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New Results in Discrete-Time Nonlinear Filtering

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ABSTRACT

Title of Thesis: New Results in Discrete-Time Nonlinear Filtering

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We consider a discrete-time linear system with correlated Gaussian plant and observation noises and non-Gaussian initial condition independent of the plant and observation noises. We firstly find a solution for the filtering problem; we find a representation for the conditional distribution of the state at time t given the observations up to time $t-1$. This representation is in terms of a finite collection of easily-computable statistics. With this solution to the filtering problem, we then find representations for the MMSE and LLSE estimates of the state given the previous observations, and the mean-square error between the two. (Of course the MMSE estimate will in general be a nonlinear function of the observations, whereas the LLSE estimate is by definition linear and is given by the Kalman filtering equations.) We then consider the asymptotic behavior of the mean-square error between the MMSE and LLSE estimates as time tends to infinity. We find conditions on the system dynamics under which the effects of the initial condition die out; under these conditions the non-Gaussian nature of the initial condition becomes unimportant as t becomes large. The practical value of this result is clear—under these conditions, the LLSE estimate, which is usually less costly to generate than the MMSE estimate, is asymptotically as good as the MMSE estimate (i.e., asymptotically optimal) in the mean-square sense.

NEW RESULTS IN DISCRETE-TIME NONLINEAR FILTERING

by

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DEDICATION

This thesis is dedicated to Svetlana Vranić, Mary Hamilton, and all who have given me their love and support throughout the years.

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CHAPTER I: INTRODUCTION

I.1. Problem Statement and Outline

We consider the one-step prediction problem associated with the stochastic discrete-time linear dynamical system

$$\begin{aligned} X^\circ_{t+1} &= A_t X^\circ_t + W^\circ_{t+1} \\ X^\circ_0 &= \xi \\ Y_t &= H_t X^\circ_t + V^\circ_{t+1} \end{aligned} \qquad t = 0, 1, \ldots \quad (1.1)$$

which is defined on some underlying probability triple $(\Omega, \mathcal{F}, P)$ carrying the $I\!R^n$-valued plant process $\{X^\circ_t\}_0^\infty$ and the $I\!R^k$-valued observation process $\{Y_t\}_0^\infty$. Throughout, we shall make the following assumptions:

(A.1): the process $\{(W^\circ_{t+1}, V^\circ_{t+1})\}_0^\infty$ is a zero-mean Gaussian White Noise (GWN) sequence with covariance structure $\{\Gamma_{t+1}\}_0^\infty$ given by

$$\Gamma_{t+1} := \mathrm{Cov} \begin{pmatrix} W^\circ_{t+1} \\ V^\circ_{t+1} \end{pmatrix} = \begin{pmatrix} \Sigma^w_{t+1} & \Sigma^{wv}_{t+1} \\ \Sigma^{vw}_{t+1} & \Sigma^v_{t+1} \end{pmatrix}, \qquad t = 0, 1, \ldots \quad (1.2)$$

(A.2): for all $t = 0, 1, \ldots$, the covariance matrix Σ^v_{t+1} is positive definite,

(A.3): the initial condition ξ has distribution F with finite first and second moments μ and Δ (resp.) and is independent of the process $\{(W^\circ_{t+1}, V^\circ_{t+1})\}_0^\infty$ and

(A.4): the covariance matrix Δ is positive definite.

Note that no *a priori* assumptions, save those on the first two moments, is enforced on F.

Define $\mathcal{Z}$ as the vector space of all bounded Borel mappings from $I\!R^n$ into $\mathcal{C}$, the complex numbers. The one-step prediction problem (hereafter referred to simply as the "prediction problem") associated with (1.1) is defined as the problem of computing, for each $t = 0, 1, \ldots$, the conditional distribution of the state X°_{t+1} given the observations $\{Y_s\}_0^t$ or, equivalently, the evaluating, for all $t = 0, 1, \ldots$ and all ϕ in $\mathcal{Z}$, the conditional expectation

$$E[\phi(X^\circ_{t+1}) | Y_s; s = 0, 1, \ldots, t]. \qquad (1.3)$$

In this thesis, we shall solve the prediction problem associated with (1.1). For each $t = 0, 1, \ldots$, once the conditional distribution of X°_{t+1} given $\{Y_s\}_0^t$ is available, it is then possible to construct $\hat{X}_{t+1} := E[X^\circ_{t+1} | Y_s;\ s = 0, 1, \ldots, t]$. In general, $\hat{X}_{t+1}$ is a *non*linear function of $\{Y_s\}_0^t$, in contrast to the 'Kalman', or LLSE, estimate of X°_{t+1} on the basis of $\{Y_s\}_0^t$, which is by definition linear, and which we denote by $\hat{X}^K_{t+1}$. We shall find representations for both $\{\hat{X}_t\}_1^\infty$ and $\{\hat{X}^K_t\}_1^\infty$ and then form the mean square error $\epsilon_t := E\|\hat{X}_t - \hat{X}^K_t\|^2]$ for $t = 1, 2, \ldots$. Simply stated, ϵ_t is a measure of the agreement between the MMSE and LLSE estimates of X°_t on the basis of $\{Y_s\}_0^{t-1}$, for $t = 1, 2, \ldots$. The final efforts of this thesis will be to analyze the asymptotic behavior of ϵ_t as t tends to infinity—the asymptotic mean-square agreement of the true conditional and wide-sense conditional expectation of the state given the observations. This analysis shall focus on the time-invariant version of (1.1), when $A_t = A$, $H_t = H$, and $\Gamma_{t+1} = \Gamma$ for all $t = 0, 1, \ldots$. Then we can parametrize the asymptotic behavior of $\{\epsilon_t\}_1^\infty$ by the system (A, H, Γ) and the initial distribution F. We are particularly interested in triples (A, H, Γ) and distributions F for which $\lim_t \epsilon_t = 0$, for then we have the important result that the LLSE estimates $\{\hat{X}^K_t\}_1^\infty$ are asymptotically as good as the MMSE estimates $\{\hat{X}_t\}_1^\infty$; the LLSE estimates are asymptotically optimal in the mean square sense. The practical value of this is clear–the LLSE estimates are usually less costly to generate than the nonlinear MMSE estimates.

The thesis is organized as follows. In the remaining section of Chapter I, we introduce notations to be used in what follows. In Chapter II, we review the discrete-time Girsanov mutually absolutely continuous change of measure, which shall enable us to solve the prediction problem for (1.1). We discuss some aspects of the infinite-horizon Girsanov transformation in the second section of Chapter II. Chapter III develops the discrete-time counterpart of [17] and [19]—the case where the plant and observation noises are uncorrelated and the observation nose has unit covariance. We call this the "uncorrelated" problem, and the calculations of Chapter III concerning the uncorrelated problem serve primarily as a point of departure for the solution to the more general "correlated" problem, which is found in Chapter IV. By the "correlated" problem, we refer to the case where the plant and observation noise jointly form a GWN sequence with *any* covariance such that $E[V^\circ_{t+1} V^{\circ\prime}_{t+1}]$ is positive-definite for all $t = 0, 1, \ldots$.

Once the solution to the prediction problem associated with (1.1) is known, we, in

Chapter V, turn to the task of finding representations for $\{\hat{X}_t\}_1^\infty$, $\{\hat{X}_t^K\}_1^\infty$ and $\{\epsilon_t\}_1^\infty$. Recall that for $t = 1, 2, \ldots$, $\hat{X}_t$ is the MMSE estimate (or conditional expectation of) X_t° on the basis of $(Y_0, Y_1, \ldots, Y_{t-1})$, while $\hat{X}_t^K$ is the LLSE estimate of X_t° on the basis of $(Y_0, Y_1, \ldots, Y_{t-1})$, and that $\epsilon_t := E\left[||\hat{X}_t - \hat{X}_t^K||^2\right]$.

Chapter VI is devoted to an analysis of the asymptotic behavior of $\{\epsilon_t\}_1^\infty$ for the time-invariant version of (1.1). In Section 2 of Chapter VI, we use a result of Caines and Mayne to find conditions on (A, H, Γ) such that for *any* initial distribution F, we have $\lim_t \epsilon_t = 0$ with bounds on the *rate* of convergence also being independent of the initial distribution F for non-Gaussian distributions F (of course if F is Gaussian, then $\epsilon_t = 0$ for all $t = 1, 2, \ldots$ and all systems (A, H, Γ), since the nonlinear and Kalman estimates coincide). We then further restrict ourselves to the scalar case in Section 3—here the plant and observation processes take values in $I\!R^n = I\!R^k = I\!R$. In the scalar case, we develop a complete characterization of the asymptotics of $\{\epsilon_t\}_1^\infty$ as parametrized by the time-invariant plant (a, h, Γ) and the initial distribution F. We shall find that if (a, h, Γ) satisfies a generalized version of the criterion presented in Section 2, then $\lim_t \epsilon_t = 0$ with the rate of decay also determined *only* by the dynamics (a, h, Γ). Conversely, we shall find that if (a, h, Γ) satisfies a certain instability criterion, then the asymptotic behavior of $\{\epsilon_t\}_1^\infty$ depends nontrivially upon F to the extent that for some distributions F, $\lim_t \epsilon_t = 0$, while for other distributions F, $\lim_t \epsilon_t > 0$. The significance of these results is the subject of Section 4.

I.2. Background of the Problem

Filtering theory is an extremely well-developed field. The Kalman filtering equations were first published in 1961 [11], and in the literature of the past three decades, a vast amount of theory has been developed. The reader may wish to consult [10] for a recent bibliography of filtering theory.

The main contributions of this thesis are twofold. Firstly, we extend the work of Makowski [18]–[19] to cover *correlated* plant and observation noises with a non-Gaussian initial condition. Secondly, we study the *asymptotic* behavior of $\{\epsilon_t\}_1^\infty$. The filtering or prediction problem for linear systems with a non-Gaussian initial condition and uncorrelated plant and observation noises has been solved in [1], [2], [17]–[19], [21], [22] and [24]. In [19], a linear system with Gaussian initial condition and observation noise, but general non-Gaussian plant noise was considered. Of course, this problem overlaps the one considered in this thesis, since in [19], the non-Gaussian plant noise may be taken to be the effect of the non-Gaussian initial condition. In [17] and [18], Makowski studied a continuous-time linear system with non-Gaussian initial conditions and uncorrelated Gaussian plant and observation noise. The discrete-time counterpart of these two papers, also analyzed in [14] and [21], is developed in Chapter III of this thesis. Beneš and Karatzas, in [2], analyzed, with a control-theoretic orientation, a continuous-time linear system with non-Gaussian initial condition and uncorrelated Gaussian plant and observation noises. In [24], a solution is presented for a generalized filtering problem in which a continuous-parameter state process takes values in some Polish space and is observed through discrete-time observations in $\mathbb{R}^m$. Finally, in [1], [21] and [22], a specific class of non-Gaussian initial distributions is considered, namely, distributions admitting a density with respect to Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by a convex combination of non-degenerate Gaussian densities.

I.3. Notation

We now define several notational conventions which will simplify future presentations. We follow the notation of [18].

Let $\Phi(\cdot,\cdot)$ be the state transition matrix associated with $\{A_t\}_0^\infty$:

$$\begin{aligned}\Phi(t,t) &= I_n \\ \Phi(s+1,t) &= A_s\Phi(s,t).\end{aligned} \qquad s=t,t+1,\ldots,\ t=0,1,\ldots \tag{3.1}$$

Similarly, let $\Psi(\cdot,\cdot)$ be the state transition matrix described by

$$\begin{aligned}\Psi(t,t) &= I_n \\ \Psi(s+1,t) &= [A_s - \Sigma_{s+1}^{wv}(\Sigma_{s+1}^{v})^{-1}H_s]\Psi(s,t).\end{aligned} \qquad s=t,t+1,\ldots,\ t=0,1,\ldots \tag{3.2}$$

For any positive integers n and m, let $\mathcal{M}_{n\times m}$ be the space of $n\times m$ real matrices and let $\mathcal{Q}_n$ be the cone of $n\times n$ symmetric positive-definite matrices.

Take Λ positive definite in $\mathcal{Q}_n$, and let $\|\cdot\|_\Lambda$ be the norm on $I\!R^n$ defined by

$$\|x\|_\Lambda := \sqrt{x'\Lambda x}. \qquad x\in I\!R^n \tag{3.3}$$

For convenience, define $\|\cdot\| := \|\cdot\|_{I_n}$.

For $n=1,2,\ldots$, let λ_n represent n-dimensional Lebesgue measure on the measurable space $(I\!R^n, \mathcal{B}(I\!R^n))$.

Finally, let $(\Omega', \mathcal{F}', P')$ be a probability triple such that for every Σ in $\mathcal{Q}_{2n}$, there are $I\!R^n$-valued RV's X_Σ, B_Σ, and ζ_Σ where (X_Σ, B_Σ) is a zero-mean Gaussian RV with covariance Σ, and where ζ_Σ has distribution F and is independent of (X_Σ, B_Σ). Let E' be the expectation operator associated with P'. Then for every ϕ in $\mathcal{Z}$, let the mapping $T\phi : I\!R^n \times I\!R^n \times \mathcal{Q}_{2n} \to \mathcal{C}$ be defined by

$$T\phi[x,b;\Sigma] := E'[\phi(x+X_\Sigma)\exp[b'B_\Sigma]] \tag{3.4}$$

and the mapping $U\phi : I\!R^n \times I\!R^n \times \mathcal{Q}_n \times \mathcal{M}_{n\times n} \times \mathcal{Q}_{2n} \to \mathcal{C}$ be defined by

$$U\phi[x,b,\Lambda,\Psi;\Sigma] := E'[T\phi[x+\Psi\zeta_\Sigma, \zeta_\Sigma; \Sigma]\exp[b'\zeta_\Sigma - \frac{1}{2}\zeta_\Sigma'\Lambda\zeta_\Sigma]]. \tag{3.5}$$

CHAPTER II: THE GIRSANOV TRANSFORMATION

II.1. The Finite-Horizon Girsanov Transformation

We here develop the discrete-time, finite-horizon Girsanov mutually absolutely continuous measure transformation, which plays a central role in Chapters III and IV of this thesis (see [8]). The arguments follow [5]. The reader is also referred to [6, Chaps. 2 & 3] for a discussion of predictable and adapted discrete-time processes and discrete-time martingales.

Consider a probability triple $(\Omega, \mathcal{F}, P)$ (not necessarily the same as the one given in Section I.1) and a filtration $\{\mathcal{F}_t\}_0^\infty$ of $\mathcal{F}$. Let $\{V_t\}_1^\infty$ be an $I\!R^n$-valued $(\mathcal{F}_t)$-zero-mean GWN process with covariance structure $\{\Lambda_t\}_1^\infty$ given by

$$\Lambda_t := \mathrm{Cov}(V_t) = E[V_t V_t'], \qquad t = 1, 2, \ldots \quad (1.1)$$

and let $\{\chi_t\}_1^\infty$ be an $I\!R^n$-valued $(\mathcal{F}_t)$-predictable process. Define a third $I\!R^n$-valued $(\mathcal{F}_t)$-adapted process $\{\bar{V}_t\}_1^\infty$ by

$$\bar{V}_t := V_t - \Lambda_t \chi_t. \qquad t = 1, 2, \ldots \quad (1.2)$$

Fix $T = 0, 1, \ldots$. Then the Girsanov transformation provides a measure $\bar{P}$ on $(\Omega, \mathcal{F})$ with the following properties:

(B.1): the measure $\bar{P}$ is a probability measure on $(\Omega, \mathcal{F})$ which is mutually absolutely continuous with P and which agrees with P on $\mathcal{F}_0$ and

(B.2): under $\bar{P}$, the process $\{\bar{V}_t\}_1^T$ is an $(\mathcal{F}_t)$ zero-mean GWN process with covariance structure $\{\Lambda_t\}_1^T$—the statistics of $\{\bar{V}_t\}_1^T$ under $\bar{P}$ are the same as the statistics of $\{V_t\}_1^T$ under P.

Note that in **(B.2)**, we make a statement only about the finite-horizon process $\{\bar{V}_t\}_1^T$, and not the infinite-horizon process $\{\bar{V}_t\}_1^\infty$. This is a *finite-horizon* measure transformation; under an *infinite-horizon* measure transformation, the entire process $\{\bar{V}_t\}_1^\infty$ would be a $\bar{P}$-GWN process. We note also that two mutually absolutely continuous measures on the same measurable space are said to be *equivalent.*

Rather than directly defining the measure $\bar{P}$, we first consider the $(\mathcal{F}_t)$-adapted process $\{L_t\}_0^\infty$ given by

$$L_t := \prod_{s=1}^{t} \exp[\chi_s' V_s - \frac{1}{2}\chi_s' \Lambda_s \chi_s] \qquad t = 1, 2, \ldots \quad (1.3)$$

$$L_0 := 1;$$

note that (1.3) may be rewritten as

$$L_t := \prod_{s=1}^{t} \exp[\chi_s' \bar{V}_s + \frac{1}{2}\chi_s' \Lambda_s \chi_s]. \qquad t = 1, 2, \ldots \quad (1.4)$$

Define the measure $\bar{P}$ by the Radon-Nikodym derivative

$$\frac{d\bar{P}}{dP} = L_T. \quad (1.5)$$

For convenience, define the mapping $J_t : I\!R^n \times I\!R^n \to I\!R$ for each $t = 1, 2, \ldots$ by

$$J_t[v, x] := \exp[x'v - \frac{1}{2}x'\Lambda_t x], \qquad t = 1, 2, \ldots \quad (1.6)$$

so that for $t = 1, 2, \ldots,$

$$L_t = \prod_{s=1}^{t} J_t[V_t, \chi_t]. \quad (1.7)$$

We shall repeatedly use the following standard result ([13, Prop. 6.1.16]).

Lemma 1.1. *Suppose that X and Y are $I\!R^n$-valued RV's and $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$. Suppose that Y is $\mathcal{G}$-measurable and X is independent of $\mathcal{G}$. For any bounded Borel mapping $\varphi : I\!R^n \times I\!R^n \to \mathcal{C}$, then*

$$E[\varphi(X, Y)|\mathcal{G}] = E[\varphi(X, y)]\Big|_{y=Y} \quad P\text{-a.s.} \quad (1.8)$$

The following results are of paramount importance in what follows.

Lemma 1.2. *For all $t = 0, 1, \ldots$ and x in $I\!R^n$,*

$$E\big[J_t[V_t, x]\big] = 1 \quad (1.9)$$

so that

$$E\left[J_t[V_t,\chi_t]\middle|\mathcal{F}_{t-1}\right] = 1. \tag{1.10}$$

Proof. Equation (1.9) follows by direct evaluation of the expectation. For each $n = 0, 1, \ldots$, Lemma 1.1 implies that

$$E\left[J_t[V_t,\chi_t] \wedge n \mid \mathcal{F}_{t-1}\right] = E\left[J_t[V_t,x] \wedge n \mid \mathcal{F}_{t-1}\right]\Big|_{x=\chi_t}, \tag{1.11}$$

and for all x in $I\!R^n$, the Monotone Convergence Theorem and (1.9) yield that

$$E\left[J_t[V_t,\chi_t] \wedge n\middle|\mathcal{F}_{t-1}\right] \nearrow 1. \tag{1.12}$$

Taking the expectation of both sides of (1.11) and using (1.12), we easily verify that $J_t[V_t,\chi_t]$ is integrable. Relation (1.10) is verified by passing to the limit in (1.11) and using (1.12). ○

To proceed with the verification of **(B.1)**, we first show

Proposition 1.1. *The process* $\{L_t\}_0^\infty$ *is an* $(\mathcal{F}_t, P)$*-martingale.*

Proof. By inspecting (1.3), we see that $\{L_t\}_0^\infty$ is $(\mathcal{F}_t)$-adapted. Fix $t = 1, 2, \ldots$ and assume that L_{t-1} is integrable. If we can show that for any A in $\mathcal{F}_{t-1}$, the relation

$$E[1_A L_{t-1}] = E[1_A L_t] \tag{1.13}$$

holds, then the conclusion readily follows. Indeed, by setting $A = \Omega$, we may verify that L_t is integrable, so by induction on t and the obvious integrability of L_0, L_t will be integrable for each $t = 0, 1, \ldots$. If (1.13) is true for all A in $\mathcal{F}_{t-1}$, then also

$$E[L_t|\mathcal{F}_{t-1}] = L_{t-1} \quad \text{P-a.s.}, \tag{1.14}$$

so $\{L_t\}_0^\infty$ will in fact be an $(\mathcal{F}_t)$-martingale.

The proof of (1.13) is straightforward. Using Lemma 1.2 and the fact that L_{t-1} is $\mathcal{F}_{t-1}$-measurable, we have that

$$\begin{aligned} E\left[1_A J_t[V_t,\chi_t](L_{t-1} \wedge n)\right] &= E\left[1_A E\left[J_t[V_t,\chi_t] \mid \mathcal{F}_{t-1}\right](L_{t-1} \wedge n)\right] \\ &= E[1_A(L_{t-1} \wedge n)] \end{aligned} \tag{1.15}$$

for each $n = 0, 1, \ldots$. Passing to the limit, we verify (1.13) by the Monotone Convergence Theorem. $\bigcirc$

Now for each $t = 0, 1, \ldots, T$, let P_t (resp. $\bar{P}_t$) be the restriction of P (resp $\bar{P}$) to $\mathcal{F}_t$; clearly $\bar{P}_t \ll P_t$ for $t = 0, 1, \ldots, T$. The following result provides an alternate characterization of the process $\{L_t\}_0^T$.

Proposition 1.2. *For $t = 0, 1, \ldots, T$,*

$$L_t = \frac{d\bar{P}_t}{dP_t}. \tag{1.16}$$

Proof. We must show that for $t = 0, 1, \ldots, T$, the relation

$$\bar{P}_t(A) = \int_A L_t dP_t \tag{1.17}$$

holds for any set A in $\mathcal{F}_t$. But for A in $\mathcal{F}_t$,

$$\bar{P}_t(A) = \bar{P}(A) \tag{1.18}$$

$$= \int_A L_T dP \tag{1.19}$$

$$= \int_A E[L_T|\mathcal{F}_t] dP \tag{1.20}$$

$$= \int_A L_t dP \tag{1.21}$$

$$= \int_A L_t dP_t, \tag{1.22}$$

so (1.17) is true. $\bigcirc$

The verification that $\bar{P}$ has property **(B.1)** is now trivial.

Proposition 1.3. *The measure $\bar{P}$ is a probability measure on $(\Omega, \mathcal{F})$ enjoying property* **(B.1)**.

Proof. Note by Proposition 1.2 that

$$\frac{d\bar{P}_0}{dP_0} = L_0 = 1, \tag{1.23}$$

so $\bar{P}$ and P agree on $\mathcal{F}_0$; in particular, $\bar{P}(\Omega) = P(\Omega) = 1$, so $\bar{P}$ is a probability measure. Since $P\{L_T = 0\} = 0$, P and $\bar{P}$ are mutually absolutely continuous (see [15, Lemma 6.8]).

$\bigcirc$

From the mutual absolute continuity of $\bar{P}$ and P and Proposition 1.2, we note that

$$L_t^{-1} = \frac{dP_t}{d\bar{P}_t}. \qquad t = 0, 1, \ldots, T \quad (1.24)$$

Let $\bar{E}$ be the expectation operator associated with $\bar{P}$. The following result indicates the structure of $\{L_t^{-1}\}_0^T$ under $\bar{P}$.

Proposition 1.4. *The process* $\{L_t^{-1}\}_0^T$ *is an* $(\mathcal{F}_t, \bar{P})$*- martingale.*

Proof. By inspection of definition (1.3), we see that $\{L_t^{-1}\}_0^T$ is both well-defined and $(\mathcal{F}_t)$-adapted, and from (1.24), L_t^{-1}, is automatically $\bar{P}$-integrable. To prove the proposition, it is then sufficient to show that for $t = 0, 1, \ldots, T$ and A in $\mathcal{F}_t$,

$$\bar{E}[1_A L_t^{-1}] = \bar{E}[1_A \frac{dP}{d\bar{P}}]; \quad (1.25)$$

then

$$L_t^{-1} = \bar{E}[\frac{dP}{d\bar{P}} | \mathcal{F}_t]. \qquad t = 0, 1, \ldots, T \quad (1.26)$$

But (1.25) is trivially true, since

$$\bar{E}[1_A L_t^{-1}] = \bar{E}\big[1_A \frac{dP_t}{d\bar{P}_t}\big] = P(A), \quad (1.27)$$

and

$$\bar{E}\big[1_A \frac{dP}{d\bar{P}}\big] = P(A). \quad (1.28)$$

Hence the proposition holds. ○

The following result relates the conditional expectation operators under P and $\bar{P}$.

Proposition 1.5. *For* $t = 1, 2, \ldots, T$ *and any bounded* $\mathcal{C}$*-valued* $\mathcal{F}_t$*-measurable RV* X,

$$\bar{E}[X | \mathcal{F}_{t-1}] = \frac{E[X L_t | \mathcal{F}_{t-1}]}{L_{t-1}} \quad (1.29)$$

$$= E\big[X J_t[V_t, \chi_t] \big| \mathcal{F}_{t-1}\big]. \quad (1.30)$$

Proof. Since X is bounded, $X L_t$ is clearly P-integrable. By Lemma 1.2, $J_t[V_t, \chi_t]$ is P-integrable, so $X J_t[V_t, \chi_t]$ is P-integrable. To prove (1.29), it is sufficient to show that for any A in $\mathcal{F}_{t-1}$,

$$E\big[1_A L_{t-1} \bar{E}[X | \mathcal{F}_{t-1}]\big] = E\big[1_A E[X L_t | \mathcal{F}_{t-1}]\big]. \quad (1.31)$$

By arguments which should now be clear, we have

$$E\big[1_A E[XL_t|\mathcal{F}_{t-1}]\big] = E[1_A X L_t] \tag{1.32}$$

$$= E\big[1_A X \frac{d\bar{P}_t}{dP_t}\big] \tag{1.33}$$

$$= \bar{E}[1_A X], \tag{1.34}$$

whereas

$$E\big[1_A L_{t-1}\bar{E}[X|\mathcal{F}_{t-1}]\big] = E\big[\bar{E}[1_A X|\mathcal{F}_{t-1}]\frac{d\bar{P}_{t-1}}{dP_{t-1}}\big] \tag{1.35}$$

$$= \bar{E}\big[\bar{E}[X1_A|\mathcal{F}_{t-1}]\big] \tag{1.36}$$

$$= \bar{E}[1_A X]; \tag{1.37}$$

thus (1.31) holds, and the proof is complete. ○

We can now verify

Proposition 1.6. *The probability measure $\bar{P}$ enjoys property* **(B.2)**.

Proof. It is sufficient to verify that for any $t = 1, 2, \ldots$ and each θ in $\mathbb{R}^n$,

$$\bar{E}\big[\exp[i\theta'\bar{V}_t]\big|\mathcal{F}_{t-1}\big] = \exp[-\frac{1}{2}\theta'\Lambda_t\theta]. \tag{1.38}$$

Now by Proposition 1.5, we see that

$$\bar{E}\big[\exp[i\theta'\bar{V}_t]\big|\mathcal{F}_{t-1}\big] = E\big[\exp[i\theta'\bar{V}_t]J_t[V_t,\chi_t]\big|\mathcal{F}_{t-1}\big], \tag{1.39}$$

where some care must be taken to ensure that the appropriate integrability conditions are satisfied. Lemma 1.2 ensures that $J_t[V_t, \chi_t]$ is P-integrable, and since the complex exponential function has magnitude 1, we see that $\exp[i\theta'\bar{V}_t]J_t[V_t,\chi_t]$ is also P-integrable. For $n = 1, 2, \ldots$, define $T_n : \mathcal{C} \to \mathcal{C}$ by

$$T_n(x) := x1_{\{|x|\le n\}}. \qquad x \in \mathcal{C} \tag{1.40}$$

Now from the Dominated Convergence Theorem for conditional expectations, we get

$$E\big[\exp[i\theta'\bar{V}_t]J_t[V_t,\chi_t]\big|\mathcal{F}_{t-1}\big] = \lim_n E\Big[T_n\big(\exp[i\theta'\bar{V}_t]J_t[V_t,\chi_t]\big)\Big|\mathcal{F}_{t-1}\Big], \qquad n = 1, 2, \ldots \tag{1.41}$$

and Lemma 1.1 gives

$$E\Big[T_n\big(\exp[i\theta'\bar{V}_t]J_t[V_t,\chi_t]\big)\Big|\mathcal{F}_{t-1}\Big] = E\Big[T_n\big(\exp[i\theta'V_t - i\theta'\Lambda_t x]J_t[V_t,x]\big)\Big]\Big|_{x=\chi_t}.$$

$$n = 1,2,\ldots \quad (1.42)$$

For each $n = 1,2,\ldots$ and each x in $I\!R^n$,

$$\Big|T_n\big(\exp[i\theta'V_t - i\theta'\Lambda_t x]J_t[V_t,x]\big)\Big| \leq J_t[V_t,x], \tag{1.43}$$

so

$$\lim_n T_n\big(\exp[i\theta'V_t - i\theta'\Lambda_t x]J_t[V_t,x]\big) = \exp[i\theta'V_t - i\theta'\Lambda_t x]J_t[V_t,x]. \tag{1.44}$$

Applying Lemma 1.2 and the Dominated Convergence Theorem, we conclude that

$$\lim_n E\Big[T_n\big(\exp[i\theta'V_t - i\theta'\Lambda_t x]J_t[V_t,x]\big)\Big]$$

$$= E\Big[\exp[i\theta'V_t - i\theta'\Lambda_t x]J_t[V_t,x]\Big] \tag{1.45}$$

$$= E\Big[\exp[(i\theta + x)'V_t - \frac{1}{2}x'\Lambda_t x - i\theta'\Lambda_t x]\Big] \tag{1.46}$$

$$= \exp[-\frac{1}{2}x'\Lambda_t x], \tag{1.47}$$

where (1.47) holds by direct evaluation of (1.46). Upon combining (1.39), (1.41), (1.42) and (1.47), we verify (1.38). ○

The following result, which we shall use in Chapters III and IV, is proved in exactly the same manner as Proposition 1.5 (see [16, Sec. 28.4]).

Proposition 1.7. *For $t = 1,2,\ldots,T$ and any bounded C-valued $\mathcal{F}_t$-measurable RV X,*

$$E[X|\mathcal{F}_{t-1}] = \frac{\bar{E}[XL_t^{-1}|\mathcal{F}_{t-1}]}{\bar{E}[L_t^{-1}|\mathcal{F}_{t-1}]}. \tag{1.48}$$

Proof. The proof of (1.48) is the same as Proposition 1.5 if we reverse the roles of P and $\bar{P}$ and note that $L_{t-1}^{-1} = \bar{E}[L_t^{-1}|\mathcal{F}_{t-1}]$, which results from Proposition 1.4. ○

The Girsanov transformation presented in [5] is slightly less general than the one presented here. In [5], it is assumed that $\{V_t\}_1^T$ is a *standard* GWN sequence (i.e., $\Lambda_t = I_n$ for $t = 1,2,\ldots$). The case of a general, non-standard, GWN process could have been considered within the framework of [5] by normalizing $\{\bar{V}_t\}_1^\infty$ to have unit variance, but the approach presented here is more direct.

II.2. The Infinite-Horizon Girsanov Transformation

In this section, we attempt to extend the results of the previous section to the infinite horizon. The notation is the one introduced in the previous section. We seek a probability measure $\bar{P}$ on $(\Omega, \mathcal{F})$ which is mutually absolutely continuous with P and which enjoys property that

(C.1): the probability measure $\bar{P}$ agrees with P on $\mathcal{F}_0$, and $\{\bar{V}_t\}_1^\infty$ is an $(\mathcal{F}_t, \bar{P})$ zero-mean GWN process with covariance structure $\{\Lambda_t\}_1^\infty$.

Our starting point shall be definition (1.4) and Proposition 1.2. For $t = 0, 1, \ldots$, we define a probability measure $\hat{P}_t$ on $(\Omega, \mathcal{F})$ through the Radon-Nikodym derivative

$$\frac{d\hat{P}_t}{dP} = L_t. \tag{2.1}$$

By Proposition 1.2, we know that $\hat{P}_{t+1}$ and $\hat{P}_t$ agree on $\mathcal{F}_t$ for each $t = 0, 1, \ldots$. The problem is then to determine if $\{\hat{P}_t\}_0^\infty$ in some sense converges to the sought-after probability measure $\bar{P}$ satisfying **(C.1)** and which may be supposed to also satisfy **(C.2)**, where

(C.2): for $t = 0, 1, \ldots$ $\hat{P}_t$ and $\bar{P}$ agree on $\mathcal{F}_t$.

Note that by the arguments of the previous section, property **(C.1)** in fact follows from property **(C.2)**. Throughout this section, $\bar{E}$ shall denote the expectation operator associated with the sought-after probability measure $\bar{P}$, and $\hat{E}_t$ shall be the expectation operator associated with $\hat{P}_t$ for $t = 0, 1, \ldots$.

We shall see that the existence of a probability measure $\bar{P}$ on $(\Omega, \mathcal{F})$ equivalent to P and satisfying properties **(C.1)** and **(C.2)** is closely related to the uniform P-integrability of $\{L_t\}_0^\infty$. We shall investigate the ramifications of the uniform P-integrability of $\{L_t\}_0^\infty$ and provide a counter-example to show that $\{L_t\}_0^\infty$ need not be uniformly P-integrable. We shall then provide a sufficient condition for uniform P-integrability which will have a pleasing interpretation in later parts of this thesis. Finally, we shall show that even when $\{L_t\}_0^\infty$ is not uniformly P-integrable, if the filtration $\{\mathcal{F}_t\}_0^\infty$ satisfies a separability condition, then the Daniell-Kolmogorov theorem enables us to construct a probability measure $\bar{P}$ on $(\Omega, \mathcal{F}_\infty)$ satisfying conditions **(C.1)** and **(C.2)**, but which need not satisfy any absolute continuity conditions with respect to P.

As a first step, from Section 2, we immediately see that $\{L_t\}_0^\infty$ is a nonnegative $(\mathcal{F}_t, P)$ martingale. Thus by well-known results ([6, Cor. 3.17]) there is a nonnegative RV L_∞ such that

$$\lim_t L_t = L_\infty \quad \text{P-a.s.} \tag{2.2}$$

and $E[L_\infty] \leq 1$ (by Fatou's Lemma).

The following classical result indicates the significance of uniform P-integrability of $\{L_t\}_0^\infty$.

Theorem 2.1. *Uniform P-integrability of $\{L_t\}_0^\infty$ is a necessary and sufficient condition for the existence of a probability measure $\bar{P}$ on $(\Omega, \mathcal{F})$ satisfying* **(C.1)** *and* **(C.2)** *and with $\bar{P} \ll P$. Furthermore, if $\{L_t\}_0^\infty$ is uniformly P-integrable, uniform $\bar{P}$-integrability of $\{L_t^{-1}\}_0^\infty$ is both a necessary and sufficient condition for the equivalence of P and $\bar{P}$.*

Proof. The theorem results from [20, Prop III-1-1] and [20, Prop IV-2-3]. ○

We now consider conditions under which we may find a probability measure $\bar{P}$ enjoying properties **(C.1)** and **(C.2)** and which is equivalent to P.

Proposition 2.1. *Suppose that $\{L_t\}_0^\infty$ is uniformly P-integrable, and let $\bar{P}$ be defined by $d\bar{P}/dP = L_\infty$. A necessary and sufficient condition for $\bar{P}$ to be equivalent to P is that for $\epsilon > 0$, there exist an $\eta > 0$ such that*

$$\sup_t P\{L_t < \eta\} < \epsilon. \tag{2.3}$$

Proof. Note that for $t = 0, 1, \ldots$ and $c > 0$,

$$\begin{aligned} \bar{E}[L_t^{-1} 1_{\{L_t^{-1} > c\}}] &= E[L_\infty L_t^{-1} 1_{\{L_t^{-1} > c\}}] \\ &= E[L_t L_t^{-1} 1_{\{L_t^{-1} > c\}}] \\ &= P\{L_t^{-1} > c\} \\ &= P\{L_t < 1/c\}. \end{aligned} \tag{2.4}$$

Consequently, condition (2.3) holds if and only if $\{L_t^{-1}\}_0^\infty$ is uniformly $\bar{P}$-integrable, so the Proposition is verified by invoking Theorem 2.1. ○

For convenience, define the processes $\{N_t\}_0^\infty$, $\{\bar{N}_t\}_0^\infty$, and $\{\langle N\rangle_t\}_0^\infty$ by

$$\begin{aligned} N_t &:= \sum_{s=1}^{t} \chi_s' V_s \\ N_0 &:= 0 \end{aligned} \qquad t = 0, 1, \ldots \quad (2.5)$$

$$\begin{aligned} \bar{N}_t &:= \sum_{s=1}^{t} \chi_s' \bar{V}_s \\ \bar{N}_0 &:= 0 \end{aligned} \qquad t = 0, 1, \ldots \quad (2.6)$$

$$\begin{aligned} \langle N\rangle_t &:= \sum_{s=1}^{t} \chi_s' \Lambda_s \chi_s \\ \langle N\rangle_0 &:= 0. \end{aligned} \qquad t = 0, 1, \ldots \quad (2.7)$$

With this notation,

$$\begin{aligned} L_t &= \exp[N_t - \frac{1}{2}\langle N\rangle_t] \\ &= \exp[\bar{N}_t + \frac{1}{2}\langle N\rangle_t]. \end{aligned} \qquad t = 0, 1, \ldots \quad (2.8)$$

We observe in passing that $\{\langle N\rangle_t\}_0^\infty$ is called the *quadratic variation* process associated with the square-integrable martingale $\{N_t\}_0^\infty$ and is the unique $(\mathcal{F}_t)$-adapted process such that $\{N_t^2 - \langle N\rangle_t\}_0^\infty$ is a martingale ([20, Chapter 8]).

Anticipating the Girsanov transformation used in Chapters III and IV, we now assume condition **(D)**, where

(D): for $t = 1, 2, \ldots$, the RV χ_t is $\mathcal{F}_0$-measurable.

If $\bar{P}$ is a probability measure on $(\Omega, \mathcal{F})$ enjoying properties **(C.1)** and **(D)**, it is not then difficult to see that the processes $\{N_t\}_0^\infty$ and $\{\langle N\rangle_t\}_0^\infty$ have the same joint statistics under P as the processes $\{\bar{N}_t\}_0^\infty$ and $\{\langle N\rangle_t\}_0^\infty$ have under $\bar{P}$.

These observations lead to the following result.

Theorem 2.2. *Under assumption* **(D)**, *uniform P-integrability of $\{L_t\}_0^\infty$ is a necessary and sufficient condition for the existence of a probability measure $\bar{P}$ equivalent to P and with properties* **(C.1)** *and* **(C.2)**.

Proof. Note, from (2.6)-(2.7), that, by symmetry, the $\bar{P}$-statistics of $\{\bar{N}_t, \langle N \rangle_t;\ t = 0, 1, \ldots\}$ are the same as those of $\{-\bar{N}_t, \langle N \rangle_t;\ t = 0, 1, \ldots\}$, so the P-statistics of $\{N_t, \langle N \rangle_t; t = 0, 1, \ldots\}$ are the same as the $\bar{P}$-statistics of $\{-\bar{N}_t, \langle N \rangle_t;\ t = 0, 1, \ldots\}$. Uniform integrability being a statistical property, uniform P-integrability of $\{L_t\}_0^\infty$ is equivalent to uniform $\bar{P}$-integrability of $\{L_t^{-1}\}_0^\infty$ by (2.8). The proof is completed with the aid of Theorem 2.1. ○

Having seen the significance of uniform P-integrability of $\{L_t\}_0^\infty$, we now provide a counter-example to show that in general, $\{L_t\}_0^\infty$ need *not* be uniformly P-integrable.

A Counter-example. Let $n = 1$, and let $\{v_t\}_1^\infty$ be a scalar zero-mean standard $(\mathcal{F}_t, P)$-GWN process (i.e., $\Lambda_t = 1$ for $t = 0, 1, \ldots$). Take a in $I\!R$ with $|a| > 1$, and let β be any square-integrable $\mathcal{F}_0$- measurable random variable with $P\{\beta = 0\} < 1$. Set

$$\chi_t = a^t \beta. \qquad t = 0, 1, \ldots \quad (2.9)$$

We shall show that the martingale

$$L_t := \exp\left[\beta \sum_{s=0}^{t} a_s v_s - \frac{1}{2}\beta^2 \sum_{s=0}^{t} a^{2s}\right] \qquad t = 1, 2, \ldots \quad (2.10)$$
$$L_0 := 1$$

is *not* uniformly P-integrable.

For convenience, define the processes $\{n_t\}_0^\infty$ and $\{[n]_t\}_0^\infty$ by

$$n_t := \sum_{s=0}^{t} a^s v_s \qquad t = 0, 1, \ldots \quad (2.11)$$
$$n_0 := 0$$

$$[n]_t := a^2 \frac{a^{2t} - 1}{a^2 - 1} \qquad t = 0, 1, \ldots \quad (2.12)$$
$$[n]_0 := 0$$

and the collection $\{H_t\}_0^\infty$ of random Borel mappings from $I\!R$ to $I\!R$ by

$$H_t(x) := \exp[x n_t - \frac{1}{2} x^2 [n]_t] \qquad t = 0, 1, \ldots \quad (2.13)$$
$$H_0(x) := 1.$$

Then we may write

$$L_t = H_t(\beta), \qquad t = 0, 1, \ldots \quad (2.14)$$

so that the representation

$$L_\infty = \lim_t H_t(\beta). \quad (2.15)$$

holds. We shall show that

$$L_\infty = 1_{\{\beta=0\}}; \quad (2.16)$$

then $E[L_\infty] = P\{\beta = 0\} < 1$, so $\{L_t\}_0^\infty$ *cannot* be uniformly P-integrable.

Clearly, $\lim_t H_t(0) = 1$. Thus it is sufficient to show that for $x \neq 0$,

$$\lim_t H_t(x) = 0 \quad \text{P-a.s.}; \quad (2.17)$$

by standard conditioning arguments, it then follows that

$$\begin{aligned} E[L_\infty 1_{\{\beta\neq 0\}}] &= E[\lim_t H_t(\beta) 1_{\{\beta \neq 0\}}] \\ &= 0, \end{aligned} \quad (2.18)$$

and $L_\infty 1_{\{\beta\neq 0\}} = 0$ P-a.s., so (2.16) is immediate. We shall use the first Borel-Cantelli lemma to verify (2.17).

Fix $x \neq 0$ and $t = 1, 2, \ldots$. Noting that n_t is a normal RV with zero mean and covariance $[n]_t$, we have

$$P\{xn_t - \frac{1}{2}x^2[n]_t \leq -x^2[n]_t\} = P\left\{\frac{xn_t}{\sqrt{x^2[n]_t}} \leq \frac{-\frac{1}{2}x^2[n]_t}{\sqrt{x^2[n]_t}}\right\} \quad (2.19)$$

$$= P\left\{\frac{\text{sgn}(x)n_t}{\sqrt{[n]_t}} \leq -\frac{1}{2}|x|\sqrt{n_t}\right\} \quad (2.20)$$

$$= P\left\{\text{sgn}(x)\frac{n_t}{\sqrt{[n]_t}} \geq \frac{1}{2}|x|\sqrt{[n]_t}\right\}, \quad (2.21)$$

where (2.21) follows from (2.20) by the symmetry of an N(0, 1)- distributed RV.

Now from a well-known result in the theory of Gaussian RV's (see [9], Appendix 2)

$$\frac{1}{\sqrt{2\pi}}\int_b^\infty \exp(-t^2/2)dt \leq \frac{1}{2}\exp(-\frac{1}{2}b^2) \quad (2.22)$$

for $b \geq 0$, so that

$$P\{xn_t - \frac{1}{2}x^2[n]_t \leq -x^2[n]_t\} \leq \frac{1}{2}\exp\left(\frac{-\frac{1}{4}x^2[n]_t}{2}\right) \\ = \frac{1}{2}\exp\left(-\frac{1}{8}x^2[n]_t\right). \tag{2.23}$$

Now $\lim_p p\exp(-p) = 0$, so there clearly exists a $B > 0$ such that for $p > B$, $p\exp(-p) < 1$, or, equivalently, $\exp(-p) < 1/p$. Since $[n]_t \nearrow \infty$, there exists an integer T such that for $t = T, T+1, \ldots$, we have $x^2[n]_t/8 > B$. Hence

$$\frac{1}{2}\exp\left(-\frac{1}{8}x^2[n]_t\right) \leq 4\frac{1}{x^2[n]_t}. \qquad t = T, T+1, \ldots \tag{2.24}$$

But by inspection of (2.12),

$$\sum_{t=1}^{\infty} \frac{1}{[n]_t} < \infty, \tag{2.25}$$

so that by Borel-Cantelli, we obtain

$$P\{xn_t - \frac{1}{2}x^2[n]_t \leq -x^2[n]_t \quad \text{i.o.} \quad\} = 0. \tag{2.26}$$

Consequently,

$$\limsup_t \left(xn_t - \frac{1}{2}x^2[n]_t\right) \leq \limsup_t(-x^2[n]_t) = -\infty \quad \text{P-a.s.}, \tag{2.27}$$

which implies that

$$\lim_t H_t(x) = 0 \quad \text{P-a.s.} \tag{2.28}$$

The analysis of the counter-example is complete. ○

We can now appreciate the following criterion which ensures that the sequence $\{L_t\}_0^\infty$ is uniformly P-integrable.

Theorem 2.3. *Under assumption* **(D)**, *if*

$$E[\langle N \rangle_\infty] < \infty, \tag{2.29}$$

then $\{L_t\}_0^\infty$ *is uniformly* P*-integrable, so the probability measures* P *and* $\bar{P}$ *are equivalent.*

Proof. It is not difficult to see (by first conditioning upon $\mathcal{F}_0$) that for any $t = 0, 1, \ldots,$ $E[N_t^2] = E[\langle N \rangle_t] \leq E[\langle N \rangle_\infty]$. Thus, by the Vallée-Poisson uniform integrability condition ([6], Cor. 1.19), $\{|N_t|\}_0^\infty$ is uniformly P-integrable. Since $\{\langle N \rangle_t\}_0^\infty$ is increasing and nonnegative, (2.29) also implies that $\{\langle N \rangle_t\}_0^\infty$ is uniformly P-integrable. Since the sum of two uniformly P-integrable processes is itself uniformly P-integrable, $\{|N_t| + \frac{1}{2}\langle N \rangle_t\}_0^\infty$ is uniformly P-integrable. We then know, using [6, Theorem 1.18], that

$$\sup_t E[|N_t| + \frac{1}{2}\langle N \rangle_t] < \infty. \tag{2.30}$$

Take $\epsilon > 0$ and $c > 0$ such that

$$\ln c > \frac{\sup_t E[|N_t| + \frac{1}{2}\langle N \rangle_t]}{\epsilon}, \tag{2.31}$$

and observe that for all $t = 0, 1, \ldots,$

$$\begin{aligned} E[L_t 1_{\{L_t > c\}}] &= \hat{E}_t[1_{\{L_t > c\}}] \\ &= \hat{P}_t\{L_t > c\} \\ &= \hat{P}_t\{\bar{N}_t + \frac{1}{2}\langle N \rangle_t > \ln c\}. \end{aligned} \tag{2.32}$$

Since $\bar{N}_t$ and $\langle N \rangle_t$ have the same joint statistics under $\hat{P}_t$ as N_t and $\langle N \rangle_t$ have under P, we see that

$$\begin{aligned} \hat{P}_t\{\bar{N}_t + \frac{1}{2}\langle N \rangle_t > \ln c\} &= P\{N_t + \frac{1}{2}\langle N \rangle_t > \ln c\} \\ &\leq P\{|N_t| + \frac{1}{2}\langle N \rangle_t > \ln c\} \\ &\leq \frac{E[|N_t| + \langle N \rangle_t]}{\ln c} \end{aligned} \tag{2.33}$$

by Markov's inequality. From (2.31), we then get that

$$E[L_t 1_{\{L_t > c\}}] < \epsilon \tag{2.34}$$

for all $t = 0, 1, \ldots,$ so the process $\{L_t\}_0^\infty$ is uniformly P-integrable. ○

Finally, we show that if for $t = 0, 1, \ldots,$ the σ-field $\mathcal{F}_t$ is separable (i.e., generated by a countable number of RV's), then the Daniell-Kolmogorov theorem enables us to construct a probability measure $\bar{P}$ on $(\Omega, \mathcal{F}_\infty)$ which enjoys properties **(C.1)** and **(C.2)**,