SRC TR 89-22

Ode and Diffusion Limits in Large Symmetric Circuit Switched Networks

By

P. Tsoucas

ODE AND DIFFUSION LIMITS IN LARGE SYMMETRIC CIRCUIT SWITCHED NETWORKS

by

Pantelis Tsoucas

Systems Research Center
University of Maryland
College Park, MD 20742

ABSTRACT

This paper considers a large symmetric star-shaped circuit-switched network where each route requires one circuit from each of two links. Interarrival and holding times of calls are exponential. The process of the number of free circuits in each link is analyzed. An ordinary differential equation limit is established along with a diffusion approximation limit conjectured by Whitt [Wh].

1. Introduction

Performance evaluation of dynamic routing strategies in circuit-switched networks has received a lot of recent attention. Most of the existing literature is concerned with either simulation (see [Ak]) or Erlang-type approximations (see [Ke]). The exact analysis of such models appears to be difficult in view of the fact that Markovian descriptions of such models are cumbersome. An alternative is to take advantage of simplifications that arise in large, symmetric models due to the law of lalrge numbers and the central limit theorem. Such asymptotic analysis has appeared in [Wh] for a network with fixed routes (see [Ke] for a different asymptotic regime). Although the stationary distribution of the network has a product form, the results are interesting because the computational burden grows quickly with the size of the network. Also, an asymptotic transient analysis is possible. Furthermore, it turns out that in the limit each link behaves independently of the rest of the network as an $M/M/C/C$ queue. The main result of this paper is a proof of the conjectured diffusion limits in [Wh]. The approach utilizes recent results of [KLS] on the weak convergence of non-Markov processes. It considerably simplifies the derivations in [Wh] and appears to be promising for the analysis of dynamic schemes.

In Section 2 we describe the model and we derive an ordinary differential equation limit for the process of the number of occupied circuits in each link. The infinite time behavior and asymptotic independence of these processes is also discussed. In Section 3 we prove a diffusion limit theorem for the normalized difference between the deterministic process and the actual process of occupied circuits in a link.

2.Deterministic limit

2.1. Model

Consider the following model of a circuit-switched network. It consists of a star-shaped undirected graph with $N+1$ nodes numbered $\{0,1,\ldots,N\}$. The set of edges is $\{(i,0)|i=1,\ldots,N\}$. Each edge is a communication link comprising C circuits. Calls establishing communication between nodes i and j in $\{1,\ldots,N\}$ via node 0 arrive as a Poisson process with rate $\lambda/(N-1)$. Each such call requests one circuit from each of the links $(i,0)$ and $(j,0)$. A call is successful if the requested circuits are available, and is lost otherwise. A successful call occupies its circuits for an exponentially distributed period with mean 1. All arrival and service processes are assumed independent. We investigate the asymptotic regime where $N\to\infty$ and all other quantities remain fixed. Note that the total traffic offered to any link has fixed rate λ.

Definitions: Let $x_{ij}^N(t)$ be the number of calls in progress from node i to node j via node 0 at time $t\geq 0$. Let $y_i^N(t)$ be the total number of occupied circuits in link $(i,0)$, i.e.,

$$y_i^N(t)=\sum_{j\neq i}x_{ij}^N(t).$$

By $q_k^N(t)$ denote the fraction of links that have exactly k circuits busy at time t, $k=0,1,\ldots,C$. Note that

$$\sum_{k=0}^{C}q_k^N(t)=1, \tag{2.1}$$

and set

$$S=_{def}\left\{q\in I\!R^C|q_k\geq 0,\ k=1,\ldots,C;\quad \sum_{k=1}^{C}q_k\leq 1\right\}. \tag{2.2}$$

For $l,m\in\{0,1,\ldots,C\}$ denote by $A_{l,m}^N(t)$ the counting process given by

$$dA_{l,m}^N(t)=\sum_{i=1}^{N}1\left\{y_i^N(t)=m\right\}1\left\{y_i^N(t-)=l\right\},$$

i.e., the process counting transitions of edges from state l into state m.

Lastly, in what follows, $(Y(\cdot))$ denotes a Poisson process of unit rate, and all stochastic integrals are understood to be in the Lebesque-Stieltjes sense.

2.2. Evolution equations

For $i, j \in \{1, \dots, N\}$, $i \neq j$, processes $(x_{ij}^N(t))$ and $(y_i^N(t))$ are represented using independent Poisson processes of unit rate $(Y_{ij}^a(\cdot))$, $(Y_{ij}^d(\cdot))$:

$$x_{ij}^N(t) = x_{ij}^N(0) + Y_{ij}^a\left(\frac{\lambda}{N-1}\int_0^t 1\left\{y_i^N(s) < C\right\} 1\left\{y_j^N(s) < C\right\} ds\right) - Y_{ij}^d\left(\int_0^t x_{ij}^N(s) ds\right), \tag{2.2}$$

$$y_i^N(t) = y_i^N(0) + \sum_{j\neq i} Y_{ij}^a\left(\frac{\lambda}{N-1}\int_0^t 1\left\{y_i^N(s) < C\right\} 1\left\{y_j^N(s) < C\right\} ds\right) - \sum_{j\neq i} Y_{ij}^d\left(\int_0^t x_{ij}^N(s) ds\right). \tag{2.3}$$

Similarly, one has for $k = 1, \dots, C-1$,

$$\begin{aligned} q_k^N(t) &= q_k^N(0) + \frac{1}{N}\int_0^t dA_{k+1,k}^N(s) + \frac{1}{N}\int_0^t dA_{k-1,k}^N(s) \\ &\quad - \frac{1}{N}\int_0^t dA_{k,k-1}^N(s) - \frac{1}{N}\int_0^t dA_{k,k+1}^N(s), \quad k = 1, \dots, C-1, \\ q_C^N(t) &= q_C^N(0) + \frac{1}{N}\int_0^t dA_{C-1,C}^N(s) - \frac{1}{N}\int_0^t dA_{C,C-1}^N(s). \end{aligned} \tag{2.4}$$

By $\left(\tilde{A}_{l,m}^N(\cdot)\right)$ denote the compensators (see [LS]) of the processes $\left(A_{l,m}^N(\cdot)\right)$ with respect to the σ-field $\mathcal{F}_t^N =_{def} \sigma\left\{q^N(s);\ 0 \le s \le t\right\}$, where $q^N(s) = \left(q_1^N(t), \dots, q_C^N(t)\right)$. From equations (2.2), (2.3), and some counting, one has

$$\begin{aligned} d\tilde{A}_{k+1,k}^N(t) &= N(k+1)q_{k+1}^N(t)dt \\ d\tilde{A}_{k-1,k}^N(t) &= \frac{N^2}{N-1}\lambda q_{k-1}^N(t)(1 - q_C^N(t))dt \\ d\tilde{A}_{k,k-1}^N(t) &= Nkq_k^N(t)dt \\ d\tilde{A}_{k,k+1}^N(t) &= \frac{N^2}{N-1}\lambda q_k^N(t)(1 - q_C^N(t))dt. \end{aligned} \tag{2.5}$$

To study equations (2.4) one defines the martingales,

$$M_{l,m}^N(t) =_{def} A_{l,m}^N(t) - \tilde{A}_{l,m}^N(t).$$

Recall that $< M >_t$ denotes the quadratic variation of a locally square integrable martingale $M(t)$ (see [LS]). From the properties of Poisson processes,

$$\begin{aligned} &< \frac{1}{N}M_{k+1,k}^N >_t = \frac{1}{N^2}\tilde{A}_{k+1,k}^N(t) = \frac{1}{N}(k+1)\int_0^t q_{k+1}^N(s)ds \\ &< \frac{1}{N}M_{k-1,k}^N >_t = \frac{1}{N^2}\tilde{A}_{k-1,k}^N(t) = \frac{1}{N}\lambda\int_0^t q_{k-1}^N(s)(1 - q_C^N(s))ds + o(\frac{1}{N}) \\ &< \frac{1}{N}M_{k,k-1}^N >_t = \frac{1}{N^2}\tilde{A}_{k,k-1}^N(t) = \frac{1}{N}k\int_0^t q_k^N(s)ds \\ &< \frac{1}{N}M_{k,k+1}^N >_t = \frac{1}{N^2}\tilde{A}_{k,k+1}^N(t) = \frac{1}{N}\lambda\int_0^t q_k^N(s)(1 - q_C^N(s))ds + o(\frac{1}{N}), \end{aligned} \tag{2.6}$$

By independence, all cross-variations are zero.

The next result shows that the right hand side of equations (2.4) is essentially deterministic for large N.

Lemma 2.1: (Lenglart) For any locally square integrable martingale $(M(t), \mathcal{F}_t)$, $\mathcal{F}_t$-stopping time τ, $\delta > 0$, and $\epsilon > 0$,

$$P\left\{\sup_{0\le s\le \tau} |M(s)| \ge \epsilon\right\} \le \frac{1}{\epsilon^2} + P\{< M >_\tau \ge \delta\}.$$

Proof: See [JS]. □

Corollary 2.1: For martingales $\left(M_{l,m}^N(\cdot)\right)$ as defined above, $\epsilon > 0$, and $t \ge 0$,

$$\lim_{N\to\infty} P\left\{\sup_{0\le s\le t} |M_{l,m}^N(s)| \ge \epsilon\right\} = 0.$$

Proof: Apply Lemma 2.1 to $\left(M_{l,m}^N(\cdot)\right)$ at time t and observe that S is compact. □

2.3. Convergence

In view of the above lemma, one anticipates that, as $N \to \infty$, equations (2.4) assume the form

$$\begin{aligned} \dot q_k(t) &= (k+1)q_{k+1}(t) + \lambda q_{k-1}(t)(1-q_C(t)) - kq_k(t) - \lambda q_k(t)(1-q_C(t)), \quad k = 1,2,\ldots,C-1. \\ \dot q_C(t) &= \lambda q_{C-1}(t)(1-q_C(t)) - Cq_C(t). \end{aligned} \tag{2.7}$$

Indeed, denoting by $\|\cdot\|$ the euclidean norm in $I\!R^C$ one has,

Theorem 2.1: For any $T \ge 0$ and $\epsilon > 0$, if $\lim_{N\to\infty} q^N(0) = q(0)$ in probability, then,

$$\lim_{N\to\infty} P\left\{\sup_{0\le s\le T} \|q^N(s) - q(s)\| \ge \epsilon\right\} = 0.$$

Proof: Subtracting (2.7) from (2.4), and setting $M_k^N(t) = M_{k+1,k}^N(t) + M_{k-1,k}^N(t) + M_{k,k+1}^N(t) + M_{k,k-1}^N(t)$ gives,

$$\begin{aligned} q_k^N(t) - q_k(t) &= q_k^N(0) - q_k(0) + \frac{1}{N}M_k^N(t) + o(\frac{1}{N}) \\ &\quad + \lambda\int_0^t \left\{\left(q_{k-1}^N(s) - q_k^N(s)\right)\left(1 - q_C^N(s)\right) - \left(q_{k-1}(s) - q_k(s)\right)\left(1 - q_C(s)\right)\right\} ds \\ &\quad + (k+1)\int_0^t \left(q_{k+1}^N(s) - q_{k+1}(s)\right) ds - k\int_0^t \left(q_k^N(s) - q_k(s)\right) ds, \quad k = 1,\ldots,C-1, \\ q_C^N(t) - q_C(t) &= q_C^N(0) - q_C(0) + \frac{1}{N}M_C^N(t) + o(\frac{1}{N}) \\ &\quad + \lambda\int_0^t \left\{q_{C-1}^N(s)\left(1 - q_C^N(s)\right) - q_{C-1}(s)\left(1 - q_C(s)\right)\right\} ds - C\int_0^t \left(q_C^N(s) - q_C(s)\right) ds. \end{aligned}$$

In shorthand notation,

$$q^N(t) - q(t) = q^N(0) - q(0) + M^N(t) + \int_0^t \left\{f(q^N(s)) - f(q(s))\right\} ds, \tag{2.8}$$

where $f(\cdot) = (f_1(\cdot), \ldots, f_C(\cdot))$. For each k, $f_k(\cdot)$ can be seen to be a Lipschitz continuous function with constant K. Taking norms in (2.8) gives

$$\|q^N(t) - q(t)\| \le \|q^N(0) - q(0)\| + \|M^N(t)\| + K\int_0^t \|q^N(s) - q(s)\| ds.$$

Setting $M_T^N = \sup_{0\leq t\leq T} \|M^N(t)\|$ gives

$$\|q^N(t) - q(t)\| \leq \|q^N(0) - q(0)\| + M_T^N + K\int_0^t \|q^N(s) - q(s)\|ds.$$

From Bellman-Gronwall,

$$\|q^N(t) - q(t)\| \leq (\|q^N(0) - q(0)\| + M_T^N)e^{Kt},$$

and thus,

$$P\left\{\sup_{0\leq s\leq T} \|q^N(s) - q(s)\| \geq \epsilon\right\} \leq P\left\{\sup_{0\leq s\leq T} \|q^N(0) - q(0)\| + M_T^N \geq \epsilon e^{-KT}\right\}.$$

Corollary 2.1, equations (2.6), and the compactness of S, imply that $\lim_{N\to\infty} M_T^N = 0$, in probability. Then, our assumption on the initial conditions establishes the result. □

2.4. Stability properties

In this section we summarize from [Wh] qualitative properties of equations (2.7). They are used in Sections 2.5 and 2.6. To obtain the stationary points of (2.7), set the left hand side equal to 0 and solve the last of equations (2.7) for q_{C-1}. Substituting backward, $(q_l)_{l=1}^{C-1}$ can be expressed in terms of q_C which, because of (2.1), must satisfy

$$q_C = \frac{\lambda^C(1-q_C)^C/C!}{\sum_{k=0}^C \lambda^k(1-q_C)^k/k!}. \tag{2.9}$$

Equation (2.9) has a unique solution because its right hand side is a decreasing function of q_C. Denote the solution by ξ, which is shown in [Wh] to be a stable stationary point. These facts are summarized in

Theorem 2.2: The system of differential equations (2.7) has a unique, asymptotically stable stationary point ξ.

Remark 2.1: The right hand side of (2.9) is the Erlang probability of loss function for an $M/M/C/C$ node with arrival rate $\lambda(1-q_C)$ and service rate 1. Some insight for this fact can be gained by the results in Section 2.6. Also, in connection with Conjecture 1 in [Wh] we remark that, from Theorem 5.1 in [Ke], ξ is the unique solution of the Erlang fixed point approximation equations for this network.

2.5. Invariant measures

We now examine the infinite time behavior of the processes $(q^N(\cdot))$ as $N\to\infty$. For each N, denote by $\mu^N(\cdot)$ the invariant probability measure of $(q^N(\cdot))$, and by $\delta_\xi(\cdot)$ denote the probability measure that assigns unit mass on ξ. One anticipates

Theorem 2.3: $\mu^N(\cdot) \Rightarrow_{N\to\infty} \delta_\xi(\cdot)$ (recall that $\Rightarrow$ denotes weak convergence.)

Proof: Since S is compact, the sequence $(\mu^N(\cdot))$ is tight. By Pohorov's theorem ([Bi], p. 37), there exists a limit point $\nu(\cdot)$ of $(\mu^N(\cdot))$, and write $\mu^{N'}(\cdot) \Rightarrow_{N'\to\infty} \nu(\cdot)$ along a subsequence. Following the scheme of Theorem 9.10, p.244 in [EK], we prove that $\nu(\cdot)$ is invariant for $(q(\cdot))$, and Theorem 2.2 then implies that $\nu(\cdot) = \delta_\xi(\cdot)$. To this end, assume that $q(0)$ is distributed according to $\nu(\cdot)$, $\hat{q}(0) = q^N(0)$ with common distribution $\mu^N(\cdot)$ Assume also that $(q(\cdot))$, $(\hat{q}(\cdot))$ satisfy (2.7) (possibly with different initial conditions,) and $(q^N(\cdot))$ satisfies (2.4). Then, for any Borel subset A of $I\!R^C$, and $t\geq 0$,

$$\begin{aligned}|P\{q(t)\in A\} - P\{q(0)\in A\}| &\leq \limsup_{N\to\infty} |P\{q(t)\in A\} - P\{\hat{q}(t)\in A\}| \\ &+ \limsup_{N\to\infty} |P\{\hat{q}(t)\in A\} - P\{q^N(t)\in A\}| \\ &+ \limsup_{N\to\infty} |P\{q^N(0)\in A\} - P\{q(0)\in A\}| \to_{N\to\infty} 0.\end{aligned}$$

The first and third terms above converge to zero since $\mu^{N'}(\cdot) \Rightarrow_{N' \to \infty} \nu(\cdot)$, the second term converges to zero by Theorem 2.1 and we have used the invariance of $(\mu^N(\cdot))$. Therefore, $\mu^{N'}(\cdot) \Rightarrow_{N' \to \infty} \delta_\xi(\cdot)$. Our result follows from [Bi], Theorem 2.3 on p.16, since the argument can be repeated for any subsequence of $(\mu^N(\cdot))$. □

2.6. Asymptotic independence

In this section we focus attention on a fixed number of links, say $\{1, \ldots, m\}$, as the size of the network grows to infinity. Assume that the initial distribution of calls $\{x_{ij}^N(0)\}_{ij}$ is invariant under index permutation for all N. Then, if Theorem 2.1 is valid, the processes of occupied circuits in these links behave, for any $t \geq 0$ and as $N \to \infty$, as m independent copies of the Markov process

$$y(t) = y(0) + Y^a\left(\lambda \int_0^t 1\{y(s) < C\} \ \Pr\{y(s) < C\}\, ds\right) - Y^d\left(\int_0^t y(s)ds\right), \tag{2.10}$$

where $y(0)$ is distributed according to $q(0)$, and $Y^a(\cdot)$, $Y^d(\cdot)$ are independent. Note that the process is somewhat non-standard since the right hand side of (2.10) depends on the distribution of $(y(t))$. This gives rise to non-linearities in the forward equations of the process. They can be seen to be equations (2.7). Results of this type are known in stochastic analysis as "propagation of chaos".

More precisely, asymptotic independence concerns the joint distribution of $\left(y_1^N(t), \ldots, y_m^N(s)\right)$, denoted by $\mathcal{L}\left(y_1^N(s), \ldots, y_m^N(s)\right)$. Set $E = \{1, \ldots, C\}$ and for $X(t) \in E$ let $\delta_{X(t)}$ be the distribution on E that assigns unit mass on $X(t)$. In this notation, Theorem 2.1 implies that

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{y_i^N(t)} \xrightarrow{w}_{N \to \infty} q(\cdot).$$

The following appears to be "folklore" in the literature of stochastic analysis. We omit the proof as it requires additional notation.

Theorem 2.4: Assume that the distribution of $\{x_{ij}^N(0)\}_{ij}$ is invariant under permutation of the indices and that Theorem 2.1 holds. Then, for any $t \geq 0$,

$$\mathcal{L}\left(y_1^N(t), \ldots, y_m^N(t)\right) \longrightarrow_{N \to \infty} \mathcal{L}\left(y(t)\right)^{\otimes m} = q(t)^{\otimes}.$$

($\otimes m$ denotes the m-fold product).

Proof: See [Sz], Lemma 3.1, where a converse to this theorem is also established. □

To recover the results in [Wh] concerning infinite time behavior we can employ the scheme of Theorem 2.1 to get

Corollary 2.2: If the distribution of $\{x_{ij}^N(0)\}_{ij}$ is invariant under permutation of the indices and if Theorem 2.1 holds, then

$$\mathcal{L}\left(y_1^N(\infty), \ldots, y_m^N(\infty)\right) \longrightarrow_{N \to \infty} \xi^{\otimes m}.$$

(Recall that $q(t) \to_{t \to \infty} \xi$.)

Note that, in this functional form, asymptotic independence is similar to the product form results in Jackson networks. However, functional versions of Theorem 2.4 as in [Sz] should be possible to establish.

3. Diffusion approximation

We now proceed to the main result of this paper, a diffusion limit for the difference

$$u^N(t) = \sqrt{N}\left(q^N(t) - q(t)\right).$$

The form of this limit can be guessed from standard results concerning diffusion limits of Markov processes (see e.g. [Ku]). To describe the limit, define matrix $\partial F(\cdot) : I\!R^C \to I\!R^{C\times C}$ whose kth row consists of the vector $\nabla f_k(\cdot)$ and matrix $\Sigma(\cdot) : I\!R^C \to I\!R^{C\times C}$ which is diagonal with $\Sigma_{kk}(\cdot) = f_k(\cdot)$. Then, the weak limit of $\left(u^N(s)\right)_{s\leq t}$ will be identified as the unique solution $(v(s))_{s\leq t}$ of the Ito stochastic differential equation

$$du(t) = \partial F(q(t))\; u(t)\; dt + \Sigma^{1/2}(q(t))\; dw(t),\; v(0) = v_0,$$

where $(w(s))_{s\leq t}$ is a standard brownian motion in $I\!R^C$. Denote the induced measure on $D\left([0,t], I\!R^C\right)$ by Q and the generated σ-field by $(\mathcal{G}_s)_{s\leq t}$. The difficulty of proving such a weak limit appears, as pointed out in [Wh], in the computation of the infinitesimal covariance matrix of the non-Markov process $\left(u^N(s)\right)_{s\leq t}$. Instead we will appeal to a recent result of Kogan et al in [KLS] where it is shown that it suffices to verify convergence of the quadratic variation of $\left(u^N(s)\right)_{s\leq t}$ to that of $(u(s))_{s\leq t}$.

To this end we will need the following characterization of process $(u(s))_{s\leq t}$: measure Q is the unique solution to the martingale problem where

$$z_t(u) = u(t) - u(0) - \int_0^t \partial F(q(s))ds$$

is a square integrable martingale with respect to $(\mathcal{G}_t)_{s\leq t}$ with quadratic variation

$$< z(u) >_t = \int_0^t \Sigma(q(s))ds,$$

From (2.8) one sees that $\left(u^N(s)\right)_{s\leq t}$ has a similar representation

$$u_k^N(t) = u_k^N(0) + \sqrt{N}\int_0^t \left\{f_k\left(q^N(s)\right) - f_k\left(q(s)\right)\right\} ds + \frac{1}{\sqrt{N}} M_k^N(t),\; k = 1,\ldots,C$$

where

$$< \frac{1}{\sqrt{N}} M_k^N >_t = \int_0^t f_k\left(q^N(s)\right)\; ds = \int_0^t \Sigma_{kk}\left(q^N(s)\right)\; ds.$$

The main result can now be stated and proved.

Theorem 3.1: If $u^N(0) \xrightarrow{w} u_0$, then, for all $t \geq 0$,

$$\left(u^N(s)\right)_{s\leq t} \longrightarrow_{N\to\infty} (u(s))_{s\leq t}$$

in the sense of weak convergence of measures on $D\left([0,t], I\!R^C\right)$.

Proof: In view of Theorem 3 in [KLS] it suffices to check the following three conditions.

(C1) $\quad \sup_{0\leq s\leq t} ||\Delta u^N(s)|| \xrightarrow{P}_{N\to\infty} 0.$

(Recall that $\Delta u^N(s) = u^N(s) - u^N(s-)$.)

(C2) $\quad \sup_{0\leq s\leq t} \left| \int_0^t \left\{\sqrt{N}\left[f_k\left(q^N(s)\right) - f_k\left(q(s)\right)\right] - \nabla f_k\left(q(s)\right)\cdot u(s)\right\} ds\right| \xrightarrow{P}_{N\to\infty} 0,\; k = 1,\ldots,C.$

(C3) $\quad < \frac{1}{\sqrt{N}} M_k^N >_t \xrightarrow{P}_{N\to\infty} \int_0^t \Sigma_{kk}\left(q(s)\right) ds,\; k = 1,\ldots,C.$

Here $\xrightarrow{P}$ denotes convergence in probability.

Condition (C1) is immediate since, with probability 1, $\Delta u_k^N(s) \leq 1/\sqrt{N}$ for all $s \leq t$. Also, condition (C3) easily follows from Theorem 2.1. It remains to check (C3). The proof follows closely the one in [KLS].

From the intermediate value theorem one obtains for $k = 1, \ldots, C$,

$$\sqrt{N}\left[f_k\left(q^N(s)\right) - f_k\left(q(s)\right)\right] = \nabla f_k\left(q(s) + \theta_k\left(q^N(s) - q(s)\right)\right) \cdot u^N(s),$$

where $0 \leq \theta_k \leq 1$, and from the Lipschitz continuity of the second derivatives of f_k,

$$\left|\sqrt{N}\left[f_k\left(q^N(s)\right) - f_k\left(q(s)\right)\right] - \nabla f_k\left(q(s)\right) \cdot u(s)\right| \;\leq\; L\, ||q^N(s) - q(s)||\; ||u^N(s)||.$$

It therefore suffices to show that, in probability,

$$\sup_{0 \leq s \leq t} ||q^N(s) - q(s)|| \sup_{0 \leq s \leq t} ||u^N(s)|| \longrightarrow_{N \to \infty} 0.$$

For any $a \geq 0$, the result follows from Theorem 2.1 on the event $\left\{\sup_{0 \leq s \leq t} ||u^N(s)|| \leq a\right\}$. It therefore suffices to show that

$$\lim_{a \to \infty} \limsup_{N} P\left\{\sup_{0 \leq s \leq t} ||u^N(s)|| > a\right\} = 0. \tag{3.1}$$

From (2.8) and Bellman Gronwall one has

$$\sup_{0 \leq s \leq t} ||u^N(s)|| \leq \left(||u^N(0)|| + \frac{1}{\sqrt{N}} \sup_{0 \leq s \leq t} ||M^N(s)||\right) e^{Kt}.$$

Then, (3.1) can be seen to hold because of Lemma 2.1. □

4. References

[Ak] Akinpelu, J. M. (1984) The overload performance for engineered networks with nonhierarchical and hierarchical routing. *AT&T Bell Lab. Tech. J.* 63, 7, 1261-1282.

[Bi] Billingsley, P. (1968) *Convergence of Probability Measures.* Wiley. New York.

[EK] Ethier, S. and Kurtz, T. (1985) *Markov Processes: Convergence and Characterization.* Wiley. New York.

[JS] Jacod, J. and Shiryaev, A. N. (1987) *Limit Theorems for Stochastic Processes.* Springer Verlag. Berlin.

[Ke] Kelly, F. P. (1986) Blocking probabilities in large circuit-switched networks. *Adv. Appl. Prob.* 18, 473-505.

[KLS] Kogan, Ya. A., Liptser, P. Sh. and Smorodinskii, A. V. (1987) Gaussian diffusion approximation of closed Markov models of computer networks. *Problems of Information Transmission* 22, 1, 38-51.

[Ku] Kurtz, T. G. (1978) Strong approximation theorems with density-dependent Markov chains. *Stoch. Proc. and Appl.* 6, 223-240.

[LS] Liptser, R. Sh. and Shiryaev, A. N. (1977) *Statistics of Random Processes II,* Springer Verlag, Berlin.

[Wh] Whitt, W. (1985) Blocking when service is required from several facilities simultaneously. *AT&T Tech. J.* 64, 1807-1856.