On Robust Stability of Linear State Space Models

by

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Introduction. The structured singular value ($\mu$), introduced by Doyle [1] allows to analyze robust stability and performance of linear systems affected by parametric as well as dynamic uncertainty. While exact computation of $\mu$ can be prohibitively complex, an efficiently computable upper bound was obtained in [2], yielding a practical sufficient condition for robust stability and performance.

In this note, the results of [2] are used to study the case of state space models of the form

$$\dot{x} = (A_0 + \sum_{i=1}^{m} \delta_i A_i)x$$  \hspace{1cm} (1)

where the $A_i$'s are $n \times n$ real matrices and the $\delta_i$'s are uncertain real parameters. The case where the $A_i$’s have low rank is given special attention. When the $A_i$’s all have rank one, (1) is equivalent to the model used by Qiu and Davison [3], which itself generalizes that used by Yedavalli [4]. By means of two examples, we compare our bound to those proposed in [3] and [4].

Preliminaries. Throughout the note, given any square complex matrix $M$, we denote by $\sigma(M)$ its largest singular value and by $M^{H}$ its complex conjugate transpose. Given any Hermitian matrix $A$, we denote by $\lambda(A)$ its largest eigenvalue. Given any integer $k$, $I_k$ denotes the $k \times k$ identity matrix and $O_k$ the $k \times k$ zero matrix. Finally, $j$ will denote $\sqrt{-1}$.

Given a $p \times p$ complex matrix $M$ and positive integers $k_1, \ldots, k_m$, with $\sum_{q=1}^{m} k_q = p$, consider the family of block diagonal $p \times p$ matrices (In this note we consider only parametric perturbations)

$$\mathcal{X} = \{ \text{block diag} (\delta_1 I_{k_1}, \ldots, \delta_m I_{k_m}) : \delta_q \in \mathbb{R} \} .$$
Definition 1. [1] The structured singular value $\mu_\mathcal{X}(M)$ of a complex $p \times p$ matrix $M$ with respect to $\mathcal{X}$ is 0 if there is no $\Delta$ in $\mathcal{X}$ such that $\det(I - \Delta M) = 0$, and

$$\mu_\mathcal{X}(M) = \left( \min_{\Delta \in \mathcal{X}} \{\overline{\sigma}(\Delta) : \det(I - \Delta M) = 0\} \right)^{-1}$$

otherwise. □

Exact computation of the structured singular value is generally intractable. In [2], the following computable upper bound was obtained.

Fact 1. [2] For any matrix $M$ and $\mathcal{X}$,

$$\mu_\mathcal{X}(M) \leq \hat{\mu}_\mathcal{X}(M) := \inf_{D \in \mathcal{D}_\mathcal{X}} \nu_\mathcal{X}(DMD^{-1})$$

where

$$\mathcal{D}_\mathcal{X} = \left\{ \text{block diag}(D_1, \ldots, D_m) : 0 < D_q = D_q^H \in \mathbb{C}^{k_q \times k_q} \right\}$$

and where, for any matrix $A$ and $\mathcal{X}$, $\nu_\mathcal{X}(A)$ is defined by

$$\nu_\mathcal{X}(A) = \sqrt{\max \left\{ 0, \inf_{G \in \mathcal{G}_\mathcal{X}} \overline{\lambda}[A^H A + j(GA - A^H G)] \right\}}$$

with

$$\mathcal{G}_\mathcal{X} = \left\{ \text{block diag}(G_1, \ldots, G_m) : G_q = G_q^H \in \mathbb{C}^{k_q \times k_q} \right\}.$$ 

□

An efficient algorithm for computing $\hat{\mu}_\mathcal{X}(M)$ is described in [5,6].

Main result. Following [7], one can easily show that system (1) is asymptotically stable for all $|\delta_i| \leq \delta$ if, and only if,

$$\delta < \left( \sup_{\omega \geq 0} \mu_\mathcal{X}(H_1(j\omega)) \right)^{-1}$$

where $H_1(s)$ is the transfer matrix defined by

$$H_1(s) = \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} (sI - A_0)^{-1} \begin{bmatrix} A_1 | A_2 | \cdots | A_m \end{bmatrix}$$


and where

\[ \mathcal{X} = \{ \text{block diag } (\delta_1 I_n, \ldots, \delta_m I_n) : \delta_q \in \mathbb{R} \} . \]

However, whenever some \( A_i \)'s are not of full rank, one can obtain a necessary and sufficient condition for robust stability involving a transfer matrix of lower size than \( H_1 \). To see this, decompose each \( A_i \) as

\[ A_i = b_i c_i^T \]

where \( b_i, c_i \in \mathbb{R}^{n \times r_i} \), with \( r_i \) the rank of \( A_i \), and define

\[ B = [b_1 | \cdots | b_m] \]

and

\[ C = \begin{bmatrix} c_1^T \\ \vdots \\ c_m^T \end{bmatrix} . \]

The following is then easily proven using [7].

**Proposition 1.** The system in (1) is asymptotically stable if, and only if,

\[ \delta < \left( \sup_{\omega \geq 0} \mu_{\mathcal{X}}(H_2(j\omega)) \right)^{-1} \]

where \( H_2(s) \) is the transfer matrix defined by

\[ H_2(s) = C(sI - A)^{-1} B \]

and where

\[ \mathcal{X} = \{ \text{block diag } (\delta_1 I_{r_1}, \ldots, \delta_m I_{r_m}) : \delta_q \in \mathbb{R} \} . \]

\( \square \)

Substituting for \( \mu_{\mathcal{X}} \) its upper bound \( \hat{\mu}_{\mathcal{X}} \) gives the following sufficient condition

**Corollary 1.** The system in (1) is asymptotically stable if

\[ \delta < \left( \sup_{\omega \geq 0} \hat{\mu}_{\mathcal{X}}(H_2(j\omega)) \right)^{-1} \]

where \( H_2(s) \) and \( \mathcal{X} \) are defined as in Proposition 1. \( \square \)
Models of the type (1) for which the $A_i$'s have low rank are of practical importance. The case where all $A_i$'s have rank one corresponds to the model used by Qiu and Davison,

$$
\dot{x} = (A + B\Delta C)x
$$

(2)

where $\Delta$ is uncertain. Yedavalli considered the model

$$
\dot{x} = (A + \Delta)x
$$

with $\Delta$ uncertain, which is clearly a special case of (2).

**Numerical examples.** We conclude this note by comparing on two examples the results obtained using Corollary 1 above to those obtained using the techniques of [3] and [4]. The examples are borrowed from [3,4,8].

**Example 1.** The following matrices were considered in [3,4].

$$
A + B\Delta C = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 + \delta_1 & -2 + \delta_2 \\ 1 & 0 \end{bmatrix},
$$

$$
A + B\Delta C = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 + \delta_1 & -2 \\ 1 + \delta_2 & 0 \end{bmatrix},
$$

$$
A + B\Delta C = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 + \delta_2 \\ 1 + \delta_1 & 0 \end{bmatrix},
$$

$$
A + B\Delta C = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 + \delta_1 & -2 + \delta_3 \\ 1 + \delta_2 & 0 \end{bmatrix}.
$$

Bounds of $\delta$ given in [4] which guarantees robust stability were 1.0, 0.48, 0.5 and 0.317, respectively. Bounds given in [3] were 1.5201, 0.9150, 0.8108 and 0.6848, respectively. Our bounds are 2,1,1 and 1, respectively. In these cases, our bounds are also exact.

**Example 2.** Consider the following system [3,8].

$$
\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 7 & 8 \\ 12 & 14 \end{bmatrix} u, \quad y = \begin{bmatrix} 7 & -8 \\ -6 & 7 \end{bmatrix} x
$$

with output feedback control

$$
u = \begin{bmatrix} -k_1 & 0 \\ 0 & -k_2 \end{bmatrix} y.
$$
The controller gains are subject to uncertainty such that $|\Delta k_1| \leq \delta/2$ and $|\Delta k_2| \leq \delta$. The nominal value of controller gains are $k_1 = k_2 = 1$. Corollary 1 implies that the closed loop system is stable if

$$
\delta < \hat{\delta} := \left( \sup_{\omega \geq 0} \mu_\nu(\mathcal{H}(j\omega)) \right)^{-1}
$$

where $H(s) = C(sI - A)^{-1}B$, with

$$
A = \begin{bmatrix}
-1 & 0 \\ 0 & -2
\end{bmatrix} + \begin{bmatrix}
7 & 8 \\ 12 & 14
\end{bmatrix} \begin{bmatrix}
-1 & 0 \\ 0 & -1
\end{bmatrix} \begin{bmatrix}
7 & -8 \\ -6 & 7
\end{bmatrix} = \begin{bmatrix}
-2 & 0 \\ 0 & -4
\end{bmatrix}
$$

$$
B = \begin{bmatrix}
3.5 & 8 \\ 6 & 14
\end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix}
7 & -8 \\ -6 & 7
\end{bmatrix}.
$$

Solution of the optimization problem in (3) yields $\hat{\delta} = 0.0816$. This result agrees with that in [3]. It turns out to be an exact bound.

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**References.**


