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Relative Equilibria for Two Rigid Bodies

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RELATIVE EQUILIBRIA FOR TWO RIGID BODIES CONNECTED BY A BALL-IN-SOCKET JOINT *

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ABSTRACT. For simple mechanical systems with symmetry, a variational principle on configuration space determines relative equilibria. Here, this principle of symmetric criticality is applied to a problem of coupled rigid bodies. Numerical optimization by CONSOLE (a package intended originally for optimization based control system design) is shown to be an effective technique to search for some of the relative equilibria.

1 Introduction

This paper is part of an ongoing program to understand the dynamics and control of multibody systems from a modern point-of-view. In recent years, engineering applications have brought into focus, questions concerning the dynamics of systems of kinematically

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coupled rigid and flexible bodies. In [16], these are referred to as Eulerian many-body problems to emphasize the role of Euler forces (or frame forces) in determining the nature of inter-body interactions. Eulerian many-body problems arise as models of robotic manipulators, high speed mechanical machinery, complex spacecraft with articulated components, space-based sensors, etc. See Wittenburg[31] and [5],[7] for treatments of engineering questions and formulation of equations of motion.

In recent work [3][4][12][15–17][21][24–27][29][30], modern geometric techniques have been brought to bear on certain classes of Eulerian many-body problems. Included among the classes of problems investigated are rigid bodies carrying rotors, planar many-body systems, three dimensional rigid bodies coupled by ball-in-socket joints, and rigid bodies with flexible attachments.

In the present paper, we investigate the structure of relative equilibria in the dynamics of two rigid bodies connected by a ball-in-socket joint. We follow the framework of [12] and obtain a variational characterization of relative equilibria using a theorem of Smale. A key geometric condition is derived. Numerical search for extremal critical points is carried out using CONSOLE, a software package originally intended for optimization-based design.

The results of this paper can be considered as part of a program to determine the phase portrait of coupled rigid body systems. In his ongoing doctoral dissertation work at Berkeley (preliminary results presented at an AMS Summer Research Conference on Control Theory and Multibody System at Bowdoin, August 1988), George Patrick has computed relative equilibria for certain *restricted classes* of coupled rigid body problems. The restrictions involve *material symmetry* and the present paper does not require such symmetries. Patrick's methods involve direct symbolic calculations while here we use a variational principle.

The model problem treated in this paper should be taken as illustrative (partly inspired by early examples of multibody communication satellite design [11][23]), but the techniques apply to other configurations as well. Attitude control of a system such as the one in this paper is under investigation and we hope to report on this in a later paper.

2 Mechanical Setting

In this section we describe the kinematics of a mechanical system. Two bodies, with masses m_1, m_2, are free to move in three dimensional Euclidean space, subject to a (three degrees of freedom) ball and socket coupling (See Figure 1).

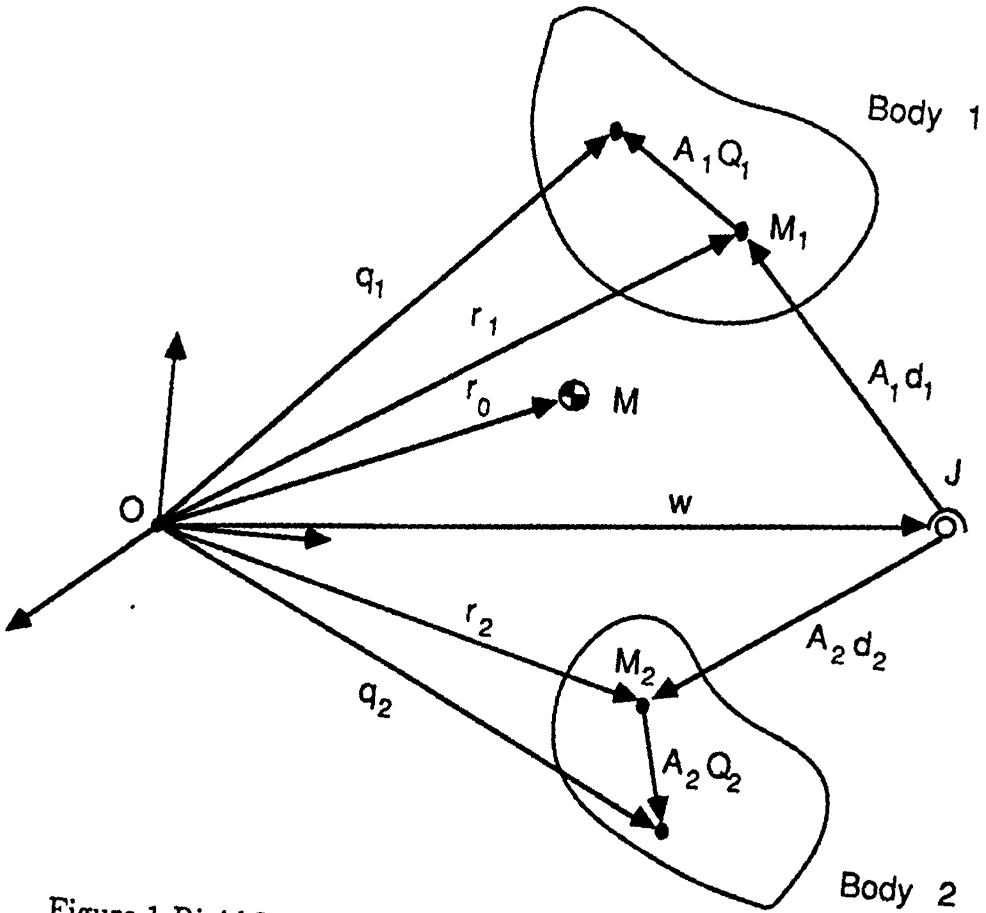

Figure 1 Rigid Bodies connected by the Ball-in-Socket Joint

We introduce the following notations.

Γ_0: inertial frame of reference in space.

O: origin of the inertial reference system.

M_1: center of mass of body 1.

M_2: center of mass of body 2.

Γ_1: orthonomal frame on body 1 with origin at M_1.

A_1: rotational coordinate transformation matrix from Γ_1 to Γ_0.

Γ_2: orthonomal frame on body 2 with origin at M_2.

A_2: rotational coordinate transformation matrix from Γ_2 to Γ_0.

d_1: vector from the joint to M_1 in the frame Γ_1.

d_2: vector from the joint to M_2 in the frame Γ_2.

r_1: vector from O to M_1 in frame Γ_0.

r_2: vector from O to M_2 in frame Γ_0.

r_0: vector from O to the system center of mass in frame Γ_0.

m: total mass $(= m_1 + m_2)$.

Q_1: vector from M_1 to a point of body 1 in the frame Γ_1.

q_1: vector from O to the same point of body 1 as Q_1 above in the frame Γ_0.

Q_2: vector from M_2 to a point of body 2 in the frame Γ_2.

q_2: vector from O to the same point of body 2 as Q_2 above in the frame Γ_0.

w: vector from O to the joint in the frame Γ_0.

We have the kinematic relations.

$$q_1 = r_1 + A_1 Q_1. \tag{1}$$

$$q_2 = r_2 + A_2 Q_2. \tag{2}$$

$$m r_0 = m_1 r_1 + m_2 r_2. \tag{3}$$

$$r_1 = w + A_1 d_1. \tag{4}$$

$$r_2 = w + A_2 d_2. \tag{5}$$

Also we know that A_1 and A_2 belong to the special orthogonal group $SO(3)$.

Let $\mu_1(\cdot)$ denote the mass measure of body 1 in the frame Γ_1 and $\mu_2(\cdot)$ denote the mass measure of body 2 in the frame Γ_2. The kinetic energy of body 1 can be thus written as

$$K_1 = \frac{1}{2}\int_{B_1} \|\dot{q}_1(Q_1)\|^2 d\mu_1(Q_1).$$

Expanding the above by using (1), (2) and the formula $\|x\|^2 = tr(xx^t)$, we have the form

$$K_1 = \frac{m_1}{2}\|\dot{r}_1\|^2 + \frac{1}{2}tr(\dot{A}_1 I_1 \dot{A}_1^T).$$

where I_1 is the coefficient of inertia of body 1, defined by

$$I_1 = \int_{B_1} Q_1 Q_1^T d\mu_1(Q_1),$$

and $tr(\cdot)$ denotes the trace of a matrix.

The kinetic energy of body 2 has a similar form. We thus have the total kinetic energy expressed as

$$\begin{aligned} K &= K_1 + K_2 \\ &= \frac{m_1}{2}\|\dot{r}_1\|^2 + \frac{1}{2}tr(\dot{A}_1 I_1 \dot{A}_1^T) + \frac{m_2}{2}\|\dot{r}_2\|^2 + \frac{1}{2}tr(\dot{A}_2 I_2 \dot{A}_2^T). \end{aligned}$$

By (3)–(5), we may write the total kinetic energy in terms of the total linear momentum $p = m\dot{r}_0$ of the system.

$$K = \frac{1}{2}tr(\dot{A}_1 I_1 \dot{A}_1^T) + \frac{1}{2}tr(\dot{A}_2 I_2 \dot{A}_2^T) + \frac{\epsilon}{2}\|\dot{A}_1 d_1 - \dot{A}_2 d_2\|^2 + \frac{1}{2m}\|p\|^2.$$

Here $\epsilon = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass. Since there is no potential assumed, this is also the Lagrangian of the system.

The configuration space is $SO(3) \times SO(3) \times R^3$. The system is invariant under translation of the inertial reference frame, i.e. we have a symmetry group action on the configuration space

$$\begin{aligned} \Phi: R^3 \times (SO(3) \times SO(3) \times R^3) &\to SO(3) \times SO(3) \times R^3 \\ (\lambda, (A_1, A_2, r)) &\mapsto (A_1, A_2, \lambda + r). \end{aligned}$$

We can symplectically reduce the system by R^3 (see Marsden and Weinstein[20], Abraham and Marsden[1]) which in turn corresponds to jumping to the center of mass frame. This is also done in [12] and for planar problem in [21][29][30]. After this reduction, the reduced Lagrangian is

$$L = \frac{1}{2}tr(\dot{A}_1 I_1 \dot{A}_1^T) + \frac{1}{2}tr(\dot{A}_2 I_2 \dot{A}_2^T) + \frac{\epsilon}{2}\|\dot{A}_1 d_1 - \dot{A}_2 d_2\|^2.$$

which is a function on $T(SO(3) \times SO(3))$.

Although the mechanical system considered here is exactly the same as in [12], the Lagrangian is expressed in terms of coefficients of inertia referred to different body frames than the one they use. Ours is based on the body frames affixed to centers of mass. By applying the formula for change of coefficient of inertia by translation, one checks that the results are the same. In the next section, we outline the general theory of simple mechanical system with symmetry to the point of characterizing relative equilibria. Standard references for the next section are Smale[28] and Abraham and Marsden[1]. See also Libermann and Marle[18], Arnold[2], and Guillemin and Sternberg[13].

3 Mechanical Systems with Symmetry

Let $(Q, \ll \cdot, \cdot \gg)$ be a Riemannian manifold with the Riemannian metric $\ll \cdot, \cdot \gg$. Let G be a Lie group. $\Phi : G \times Q \to Q$ is a group action on the manifold Q that leaves the Riemannian metric invariant. Let $V : Q \to R$ be a G-invariant function on the manifold, i.e.

$$V(\Phi_g(x)) = V(x) \qquad \forall g \in G,$$

where $\Phi_g : M \to M$ is defined by

$$\Phi_g(x) \equiv \Phi(g, x).$$

Let $\tau : TQ \to Q$ be the canonical projection. We now define a Lagrangian $L : TQ \to R$ to be, for $v_x \in TQ$ with $\tau(v_x) = x$,

$$L(v_x) = \frac{1}{2} \ll v_x, v_x \gg - V(x).$$

It follows that the Lagrangian is G-invariant and hyperregular. We thus have the corresponding invertible Legendre transformation $FL : TQ \to T^*Q$ given by

$$FL(v_x)(w_x) = \ll v_x, w_x \gg .$$

Let ω_0 be the canonical symplectic two form on T^*Q, and define

$$\omega_L = (FL)^* \omega_0.$$

Then it is easy to see that ω_L is a symplectic form on TQ. The *action* $A: TQ \to R$ is now

$$A(v_x) = \ll v_x, v_x \gg$$

and the energy $E = A - L$ can be found subsequently. Thus we have two equivalent hamiltonian systems (TQ, ω_L, E) and $(T^*Q, \omega_0, H = E \circ FL^{-1})$. These are simple mechanical systems with symmetry in the sense of Smale[28][1]. The hamiltonian function on T^*Q can be written as (for $\alpha_x \in T^*Q$ with $\tau^*(\alpha_x) = x$ where τ^* is the canonical projection on T^*Q),

$$H(\alpha_x) = \frac{1}{2} \ll FL^{-1}(\alpha_x), FL^{-1}(\alpha_x) \gg + V(x).$$

We define an inner product on T_x^*Q by

$$< \alpha_x, \beta_x >_{T^*Q} = \ll FL^{-1}(\alpha_x), FL^{-1}(\beta_x) \gg .$$

Then we may write

$$H(\alpha_x) = \frac{1}{2} < \alpha_x, \alpha_x >_{T^*Q} + V(x).$$

Let $\Im$ denote the Lie algebra of G and $\Im^*$ is the dual of the Lie algebra. By the Corollary 4.2.11 in [1], we know that the lifted action

$$\Phi^{T^*}: G \times T^*Q \to T^*Q$$

is symplectic and has an Ad^*-equivariant momentum mapping given by $J: T^*Q \to \Im^*$,

$$J(\alpha_x)(\xi) = \alpha_x(\xi_Q(x)) \tag{6}$$

where $\xi \in \Im$ and ξ_Q is the associated infinitesimal generator of Φ on Q.

Now we introduce the notion of relative equilibrium in a general setting. Assume G acts on a symplectic manifold (P, ω) freely and properly. Then the quotient space P/G is a smooth manifold with an induced Poisson structure. For any G-invariant hamiltonian function H on P, we find the induced function $\tilde{H}: P/G \to R$ in the following way. Letting $\tilde{\tau}$ be the projection from P to P/G, we have

$$\tilde{H} \circ \tilde{\tau}(x) = H(x).$$

Since H is G-invariant, $\tilde{H}$ is well-defined. The quotient P/G carries an induced Poisson structure[19]. Given $f, g \in C^\infty(P/G)$, the induced Poisson bracket of f and g is

$$\{f, g\} \circ \tilde{\tau} = \{f \circ \tilde{\tau}, g \circ \tilde{\tau}\}_0.$$

where $\{,\}_0$ is the standard Poisson structure on the symplectic manifold (P, ω).

With this induced Poisson structure and the induced hamiltonian function, we define the projected hamiltonian vector field $X_{\tilde{H}}$ on P/G. For any $f \in C^\infty(P/G)$,

$$X_{\tilde{H}}[f] = \{f, \tilde{H}\}.$$

Definition (Relative Equilibrium)

$z_e \in P$ is a relative equilibrium for X_H if

$$X_{\tilde{H}}(\tilde{\tau}(z_e)) = 0.$$

Assume there is an Ad^*-equivariant momentum mapping J on P. Then we have the following characterization of a relative equilibrium.

Theorem (Relative Equilibrium)

$z_e \in P$ is a relative equilibrium for X_H iff there exists a $\xi \in \mathfrak{I}$ such that z_e is a critical point of

$$H_\xi \equiv H - < J, \xi >,$$

where $< J, \xi >: P \to R$ is given by $x \mapsto J(x)(\xi)$.

Proof

Let $F_t : P \to P$ be the flow of H on P. We have the induced flow $\tilde{F}_t : P/G \to P/G$ satisfying

$$\tilde{F}_t \circ \tilde{\tau} = \tilde{\tau} \circ F_t.$$

Thus z_e is a relative equilibrium *iff* $X_{\tilde{H}}(\tilde{\tau}(z_e)) = 0$ *iff*

$$\frac{d}{dt}\tilde{F}_t(\tilde{\tau}(z_e)) = 0$$

iff

$$\frac{d}{dt}\tilde{\tau} \circ F_t(z_e) = 0.$$

That is

$$\tilde{\tau} \circ F_t(z_e) = \tilde{\tau}(z_e). \qquad \forall t$$

Thus, for all t, $F_t(z_e)$ must belong to the same orbit. And there exists a one-parameter subgroup $g(t) \in G$ such that

$$F_t(z_e) = \Phi_{g(t)}(z_e).$$

By the one-to-one correspondence between one-parameter subgroups in G and its Lie algebra, we know that there exists a $\xi \in \Im$ such that

$$F_t(z_e) = \Phi_{expt\xi}(z_e).$$

Differentiate both sides with respect to t and set $t = 0$, we get

$$X_H(z_e) = \xi_P(z_e).$$

On the other hand, by the definition of a momentum mapping, $X_{<J,\xi>} = \xi_P$. Thus

$$X_H(z_e) = X_{<J,\xi>}(z_e),$$

which implies

$$X_{H-<J,\xi>}(z_e) = 0.$$

By the nondegeneracy of ω, we have

$$d(H- < J, \xi >)(z_e) = 0,$$

i.e. z_e is a critical point of $H_\xi = H- < J, \xi >$.

Conversely, if z_e is a critical point of H_ξ, also from the nondegeneracy of ω,

$$X_{H-<J,\xi>}(z_e) = 0.$$

which implies

$$X_H(z_e) = X_{<J,\xi>}(z_e) = \xi_P(z_e).$$

Thus, by uniqueness of the integral curve,

$$F_t(z_e) = \Phi_{expt\xi}(z_e).$$

It follows that z_e is a relative equilibrium for X_H from the previous arguments.

QED ∎

Remark 1.

This theorem can be considered a corollary to the Souriau-Smale-Robbin Theorem (see Abraham and Marsden[1]). It is also discussed in [27].

Remark 2.

We note that if a point z_e is a relative equilibrium point, the motion $F^t_{X_H}(z_e)$ is a stationary motion, i.e. it corresponds to a group orbit. If for instance the group $G = SO(3)$, this would imply that $F^t_{X_H}(z_e)$ corresponds to a uniform rotation about a fixed axis ξ in space.

We now apply this theorem to the setting of simple mechanical systems with symmetry discussed before. By the hyperregularity of FL, for any $\alpha_x \in T^*Q$, there is an element $v_x \in TQ$ such that $\alpha_x = FL(v_x)$ and

$$\alpha_x(w) = \ll v_x, w \gg \qquad \forall w \in T_xQ.$$

Thus the momentum mapping, which is given by (6), can be written as

$$\alpha_x(\xi_Q(x)) = \ll FL^{-1}(\alpha_x), \xi_Q(x) \gg .$$

Defining $B_\xi(x) = FL(\xi_Q(x))$, we have

$$< J, \xi > (\alpha_x) = < \alpha_x, B_\xi(x) >_{T^*Q} .$$

We can now write

$$H_\xi(\alpha_x) = \frac{1}{2} < \alpha_x, \alpha_x >_{T^*Q} + V(x) - < \alpha_x, B_\xi(x) >_{T^*Q} .$$

By determining the norm on T^*_xQ through the inner product and completion of squares, we may express H_ξ as

$$H_\xi(\alpha_x) = \frac{1}{2}\|\alpha_x - B_\xi(x)\|^2 + V(x) - \frac{1}{2} < B_\xi(x), B_\xi(x) >_{T^*Q} .$$

By the above theorem, computing the relative equilibria is equivalent to finding critical points of H_ξ. Letting $\alpha_x = (x, p)$ we have

$$H_\xi(x,p) = \frac{1}{2}\|p - B_\xi(x)\|^2 + V(x) - \frac{1}{2} < B_\xi(x), B_\xi(x) >_{T^*Q} .$$

It is then easy to check that the necessary conditions for (x_e, p_e) to be a critical point of H_ξ are

$$p_e = B_\xi(x_e) \tag{7}$$

and

$$d_{x_e}[V(x) - \frac{1}{2} < B_\xi(x), B_\xi(x) >_{T^*Q}] = 0$$

We summarize the algorithm (*principle of symmetric criticality*) to find relative equilibria.

Algorithm

0. Pick $\xi \in \Im$.
1. Search for the critical points x_e of the function

$$V_\xi : Q \to R$$
$$V_\xi(x) = V(x) - \frac{1}{2} \ll \xi_Q(x), \xi_Q(x) \gg$$

2. Put x_e in (7) to find the corresponding $p_e = B_\xi(x_e)$.

We note that the computation in step 1 is fully on the configuration space.

Remark 3. (Historical)

The principle of symmetric criticality as stated here appears as Theorem 1.1 in Part II of Smale[28]. Smale also notes that special versions have been known earlier, e.g. in the study of symmetric geodesics. See also pp. 355 of [1], Theorem 16.7 in Hermann[14], and Palais[22].

Remark 4. (Symmetry of V_ξ)

It should be noted, for a given $\xi \in \Im$, V_ξ has the symmetry,

$$V_\xi(\Phi_g(x)) = V_\xi(x),$$

for all $g \in G_\xi = \{g \in G | Ad_g(\xi) = \xi\}$, the stabilizer of ξ. Thus V_ξ induces a function $\hat{V}_\xi$ such that the diagram in Fig. 2 commutes.

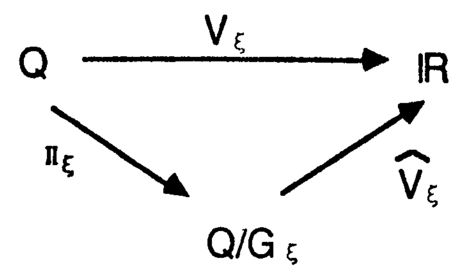

Figure 2 Symmetry of V_ξ

Typically, $\hat{V}_\xi$ is a Morse function on Q/G_ξ and $\pi_\xi^{-1}(\hat{x}_e)$ is a nondegenerate critical manifold in the sense of Bott[6], if $\hat{x}_e$ is a critical point of $\hat{V}_\xi$.

Example

One application of the principle here is to find the relative equilibria of the planar three-body system discussed in [29][30]. If we plot the function $\hat{V}_\xi$ (for particular kinematic parameters) on the joint space, we get the picture in Fig. 3, from which the *fundamental equilibria* defined in [29][30] can be easily seen. These are the relative (joint) configurations $(0,0)$, $(0,\pi)$, $(\pi,0)$, (π,π).

4 Relative Equilibria for Coupled Rigid Bodies

For the mechanical system described in section 2, the Riemannian metric on $T(SO(3)\times SO(3))$ is given by the (symplectically) reduced Lagrangian as

$$\ll (W_1,W_2),(W_1,W_2)\gg = tr(W_1 I_1 W_1^T) + tr(W_2 I_2 W_2^T) + \epsilon \|W_1 d_1 - W_2 d_2\|^2.$$

where (W_1, W_2) belongs to $T(SO(3)\times SO(3))$. We know that every element in $T(SO(3)\times SO(3))$ can be represented as

$$T(SO(3)\times SO(3)) = \{(A_1, A_2, \hat{w}_1 A_1, \hat{w}_2 A_2) : A_1, A_2 \in SO(3), w_1, w_2 \in R^3\},$$

where $\hat{}: R^3 \to so(3)$ is the one-to-one map from R^3 to the skew-symmetric matrices,

$$\hat{w} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}.$$

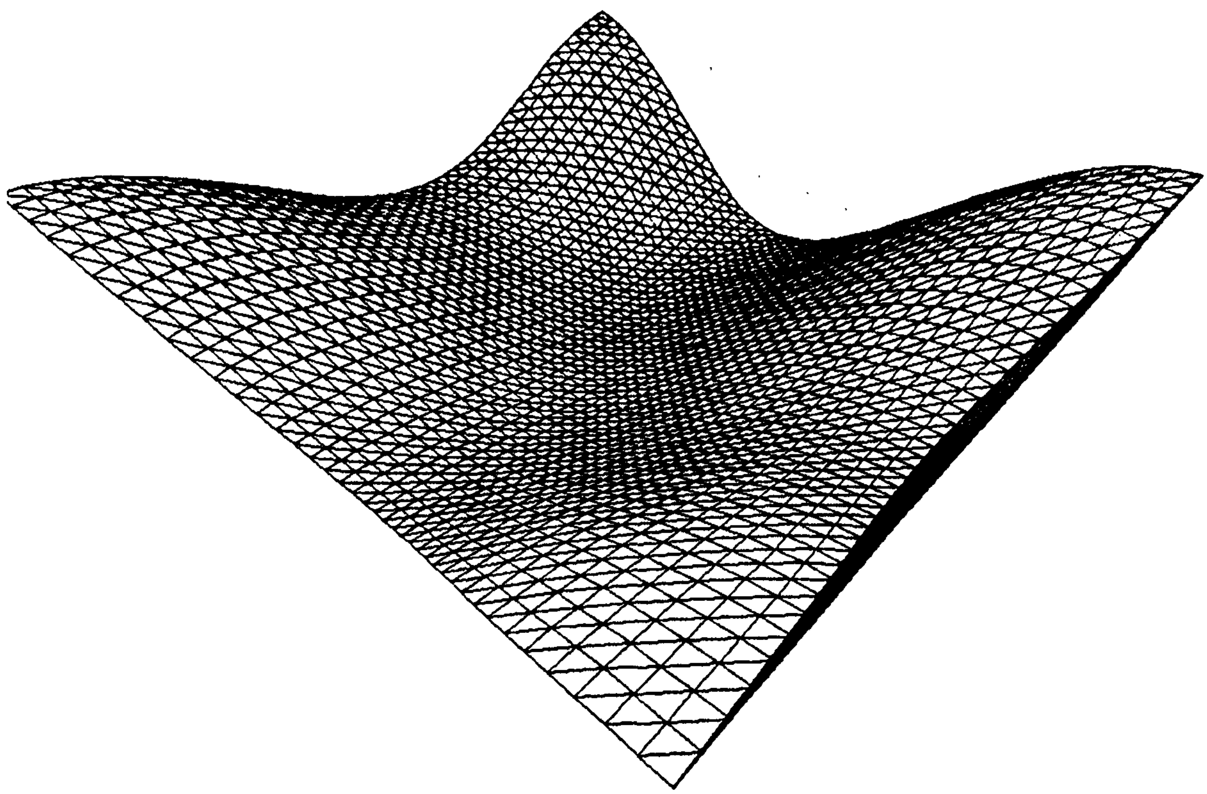

Figure 3 Function $\hat{V}_\xi$ for the planar 3-body problem

In terms of w_1, w_2, we have

$$\ll (W_1, W_2), (W_1, W_2) \gg = tr(\hat{w}_1 A_1 I_1 A_1^T \hat{w}_1^T) + tr(\hat{w}_2 A_2 I_2 A_2^T \hat{w}_2^T) + \epsilon \|\hat{w}_1 A_1 d_1 - \hat{w}_2 A_2 d_2\|^2.$$

It is a straightforward calculation to show that

$$tr(\hat{\omega} I \hat{\omega}^T) = < \omega, \mathring{I}\ \omega >_E$$

where I is a coefficient of inertia tensor and $\mathring{I}$ is the associated moment of inertia tensor and $<,>_E$ is the Euclidean inner product. Upon further simplifications and rearrangements, we get

$$\ll (W_1, W_2), (W_1, W_2) \gg = \begin{pmatrix} (A_1^T w_1)^T & (A_2^T w_2)^T \end{pmatrix} \begin{pmatrix} J_1 & J_{12} \\ J_{12}^T & J_2 \end{pmatrix} \begin{pmatrix} A_1^T w_1 \\ A_2^T w_2 \end{pmatrix},$$

where

$$J_1 = \mathring{I}_1 + \epsilon \hat{d}_1^T \hat{d}_1$$

$$J_2 = \mathring{I}_2 + \epsilon \hat{d}_2^T \hat{d}_2$$

$$J_{12} = \epsilon \hat{d}_1 A_1^T A_2 \hat{d}_2$$

The group action to consider is defined on $SO(3)\times SO(3)$, the configuration space, ***relative to an observer at the system center of mass.*** The diagonal action of the group $G = SO(3)$ is given by

$$\Psi: G \times (SO(3) \times SO(3)) \to SO(3) \times SO(3)$$
$$(R,(A_1,A_2)) \mapsto (RA_1, RA_2).$$

Letting $\hat{\xi} \in \Im$, the corresponding infinitesimal generator can be found as

$$\begin{aligned}\xi_Q(A_1,A_2) &= \left.\frac{d}{dt}\right|_{t=0} (\exp t\hat{\xi})(A_1,A_2) \\ &= \left.\frac{d}{dt}\right|_{t=0} \Big((\exp t\hat{\xi})A_1, (\exp t\hat{\xi})A_2\Big) \\ &= (\hat{\xi}A_1, \hat{\xi}A_2).\end{aligned}$$

Since here the potential energy V is identically 0, the function V_ξ is

$$V_\xi(A_1,A_2) = -\frac{1}{2}\left((A_1^T\xi)^T \quad (A_2^T\xi)^T\right)\begin{pmatrix} J_1 & J_{12} \\ J_{12}^T & J_2 \end{pmatrix}\begin{pmatrix} A_1^T\xi \\ A_2^T\xi \end{pmatrix}.$$

It is clear that V_ξ is invariant under $G_\xi = \{R \in G: R\xi = \xi\}$ which is isomorphic to S^1.

By the compactness of $SO(3) \times SO(3)$ we know that for each ξ, V_ξ has critical points. We need to find the conditions on A_1, A_2 so that the gradient of V_ξ with respect to A_1, A_2 is 0. Equivalently one can check the vanishing of the differential dV_ξ on the space $T(SO(3) \times SO(3))$.

Given $f : M \to R$ a smooth function on a smooth manifold M, $v_x \in T_xM$ which generates a curve $\phi : R \to M$ with $\phi(0) = x$, the differential df at x is defined by

$$df(x)(v_x) \equiv \left.\frac{d}{dt}\right|_{t=0} f(\phi(t)).$$

Let $W \in T(SO(3) \times SO(3))$,

$$W = (A_1, A_2, \hat{w}_1 A_1, \hat{w}_2 A_2).$$

The curve in $SO(3) \times SO(3)$ generated by W is $(e^{t\hat{w}_1}A_1, e^{t\hat{w}_2}A_2)$. Thus we have the formula

$$dV_\xi(A_1,A_2)(W) = \left.\frac{d}{dt}\right|_{t=0} V_\xi(e^{t\hat{w}_1}A_1, e^{t\hat{w}_2}A_2),$$

Explicitly, we get the following final form (here $A = A_1^T A_2$),

$$\begin{aligned} dV_\xi(A_1, A_2)(W) = &< w_1, \hat{\xi} A_1 J_1 A_1^T \xi >_E + < w_2, \hat{\xi} A_2 J_2 A_2^T \xi >_E \\ &+ \epsilon < w_1, \hat{\xi} A_1 \hat{d}_1 A \hat{d}_2 A_2^T \xi >_E + \epsilon < w_2, \hat{\xi} A_2 \hat{d}_2 A^T \hat{d}_1 A_1^T \xi >_E \\ &+ \epsilon < w_1, \widehat{A_1 d_1 \xi}\, A_2 \hat{d}_2 A_2^T \xi >_E + \epsilon < w_2, \widehat{A_2 d_2 \xi}\, A_1 \hat{d}_1 A_1^T \xi >_E \end{aligned}$$

Thus we know that the necessary conditions for a critical point of V_ξ are

$$\hat{\xi} A_1 J_1 A_1^T \xi + \epsilon \hat{\xi} A_1 \hat{d}_1 A \hat{d}_2 A_2^T \xi + \epsilon \widehat{A_1 d_1 \xi}\, A_2 \hat{d}_2 A_2^T \xi = 0,$$
$$\hat{\xi} A_2 J_2 A_2^T \xi + \epsilon \hat{\xi} A_2 \hat{d}_2 A^T \hat{d}_1 A_1^T \xi + \epsilon \widehat{A_2 d_2 \xi}\, A_1 \hat{d}_1 A_1^T \xi = 0.$$

Now if we define $\Omega_1 \equiv A_1^T \xi$, and $\Omega_2 \equiv A_2^T \xi$, we get the conditions (in terms of cross products in R^3)

$$\Omega_1 \times J_1 \Omega_1 + \epsilon d_1 \times (\Omega_1 \times A(d_2 \times \Omega_2)) = 0, \tag{8}$$
$$\Omega_2 \times J_2 \Omega_2 + \epsilon d_2 \times (\Omega_2 \times A^T(d_1 \times \Omega_1)) = 0, \tag{9}$$

which are exactly the conditions found by Poisson reduction in [unpublished notes of P.S. Krishnaprasad].

In step 2 of the algorithm of section 3, we put in the A_1, A_2 found by solving the above conditions into

$$\begin{aligned} p &= B_\xi(A_1, A_2) \\ &= FL(\xi_Q(A_1, A_2)). \end{aligned}$$

Let $p \in T^*(SO(3) \times SO(3))$ be represented as

$$p = (\hat{\alpha}_1 A_1, \hat{\alpha}_2 A_2).$$

We find that α_1, α_2 can be expressed as

$$\alpha_1 = A_1 J_1 \Omega_1 + \epsilon (A_1 d_1 \times A_2 (d_2 \times \Omega_2)),$$
$$\alpha_2 = A_2 J_2 \Omega_2 + \epsilon (A_2 d_2 \times A_1 (d_1 \times \Omega_1)).$$

Also we know that the relation between Ω_1 and Ω_2 is

$$\Omega_1 = A\Omega_2.$$

If we now let $s_1 = A_1 d_1$, $s_2 = A_2 d_2$, from (8), we get

$$A_1^T \xi \times J_1 A_1^T \xi + \epsilon A_1^T s_1 \times \left(A_1^T \xi \times A(A_2^T s_2 \times A_2^T \xi)\right) = 0,$$

which implies

$$\xi \times A_1 J_1 A_1^T \xi + \epsilon s_1 \times (\xi \times (s_2 \times \xi)) = 0. \tag{10}$$

Taking the inner product of (10) with ξ, we obtain a key necessary condition for a relative equilibrium

$$\xi \cdot (s_1 \times s_2) = 0. \tag{11}$$

We note that ξ is the axis of rotation of the whole body, s_1, s_2 are the spatial vectors from joint to body 1 and 2, respectively. From (11), we conclude that, at relative equilibria, ξ, s_1, s_2 *must lie on the same plane, no matter what the inertias are.*

5 Numerical Method

Although we can get the same critical conditions (8) (9) by other methods, the principle of symmetric criticality provides more information. Notice that in the first step of the algorithm in Section 2, we simply try to find the critical points of V_ξ on $SO(3) \times SO(3)$ without any additional constraint. Thus one has an associated unconstrained optimization problem. Numerical optimization schemes can be used to find extremal relative equilibria. This issue is discussed in this section.

By the symmetry of the system, we know that the function V_ξ is invariant in the direction tangent to the orbit of G_ξ. Thus in the search for critical points, we should avoid these directions. It turns out that the usual gradient-type method is a good choice. Here, we use an optimization package named CONSOLE which was developed at the University of Maryland[8]. The current version of CONSOLE basically uses the steepest descent method and is thus applicable to our circumstances.

In formulating the optimization problem, in order to avoid other constraints arising from the restrictions on $SO(3)$, e.g. $A^T A = Identity$, we use Cayley's parametrization. That is, any element $A \in SO(3)$ can be represented by

$$A = \frac{1}{1+a_1^2+a_2^2+a_3^2}\begin{pmatrix} 1+a_1^2-a_2^2-a_3^2 & 2(a_1a_2-a_3) & 2(a_1a_3+a_2) \\ 2(a_1a_2+a_3) & 1-a_1^2+a_2^2-a_3^2 & 2(a_2a_3-a_1) \\ 2(a_1a_3-a_2) & 2(a_2a_3+a_1 & 1-a_1^2-a_2^2+a_3^2) \end{pmatrix}$$

where $a_1, a_2, a_3 \in R$. The problem can now be written as

$$\underset{\left\{\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2, b_3 \end{matrix}\right\}}{extremize} \quad V_\xi(A_1, A_2)$$

where (a_1, a_2, a_3), (b_1, b_2, b_3) are the parameters for A_1, A_2, respectively.

The CAD package CONSOLE is composed of two main programs: CONVERT, SOLVE. CONVERT reads a problem description file which describes the optimization problem to be solved. SOLVE then performs the optimization process with the interaction of user and/or some simulator. For more details, see Fan et el.[9][10]. The problem description file for our problem is easily formulated as follows.

```
design_parameter a1 init=0
design_parameter a2 init=0
design_parameter a3 init=0

design_parameter b1 init=0
design_parameter b2 init=0
design_parameter b3 init=1

objective "V-xi"
   minimize {
      import a1, a2, a3;
      import b1, b2, b3;

      double cost();

      return  cost( a1, a2, a3, b1, b2, b3 );
      }
   good_value=0
   bad_value=100
```

where the subroutine `cost()` reads a system description file containing the information of I_1, I_2, d_1, d_2, ξ, m_1, m_2 and then returns the value of the function V_ξ. By choosing different moments of inertia and initial structure, we can perform the optimization. In the process, one thing we learned is that if the augmented inertia is diagonal, the rate of convergence is faster. Thus preliminary diagonalizations should be performed to get speed up.

In the particular case that

$$m_1 = 3.0$$

$$m_2 = 2.0$$

$$d_1 = (0\ 0\ 1)$$

$$d_2 = (-1\ 1\ 1)$$

$$\xi = (0\ 0\ 1)$$

$$I_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$I_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

the relative equilibrium we found by numerical methods was

$$A_1 = \begin{pmatrix} 0.0 & -0.939 & 0.344 \\ 0.0 & -0.344 & -0.939 \\ 1.0 & 0.0 & 0.0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0.007 & 0.350 & -0.937 \\ -0.528 & 0.796 & 0.294 \\ 0.849 & 0.493 & 0.191 \end{pmatrix}$$

$$s_1 = (0.344\ -0.939\ 0.0)$$

$$s_2 = (-0.593\ 1.618\ -0.165)$$

$$\Omega_1 = (1.0\ 0.0\ 0.0)$$

$$\Omega_2 = (0.849\ 0.493\ 0.191)$$

$$\alpha_1 = (0.0\ 0.0\ 10.269)$$

$$\alpha_2 = (-0.004\ 0.007\ 14.321).$$

Several relative equilibria corresponding to different choices of parameters are shown in Fig. 4. Case 1 in that figure corresponds to the above numerical result.

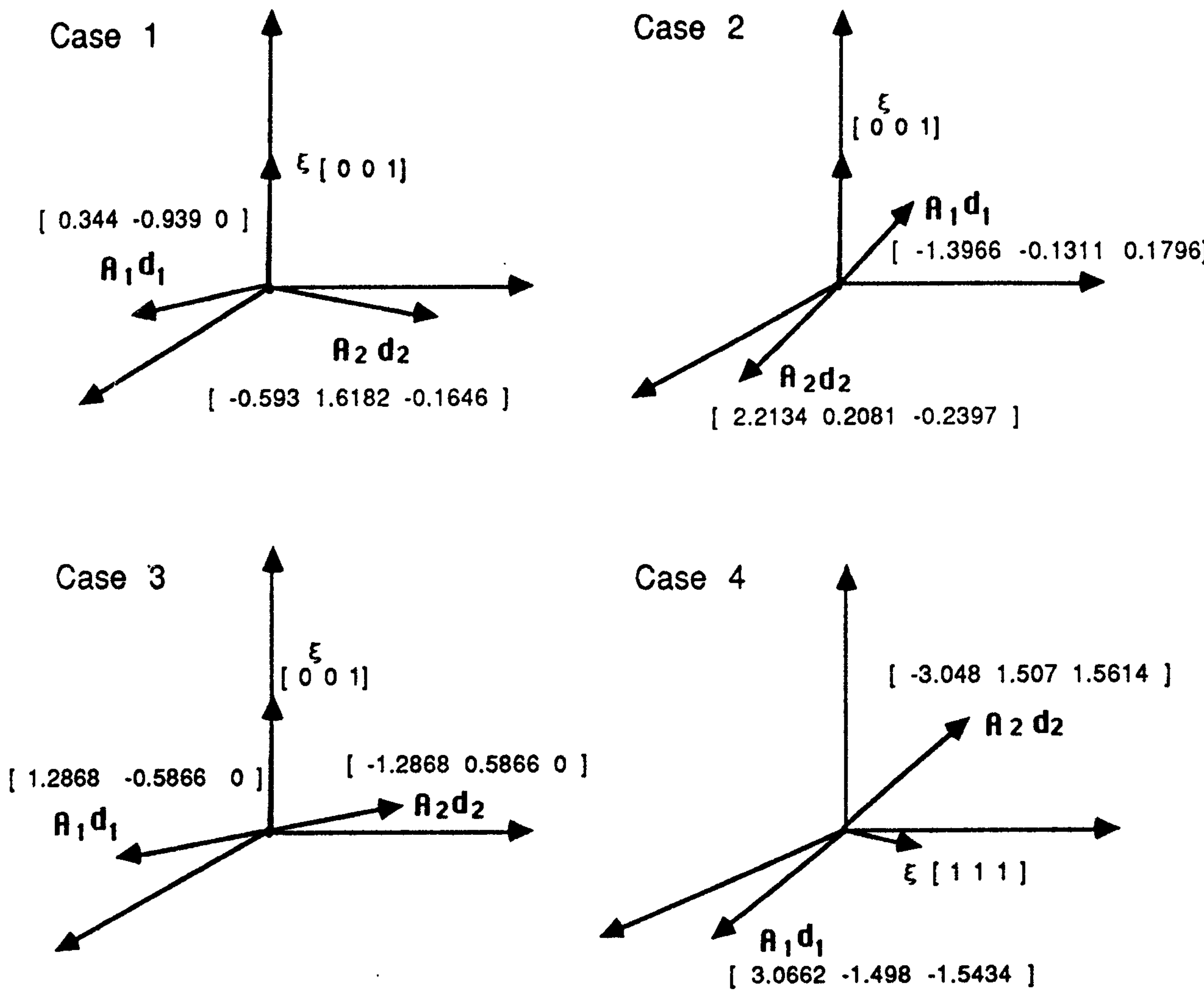

Figure 4 Relative Equilibria

Note that in all cases, either $s_1 = A_1 d_1$, $s_2 = A_2 d_2$ are on a straight line or they and ξ are on one plane. It matches the conclusion we made at the end of section 4.

6 Conclusion

We have shown how to determine relative equilibria by numerical search. A key geometric condition (Equation 11) appears as a consequence of the variational formulation. Dynamic simulations are being carried out to determine further details of the phase-portrait.

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REFERENCES

1. R. Abraham and J.E. Marsden (1978), *Foundations of Mechanics*, Second Edition, revised, enlarged, reset. Benjamin/Cummings, Reading.

2. V.I. Arnold (1978), *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York.

3. J. Baillieul (1988). "Linearized Models for the Control of Rotating Beams", *Proc. 27th IEEE Conf. Dec. Control,* New York, pp. 1726-1731.

4. J. Baillieul and M. Levi (1987), "Rotational Elastic Dynamics", *Physica D*, 27D, pp. 43-62.

5. G. Bianchi and W. Schielen (1985), eds. *Dynamics of Multibody Systems*, Proceedings of IUTAM/IFToMM Symposium Udine/Italy 1985, Springer Verlag, Berlin.

6. R. Bott (1960), *Morse Theory and its Application to Homotopy Theory, Lecture Notes,* University of Bonn.

7. CIME (1972), *Stereodynamics*, edizione cremonese, Roma.

8. M.K.H. Fan, L.-S. Wang, Jan Koninckx, A.L. Tits (1987), "CONSOLE User's Manual," *SRC TR-87-212,* Systems Research Center, University of Maryland, 1987.

9. M.K.H. Fan, L.-S. Wang, Jan Koninckx, A.L. Tits (1988), "CONSOLE: A CAD Tandem for Optimization-based Design Interacting with User-Supplied Simulators," *Proceedings of the 1988 American Control Conference,* Atlanta, GA, pp. 701–706.

10. M.K.H. Fan, L.-S. Wang, Jan Koninckx, A.L. Tits (1989), "Software Package for Optimization-based Design with User-Supplied Simulators," *IEEE Control Systems Magazine,* Vol. 9, No. 1, pp. 66–71.

11. H.J. Fletcher, L. Rongved, E.Y. Yu (1963), "Dynamics Analysis of a Two-body Gravitationally Oriented Satellite," *The Bell System Technical Journal,* September, 1963, pp. 2239–2266.

12. R. Grossman, P.S. Krishnaprasad and J.E. Marsden (1987) "The Dynamics of Two Coupled Three Dimensional Rigid Bodies" in F. Salam & M. Levi, eds. *Dynamical Systems Approaches to Nonlinear Problems in Systems and Circuits,* pp. 373-378, SIAM Publ., Philadelphia, 1988.

13. V. Guillemin and S. Sternberg (1984), *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge.

14. R. Hermann (1977), *Differential Geometry and the Calculus of Variation, 2nd Ed.*, Math. Sci. Press.

15. P.S. Krishnaprasad (1985), "Lie-Poisson Structures Dual-Spin Spacecraft and Asymptotic Stability", *Nonlinear Analysis: Theory, Methods and Applications,* Vol. 9, No. 10, pp. 1011-1035.

16. P.S. Krishnaprasad (1989), "Eulerian Many-Body Problems," to appear in *Contemporary Mathematics,* AMS.

17. P.S. Krishnaprasad and J.E. Marsden (1987) "Hamiltonian Structures & Stability for Rigid Bodies with Flexible Attachments." *Arch. Rat. Mech. Anal.*, 98, pp. 71-93.

18. P. Libermann and C-M. Marle (1987), *Symplectic Geometry and Analytical Mechanics*, D. Reidel Publ., Dordrecht.

19. J.E. Marsden and T. Ratiu (1986), "Reduction of Poisson Manifolds", *Letters in Math. Phys.,* Vol. 11, pp. 161-169.

20. J. E. Marsden and A. Weinstein (1974), "Reduction of Symplectic Manifolds with Symmetry", *Reports in Math. Phys.,* Vol. 5, pp. 121-130.

21. Y.G. Oh, N. Sreenath, P.S. Krishnaprasad and J.E. Marsden (1988), "The Dynamics of Coupled Planar Rigid Bodies Part II: Bifurcation, Periodic Orbits, and Chaos", (in press) *J. Dynamics & Differential Equations.*

22. R.S. Palais (1979), "The Principle of Symmetric Criticality," *Communications in Math. Physics,* Vol. 69, No. 1, pp. 19–30.

23. B. Paul, J.W. West, E.Y. Yu (1963), "A Passive Gravitational Attitude Control System for Satellites," *The Bell System Technical Journal,* September, 1963, pp. 2195–2238.

24. T. Posbergh, P.S. Krishnaprasad and J.E. Marsden (1987), "Stability Analysis of a Rigid Body with a Flexible Attachment using the Energy-Casimir Method" in M. Luksic, C. F. Martin, W. Shadwick eds. *Differential Geometry : The Interface between Pure and Applied Mathematics*, in series, *Contemporary Math.*, Vol. 68, pp. 253-273. AMS, Providence.

25. T. Posbergh (1988) Ph.D. Thesis, University of Maryland, *Modeling and Control of Mixed and Flexible Structures,* Also, Systems Research Center Technical Report *SRC TR88-58.*

26. J.C. Simo, J.E. Marsden and P.S. Krishnaprasad (1988), "The Hamiltonian Structure of Nonlinear Elasticity: The Material and Convective Representation of Rods, Plates and Shells", *Arch. Rat. Mech. & Anal.*, Vol. 104, No. 2, pp. 125-183.

27. J.C. Simo, T. Posbergh and J.E. Marsden (1988), "Nonlinear Stability of Geometrically Exact Rods by the Energy-Momentum Method". *Preprint,* Stanford University, Division of Applied Mechanics.

28. S. Smale (1970) "Topology and Mechanics, I, II," *Invent. Math.*, Vol. 10, pp. 305-331 and Vol. 11, pp. 45-64.

29. N. Sreenath (1987) Ph.D. Thesis, University of Maryland, *Modeling and Control of Multibody Systems.* Also, Systems Research Center Technical Report *SRC TR87-163.*

30. N. Sreenath, Y.G. Oh, P.S. Krishnaprasad and J.E. Marsden (1988), "The Dynamics of Coupled Planar Rigid Bodies Part I: Reduction, Equilibria & Stability," *Dynamics & Stability of Systems,* Vol. 3, No. 1&2, pp. 25–49.

31. J. Wittenburg (1977), *Dynamics of Multibody Systems*, B.G. Teubner, Stuttgart.