, readers are referred to machine design books such as (Spotts, 1985).

Modeling a perfect revolute bearing contact between two links requires five unknown constraint quantities: 3 forces and 2 torques. Frictional torque should be treated as known or related to the other forces and torques.

Applying the above arguments on the graph representation of mechanism, such as Fig. 3, one can say that breaking each heavy edge requires the introduction of one unknown quantity while breaking each thin edge requires five unknown quantities. This observation can be useful in guiding the evaluation of constraint forces and/or torques. For example, it can be seen in Fig. 3 that link 4 is only connected to other links with one heavy and one thin edges. Free-body balance of link 4 will therefore yield six equations in six unknown constraint quantities arising from between links 4 and 3, and between links 4 and 11. Once that is done, the balance equations for link 11 can be used to solve for the constraint quantities due to connections between links 11 and 3, and between links 11 and 10. This is true because the action of link 4 on link 11 is equal to the opposite of that applied by link 11 on 4, which has been evaluated in the first step. This procedure can go on in the sequence of 4, 11, 3, 10, 2, 7, 9, 5, 8, 6. Alternatively, the sequence can be arranged as three independent paths, namely, <5>, <6,7> and <8,9,10,11> to solve all the constraint forces and torques on the carried links. Then the constraint forces and torques on the major links can be solved backward from link 4 to link 2.

The free-body analysis for link 4 will be given in the following as an example. In this analysis the torques due to friction at either the bearings or the meshing gears are assumed to be negligible.

Fig. 5 shows a schematic drawing of link 4 with a payload on it. Orthogonal dextral unit vectors  $\underline{i}_4$ ,  $\underline{j}_4$  and  $\underline{k}_4$  fixed on link 4 are introduced for expressing the force and torque vectors. The combined mass center of link 4 and the payload is denoted as Q. The action of link 3 on link 4 through the bearings can be replaced by a force  $\underline{F}_{3/4}$  through Q and a couple whose torque is  $\underline{T}_{3/4}$ . The action of link 11 on link 4 through spur gear meshing can be replaced by a single force  $\underline{F}_{11/4}$  passing through point P. These force and torque vectors can be expressed as

$$\underline{F}_{3/4} = f_x \underline{i}_4 + f_y \underline{j}_4 + f_z \underline{k}_4 \quad (14)$$

$$\underline{T}_{3/4} = T_x \underline{i}_4 + T_y \underline{j}_4 \quad (15)$$

$$\underline{F}_{11/4} = w_{t1} \underline{i}_4 + |w_{t1}| \tan(\phi) \underline{j}_4 \quad (16)$$

where  $w_{t1}$ ,  $f_x$ ,  $f_y$ ,  $f_z$ ,  $T_x$  and  $T_y$  are the scalar quantities to be evaluated;  $w_{t1}$  is the tangential measure number of the gear force at gear pair (4,11),  $|w_{t1}|$  is the absolute value of  $w_{t1}$ , and  $\phi$  is the gear pressure angle which is always positive.

In the case of bevel gear meshing, the force of one gear on the other has additional component due to the pitch cone angle. For instance, in Fig. 3, link 3 is the carrier of gear pair (10,11). Thus, the force  $\underline{F}_{10/11}$ , exerted on link 11 by link 10, as shown in Fig. 6, can be expressed as

$$\underline{F}_{10/11} = w_{t2} \underline{i}_3 + |w_{t2}| \tan(\phi) \cos(\varphi_{11}) \underline{j}_3 + |w_{t2}| \tan(\phi) \sin(\varphi_{11}) \underline{k}_3 \quad (17)$$

where unit vectors  $\underline{i}_3$ ,  $\underline{j}_3$  and  $\underline{k}_3$  for this expression are fixed in link 3,



and  $\varphi_{11}$  is the pitch cone angle of link 11.

Assuming the absolute acceleration of link 4 at Q and the inertia torque  $\underline{T}_4^*$  (Kane and Levinson, 1985) of the combined rigid body can be worked out, one can write the equations of balance as

$$\underline{F}_{11/4} + \underline{F}_{3/4} = m_4 \underline{a}^Q \quad (18)$$

$$\underline{r}^{QP} \times \underline{F}_{11/4} + \underline{T}_{3/4} + \underline{T}_4^* = 0 \quad (19)$$

where  $\underline{r}^{QP}$  is the position vector from Q to P.

Equations (17) and (18) contain six linear scalar equations which can be readily solved for the six scalar unknowns.

#### Summary

It has been shown that the dynamics of a class of articulated gear-mechanisms commonly used in robot wrists can be analyzed in a systematic manner. The approach combines recently published results related to the kinematic and dynamic analyses of such gear systems in a uniform framework. Efficient method for constraint forces and torques evaluation has also been developed. The procedure can be the basis for a computer program which can provide automated analysis of a general class of articulated gear-mechanisms.

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Figure Captions:

Fig. 1: An articulated-gear-mechanism example.

Fig. 2: Graph representation of the example mechanism.

Fig. 3: Canonical graph representation of the example mechanism.

Fig. 4: The equivalent open-loop chain of the example mechanism.

Fig. 5: Free-body diagram of link 4 of the example mechanism.

Fig. 6: Free-body diagram of link 11 of the example mechanism.

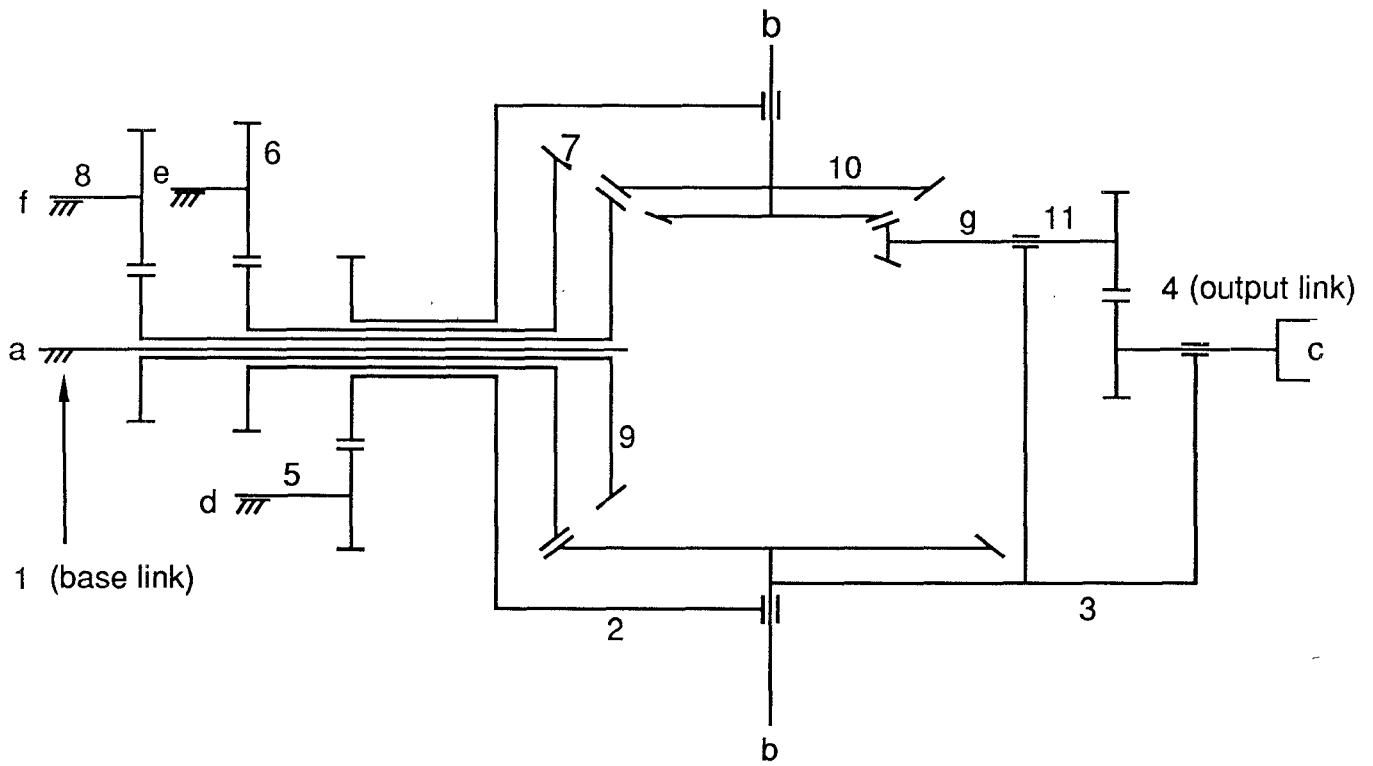


Fig. 1

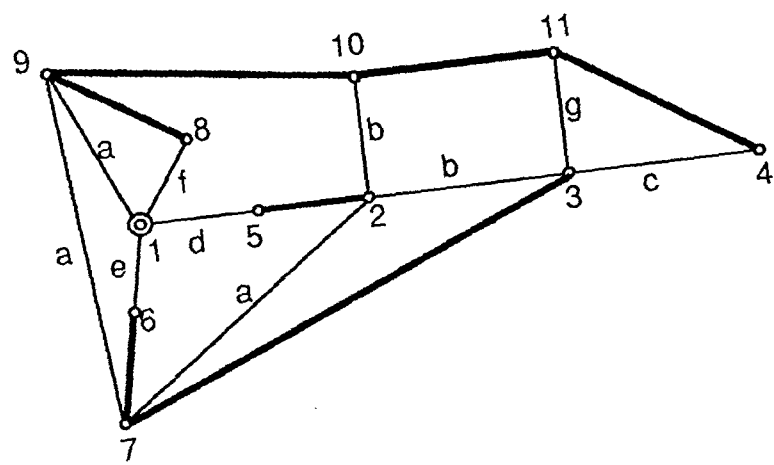


Fig. 2

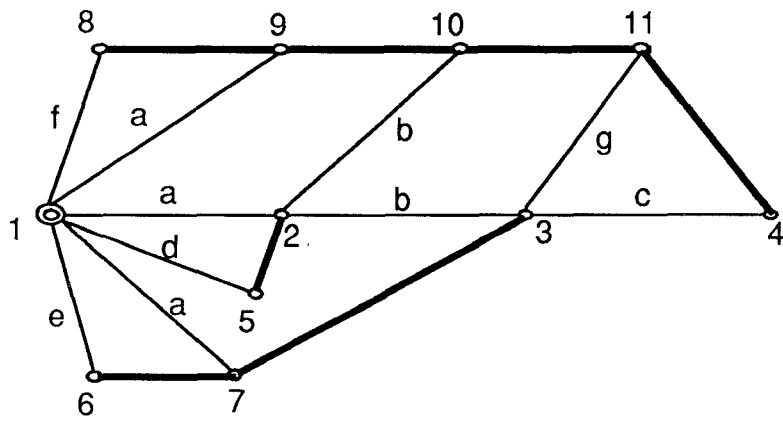


Fig. 3

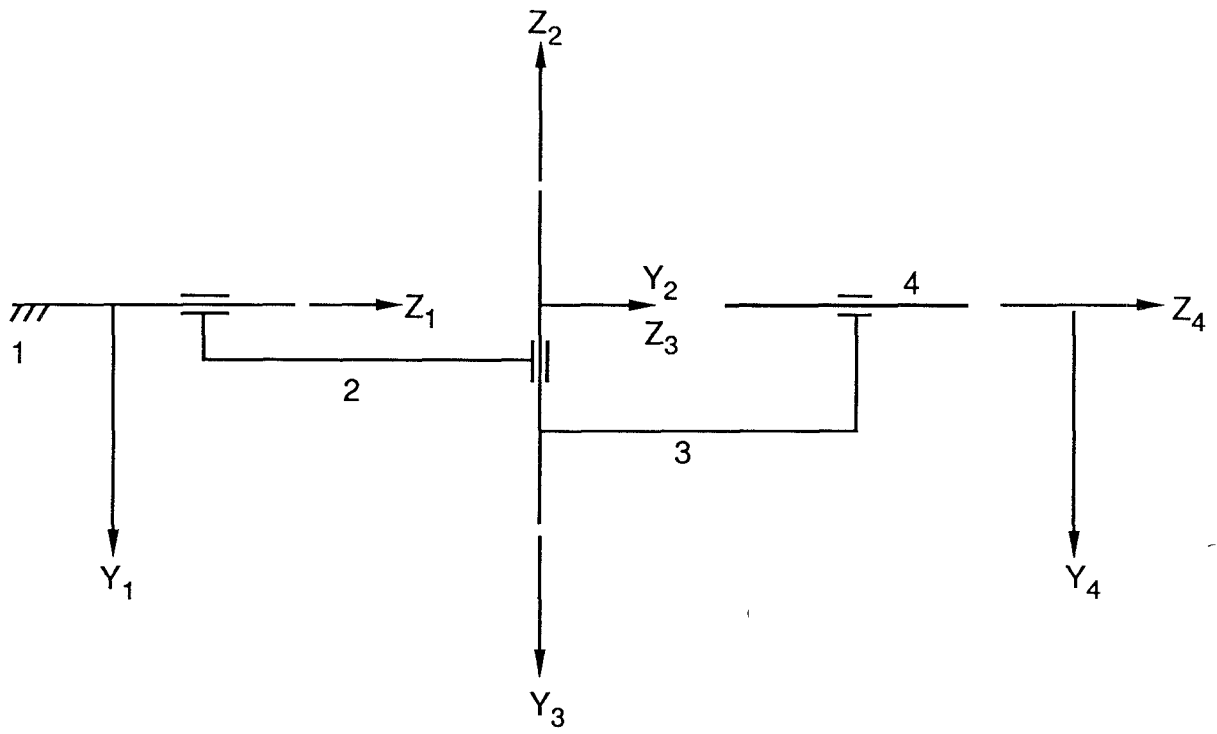


Fig. 4

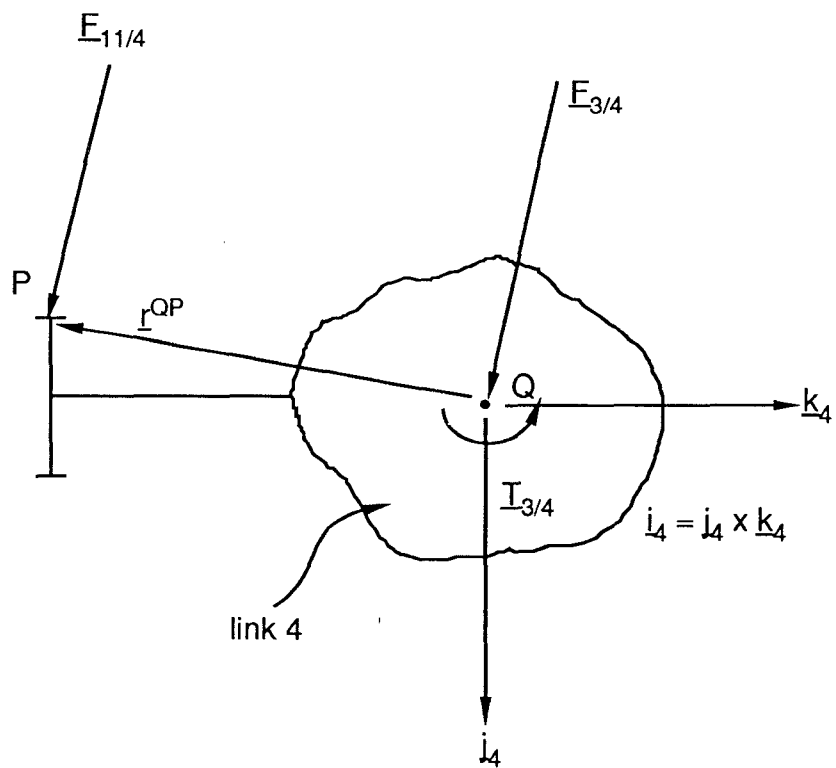


Fig. 5



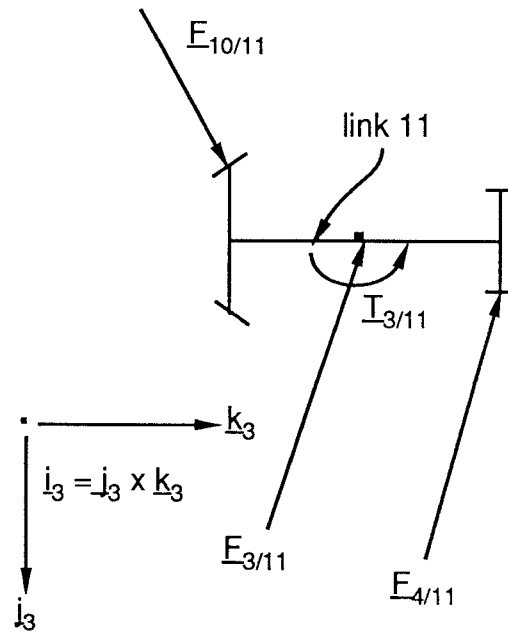


Fig. 6