SRC TR 88-1

The Conditional Adjoint Process

by

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Abstract

The adjoint and minimum principle for a partially observed diffusion can be obtained by differentiating the statement that a control u^* is optimal. Using stochastic flows the variation in the cost resulting from a change in an optimal control can be computed explicitly. The technical difficulty is to justify the differentiation.

[1] Partially supported by US Army contract DAAL03-86-C-0014 and by NSF grant CDR-85-00108

[2] Partially supported by the Natural Sciences and Engineering Research Council of Canada under grant A-7964 and the Air Force Office of Scientific Research, United States Air Force, under grant AFOSR-86-0332.

[3] Partially supported by the Natural Sciences and Engineering Research Council of Canada under grant A-7964.

1. INTRODUCTION.

Using stochastic flows we calculate below the change in the cost due to a 'strong' variation of an optimal control. Differentiating this quantity enables us to identify the adjoint, or co-state variable, and give a partially observed minimum principle. If the drift coefficient is differentiable in the control variable the related result of Bensoussan [2] follows from our theorem. Full details will appear in [1]. The method appears simpler than that employed in Haussman [4].

2. DYNAMICAL EQUATIONS.

Suppose the state of a stochastic system is described by the equation

$$d\xi_t = f(t, \xi_t, u)dt + g(t, \xi_t)dw_t,$$

$$\xi_t \in R^d, \qquad \xi_0 = x_0, \qquad 0 \leq t \leq T. \tag{2.1}$$

The control variable u will take values in a compact subset U of some Euclidean space R^k. We shall assume

A_1: $x_0 \in R^d$ is given.

A_2: $f : [0,T] \times R^d \times U \to R^d$ is Borel measurable, continuous in u for each (t,x), continuously differentiable in x for each (t,u) and

$$(1+|x|)^{-1}|f(t,x,u)| + |f_x(t,x,u)| \leq K_1.$$

A_3: $g : [0,T] \times R^d \to R^d \otimes R^n$ is a matrix valued function, Borel measurable, continuously differentiable in x, and for some K_2:

$$|g(t,x)| + |g_x(t,x)| \leq K_2.$$

The observation process is defined by

$$dy_t = h(\xi_t)dt + d\nu_t \tag{2.2}$$

$$y_t \in R^m, \qquad y_0 = 0, \qquad 0 \leq t \leq T.$$

In (2.1) and (2.2) $w = (w^1, \dots, w^n)$ and $\nu = (\nu^1, \dots, \nu^m)$ are independent Brownian notions defined on a probability space (Ω, F, P).

Furthermore, we assume

A_4: $h : R^d \to R^m$ is Borel measurable, continuously differentiable in x and

$$|h(t,x)| + |h_x(t,x)| \le K_3.$$

REMARK 2.1. These hypotheses can be weakened to those discussed by Haussman [4]. See [1].

Write $\hat{P}$ for the Wiener measure on $C([0,T], R^n)$ and μ for the Wiener measure on $C([0,T], R^m)$.

$$\Omega = C([0,T], R^n) \times C([0,T], R^m)$$

and the coordinate functions in Ω will be denoted (x_t, y_t). Wiener measure P on Ω is

$$P(dx, dy) = \hat{P}(dx)\mu(dy).$$

DEFINITION 2.2. *$Y = \{Y_t\}$ will be the right continuous, complete filtration on $C([0,T], R^m)$ generated by*

$$Y_t^0 = \sigma\{y_s : s \le t\}.$$

The set of admissible control functions $\underline{U}$ will be the Y-predictable functions defined on $[0,T] \times C([0,T], R^m)$ with values in U.

For $u \in \underline{U}$ and $x \in R^d$, $\xi^u_{s,t}(x)$ will denote the strong solution of (2.1) corresponding to u with $\xi^u_{s,s} = x$.

Define

$$Z^u_{s,t}(x) = \exp\Big(\int_s^t h(\xi^u_{s,r}(x))' dy_r - \frac{1}{2}\int_s^t h(\xi^u_{s,r}(x))^2 dr\Big). \tag{2.3}$$

Note a version of Z defined for every trajectory y can be obtained by integrating the stochastic integral in the exponential by parts.

If a new probability measure P^u defined on Ω by putting

$$\frac{dP^u}{dP} = Z^u_{0,T}(x_0),$$

under P^u $(\xi^u_{0,t}(x_0), y_t)$ is a solution of the system (2.1) and (2.2). That is, under P^u, $\xi^u_{0,t}(x_0)$ remains a strong solution of (2.1) and there is an independent Brownian motion ν such that y_t satisfies (2.2).

Because of hypothesis A_4, for $0 \leq t \leq T$ easy applications of Burkholder's and Gronwall's inequalities show that

$$E[(Z^u_{0,t}(x_0))^p] < \infty \tag{2.4}$$

for all $u \in \underline{U}$ and all p, $1 \leq p < \infty$.

COST 2.3. We shall suppose the cost is purely terminal and equals

$$c(\xi^u_{0,T}(x_0))$$

where c is a bounded, differentiable function. If control $u \in \underline{U}$ is used the expected cost is

$$J(u) = E_u[c(\xi^u_{0,T}(x_0))].$$

With respect to P, under which y_t is a Brownian motion

$$J(u) = E[Z^u_{0,T}(x_0)c(\xi^u_{0,T}(x_0))]. \tag{2.5}$$

A control $u^* \in \underline{U}$ is optimal if

$$J(u^*) \leq J(u)$$

for all $u \in \underline{U}$. We shall suppose there is an optimal control u^*.

3. FLOWS.

For $u \in \underline{U}$ and $x \in R^d$ consider the strong solution

$$\xi^u_{s,t}(x) = x + \int_s^t f(r, \xi^u_{s,r}(x), u_r)dr + \int_s^t g(r, \xi^u_{s,r}(x))dw_r. \qquad (3.1)$$

We wish to consider the behaviour of $\xi^u_{s,t}(x)$ for each trajectory y of the observation process. In fact the results of Bismut [3] and Kunita [6] extend and show the map

$$\xi^u_{s,t} : R^d \to R^d$$

is, almost surely, a diffeomorphism for each $y \in C([0,T], R^m)$.

Write

$$\|\xi^u(x_0)\|_t = \sup_{0 \leq s \leq t} |\xi^u_{0,s}(x_0)|.$$

Then, using Gronwall's and Jensen's inequalities, for any p, $1 \leq p < \infty$

$$\|\xi^u(x_0)\|_T^p \leq C\Big(1 + |x_0|^p + \Big|\int_0^T g(r, \xi^u_{0,r}(x_0))dw_r\Big|^p\Big)$$

almost surely, for some constant C.

Using A_3 and Burkholder's inequality

$$\|\xi^u(x_0)\|_T \in L^p \quad \text{for} \quad 1 \leq p < \infty.$$

Suppose u^* is an optimal control, and write

$$\xi^*_{s,t}(\cdot) \quad \text{for} \quad \xi^{u^*}_{s,t}(\cdot).$$

The Jacobian $\frac{\partial \xi^*_{s,t}}{\partial x}$ is the matrix solution C_t of the equation

$$dC_t = f_x(t, \xi^*_{s,t}(x), u^*)C_t dt + \sum_{i=1}^n g_x^{(i)}(t, \xi^*_{s,t}(x))C_t dw^i_t. \qquad (3.2)$$

with $C_s = I$.

Here $g^{(i)}$ is the i^{th} column of g and I is the $n \times n$ identity matrix. Writing $\|C\|_T = \sup_{0 \le s \le t} |C_s|$ and using Burkholder's, Jensen's and Gronwall's inequalities we see $\|C\|_T \in L^p, \ 1 \le p < \infty$.

Consider the matrix valued process D defined by

$$D_t = I - \int_s^t D_r f_x(r, \xi^*_{s,r}(x), u^*_r)dr$$
$$- \sum_{i=1}^n \int_s^t D_r g_x^{(i)}(r, \xi^*_{s,r}(x))dw^i_r + \sum_{i=1}^n \int_s^t D_r (g_x^{(i)}(r, \xi^*_{s,r}(x)))^2 dr \qquad (3.3)$$

Then as in [5] or [6] $d(D_t C_t) = 0$ and $D_s C_s = I$ so

$$D_t = C_t^{-1} = \left(\frac{\partial \xi^*_{s,t}}{\partial x}\right)^{-1}.$$

Furthermore, $\|D\|_t \in L^p, \ 1 \le p < \infty$.

Suppose $z_t = z_s + A_t + \sum_{i=1}^n \int_s^t H_i dw^i_r$ is a d-dimensional semimartingale. Bismut [3] shows one can consider the process $\xi^*_{s,t}(z_t)$ and in fact:

$$\xi^*_{s,t}(z_t) = z_s + \int_s^t \Big(f(r, \xi^*_{s,r}(z_r), u^*_r)$$
$$+ \sum_{i=1}^n g_x^{(i)}(r, \xi^*_{s,r}(z_r), u^*_r) \ \frac{\partial \xi^*_{s,r}}{\partial x} H_i$$
$$+ \frac{1}{2} \sum_{i=2}^n \frac{\partial^2 \xi^*_{s,r}}{\partial x^2}(H_i, H_i)\Big) dr$$
$$+ \int_s^t \frac{\partial \xi^*_{s,r}}{\partial x}(z_r)dA_r + \sum_{i=1}^n \int_s^t \Big(g^{(i)}(r, \xi^*_{s,r}(z_r)) + \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) H_i \Big) dw^i_r. \qquad (3.4)$$

DEFINITION 3.1. *For $s \in [0,T]$, $h > 0$ such that $0 \le s < s+h \le T$, for any $\tilde{u} \in U$, and $A \in Y_s$ consider a 'strong' variation u of u^* defined by*

$$u(t,w) = \begin{cases} u^*(t,w) & \text{if } (t,w) \notin [s, s+h] \times A \\ \tilde{u}(t,w) & \text{if } (t,w) \in [s, s+h] \times A. \end{cases}$$

THEOREM 3.2. *For any strong variation u of u^* consider the process*

$$z_t = x + \int_s^t \left(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\right)^{-1} (f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r))dr. \tag{3.5}$$

Then the process $\xi^*_{s,t}(z_t)$ is indistinguishable from $\xi^u_{s,t}(x)$.

PROOF We shall substitute in (3.4), (noting $H_i = 0$ for all i). Therefore,

$$\begin{aligned}\xi^*_{s,t}(z_t) &= x + \int_s^t f(r, \xi^*_{s,r}(z_r), u^*_r)dr \\ &+ \int_s^t \left(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\right)\left(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\right)^{-1} (f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r))dr \\ &+ \int_s^t g(r, \xi^*_{s,r}(z_r))dw_r.\end{aligned}$$

The solution of (3.1) is unique, so $\xi^*_{s,t}(z_t) = \xi^u_{s,t}(x)$. Note $u(t) = u^*(t)$ if $t > s + h$ so $z_t = z_{s+h}$ if $t > s + h$ and

$$\begin{aligned}\xi^*_{s,t}(z_t) &= \xi^*_{s,t}(z_{s+h}) \\ &= \xi^*_{s+h,t}(\xi^u_{s,s+h}(x)).\end{aligned} \tag{3.6}$$

4. THE EXPONENTIAL DENSITY.

Consider the $(d+1)$-dimensional system

$$\begin{aligned}\xi^*_{s,t}(x) &= x + \int_s^t f(r, \xi^*_{s,r}(x), u^*_r)dr + \int_s^t g(r, \xi^*_{s,r}(x))dw_r \\ Z^*_{s,t}(x,z) &= z + \int_s^t Z^*_{s,r}(x,z)h(\xi^*_{s,r}(x))'dy_r.\end{aligned} \tag{4.1}$$

That is, we are considering an augmented flow (ξ, Z) in R^{d+1} in which Z^* has a variable initial condition $z \in R$. Note:

$$Z^*_{s,t}(x,z) = zZ^*_{s,t}(x).$$

The map $(x,z) \to (\xi^*_{s,t}(x), Z^*_{s,t}(x,z))$ is, almost surely, a diffeomorphism of R^{d+1}. Clearly,

$$\frac{\partial \xi^*_{s,t}}{\partial z} = 0, \qquad \frac{\partial f}{\partial z} = 0 \quad \text{and} \quad \frac{\partial g}{\partial z} = 0.$$

The Jacobian of this augmented map is represented by the matrix

$$\tilde{C}_t = \begin{pmatrix} \frac{\partial \xi^*_{s,t}}{\partial x} & 0 \\ \frac{\partial Z^*_{s,t}}{\partial x} & \frac{\partial Z^*_{s,t}}{\partial z} \end{pmatrix}.$$

In particular, from (4.1), for $1 \le i \le d$

$$\frac{\partial Z^*_{s,t}}{\partial x_i} = \sum_{j=1}^{m} \int_s^t \Big(Z^*_{s,r}(x,z) \sum_{k=1}^{n} \frac{\partial h^j}{\partial \xi_k} \cdot \frac{\partial \xi^*_{k,s,r}}{\partial x_i} + h^j\left(\xi^*_{s,r}(x)\right) \frac{\partial Z^*_{s,r}}{\partial x_i} \Big) dy^j_r . \tag{4.2}$$

We are interested in solutions of (4.1) and (4.2) only when $z = 1$, so as above we write

$$Z^*_{s,t}(x) \quad \text{for} \quad Z^*_{s,t}(x,1) \quad \text{etc.}$$

LEMMA 4.1.

$$\frac{\partial Z^*_{s,t}}{\partial x} = Z^*_{s,t}(x)\Big(\int_s^t h_x(\xi^*_{s,t}(x)) \cdot \frac{\partial \xi^*_{s,r}}{\partial x} d\nu_r\Big)$$

where, as in (2.2), $d\nu_t = dy_t - h(\xi^*_{s,t}(x))dt$.

PROOF From (4.2)

$$\frac{\partial Z^*_{s,t}}{\partial x} = \int_s^t \Big(\frac{\partial Z^*_{s,r}}{\partial x} h'(\xi^*_{s,r}(x)) + Z^*_{s,r}(x) h_x(\xi^*_{s,r}(x)) \frac{\partial \xi^*_{s,r}}{\partial x} \Big) dy_r . \tag{4.3}$$

Write

$$L_{s,t}(x) = Z^*_{s,t}(x)\Big(\int_s^t h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x} d\nu_r\Big).$$

Then

$$Z^*_{s,t}(x) = 1 + \int_s^t Z^*_{s,r}(x) h'(\xi^*_{s,r}(x)) dy_r$$

and the product rule gives

$$L_{s,t}(x) = \int_s^t L_{s,r}(x) h'(\xi^*_{s,r}(x)) dy_r$$

$$+ \int_s^t Z^*_{s,r}(x) h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x} dy_r .$$

The minimum cost is

$$J(u^*) = E[Z^*_{0,T}(x_0)c(\xi^*_{0,T}(x_0))]$$

$$= E[Z^*_{0,s}(x_0)Z^*_{s,T}(x)c(\xi^*_{s,T}(x))].$$

Also,

$$J(u) = E[Z^*_{0,s}(x_0)Z^u_{s,T}(x)c(\xi^u_{s,T}(x))]$$

$$= E[Z^*_{0,s}(x_0)Z^*_{s,T}(z_{s+h})c(\xi^*_{s,T}(z_{s+h}))]$$

by (3.6) and (4.5). Recall $Z^*_{s,T}(\cdot)$ and $c(\xi^*_{s,T}(\cdot))$ are differentiable almost surely, with continuous and uniformly integrable derivatives. Consequently, writing

$$\Gamma(s, z_r) = Z^*_{0,s}(x_0)Z^*_{s,T}(z_r)\Big\{c_\xi(\xi^*_{s,T}(z_r))\ \frac{\partial \xi^*_{s,T}}{\partial x}\ (z_r)$$

$$+ c(\xi^*_{s,T}(z_r))\Big(\int_s^T h_\xi(\xi^*_{s,\sigma}(z_r))\ \frac{\partial \xi^*_{s,\sigma}}{\partial x}\ (z_r)d\nu_\sigma\Big)\Big\}\Big(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\Big)^{-1}$$

for $s \le r \le s+h$, we have

$$J(u) - J(u^*) = E[Z^*_{0,s}(x_0)\{Z^*_{s,t}(z_{s+h})c(\xi^*_{s,t}(z_{s+h})) - Z^*_{s,T}(x)c(\xi^*_{s,T}(x))\}]$$

$$= E\Big[\int_s^{s+h} \Gamma(s, z_r)(f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(x), u^*_r))dr\Big]. \tag{5.1}$$

This formula describes the change in the expected cost arising from the perturbation u of the optimal control. However, $J(u) \ge J(u^*)$ for all $u \in \underline{U}$ so the right hand side of (5.1) is non-negative for all $h > 0$. We wish to divide by $h > 0$ and let $h \to 0$. This requires some careful arguments using the uniform boundedness of the random variables and the monotone class theorem. It can be shown that there is a set $S \subset [0, T]$ of zero Lebesgue measure such that if $s \notin S$

$$E[\Gamma(s, x)(f(s, \xi^*_{0,s}(x_0), u) - f(s, \xi^*_{0,s}(x_0), u^*_s))I_A] \ge 0 \tag{5.2}$$

for any $u \in U$ and $A \in Y_s$.

Details of this argument can be found in [1]. Define

$$p_s(x) = E^*\Big[c_\xi(\xi^*_{0,T}(x_0))\ \frac{\partial \xi^*_{s,T}}{\partial x}(x)$$

$$+ c(\xi^*_{0,T}(x_0))\Big(\int_s^T h_\xi(\xi^*_{0,\sigma}(x_0))\ \frac{\partial \xi^*_{s,\sigma}}{\partial x}\ (x)d\nu_\sigma\Big)\Big|\ Y_s \vee \{x\}\Big]$$

where $x = \xi^*_{0,s}(x_0)$ and E^* is the expectation under $P^* = P^{u^*}$.

In (5.2) we have established the following:

THEOREM 5.1. *$p_s(x)$ is the adjoint process for the partially observed optimal control problem. That is, if $u^* \in \underline{U}$ is optimal there is a set $S \subset [0,T]$ of zero Lebesgue measure such that for $s \notin S$*

$$E^*[p_s(x)f(s,x,u^*) \mid Y_s] \geq E^*[p_{\bar{s}}(x)f(s,x,u) \mid Y_s] \quad \text{a.s.} \tag{5.3}$$

so the optimal control u^* almost surely minimizes the conditional Hamiltonian.

If $x = \xi^*_{0,s}(x_0)$ has a conditional density $q_s(x)$ under P^*, and if f is differentiable in u, (5.3) implies

$$\sum_{i=1}^{k}(u_i(s) - u_i^*(s)) \int_{R^d} \Gamma(s,x)\ \frac{\partial f}{\partial u_i}\ (s,x,u^*)q_s(x)dx \geq 0.$$

This is the result of Bensoussan [2].

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