SRC TR 88-23

On the Generalized Stability of the Convex Hull of Two Matrices

by

L. Saydy, A.L. Tits, and E.H. Abed

27^{th} CDC

On the Generalized Stability of the Convex Hull of Two Matrices

Lahcen Saydy *André L. Tits*
Eyad H. Abed

Electrical Engineering Department and Systems Research Center
University of Maryland, College Park, MD 20742.

Abstract. Contingent on the existence of certain affine maps, necessary and sufficient conditions for the eigenvalues of all matrices belonging to the convex hull of two given matrices to lie in a subset of the complex plane are obtained. Such maps are identified for every balanced convex domain $\mathcal{D}$ with polygonal boundary and conclusive *$\mathcal{D}$-stability* criteria are obtained. In the case when $\mathcal{D}$ is the open left half plane, the computational complexity of the new test is somewhat lower than that of a previously proposed criterion. For nonpolygonal balanced convex domains, conditions that are as close to being necessary and sufficient as desired may be obtained via a suitable approximation of these domains by polygonal ones.

Key words. $\mathcal{D}$-stability, convex hull, matrix polytopes, robust stability.

1. Introduction

Prompted by a result of Kharitonov ([1]), several contributions on robust stability of linear time invariant systems subject to parameter variations have been made recently. The robust stability question amounts to investigating the stability of polynomials and matrices with variable coefficient and entries.

In [1], Kharitonov showed that in order for every member of the family of polynomials $\{\ p(s) = a_n s^n + \ldots + a_1 s + a_0,\ a_i \in [\underline{a}_i, \bar{a}_i]\ \}$ to be Hurwitz stable (roots in open left half plane), it is necessary and sufficient that only certain four "corner polynomials" be Hurwitz stable. In the discrete stability case (roots inside the open unit disk), this powerful result was shown not to hold and a weaker result was established in [2]. In [3], it was proved that in order for the roots of an entire polytope of polynomials to lie in a given domain of the complex plane, it is enough that the roots of the polynomials on the edges of the polytope lie in this domain.

A more complex problem is that of stability of polytopes of matrices, i.e. sets of the form $\{A = \sum_{i=1}^{m} r_i A_i \ :\ \sum_{i=1}^{m} r_i = 1,\ r_i \geq 0,\ i = 1, \ldots, m\}$([4]). In [5], Fu and Barmish gave necessary and sufficient conditions for Hurwitz stability (eigenvalues in open left half plane) of matrix polytopes in the case of two matrices and showed by means of an example that, unlike the polynomial case, checking the edges is not enough in the case of several matrices. More specifically, Fu and Barmish showed that if A_0 is Hurwitz stable, then a necessary and sufficient condition for $A_r := (1-r)A_0 + rA_1$ to be stable $\forall r \in [0,1]$ is that the $n^2 \times n^2$ matrix $(A_0 \oplus A_0)^{-1}(A_1 \oplus A_1)$ have no eigenvalues in $(-\infty, 0\,]$, where $\oplus$ denotes the Kronecker sum. This paper focusses on the extension of this result to a broader class of stability criteria.

Given a domain or region $\mathcal{D}$ of the complex plane which is symmetric with respect to the real axis, we say that a square matrix M is *$\mathcal{D}$-stable* if all its eigenvalues lie within $\mathcal{D}$. Contingent on finding certain affine maps, necessary and sufficient conditions for *$\mathcal{D}$-stability* of a two matrix polytope, extending that of [5], are obtained in Section 2. Several types of domains for which this result yields a conclusive criterion for stability are given. For more general domains, these conditions become sufficient but may be rendered arbitrarily close to being necessary as desired, by suitably approximating the domains under consideration. It was pointed out by Fu and Barmish that from the point of view of numerical computations, the stability criterion in [5] becomes difficult to implement as the dimension n grows. To check for Hurwitz stability of a two matrix polytope, it is indeed required to invert an $n^2 \times n^2$ matrix as mentioned above. This numerical issue becomes more acute when more general stability criteria are sought. This problem is briefly addressed in Section 3 where it is pointed out that in some cases, a substantial computational gain is possible. In particular, it is shown that the necessary and sufficient condition of Fu and Barmish remains true if the kronecker sum $A \oplus A$ is substituted by a certain matrix $A_{[2]}$, the size of which is $\frac{n(n+1)}{2}$. In the last section, the $\mathcal{D}$-stability of a three matrix polytope is shown to be equivalent to $\mathcal{D}_1$-stability of a corresponding two matrix polytope, where $\mathcal{D}_1 := \{s \in \mathbb{C} : \ \Im m(s) = 0 \Rightarrow \Re e(s) > 0\}$.

Notation

$\bar{\mathcal{D}}$ = closure of $\mathcal{D}$

$\partial\mathcal{D}$ = boundary of $\mathcal{D}$

$\mathrm{Arg}(s)$ = argument of s in $[-\pi, \pi]$

$I_m = m \times m$ identity

$\sigma\{A\}$ = spectrum of A

$A \ominus B = A \oplus (-B)$

Throughout this paper, we consider domains in the complex plane which are symmetric with respect to the real axis. These are refered to as 'balanced'.

2. Necessary and sufficient conditions for generalized stability

Given m real $n \times n$ matrices $A_1, \ldots, A_m$, we denote their convex hull by

$$\mathrm{co}(A_1, \ldots, A_m) := \{\ A = \sum_{i=1}^{m} r_i A_i\ :\ \sum_{i=1}^{m} r_i = 1,\ r_i \geq 0,\ i = 1, \ldots, m\ \}$$

and we say that $\mathrm{co}(A_1, \ldots, A_m)$ is $\mathcal{D}$*-stable* if all the matrices it includes are $\mathcal{D}$*-stable* .
Let $\mathcal{D}$ be a balanced open subset of the complex plane.

Definition 1: We say that $\mathcal{D}$ satisfies property $(\mathcal{P})$ if there exist an integer $m = m(\mathcal{D})$ and an affine map $\mathcal{F}_\mathcal{D} : \mathbb{R}^{n\times n} \to \mathbb{C}^{m\times m}$ such that the following holds for every $\bar{\mathcal{D}}$*-stable* $n \times n$ real matrix A

$$A \text{ is } \mathcal{D}\text{-stable} \iff \mathcal{F}_\mathcal{D}(A) \text{ is nonsingular} \qquad (\star)$$

The following Lemma slightly generalizes a result by Fu and Barmish ([5]).

Lemma 1: Suppose A_0 is $\mathcal{D}$*-stable* and $\mathcal{D}$ satisfies property $(\mathcal{P})$ and let $\mathcal{F}_\mathcal{D}$ be a corresponding affine map. Then

$$\mathrm{co}(A_0, A_1) \text{ is } \mathcal{D}\text{-stable} \iff M_0 := \left(\mathcal{F}_\mathcal{D}(A_0)\right)^{-1}\mathcal{F}_\mathcal{D}(A_1) \text{ is } \mathcal{D}_1\text{-stable}$$

where $\mathcal{D}_1 := \{s \in \mathbb{C} :\ \Im m(s) = 0 \implies \Re e(s) > 0\}$.

□

Proof: First suppose that $\mathrm{co}(A_0, A_1)$ is not $\mathcal{D}$*-stable* . Then there exists $\lambda^* \in (0,1]$ such that A_{λ^*} is not $\mathcal{D}$*-stable* , where, $A_\lambda := (1-\lambda)A_0 + \lambda A_1$. Since A_0 is $\mathcal{D}$*-stable* and the eigenvalues of A_λ depend continuously on λ, then, λ^* can always be picked so that A_{λ^*} has all its eigenvalues inside $\bar{\mathcal{D}}$ with at least one on $\partial\mathcal{D}$. It follows from $(\star)$ that $\mathcal{F}_\mathcal{D}(A_{\lambda^*})$ is singular. Since $\mathcal{F}_\mathcal{D}$ is affine, we have:

$$\begin{aligned} \mathcal{F}_\mathcal{D}(A_{\lambda^*}) &= (1-\lambda^*)\mathcal{F}_\mathcal{D}(A_0) + \lambda^*\mathcal{F}_\mathcal{D}(A_1) && (1)\\ &= \lambda^*\mathcal{F}_\mathcal{D}(A_0)\left[\left(\mathcal{F}_\mathcal{D}(A_0)\right)^{-1}\mathcal{F}_\mathcal{D}(A_1) - \frac{\lambda^*-1}{\lambda^*}I_m\right] && (2)\end{aligned}$$

It follows from (2) that M_0 has a negative real eigenvalue $\frac{\lambda^*-1}{\lambda^*}$, thus is not $\mathcal{D}$*-stable* . Conversely, if M_0 has a negative eigenvalue α, then by writing $\alpha = \frac{\lambda^*-1}{\lambda^*}$ for some $\lambda^* \in (0,1]$ and reversing the steps in the argument above, we get that $\mathcal{F}_\mathcal{D}(A_{\lambda^*})$ is singular. This implies that A_{λ^*} is not $\mathcal{D}$*-stable* , for if A_{λ^*} were $\mathcal{D}$*-stable* (thus $\bar{\mathcal{D}}$*-stable*), property $(\mathcal{P})$ would imply that $\mathcal{F}_\mathcal{D}(A_{\lambda^*})$ is nonsingular, a contradiction.

□

The result above states that, if $\mathcal{D}$ satisfies property $(\mathcal{P})$,then in order for the family of matrices $\{A_\lambda = (1-\lambda)A_0 + \lambda A_1 :\ \lambda \in [0,1]\}$ to be $\mathcal{D}$*-stable* , it is necessary and sufficient that M_0 be $\mathcal{D}_1$*-stable*. Despite the fact that one is required to exhibit an affine map $\mathcal{F}_\mathcal{D}$ as in

Definition 1, it will be seen that Lemma 1 has the advantage of being applicable in many cases in which such a map may not exist.

As direct consequences of Lemma 1, we state the following two corollaries (Lemma 4.2 and Theorem 3.2 in [5], respectively) obtained by setting $\mathcal{D} = \mathbb{C} \setminus \{0\}$, $\mathcal{F}_{\mathcal{D}}$ equal to the identity map and $\mathcal{D} = \{s : \Re e(s) < 0\}$, $\mathcal{F}_{\mathcal{D}}(A) = A \oplus A$, $\forall A \in \mathbb{R}^{n \times n}$, respectively.

Corollary 1: Suppose A_0 in nonsingular. Then $A_\lambda := (1-\lambda)A_0 + \lambda A_1$ is nonsingular for every $\lambda \in [0,1]$ if and only if $A_0^{-1}A_1$ is $\mathcal{D}_1$*-stable.*

Corollary 2: Assume A_0 is Hurwitz stable. Then

$$\text{co}(A_0, A_1) \text{ is Hurwitz stable} \quad \Longleftrightarrow \quad M_0 := (A_0 \oplus A_0)^{-1}(A_1 \oplus A_1) \text{ is } \mathcal{D}_1\text{-stable}$$

3. Applications

In this section, two basic domains satisfying property $(\mathcal{P})$ are provided. They are in turn used to show that any convex balanced open polygon satisfies property $(\mathcal{P})$.

3.1. Two basic domains

For $\theta \in [\frac{\pi}{2}, \pi]$ and $\beta > 0$, let $\mathcal{D}_\theta$ and $\mathcal{D}^\beta$ denote the sets depicted in Fig. 1 and 2, respectively, That is, let $\mathcal{D}_\theta = \{s : |\text{Arg}(s)| > \theta\}$ and $\mathcal{D}^\beta = \{s : |\Im m(s)| < \beta\}$.

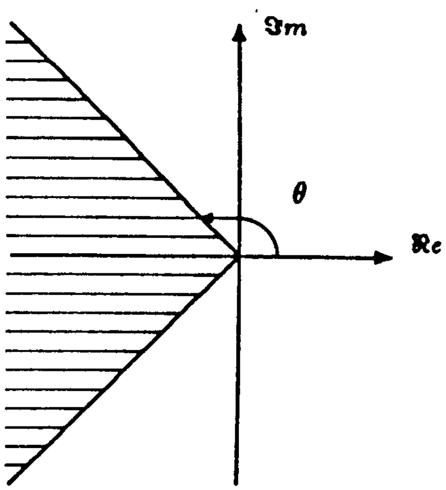

Fig. 1. $\mathcal{D}_\theta$

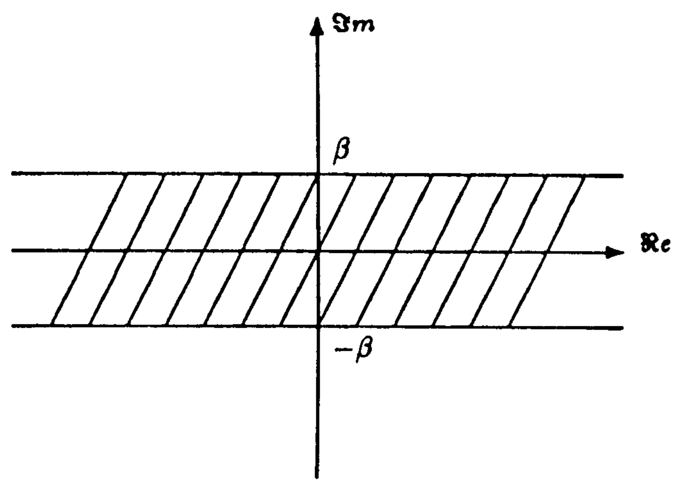

Fig. 2. $\mathcal{D}^\beta$

Proposition 1: If $\theta \in [\frac{\pi}{2}, \pi]$, then $\mathcal{D}_\theta$ satisfies property $(\mathcal{P})$. Furthermore, we can take

$$\mathcal{F}_{\mathcal{D}_\theta}(A) := \mathcal{F}_\theta(A) = e^{j\theta}A \ominus e^{-j\theta}A \quad \forall A \in \mathbb{R}^{n \times n}$$

□

Proof: From a well known property of the Kronecker sum, we obtain that

$$\sigma\{\ \mathcal{F}_\theta(A)\ \} = \{\ e^{j\theta}\lambda_1 - e^{-j\theta}\lambda_2\ :\ \lambda_1,\ \lambda_2 \in \sigma\{A\}\ \}.$$

Assume A is a $\bar{\mathcal{D}}_\theta$-*stable* real $n \times n$ matrix and let $\lambda_i := r_i e^{j\theta_i}$ with $|\theta_i| \geq \theta$, $i = 1, 2$, be two of its eigenvalues. Then $e^{j\theta}\lambda_1 - e^{-j\theta}\lambda_2 = 0$ if and only if $\theta_1 - \theta_2 = 2\theta \pmod{2\pi}$ and $r_1 = r_2$. For $\theta \geq \frac{\pi}{2}$, this is possible if and only if λ_1 and λ_2 are conjugate members of $\partial\mathcal{D}_\theta$. Thus we have shown that for every $\bar{\mathcal{D}}$-*stable* matrix A, $\mathcal{F}_\theta(A)$ is singular if and only if $\sigma\{A\} \cap \partial\mathcal{D}_\theta \neq \emptyset$. It follows by definition that $\mathcal{D}_\theta$ satisfies property $(\mathcal{P})$.

□

A similar result for $\mathcal{D}^\beta$ is stated below. Its proof is trivial and is omitted.

Proposition 2: $\mathcal{D}^\beta$, $\beta > 0$, satisfies property $(\mathcal{P})$ with

$$\mathcal{F}_\mathcal{D}(A) := \mathcal{F}^\beta(A) = (A + j\beta I) \ominus (A - j\beta I)$$

3.2. Polygonal domains

As already mentioned, $\mathcal{D}_\theta$ and $\mathcal{D}^\beta$ may be used to construct new domains satisfying property $(\mathcal{P})$. This construction relies on the following simple facts.

Proposition 3:

(i) $\mathcal{D}$ satisfies property $(\mathcal{P})$ if and only if $-\mathcal{D} := \{s : \ -s \in \mathcal{D}\}$ satisfies property $(\mathcal{P})$. Furthermore, we can take $\mathcal{F}_{-\mathcal{D}} \equiv \mathcal{F}_\mathcal{D}$, when $\mathcal{F}_\mathcal{D}$ is linear.

(ii) $\mathcal{D}$ satisfies property $(\mathcal{P})$ if and only if $\mathcal{D}^{(\alpha)} := \{s + \alpha : \ s \in \mathcal{D}\}$, $\alpha \in \mathbb{R}$, satisfies property $(\mathcal{P})$. Furthermore, we can take $\mathcal{F}_{\mathcal{D}^{(\alpha)}}(A) = \mathcal{F}_\mathcal{D}(A - \alpha I)$.

(iii) If $\mathcal{D} := \bigcap_{i=1}^m \mathcal{D}_i \neq \emptyset$, and $\mathcal{D}_i$, $i = 1, \ldots, m$ all satisfy property $(\mathcal{P})$, then so does $\mathcal{D}$.

□

Proof: (i) and (ii) are straightforward. For (iii), take $\mathcal{F}_\mathcal{D} = \mathrm{diag}(\mathcal{F}_{\mathcal{D}_1}, \ldots, \mathcal{F}_{\mathcal{D}_m})$.

□

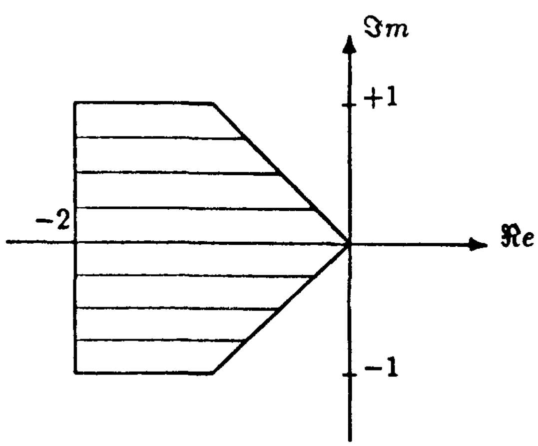

Fig. 3.

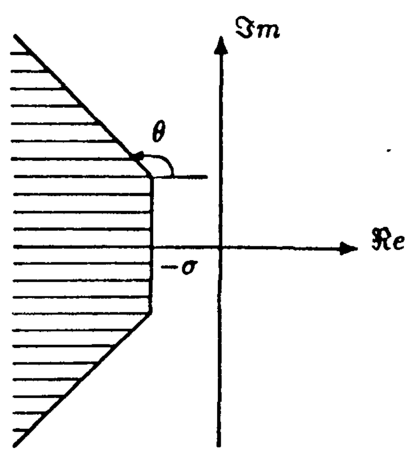

Fig. 4. $\mathcal{D}^*$

Example: The domain $\mathcal{D}$ in Fig. 3 is equal to $\mathcal{D}^1 \cap \mathcal{D}_{\frac{3\pi}{4}} \cap (-\mathcal{D}_{\frac{\pi}{4}}^{(3)})$. Consequently (see the proof of Proposition 3)

$$\mathcal{F}_\mathcal{D}(A) = \mathrm{diag}\{(A + jI) \ominus (A - jI), e^{j\frac{3\pi}{4}}A \ominus e^{-j\frac{3\pi}{4}}A, (A - 3I) \oplus (A + 3I)\}$$

Thus, a direct consequence of Propositions 1-3 is that any convex balanced domain with polygonal boundary satisfies property $(\mathcal{P})$; moreover, these same propositions provide one with a means for finding the corresponding affine map.

A domain $\mathcal{D}^*$ of particular interest in control systems design is that of Fig. 4. Indeed ([6]), compensated linear systems with eigenvalues (poles) in $\mathcal{D}^*$ exhibit step responses with settling times no greater than $\frac{4}{\sigma}$ and maximum overshoots corresponding to θ. For this domain, Lemma 1 may be restated as follows:

Theorem 1: Assume A_0 is $\mathcal{D}^*$*-stable*. Then co(A_0,A_1) is $\mathcal{D}^*$*-stable* if and only if both M_0 and M_1 are $\mathcal{D}_1$*-stable*, where $\mathcal{D}_1$ is as in Lemma 1, M_0 and M_1 are given by:

$$M_0 = \left[(A_0 + \sigma I) \oplus (A_0 + \sigma I)\right]^{-1}\left[(A_1 + \sigma I) \oplus (A_1 + \sigma I)\right]$$
$$M_1 = \left[e^{j\theta} A_0 \ominus e^{-j\theta} A_0\right]^{-1}\left[e^{j\theta} A_1 \ominus e^{-j\theta} A_1\right]$$

□

Proof: Clearly, $\mathcal{D}^* = \mathcal{D}_\theta \cap \mathcal{D}_{\frac{\pi}{2}}^{(-\sigma)}$. It follows from Proposition 3 that $\mathcal{D}^*$ satisfies property $(\mathcal{P})$ and $\mathcal{F}_\mathcal{D} = \text{diag}\{\mathcal{F}_\theta, \mathcal{F}_{\frac{\pi}{2}}^{(-s)}\}$, with obvious notations. Using Propositions 1-3 to find the expression of $\mathcal{F}_\mathcal{D}$ and applying Lemma 1 yields the result.

□

Remark: Since any convex domain, e.g. the unit disk, can be approximated arbitrarily closely by a polygon, it is clear that Lemma 1 may be used to address the $\mathcal{D}$*-stability* question for such domains as well. The condition in Lemma 1 then becomes sufficient (resp. necessary) but may be rendered, conceptually at least, as close to being necessary (resp. sufficient) as desired depending on how finely the aforementioned approximation has been carried out.

3.3 Implementation issues

From the point of view of numerical computations, the $\mathcal{D}$-stability criteria discussed above are expensive to use due to the size of the matrices involved. It can be seen from the way $\mathcal{F}_\mathcal{D}$ is constructed, e.g. see example 1, that the more faces a polygon $\mathcal{D}$ has, the greater the computation burden. For instance, in the case of $\mathcal{D}^*$-stability (see Fig. 3), the eigenvalues of two matrices, each of which has size $n^2 \times n^2$, need to be computed. For a general domain with k (pairs of conjuguate) faces, k such matrices need to be dealt with. This raises the question of whether it is possible to obtain generalized stability criteria involving matrices of a smaller size. It turns out that the answer is affirmative in some cases, though the size reduction is, by all means, not dramatic. Nevertheless, we think it is worth mentioning how this could be accomplished. We first give some definitions.

If $x = (x_1, \ldots, x_n)^T$ and $p \geq 2$, then $x^{[p]}$ is, by definition, the N_p^n-dimensional vector ($N_p^n := \binom{n+p-1}{p}$) formed by the lexicographic listing of all linearly independent terms of the form

$$\frac{p!}{p_1! \ldots p_n!}\, x_1^{p_1} \ldots x_n^{p_n}, \quad \sum_{\substack{i=1 \\ p_i \geq 0}} p_i = p$$

For an $n \times n$ matrix A, the Schläflian matrix of order p, denoted $A^{[p]}$, is implicitly defined by the relationship $(Ax)^{[p]} = A^{[p]}x^{[p]}$, $\forall x \in \mathbb{R}^n$. The map of interest, $A_{[p]}$, is then defined in terms of $A^{[p]}$ by

$$A_{[p]} := \lim_{h \to 0} \frac{1}{h}\left[(I_n + hA)^{[p]} - I_{N_p^n}\right]$$

In other words $A_{[p]}$ is the Gateaux derivative of the of the nonlinear map $A \mapsto A^{[p]}$ evaluated at the identity I_n and acting on A. As such, $A_{[p]}$ is linear in A.

Lemma 2: ([7], [8])

The eigenvalues of $A_{[p]}$ (resp. $A^{[p]}$) consist of the N_p^n sums (products) over distinct unordered index sets of the form

$$\lambda_{i_1}(A) + \cdots + \lambda_{i_p}(A) \quad (\text{resp. } \lambda_{i_1}(A) \times \cdots \times \lambda_{i_p}(A))$$

□

By contrast, we recall that the eigenvalues of the Kronecker sum $A \oplus A$ consist of the n^2 sums $\lambda_i(A) + \lambda_j(A)$ over ordered pairs (i,j). In the light of this Lemma, it is clear that $\sigma\{A \oplus A\} = \sigma\{A_{[2]}\}$ and that the $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ matrix $A_{[2]}$ is nothing but the redundancy-free version of the $n^2 \times n^2$ matrix $A \oplus A$, as far as the eigenvalues are concerned. Consequently, $A_{[2]}$ may be used advantageously instead of $A \oplus A$, whenever possible (see Section 3.4.). It should be pointed out that programs that generate the expression of $A_{[p]}$ in a symbolic form and up to any order p are currently available ([9]).

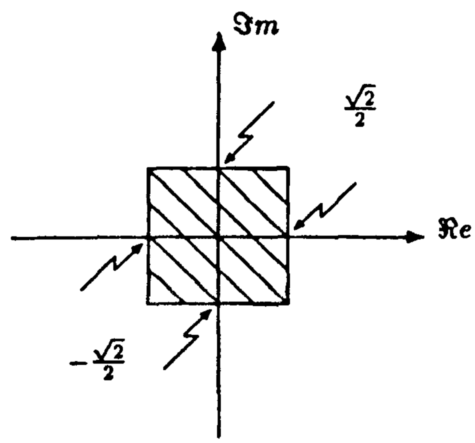

Fig. 5.

3.4. Example:

Given

$$A_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

we seek to determine whether the matrices $\{A_\lambda := (1-\lambda)A_0 + \lambda A_1 : \lambda \in [0,1]\}$ all have their eigenvalues inside the unit open disk. One way to do this, is to approximate the unit disk by a polygon from within. One such a polygon is (see Fig. 5)

$$\mathcal{D} = \{s \in \mathbb{C} : |\Re e(s)| < \frac{\sqrt{2}}{2}, |\Im m(s)| < \frac{\sqrt{2}}{2}\}$$

Clearly, $\mathcal{D}$ satisfies property $(\mathcal{P})$. To obtain a corresponding map, we write $\mathcal{D}$ as the intersection of basic domains, namely

$$\mathcal{D} = \mathcal{D}^{\frac{\sqrt{2}}{2}} \cap \mathcal{D}_{\frac{\pi}{2}}^{(\frac{\sqrt{2}}{2})} \cap (-\mathcal{D}_{\frac{\pi}{2}})^{(-\frac{\sqrt{2}}{2})}$$

Therefore

$$\mathcal{F}_{\mathcal{D}}(A) = \mathrm{diag}\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$$

where

$$\mathcal{F}_1(A) = (A + j\frac{\sqrt{2}}{2}I) \ominus (A - j\frac{\sqrt{2}}{2}I)$$
$$\mathcal{F}_2(A) = (A + \frac{\sqrt{2}}{2}I)_{[2]}$$
$$\mathcal{F}_3(A) = (A - \frac{\sqrt{2}}{2}I)_{[2]}$$

Thus $\mathrm{co}(A_0,A_1)$ is *$\mathcal{D}$-stable* if and only if the matrices $M_0 := (\mathcal{F}_1(A_0))^{-1}\mathcal{F}_1(A_1)$, $M_1 := (\mathcal{F}_2(A_0))^{-1}\mathcal{F}_2(A_1)$ and $M_2 := (\mathcal{F}_3(A_0))^{-1}\mathcal{F}_3(A_1)$ are all *$\mathcal{D}_1$-stable.* Using the fact that for $n = 2$

$$A_{[2]} = \begin{pmatrix} 2a_{11} & \sqrt{2}a_{12} & 0 \\ \sqrt{2}a_{21} & a_{11}+a_{22} & \sqrt{2}a_{12} \\ 0 & \sqrt{2}a_{21} & 2a_{22} \end{pmatrix}$$

we obtain that $\sigma(M_0) = \{\frac{9+2j}{10}, \frac{9-2j}{10}, 1\}$, $\sigma(M_1) = \{0.65, 0.59, 1\}$ and $\sigma(M_2) = \{3.41, 1.35, 1\}$. It follows that $\mathrm{co}(A_0,A_1)$ is *$\mathcal{D}$-stable* .

4. Three matrix case

In this section we show that the *$\mathcal{D}$-stability* problem of a three matrix polytope can be reduced to $\mathcal{D}_1$-stability of a two matrix polytope where $\mathcal{D}_1$ is as in Lemma 1.

More precisely, let A_1, A_2 and A_3 be three $n \times n$ real matrices, $\mathcal{D}$ a domain of the complex plane satifying property $(\mathcal{P})$, with corresponding affine map $\mathcal{F}_{\mathcal{D}}$.

Proposition 4: Suppose A_1 is *$\mathcal{D}$-stable.* Then

$$co\{A_1,A_2,A_3\} \text{ is } \mathcal{D}\text{-stable} \iff co\{M_0,M_1\} \text{ is } \mathcal{D}_1\text{-stable}$$

where $M_0 := (\mathcal{F}_{\mathcal{D}}(A_1))^{-1}\mathcal{F}_{\mathcal{D}}(A_2)$ and $M_1 := (\mathcal{F}_{\mathcal{D}}(A_1))^{-1}\mathcal{F}_{\mathcal{D}}(A_3)$.

□

Proof: It can be easily checked that $A_r := r_1 A_1 + r_2 A_2 + r_3 A_3$ is $\mathcal{D}$-*stable* $\forall r = (r_1, r_2, r_3)$, such that, $\sum_{i=1}^{3} r_i = 1,\ r_i \geq 0,\ i = 1,2,3$ if and only if $A(\lambda,\alpha) := (1-\lambda)A_1 + \lambda[(1-\alpha)A_2 + \alpha A_3]$ is $\mathcal{D}$-*stable* $\forall \lambda, \alpha \in [0,1]$. In view of Lemma 1, this is equivalent to

$$\left(\mathcal{F}_D(A_1)\right)^{-1}\mathcal{F}_D\left((1-\alpha)A_2 + \alpha A_3\right) = (1-\alpha)M_0 + \alpha M_1$$

being $\mathcal{D}_1$-*stable* $\forall \alpha \in [0,1]$.

□

Clearly, we would now be in a position to supply a complete answer to the problem under consideration if $\mathcal{D}_1$ were to satisfy property $(\mathcal{P})$. It turns out that this is not so.

Proposition 5: $\mathcal{D}_1$ does not satisfy property $(\mathcal{P})$

□

Proof: If $\mathcal{D}_1$ were to satisfy property $(\mathcal{P})$, then there would exist an affine map $\mathcal{F}$ such that $\mathcal{F}(\alpha I) = \alpha L_1 + L_0$ is singular $\forall \alpha \leq 0$, L_0, L_1 being equal to $\mathcal{F}(0), \mathcal{F}(I) - \mathcal{F}(0)$ respectivly. This means that the n^{th} order polynomial $p(\alpha) := \det(\alpha L_1 + L_0)$ is identically zero. It follows that $\mathcal{F}(\alpha I)$ is singular for $\alpha > 0$ as well, a contradiction.

□

5. Conclusion

The generalized stability ($\mathcal{D}$-*stability*) question for the convex hull of two matrices was investigated. Necessary and sufficient conditions were obtained for any convex domain with polygonal boundary. For more general domains, these conditions become sufficient and as close to being necessary as desired.

References

[1] V.L. Kharitonov, "Asymptotic Stability of an Equilibrium Position of a Family of Systems of Linear Differential Equations," *Differentsial'nye Uraveniya* 14 (1978), 2086–2088, *Differential Equations* 14 (1979), 1483–1485.

[2] F.J. Kraus, M. Mansour & B.D.O. Anderson, "Robust Schur Polynomial Stability and Kharitonov's Theorem," *Proc. of the 26th Conf. on Decision and Control*, Los Angeles (Dec. 1987).

[3] A.C. Bartlett, C.V. Hollot & H. Lin, "Root Location of an Entire Polytope of Polynomials: It suffices to check the edges," *Proc. of the American Control Conf.*, Minneapolis (1987).

[4] B.R. Barmish & L. DeMarco, "Criteria for Robust Stability of Systems with Structured Uncertainty: A Perspective," *Proc. of the American Control Conf.*, Minneapolis (1987).

[5] M. Fu & B.R. Barmish, "Stability of Convex and Linear Combinations of Polynomials and Matrices Arising in Robustness Problems," *Proc. of the CISS, John Hopkins University*, Baltimore (1987).

[6] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, The MIT Press, 1985.

[7] R. W. Brockett, "Lie Algebras and Lie Groups in Control Theory," *in Geometric Methods in System Theory*, D.Q Mayne, R. W. Brockett, Eds, (Reidel, Dordrecht) (1973).

[8] A. I. Barkin & A. L. Zelentsovsky, "Method of Power Transformations for Analysis of Stability of Nonlinear Systems," *Systems and Control Letters* 3 (1983), 303–310.

[9] L. Saydy & G. L. Blankenship, "Symbolic Generation of Carleman Linearizations of Arbitrary Order for Smooth Nonlinear Dynamical Systems," *SRC Technical Report*, Electrical Engineering Department, University of MD (1988).