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On the Dynamics of Floating Four-Bar Linkages

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ON THE DYNAMICS OF FLOATING FOUR-BAR LINKAGES *

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ABSTRACT

The hamiltonian structure of *floating*, planar four-bar linkages is discussed. The geometry of configuration space is related to the classical theory of mechanisms due to Grashof. For generic value of kinematic parameters, the techniques of symplectic (and Poisson) reduction apply.

1. INTRODUCTION

There has been significant progress in our understanding of the hamiltonian structure of serial-link (or open chain) multibody systems [2,3,9,11,12,15–18,21–23]. The use of geometric methods, symmetry principles and reduction has led to deeper knowledge of the phase portraits of model problems. This insight has been helpful in developing appropriate control-theoretic tools. The primary source of motivation for these problems has been

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in aerospace engineering where imaginative designs of multibody spacecraft have been proposed and on occasion realized [4,5,24].

In contrast, multibody systems with kinematic loops present serious challenges. The loop constraint may lead to singular configuration spaces. The Dirac theory of constraints [6,20] applies in the smooth setting. Little or nothing is known about hamiltonian structure and phase portraits in concrete cases. However, engineering applications suggest that multibody systems with kinematic loops are of practical importance [10]. Parallel linkage based robot manipulators are contemplated for space applications.

Here we discuss the geometry and dynamics of *floating* four-bar linkages. In this model problem, the classical Grashof criterion [8,10,14] appears through conditions for the configuration space to be a smooth manifold. The topology of the configuration space is also related to the Grashof criteria. We explore symmetry properties, hamiltonian structure and reduction of four bar linkage dynamics. Explicit computation of constrained dynamics is difficult. Yet in the present setting, using geometric techniques, one can infer qualitative properties without recourse to explicit analytic representation of the constrained dynamics. We use a theorem of Smale to determine relative equilibria for four-bar linkage dynamics.

2. NOTATIONS & GEOMETRIC CONSTRAINTS

The structure of a closed floating four-bar linkage is represented in Fig. 1. The bars are labeled clockwise from 0 to 3 as shown. We define the following quantities.

$\mathbf{d}_{ij}$ — the vector of hinge point which connects i-th bar with j-th bar relative to a body-fixed frame with origin at the center of mass of the i-th body;

$\mathbf{r}_i$ — the position vector of the center of mass of i-th bar relative to an inertial observer;

$\mathbf{r}_i^c$ — the vector from the system center of mass to the center of mass of i-th bar;

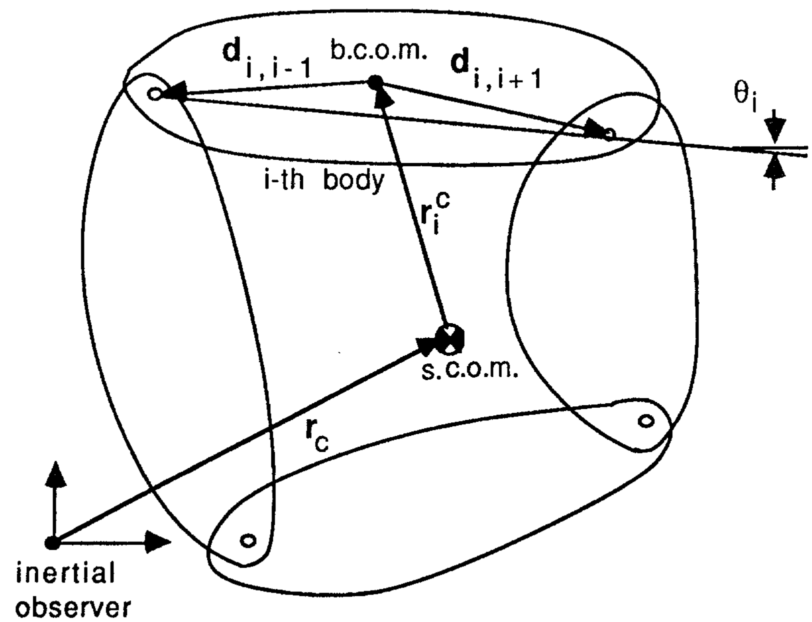


Fig. 1 The general structure of four-bar linkage

$\mathbf{r}_c$ — the position of the system center of mass relative to the reference point of the inertial observer;

$R(\theta_i)$ — the rotation through angle θ_i giving the orientation of i-th bar relative to the inertial space;

$$R(\theta_i) = \begin{pmatrix} cos(\theta_i) & -sin(\theta_i) \\ sin(\theta_i) & cos(\theta_i) \end{pmatrix};$$

$R(\theta_{ij})$ — joint rotation between i-th and j-th bar,

$$R(\theta_{ij}) = R(\theta_i - \theta_j) = R(\theta_i)R(-\theta_j);$$

l_i — the distance between the joints on i-th bar, (or "length" of i-th bar); all $l_i > 0$; $l_i = \|\mathbf{d}_{i,i+1} - \mathbf{d}_{i,i-1}\|$;

I_i the moment of inertia of i-th bar about its center of mass;

m the total mass of the system, i.e.

$$m = \sum_{i=0}^{3} m_i.$$

With the above notations, any pair of adjacent bodies is connected by the following relation, so called *hinge constraint*,

$$\mathbf{r}_{i+1}^c = \mathbf{r}_i^c + R(\theta_i)\mathbf{d}_{i,i+1} - R(\theta_{i+1})\mathbf{d}_{i+1,i} \qquad i = 0, 1, 2, 3 \pmod 4. \tag{2.1}$$

By eliminating $\mathbf{r}_i^c$ in Eq.(2.1) we find the loop constraint or *closure constraint*,

$$\sum_{i=0}^{3} R(\theta_i)(\mathbf{d}_{i,i+1} - \mathbf{d}_{i,i-1}) = 0, \tag{2.2}$$

where we adopt the convention that $\mathbf{d}_{i,4} = \mathbf{d}_{i,0}$ and $\mathbf{d}_{4,i} = \mathbf{d}_{0,i}$.

3. THE CONFIGURATION SPACE

In this section we investigate the conditions under which the loop constraint, Eq.(2.2), describes a submanifold with respect to the configuration manifold of an open four-bar chain.

For a planar floating four-bar *open* chain, the configuration space is

$$M = R^2 \times S^1 \times S^1 \times S^1 \times S^1,$$

where M is a 6-dimensional smooth manifold with local coordinates of the form,

$$\mathbf{q} = (x_0, y_0, \theta_0, \theta_1, \theta_2, \theta_3).$$

This corresponds to keeping track of a material point (say center of mass on one of the bodies) and the four absolute orientations. See [12,21,22] for the hamiltonian mechanics of such open chains.

For a *closed* four bar mechanism as considered in this paper, the configuration space denoted by Q is a subset of M determined by Eq.(2.2), or simply

$$Q = \{\mathbf{q} \in M | \mathbf{F}(\mathbf{q}) = \sum_{i=0}^{3} R(\theta_i)(\mathbf{d}_{i,i+1} - \mathbf{d}_{i,i-1}) = \mathbf{0}\}. \tag{3.1}$$

Note that $\mathbf{F} : M \longrightarrow R^2$, and from [1] we know that if $\mathbf{0}$ is a regular value of $\mathbf{F}$, i.e. $\partial\mathbf{F}/\partial\mathbf{q}$ has full rank for all $\mathbf{q} \in Q$, then Q is a submanifold of M.

From (3.1) we have,

$$\frac{\partial \mathbf{F}}{\partial \mathbf{q}} = \begin{pmatrix} 0 & 0 & -l_0 sin(\theta_0) & -l_1 sin(\theta_1) & -l_2 sin(\theta_2) & -l_3 sin(\theta_3) \\ 0 & 0 & l_0 cos(\theta_0) & l_1 cos(\theta_1) & l_2 cos(\theta_2) & l_3 cos(\theta_3) \end{pmatrix}.$$

Then it is easy to check that all the nontrivial determinants of 2×2 submatrices are given by the set of expressions,

$$\begin{aligned} g_1(\mathbf{q}) &= l_0 l_1 sin(\theta_1 - \theta_0) \\ g_2(\mathbf{q}) &= l_0 l_2 sin(\theta_2 - \theta_0) \\ g_3(\mathbf{q}) &= l_0 l_3 sin(\theta_3 - \theta_0) \\ g_4(\mathbf{q}) &= l_1 l_2 sin(\theta_2 - \theta_1) \\ g_5(\mathbf{q}) &= l_1 l_3 sin(\theta_3 - \theta_1) \\ g_6(\mathbf{q}) &= l_2 l_3 sin(\theta_3 - \theta_2). \end{aligned} \tag{3.2}$$

Therefore for each $\mathbf{q} \in Q$, if there exists an i such that $g_i(\mathbf{q}) \neq 0$, Q is a submanifold of M.

It is obvious that the above condition depends on the lengths of the links. To find the condition on the links, we can use an equivalent way, that is, find necessary conditions on the lengths of the links such that $g_i(\mathbf{q}) = 0$ for all i.

From Eqs.(3.2) it can be seen that if

$$\theta_1 - \theta_0 = 0 \ or \ \pi \tag{3.3a}$$

and

$$\theta_3 - \theta_2 = 0 \ or \ \pi \tag{3.3b}$$

and

$$\theta_3 - \theta_0 = 0 \; or \; \pi \tag{3.3c}$$

then $g_i(\mathbf{q}) = 0$ for all i.

Premultiplying Eq.(2.2) by $R(\theta_0)$, we get following equivalent closure constraint equations

$$l_0 + l_1 cos(\theta_1 - \theta_0) + l_2(cos(\theta_3 - \theta_2)cos(\theta_3 - \theta_0) + sin(\theta_3 - \theta_2)sin(\theta_3 - \theta_0))$$

$$+ l_3 cos(\theta_3 - \theta_2) = 0 \tag{3.4a}$$

$$l_1 sin(\theta_1 - \theta_0) + l_2(cos(\theta_3 - \theta_2)sin(\theta_3 - \theta_0) - sin(\theta_3 - \theta_2)cos(\theta_3 - \theta_0))$$

$$+ l_3 sin(\theta_3 - \theta_2) = 0 \tag{3.4b}$$

The conditions given in Eq.(3.3) make Eq.(3.4)b trivial. By choosing all the possible value of $\theta_1 - \theta_0$, $\theta_3 - \theta_2$ and $\theta_3 - \theta_0$ given in Eq.(3.3), from Eq.(3.4)a, we can find the link conditions, which are summarized in Table 1.

Theorem 3.1: If $l_0 \pm l_1 \pm l_2 \pm l_3 \neq 0$, Q is a submanifold of M.

From Table 1, it is easy to observe that case (i) can never happen since l_i are assumed to be positive. In addition, cases (iii), (iv), (v) and (vi) are trivial cases since none of them can be formed by any *general* four-bar closed loop. In these cases the configuration spaces lose one degree of freedom and are three dimensional.

Furthermore, Theorem 3.1 can be simplified by ignoring the labels on the bars. To do this, we first recall some definitions and results in the classical theory of mechanisms [8,13,14]. We define following quantities

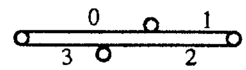
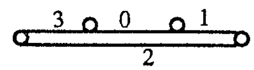
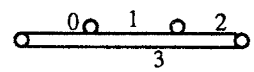
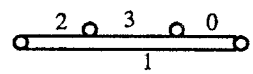
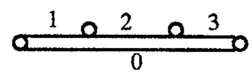
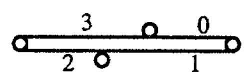
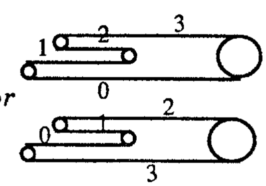
$$s = \text{length of shortest bar}$$

$$l = \text{length of longest bar}$$

$$p, q = \text{lengths of intermediate bars.}$$

A link which is free to rotate through 2π with respect to a second link is said to *revolve* relative to the second bar and is referred to as a *crank*. Any bar which does not revolve is called a *rocker*. If it is possible for all bars to become simultaneously aligned, such a state is called a *change point* and the linkage is said to be a *change-point mechanism.*

Table 1.

case	$\theta_1-\theta_0$	$\theta_3-\theta_2$	$\theta_3-\theta_0$	link condition	structure
i	0	0	0	$l_0+l_1+l_2+l_3=0$	—
ii	0	0	π	$l_0+l_1-l_2-l_3=0$	
iii	0	π	0	$l_0+l_1-l_2+l_3=0$	
iv	0	π	π	$l_0+l_1+l_2-l_3=0$	
v	π	0	0	$l_0-l_1+l_2+l_3=0$	
vi	π	0	π	$l_0-l_1-l_2-l_3=0$	
vii	π	π	0	$l_0-l_1-l_2+l_3=0$	
viii	π	π	π	$l_0-l_1+l_2-l_3=0$	or 

Theorem 3.2:(*Grashof*) (1) A four-bar mechanism has at least one revolving link if

$$s+l \leq p+q$$

and all three will rock if

$$s+l > p+q.$$

(2) A four-bar mechanism is a change-point mechanism iff

$$s+l = p+q.$$

Remark: It is easy to check that the cases (ii), (vii) and (viii) in Table 1. correspond to $s+l=p+q$ i.e. they correspond to change-point mechanisms.

Corollary 3.3: If $l \leq s+q+p$ and $s+l \neq p+q$, Q is a submanifold of M.

Remark: Note that if $l > s+q+p$, the mechanism is *not constructible.*

In order to find a topological description of the configuration manifold Q, we first introduce following fact.

Proposition 3.4: If $s+l < p+q$ and $l_1 = s$, i.e. $l_1 = \min_{0 \leq i \leq 3}(l_i)$, then $\theta_3 - \theta_2 \not\equiv k\pi$ for $k \in Z$.

Proof: The mechanism can be assembled with s adjacent to l or with s opposite l. And, l can be l_0, l_2 or l_3. If $\theta_3 - \theta_2 = k\pi$, the whole structure attains a triangular shape which has the property that the sum of two sides is larger than the third one. Then it is easy to check that all possible cases will lead to

$$s + l > p + q.$$

This is a contradiction.

∎

From Grashof's theorem and the above proposition we can get a topological description for Q.

Corollary 3.5: For a crank mechanism with $s+l < p+q$, each connected component is diffeomorphic to

$$Q = R^2 \times S^1 \times S^1$$

with parametrization $(x_0, y_0, \theta_0, \theta_1)$ if $l_1 = s$ is assumed.

Proof: We just need to prove that θ_2 and θ_3 can be uniquely determined by θ_0 and θ_1. The constraints given in Eq.(3.1) give the relations among θ_i's and they have continuous partial derivatives with respect to θ_i. Since

$$det(\frac{\partial \mathbf{F}}{\partial(\theta_2, \theta_3)}) = l_2 l_3 sin(\theta_3 - \theta_2),$$

it follows from the Implicit Function Theorem that if

$$\theta_3 - \theta_2 \neq k\pi \qquad \text{for } k \in Z$$

in some neighborhood of a point $(\theta_0, \theta_1, \theta_2, \theta_3)$, then there exists a unique pair of functions f_1 and f_2 such that

$$\theta_2 = f_1(\theta_0, \theta_1) \text{ and } \theta_3 = f_2(\theta_0, \theta_1)$$

and

$$\mathbf{F}(\theta_0, \theta_1, f_1(\theta_0, \theta_1), f_2(\theta_0, \theta_1)) = 0.$$

Using the result in above proposition, the proof is complete. In addition, as seen from [13], there exists a pair of function $\hat{f}_1$ and $\hat{f}_2$ such that

$$\theta_2 = \theta_0 + \hat{f}_1(\theta_1 - \theta_0) \text{ and } \theta_3 = \theta_0 + \hat{f}_2(\theta_1 - \theta_0). \tag{3.5}$$

∎

Remark: Following Gibson and Newstead [7] one can show that if $l + s < p + q$, then there are two connected components of Q and if $s + l > p + q$, the space Q is connected and diffeomorphic to $R^2 \times S^1 \times S^1$.

Since the configuration space under the condition given in Corollary 3.5 has an explicit parametrization, in the rest of this paper we shall study the system on this space mainly.

4. KINETIC ENERGY

In this section we shall derive the kinetic energy, or Lagrangian since we assumed that no potential energy is involved, for the whole system. The basic idea is to write the kinetic energy for each individual body first and then use the constraint equations to eliminate extra variables.

The kinetic energy of the i-th bar is

$$T_i = \frac{1}{2}\omega_i^2 I_i + \frac{1}{2}m_i\|\dot{\mathbf{r}}_i\|^2$$

where $\omega_i = \dot{\theta}_i$. The total kinetic energy is

$$T = \frac{1}{2}\sum_{i=0}^{3}\omega_i^2 I_i + \frac{1}{2}\sum_{i=0}^{3} m_i\|\dot{\mathbf{r}}_i\|^2. \tag{4.1}$$

To describe the kinetic energy relative to the center of mass, we have following useful equations,

$$\mathbf{r}_i = \mathbf{r}_c + \mathbf{r}_i^c \qquad i = 0, 1, 2, 3 \tag{4.2}$$

$$\sum_{i=0}^{3} m_i \mathbf{r}_i^c = 0. \tag{4.3}$$

By applying Eqs.(4.2) and (4.3), Eq.(4.1) becomes,

$$T = \frac{1}{2}\sum_{i=0}^{3}\omega_i^2 I_i + \frac{1}{2}\sum_{i=0}^{3} m_i \|\dot{\mathbf{r}}_i^c\|^2 + \frac{1}{2} m \|\dot{\mathbf{r}}_c\|^2. \tag{4.4}$$

Applying Eq.(2.1) and Eq.(4.2), we get

$$\begin{aligned}\mathbf{r}_i^c = \frac{1}{m}[&R(\theta_{i-1})m_{i-1}\mathbf{d}_{i-1,i}\\ &- R(\theta_i)(m_{i-1}\mathbf{d}_{i,i-1} + (m_{i+1}+m_{i+2})\mathbf{d}_{i,i+1})\\ &+ R(\theta_{i+1})((m_{i+1}+m_{i+2})\mathbf{d}_{i+1,i} - m_{i+2}\mathbf{d}_{i+1,i+2})\\ &+ R(\theta_{i+2})m_{i+2}\mathbf{d}_{i+2,i+1}]\end{aligned}$$

for $i = 0, 1, 2, 3 \pmod 4$. Furthermore,

$$\begin{aligned}\dot{\mathbf{r}}_i^c = \frac{1}{m}[&m_{i-1}\hat{\omega}_{i-1}R(\theta_{i-1})\mathbf{d}_{i-1,i}\\ &- \hat{\omega}_i R(\theta_i)(m_{i-1}\mathbf{d}_{i,i-1} + (m_{i+1}+m_{i+2})\mathbf{d}_{i,i+1})\\ &+ \hat{\omega}_{i+1}R(\theta_{i+1})((m_{i+1}+m_{i+2})\mathbf{d}_{i+1,i} - m_{i+2}\mathbf{d}_{i+1,i+2})\\ &+ \hat{\omega}_{i+2}R(\theta_{i+2})m_{i+2}\mathbf{d}_{i+2,i+1}]\end{aligned} \tag{4.5}$$

for $i = 0, 1, 2, 3 \pmod 4$, where

$$\hat{\omega}_i = \begin{pmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{pmatrix}.$$

By substituting the formula for $\dot{\mathbf{r}}_i^c$ into Eq.(4.4), we get a more compact expression for the kinetic energy

$$T = \frac{1}{2} < \tilde{\omega}, \tilde{\mathbf{J}}\tilde{\omega} > + \frac{1}{2} m \|\dot{\mathbf{r}}_c\|^2 \tag{4.6}$$

where $\tilde{\omega} = (\omega_0, \omega_1, \omega_2, \omega_3)^T$ and $\tilde{\mathbf{J}} = (\tilde{J}_{i,j}, i, j = 0, 1, 2, 3)$ is a 4×4 symmetric matrix, with elements given as follows.

Let

$$\begin{aligned}M_i^I = \frac{1}{m^2}[&m_i m_{i+1}(m_i + m_{i+1})\\ &+ m_{i+1}(m_{i+1}m_{i-1} + m_i m_{i+2})\\ &+ m_i m_{i+2}(m_{i+1} + m_{i+2})]\end{aligned} \tag{4.7a}$$

$$M_i^{II} = \frac{m_i}{m^2}(m_{i+2}^2 - m_{i+1}m_{i-1}) \tag{4.7b}$$

$$M_i^{III} = \frac{m_i m_{i+2}}{m^2}(m_{i+1} + m_{i-1}). \tag{4.7c}$$

Then

$$\tilde{J}_{ii} = I_i + M_i^I \|\mathbf{d}\|_{i,i+1}^2 + M_{i-1}^I \|\mathbf{d}\|_{i,i-1}^2$$
$$- 2M_{i-1}^{II} < \mathbf{d}_{i,i+1}, \mathbf{d}_{i,i-1} > \qquad (4.8)a$$
$$\tilde{J}_{i,i+1} = -M_i^I < \mathbf{d}_{i,i+1}, R(\theta_{i+1,i})\mathbf{d}_{i+1,i} >$$
$$+ M_i^{II} < \mathbf{d}_{i,i+1}, R(\theta_{i+1,i})\mathbf{d}_{i+1,i+2} >$$
$$+ M_{i-1}^{II} < \mathbf{d}_{i,i-1}, R(\theta_{i+1,i})\mathbf{d}_{i+1,i} >$$
$$+ M_i^{III} < \mathbf{d}_{i,i-1}, R(\theta_{i+1,i})\mathbf{d}_{i+1,i+2} > \qquad (4.8)b$$
$$\tilde{J}_{i,i+2} = -M_i^{II} < \mathbf{d}_{i,i+1}, R(\theta_{i+2,i})\mathbf{d}_{i+2,i+1} >$$
$$- M_{i+1}^{III} < \mathbf{d}_{i,i+1}, R(\theta_{i+2,i})\mathbf{d}_{i+2,i-1} >$$
$$- M_{i+2}^{II} < \mathbf{d}_{i,i-1}, R(\theta_{i+2,i})\mathbf{d}_{i+2,i-1} >$$
$$- M_i^{III} < \mathbf{d}_{i,i-1}, R(\theta_{i+2,i})\mathbf{d}_{i+2,i+1} > \qquad (4.8)c$$

for $i = 0, 1, 2, 3 \pmod 4$.

Up to now, we have not applied the condition on the lengths of links of a closed loop. It is clear that, in Eq.(4.6), θ_i and the velocities ω_i, $i = 0, 1, 2, 3$ are involved. Since the chain is closed, these variables are not independent. As shown in the proof of Corollary 3.5, if $s + l < p + q$ and $l_1 = s$, we have an one-to-one map from (θ_0, θ_1) to (θ_2, θ_3). Therefore every element of the matrix $\tilde{\mathbf{J}}$ can be expressed as a function of $\theta_1 - \theta_0$ uniquely.

Under the same conditions on links, i.e. $s + l < p + q$, the loop constraints (2.2) yield a relation between (ω_0, ω_1) and (ω_2, ω_3):

$$\begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix} = \Omega \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix} \qquad (4.9)a$$

where

$$\Omega = \begin{pmatrix} -\frac{l_0 sin(\theta_3 - \theta_0)}{l_2 sin(\theta_3 - \theta_2)} & -\frac{l_1 sin(\theta_3 - \theta_1)}{l_2 sin(\theta_3 - \theta_2)} \\ \frac{l_0 sin(\theta_2 - \theta_0)}{l_3 sin(\theta_3 - \theta_2)} & \frac{l_1 sin(\theta_2 - \theta_1)}{l_3 sin(\theta_3 - \theta_2)} \end{pmatrix}. \qquad (4.9)b$$

Here, the matrix Ω is well defined because of the result in Proposition 3.4. Again, since $s + l < p + q$ and $s = l_1$, elements of matrix Ω are functions of $\theta_1 - \theta_0$.

We summarize the above discussion in the following theorem.

Theorem 4.1: If $s + l < p + q$ and $l_1 = s$, the kinetic energy, or Lagrangian, can be represented as

$$T = L = \frac{1}{2} < \omega, \mathbf{J}\omega > + \frac{1}{2} m \|\dot{\mathbf{r}}_c\|^2 \tag{4.10}$$

where $\omega = (\omega_0, \omega_1)^T$ and

$$\mathbf{J} = (I \quad \Omega^T) \tilde{\mathbf{J}} \begin{pmatrix} I \\ \Omega \end{pmatrix}$$

for $\tilde{\mathbf{J}}$ given in (4.6)-(4.8) and Ω given in (4.9). In addition, the elements of $\mathbf{J}$ are the functions of $(\theta_1 - \theta_0)$.

Before ending this section, we give a property of the matrix Ω which will be used in section 6.

Proposition 4.2: Under the assumptions of Theorem 4.1,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \Omega \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Proof: Premultiplying Eq.(2.2) by $R(\theta_2)$ and $R(\theta_3)$, we get

$$l_0 sin(\theta_0 - \theta_2) + l_1 sin(\theta_1 - \theta_2) + l_3 sin(\theta_3 - \theta_2) = 0$$

and

$$l_0 sin(\theta_0 - \theta_3) + l_1 sin(\theta_1 - \theta_3) + l_2 sin(\theta_2 - \theta_3) = 0$$

respectively. They immediately imply the claim.

∎

5. SYMMETRY AND INTEGRAL

We shall show here that a floating four-bar linkage is a simple mechanical system with symmetry in the sense of Smale [1,19].

A simple mechanical system with symmetry is a 4-tuple (Q, K, V, G), where,

(i) (Q, K) is a Riemannian configuration manifold with metric K;

(ii) G is a Lie group acting on Q on the left,

$$\Phi : G \times Q \to Q$$
$$(g, q) \mapsto \Phi_g(q) \triangleq \Phi(g, q)$$

such that for each $g \in G$, Φ_g is an isometry of (Q, K);

(iii) $V : Q \to R$ is a G-invariant potential function.

The associated Lagrangian is defined by

$$\begin{aligned} L &: TQ \to R \\ v_q &\mapsto L(v_q) = \frac{1}{2}K(v_q, v_q) - V \circ \tau(v_q) \end{aligned} \tag{5.1}$$

where $\tau : TQ \to Q$ is the canonical tangent projection. The Legendre transform FL of L is given here by the vector bundle isomorphism

$$K^\flat : TQ \to T^*Q$$

satisfying

$$K^\flat(v_q) \cdot w_q = K(v_q, w_q) \quad \forall v_q,\ w_q\ \in T_qQ.$$

The hamiltonian $H : T^*Q \to R$ is defined to be

$$H(\alpha_q) = \frac{1}{2}K((K^\flat)^{-1}(\alpha_q), (K^\flat)^{-1}(\alpha_q)) + V \circ \tau^*(\alpha_q) \tag{5.2}$$

where $\tau^* : T^*Q \to Q$ is the canonical cotangent projection. Letting Ω_0 denote the canonical symplectic structure [1] on T*Q, the hamiltonian dynamics on T^*Q is given by the unique vector field X_H on T^*Q such that

$$dH(Y) = \Omega_0(X_H, Y)$$

for all vector field Y on T^*Q.

Let $\Phi^{T^*} : G \times T^*Q \to T^*Q$ be the cotangent lift of the action Φ. Denoting $\Phi_g^{T^*}(\cdot) = \Phi^{T^*}(g, \cdot)$, we have

$$H \circ \Phi_g^{T^*} = H, \tag{5.3}$$

i.e. the group G is a symmetry group of the hamiltonian system (T^*Q, Ω_0, X_H).

The modern setting of Nother's theorem relating symmetry to the existence of integrals of motion is given by the concept of momentum mapping. Let $\mathcal{G}$ denote the Lie algebra of G and $\mathcal{G}^*$ its dual. The map

$$J : T^*Q \to \mathcal{G}^*$$

given by

$$J(\alpha_q) \cdot \xi = \alpha_q \cdot \xi_Q(q) \quad \forall \alpha_q \in T_q^* Q \tag{5.4}$$

is Ad^*-equivariant, where $\xi \in \mathcal{G}$ and ξ_Q is the infinitesimal generator of Φ on Q associated to ξ (see [1] corollary 4.2.11). J is a momentum mapping and the G-invariance of the hamiltonian H (Eq.(5.3)) implies that J is an integral, i.e. it is conserved along trajectories of X_H.

Returning to four-bar linkages, we restrict attention for the moment to linkages of Grashof type, i.e. the condition $s + l < p + q$ holds. To fix the parameterization, we let $l_1 = s$. We also restrict ourselves to a connected component of the configuration space. Then

$$Q = R^2 \times S^1 \times S^1;$$

the kinetic energy metric is K defined by

$$K((v, w), (\tilde{v}, \tilde{w})) = \frac{1}{2} m < v, \tilde{v} > + \frac{1}{2} < w, \mathbf{J}\tilde{w} >$$

(see Eq.(4.10) of the previous section). The group of rigid motions in the plane is the symmetry group:

$$G = S^1 \times R^2.$$

Denoting a point in Q by $(\mathbf{r}_c, \theta_0, \theta_1)$, the action Φ of $G = S^1 \times R^2$ on Q is given by

$$\Phi((\phi, \mathbf{r}), (\mathbf{r}_c, \theta_0, \theta_1)) = (\mathbf{r} + \mathbf{r}_c, \theta_0 + \phi, \theta_1 + \phi).$$

For our purposes it is convenient to eliminate the effect of translations altogether by putting the inertial observer at the center of mass of the system, i.e. $\mathbf{r}_c \equiv \mathbf{0}$. In [22] this process is explained via *symplectic reduction* by the translation group R^2. The effect of taking this step is that now,

$$Q = S^1 \times S^1, \qquad G = S^1 \tag{5.5}$$

and,

$$\Phi(\phi, (\theta_0, \theta_1)) = (\theta_0 + \phi, \theta_1 + \phi). \tag{5.6}$$

The kinetic energy metric on Q is

$$K(w_1, w_2) = < w_1, \mathbf{J} w_2 > \tag{5.7}$$

for $w_1,\ w_2 \in T_q Q$.

The hamiltonian is given by

$$H = \frac{1}{2} < \mu, \mathbf{J}^{-1} \mu > \tag{5.8}$$

for $\mu = K^\flat(w)$. Using the abstract formula (5.4) and the action (5.6), one can show that a momentum mapping for the action Φ is

$$J(\theta, \mu) = \mu_0 + \mu_1. \tag{5.9}$$

Of course, $\nu = \mu_0 + \mu_1$ is conserved along trajectories of X_H for the hamiltonian H in (5.8) and it is simply the net angular momentum of the floating four bar linkage relative to an observer at the system center of mass.

The dynamical trajectories are confined to level set of the form $J^{-1}(\nu)$. The group S^1 viewed as the isotropy subgroup of the momentum value ν, acts freely on $J^{-1}(\nu)$ and one gets the symplectically reduces dynamics X_{H_ν} on the reduced phase space $P_\nu = J^{-1}(\nu)/S^1 \simeq S^1 \times R^1$. We discuss this further in the next section.

6. (REDUCED) DYNAMICS & RELATIVE EQUILIBRIA

As in [22] it is possible to Poisson-reduce the dynamics. We recall that given a symplectic manifold (M, ω), and a smooth, free, proper, symplectic action of Lie group G on M, the canonical Poisson structure on M defined by

$$\{f, g\}_M = \omega(X_f, X_g) \quad \forall f,\ g \in C^\infty(M)$$

descends to a Poisson structure on the quotient $P = M/G$. The latter is defined by

$$\{\hat{f}, \hat{g}\}_{M/G} \circ \pi = \{\hat{f} \circ \pi, \hat{g} \circ \pi\}_M \tag{6.1}$$

where $\hat{f}, \hat{g} \in C^\infty(M/G)$ and $\pi : M \to M/G$ is the canonical projection. If $H : M \to R$ is a G-invariant hamiltonian, it induces $\hat{H} : M/G \to R$ defined by $\hat{H} \circ \pi = H$. We refer to the dynamics (vector field) $X_{\hat{H}}$ defined by

$$X_{\hat{H}}(\hat{f}) = \{\hat{f}, \hat{H}\} \quad \forall \hat{f} \in C^\infty(M/G) \tag{6.2}$$

as the Poisson reduction of the dynamics X_H.

In the present context, with Q, K, H as in (5.5)-(5.8), the space $M = T^*(S^1 \times S^1)$ with parameterization $(\theta_0, \theta_1, \mu_0, \mu_1)$ carries the Poisson structure,

$$\{f, g\} = \sum_{i=0}^{1} \left(\frac{\partial f}{\partial \theta_i} \cdot \frac{\partial g}{\partial \mu_i} - \frac{\partial f}{\partial \mu_i} \cdot \frac{\partial g}{\partial \theta_i} \right) \tag{6.3}$$

for all $f,\ g \in C^\infty(T^*(S^1 \times S^1))$. The action of $G = S^1$ on Q given by (5.6) is free and proper. The quotient $P = T^*(S^1 \times S^1)/S^1 \simeq S^1 \times R^2$ carries a reduced Poisson structure. Parameterizing $P = T^*(S^1 \times S^1)/S^1$ by $\theta_{10} = (\theta_1 - \theta_0, \mu_0, \mu_1)$, the Poisson bracket on P is given by,

$$\{\hat{f}, \hat{g}\}_{T^*(S^1 \times S^1)/S^1} = \frac{\partial \hat{f}}{\partial \theta_{10}} \cdot \left(\frac{\partial \hat{g}}{\partial \mu_1} - \frac{\partial \hat{g}}{\partial \mu_0} \right) - \frac{\partial \hat{g}}{\partial \theta_{10}} \cdot \left(\frac{\partial \hat{f}}{\partial \mu_1} - \frac{\partial \hat{f}}{\partial \mu_0} \right) \tag{6.4}$$

which is a noncanonical structure. The reduced hamiltonian $\hat{H}$ is given by

$$\hat{H}(\theta_{10}, \mu_0, \mu_1) = H(\theta_0, \theta_1, \mu_0, \mu_1), \tag{6.5}$$

since the matrix $\mathbf{J}$ in (5.8) is a function of the difference $\theta_{10} = \theta_1 - \theta_0$ only. The reduced dynamics is then immediately given:

$$\begin{aligned} \dot{\mu}_0 &= \frac{\partial \hat{H}}{\partial \theta_{10}} \\ \dot{\mu}_1 &= -\frac{\partial \hat{H}}{\partial \theta_{10}} \\ \dot{\theta}_{10} &= \frac{\partial \hat{H}}{\partial \mu_1} - \frac{\partial \hat{H}}{\partial \mu_0}. \end{aligned} \tag{6.6}$$

Equation (6.6) involves complicated analytic expression resulting from the substitutions for θ_3 and θ_2 in terms of θ_1 and θ_1 as in [13]. Certain qualitative aspects of the reduced dynamics can still be explored, sidestepping analytic difficulties. For instance one can investigate relative equilibria.

Definition 6.1: $z_e \in M$ is a relative equilibrium for X_H if

$$X_{\hat{H}}(\pi(z_e)) = 0. \tag{6.7}$$

Remark: Let $F^t_{X_H}$ be the flow of X_H on M. Then z_e is a relative equilibrium iff $F^t_{X_H}(z_e)$ is a stationary motion, i.e. there exists $\xi \in \mathcal{G}$ such that

$$F^t_{X_H}(z_e) = exp(t\xi)(z_e).$$

Theorem 6.2: Let J be an Ad^*-equivariant momentum mapping on M. $z_e \in M$ is a relative equilibrium of X_H iff there exists a $\xi \in \mathcal{G}$ such that z_e is a critical point of

$$H_\xi = H - \hat{J}(\xi) \tag{6.8}$$

where $\hat{J}(\xi) : M \to R : x \mapsto J(x)(\xi)$.

For proof, see chapter 4 of [1]. This theorem can be applied to a simple mechanical system with symmetry, (Q, K, V, G) [19].

Theorem 6.3:(*Smale*) For simple mechanical system with symmetry (Q, K, V, G), define

$$V_\xi : Q \to R : q \mapsto V(q) - \frac{1}{2}K(\xi_Q(q), \xi_Q(q)) \tag{6.9}$$

for each $\xi \in \mathcal{G}$. Then $z_e = (q_e, p_e) \in T^*Q$ is a relative equilibrium iff q_e is a critical point of V_ξ for some $\xi \in \mathcal{G}$ and $p_e = K^\flat(\xi_Q(q_e))$.

Remark: It can be shown that, for a given $\xi \in \mathcal{G}$, V_ξ has the symmetry,

$$V_\xi(\Phi_g(x)) = V_\xi(x) \tag{6.10}$$

for all $g \in G_\xi := \{g \in G | Ad_g\xi = \xi\}$. If $G = S^1, G_\xi = G$ and action Φ is free and proper. Then Q/G_ξ is a smooth manifold and $\pi_\xi : Q \to Q/G_\xi$ is a submersion. Thus V_ξ induces a function $\hat{V}_\xi$ on Q/G_ξ such that

$$V_\xi = \hat{V}_\xi \circ \pi_\xi.$$

Recalling that $V = 0$, this theorem can be directly applied to the floating four-bar linkage system. Rewriting the kinetic energy given in section 4 by setting the origin of inertial space at the system of center of mass, we have

$$T = \frac{1}{2} < \omega, \mathbf{J}\omega > \tag{6.11}$$

where $\omega = (\omega_0, \omega_1)$. It can be shown that the infinitesimal generator on $Q = S^1 \times S^1$ is $\xi_Q(q) = (1,1)^T$ [21]. Hence, by theorem 6.3, (θ^*, μ^*) is a relative equilibrium point on T^*Q iff θ^* is a critical point of function

$$V_\xi(\theta_0, \theta_1) = -(1,1)\mathbf{J}\begin{pmatrix}1\\1\end{pmatrix}$$

Applying Proposition 4.2 and the definition of $\tilde{\mathbf{J}}$ given in Theorem 4.1, we have

$$V_\xi(\theta_0, \theta_1) = -\mathbf{e}^T\tilde{\mathbf{J}}\mathbf{e} \tag{6.12}$$

where $\mathbf{e} = (1\ 1\ 1\ 1)^T$. From Eq.(6.12), we observe that the diagonal terms, which include the inertia of bars, of matrix $\tilde{\mathbf{J}}$ do not effect the positions of critical points of function V_ξ. Since the elements of matrix $\tilde{\mathbf{J}}$ are only functions of $\theta_1 - \theta_0$, above V_ξ satisfies (6.10) for all $g \in S^1$. It follows that $\hat{V}_\xi(\theta_{10}) = V_\xi(\theta_0, \theta_1)$. Then, the critical points of V_ξ, $(\theta_0^*,\ \theta_1^*)$, will make $(\theta_1^* - \theta_0^*)$ to be the critical points of $\hat{V}_\xi$. At relative equilibrium, the relative angles between bars are fixed and the whole system rotates around the system center of mass with constant angular velocity. Unlike the planar two-body case, the relative equilibrium shapes depend on the values of the masses and the lengths of bars. From the expression of matrices $\tilde{\mathbf{J}}$ and the relations between relative angles given in [13], it is very difficult to find critical points of function $\hat{V}_\xi$ analytically. However, numerically searching for critical points is easy since now $\hat{V}_\xi$ is only a function of one variable. In the following, we give an example to computing the relative equilibria by applying above theorem.

Example:

An assembly in Fig. 2 is a possible structure of a robot arm. The parameters are given as follows

$$m_0 = 3, \quad m_1 = 1, \quad m_2 = 30, \quad m_3 = 1;$$

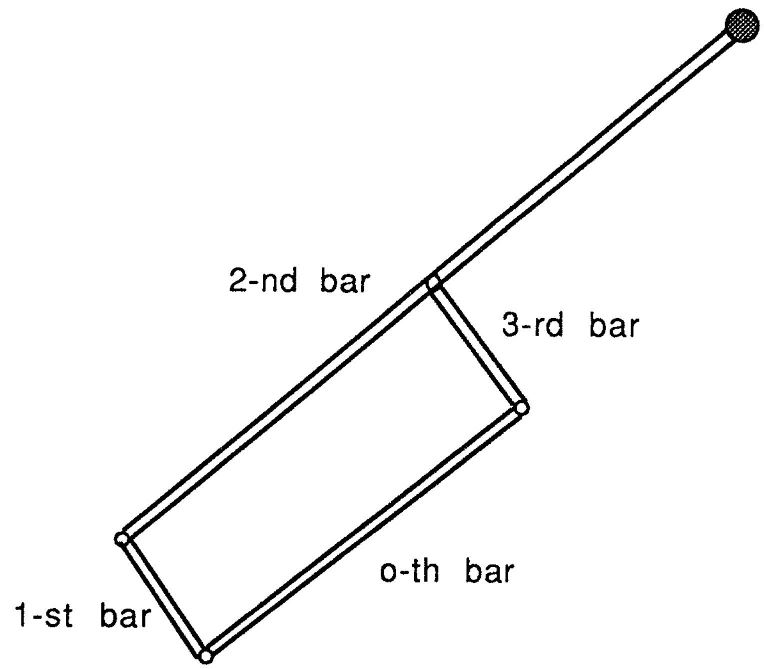


Fig. 2

$$\mathbf{d}(0,3) = (-1.5, 0), \quad \mathbf{d}(0,1) = (1.5, 0);$$

$$\mathbf{d}(1,0) = (-0.5, 0), \quad \mathbf{d}(1,2) = (0.5, 0);$$

$$\mathbf{d}(2,1) = (-6, 0), \quad \mathbf{d}(2,3) = (-3, 0);$$

$$\mathbf{d}(3,2) = (-0.55, 0), \quad \mathbf{d}(3,0) = (0.55, 0).$$

Thus, the lengths of links are

$$l_0 = 3, \quad l_1 = 1, \quad l_2 = 3, \quad l_3 = 1.1$$

It is clear that $s + l < p + q$ and $s = l_1$ are satisfied.

The graph of $\hat{V}_\xi$ is given in Fig. 3. From this figure we can see two critical points appear at $\theta_{10} = 7\ deg$ and $\theta_{10} = 186\ deg$. The shape determined by these two angles are shown in Fig. 4.

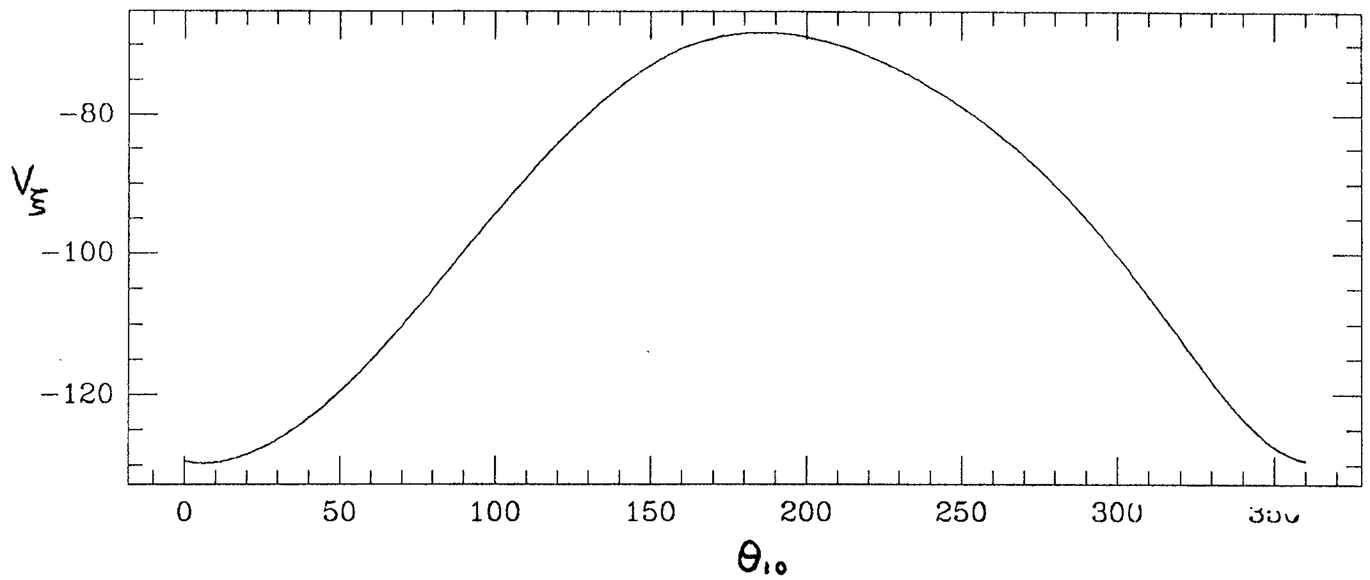


Fig. 3

7. CONCLUSIONS

In this paper, we gave sufficient conditions which make the configuration space of a closed four-bar mechanism be a smooth manifold by applying the classical theory of a mechanisms due to Grashof. Under one of them, i.e. $s+l<p+q$ we derived well defined expression of kinetic energy, or Lagrangian. This Lagrangian is invariant under the action of $SE(2)$, the rigid motion group in plane. It turns out that the four-bar mechanism is a simple mechanical system with symmetry. Applying Poisson reduction we obtained the reduced dynamics. Furthermore, by using Smale's theorem on relative equilibria we found a function whose critical points give the relative equilibria for our system. An example of its application was given.

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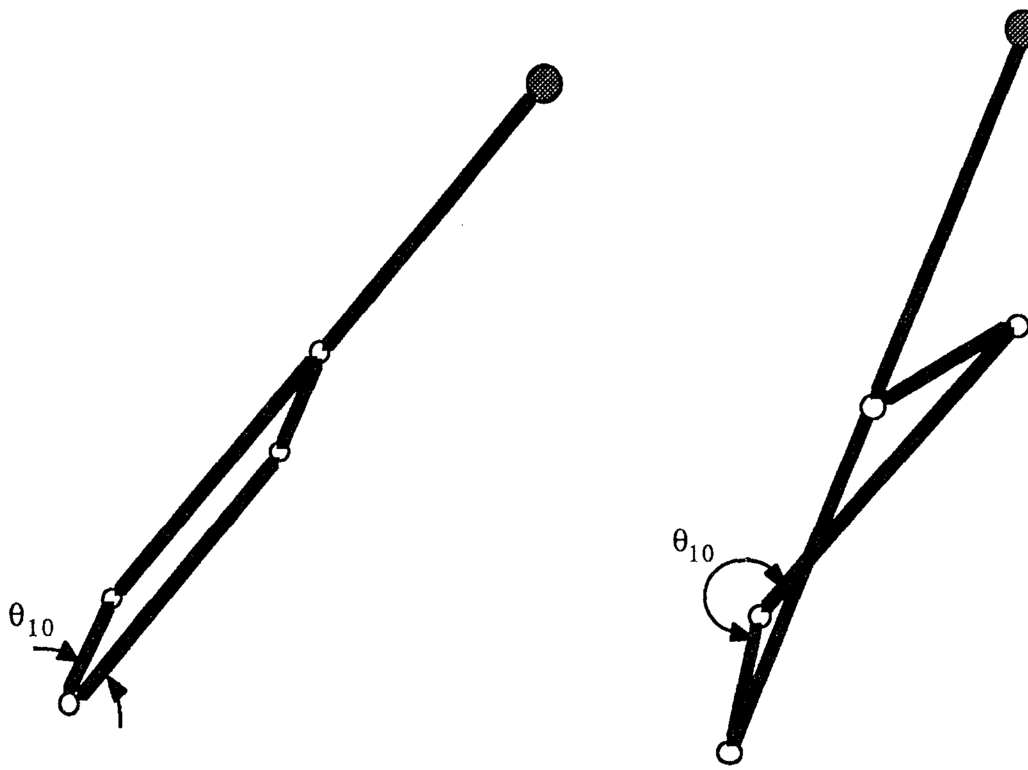

Fig. 4

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