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On the Stability of Polynomials with Uncoupled Perturbations in the Coefficients of Even and Odd Powers

by

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Abstract

In this note, we present some results concerning the stability (in Hurwitz' sense) of a family of polynomials with even and odd coefficients subject to uncoupled perturbations. It is shown that the stability of an appropriate small subset of *extreme* polynomials guarantees the stability of the entire family. In particular, a *polytope* of polynomials with the even-odd uncoupling property is stable provided a certain small subset of its *vertices* is. For the case of an arbitrary subset of polynomials, this result gives a less conservative sufficient condition than that provided by Kharitonov's theorem.

1 Introduction

In this note we study the stability (in Hurwitz' sense) of a family of polynomials of degree $2k-1$,

$$P(s,e,o) = o_k s^{2k-1} + e_k s^{2k-2} + o_{k-1} s^{2k-3} + e_{k-1} s^{2k-4} + \ldots + o_1 s + e_1$$

where the coefficient vector $[e\ o] = [e_1 \ldots e_k\ o_1 \ldots o_k]$ (split into coefficients of even and odd powers of s) belongs to a certain set $\Omega \subset \mathbb{R}^{2k}$. While, for notational convenience, $P(s,e,o)$ is assumed to have odd degree, all the results discussed here extend to even degree polynomials in a straightforward manner. A vector $[e\ o] \in \mathbb{R}^{2k}$ is said to be *strictly Hurwitz* (or *stable*) if the corresponding

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polynomial $P(s,e,o)$ is strictly Hurwitz (stable in Hurwitz' sense), i.e., if it has all its roots in the open left half of the complex plane. A subset of $\mathbb{R}^{2k}$ is said to be strictly Hurwitz (or stable) if all its points are strictly Hurwitz. The set of all strictly Hurwitz vectors is denoted by H.

In [1], Kharitonov studied the particular case when Ω is a hypercube with edges parallel to the coordinate axes, i.e.,

$$\Omega = \{[e\ o] \mid \underline{e}_i \leq e_i \leq \bar{e}_i,\ \underline{o}_i \leq o_i \leq \bar{o}_i,\ i = 1, \ldots, k\},$$

for some positive numbers $\underline{e}_i$, $\bar{e}_i$, $\underline{o}_i$, $\bar{o}_i$, $i = 1, \ldots, k$. He showed that, if a certain subset of Ω of cardinality 4 is in H, then the entire hypercube Ω is in H. Lin *et al.* [2] later showed that, if instead Ω is the Cartesian product of two parallelograms in $\mathbb{R}^k$, it is sufficient to check the stability of all the polynomials with coefficients corresponding to vertices of these parallelograms. If Ω is a general polytope, stability of the vertices is not sufficient anymore [3] but stability of all the edges is [4]. A survey of many recent results on this question can be found in [5].

In this note, it is shown that a Kharitonov type result still holds whenever perturbations of even and odd coefficients are uncoupled, i.e., when the set of coefficients Ω can be expressed as the Cartesian product of two compact sets $\mathcal{E}$ and $\mathcal{O}$ in the positive orthant of $\mathbb{R}^k$. More specifically, it is proven that, in such case, stability of an appropriate *small* subset of 'extreme' polynomials guarantees stability of the entire family. In particular, a *polytope* of polynomials with the even-odd uncoupling property is stable provided a certain small subset of its *vertices* is. For the case of an arbitrary subset of polynomials, this result yields a sufficient condition that is less conservative than that provided by Kharitonov's theorem.

In the sequel, the real and imaginary parts of a polynomial $P(j\omega, e, o)$ (with $j = \sqrt{-1}$, $\omega \in \mathbb{R}$) are denoted respectively by $R(\omega, e)$ and $I(\omega, o)$, i.e.,

$$R(\omega, o) = (-1)^{k-1} e_k \omega^{2k-2} + (-1)^{k-2} e_{k-1} \omega^{2k-4} + \ldots + e_1$$

$$I(\omega, o) = (-1)^{k-1} o_k \omega^{2k-1} + (-1)^{k-2} o_{k-1} \omega^{2k-3} + \ldots + o_1 \omega.$$

Throughout, we suppose that we have at our disposal bounds $\bar{\omega}_{\mathcal{E}}$ and $\bar{\omega}_{\mathcal{O}}$ on the real zeros of these polynomials when $[e\ o]$ varies in $\Omega = \mathcal{E} \times \mathcal{O}$, i.e.,

$$R(\omega, e) \neq 0, \quad \forall\ \omega > \bar{\omega}_{\mathcal{E}} \text{ and } \forall\ e \in \mathcal{E}$$

and

$$I(\omega, o) \neq 0, \quad \forall\ \omega > \bar{\omega}_{\mathcal{O}} \text{ and } \forall\ o \in \mathcal{O}.$$

Such bounds can be easily obtained since $e_k > 0$, $\forall\ e \in \mathcal{E}$ and $o_k > 0$, $\forall\ o \in \mathcal{O}$. Given such bounds, a set $\mathcal{E}'$ is said to be *$\mathcal{E}$-sufficient* if $\mathcal{E}' \subset \mathcal{E}$ and, for any $\omega \in [0, \bar{\omega}_{\mathcal{O}}]$,

$$\max_{e \in \mathcal{E}} R(\omega, e) = \max_{e \in \mathcal{E}'} R(\omega, e) \tag{1.a}$$

and

$$\min_{e\in\mathcal{E}} R(\omega,e) = \min_{e\in\mathcal{E}'} R(\omega,e). \tag{1.b}$$

Similarly, a set $\mathcal{O}$ is said to be *$\mathcal{O}$-sufficient* if $\mathcal{O}' \subset \mathcal{O}$ and, for any $\omega \in [0, \bar{\omega}_{\mathcal{E}}]$,

$$\max_{o\in\mathcal{O}} I(\omega,o) = \max_{o\in\mathcal{O}'} I(\omega,o)$$

and

$$\min_{o\in\mathcal{O}} I(\omega,o) = \min_{o\in\mathcal{O}'} I(\omega,o).$$

We call *extreme* point of a set S any point $x \in S$ such that, if $x = \lambda y + (1-\lambda)z$ with $y, z \in S$ and $\lambda \in (0,1)$, then $y = z$. We denote by $E(\mathcal{E})$ and $E(\mathcal{O})$ the set of all extreme points of $\mathcal{E}$ and $\mathcal{O}$ respectively, and by $\mathrm{Co}(\mathcal{E})$ and $\mathrm{Co}(\mathcal{O})$ their convex hulls. Finally, we make use of the 'sign' function

$$\mathrm{Sgn}(\alpha) = \begin{cases} -1 & \alpha < 0 \\ 0 & \alpha = 0 \\ 1 & \alpha > 0. \end{cases}$$

Below, we will invoke a well known theorem of Hermite and Biehler [6,7] (see also [8, p. 228]), stated here using the notations just introduced.

Lemma 1 (Hermite-Biehler Theorem). A vector $[e\ o]$ of the positive orthant of $\mathbb{R}^{2k}$ is strictly Hurwitz if, and only if, (i) $R(\omega,e)$ has $k-1$ nonnegative real roots $\omega_{e,1},\ldots,\omega_{e,k-1}$, (ii) $I(\omega,o)$ has k nonnegative real roots $\omega_{o,1},\ldots,\omega_{o,k}$, and (iii) these roots satisfy the interlacing property

$$\omega_{o,1} = 0 < \omega_{e,1} < \omega_{o,2} < \ldots < \omega_{e,k-1} < \omega_{o,k}. \square$$

2 Main results

Theorem 1 below shows that the $\mathcal{E}$-sufficient and $\mathcal{O}$-sufficient sets play a crucial role in the study of stability. The following lemma will be used in the proof of that theorem.

Lemma 2. Suppose that $\mathcal{E}'$ is $\mathcal{E}$-sufficient and $\mathcal{O}'$ is $\mathcal{O}$-sufficient. Then,

(i)

$$\forall\, o \in \mathcal{O}, \quad \mathcal{E} \times \{o\} \subset H \text{ iff } \mathcal{E}' \times \{o\} \subset H$$

and

(ii)

$$\forall\, e \in \mathcal{E}, \quad \mathcal{O} \times \{e\} \subset H \text{ iff } \mathcal{O}' \times \{e\} \subset H.$$

Proof. We prove that (i) holds. Proving that (ii) is satisfied can be done similarly. Stability of $\mathcal{E}' \times \{o\}$ is obviously necessary for the entire set $\mathcal{E} \times \{o\}$ to be stable. We show that it is also sufficient. Thus, let us suppose that $\mathcal{E}' \times \{o\}$ is stable. From Lemma 1, $I(\omega, o)$ has k nonnegative real roots $\omega_{o,1} = 0 < \omega_{o,2} \ldots < \omega_{o,k}$. Also, since $\mathcal{E}$ is in the positive orthant of $\mathbb{R}^k$, for all $e' \in \mathcal{E}'$, $\text{Sgn}(R(\omega_{o,i}, e')) = (-1)^{i+1}$. Consider now $e \in \mathcal{E}$. We want to show that $[e\ o]$ belongs to H. Since $\mathcal{E}'$ is $\mathcal{E}$-sufficient, we have

$$\min_{e' \in \mathcal{E}'} R(\omega_{o,i}, e') \leq R(\omega_{o,i}, e) \leq \max_{e' \in \mathcal{E}'} R(\omega_{o,i}, e'),$$

so that $\text{Sgn}(R(\omega_{o,i}, e)) = (-1)^{i+1}$. The latter shows that $R(\omega, e)$ has at least one zero in each interval $[\omega_{o,i}, \omega_{o,i+1}], i = 1, \ldots, k-1$. Therefore, in view of Lemma 1, $[e\ o] \in H$. □

Theorem 1. Suppose that $\mathcal{E}'$ is $\mathcal{E}$-sufficient and $\mathcal{O}'$ is $\mathcal{O}$-sufficient. Then $\mathcal{E} \times \mathcal{O} \subset H$ iff $\mathcal{E}' \times \mathcal{O}' \subset H$.

Proof. If $\mathcal{E}' \times \mathcal{O}' \subset H$, then, in view of Lemma 2 (i), $\mathcal{E} \times \mathcal{O}' \subset H$, and, from Lemma 2 (ii), $\mathcal{E} \times \mathcal{O} \subset H$. The converse holds trivially. □

Clearly, $\mathcal{E}$ is $\mathcal{E}$-sufficient and $\mathcal{O}$ is $\mathcal{O}$-sufficient. The remainder of the section is devoted to identifying ever smaller $\mathcal{E}$-sufficient and $\mathcal{O}$-sufficient subsets of $\mathcal{E}$ and $\mathcal{O}$.

Proposition 1. Let $\mathcal{E}'$ and $\mathcal{O}'$ be $\mathcal{E}$-sufficient and $\mathcal{O}$-sufficient, respectively. Then, $E(\mathcal{E}')$ and $E(\mathcal{O}')$ are also $\mathcal{E}$-sufficient and $\mathcal{O}$-sufficient, respectively.

Proof. Given any $\omega \in \mathbb{R}$, $R(\omega, e)$ and $I(\omega, o)$ are linear in e and o, respectively. Thus, clearly, their extremum values are reached on $E(\mathcal{E}')$ and $E(\mathcal{O}')$ respectively. □

In view of Theorem 1, we have the following.

Corollary 1. $\mathcal{E} \times \mathcal{O} \subset H$ iff $\text{E}(\mathcal{E}) \times \text{E}(\mathcal{O}) \subset H$. □

Corollary 2. $\mathcal{E} \times \mathcal{O} \subset H$ iff $\text{Co}(\mathcal{E}) \times \text{Co}(\mathcal{O}) \subset H$. □

Corollaries 1 and 2 can also be obtained from the result in [3] which states that any convex combination of two polynomials that are strictly Hurwitz and have either identical even parts or identical odd parts is also strictly Hurwitz.

Two possible sets $\mathcal{E} \subset \mathbb{R}^2$ are represented in Figures 1.a and 2. In view of Corollary 1, only their extreme points (points on the thick line on Figure 1.a, points $e^1 - e^8$ on Figure 2) need to be considered in the check for stability.

Barlett *et al.* [4], showed that, to check if a polytope is strictly Hurwitz, it is sufficient to explore the edges of that polytope. Corollary 1 shows that in the case of the Cartesian product of two polytopes corresponding to coefficients of even and odd degrees respectively, it is sufficient to check the vertices. For the case when $\mathcal{E} \times \mathcal{O}$ is a hypercube with edges parallel to the coordinates axis, Kharitonov's result [1] states that it is sufficient to check four vertices. The result obtained from Corollary 1, in that case, is much weaker. This suggests that it may be possible to find $\mathcal{E}$-sufficient and $\mathcal{O}$-sufficient sets which are smaller than the sets of extreme points. Some simple tests for finding such smaller $\mathcal{E}$-sufficient sets are now presented. Smaller $\mathcal{O}$-sufficient sets can be obtained similarly.

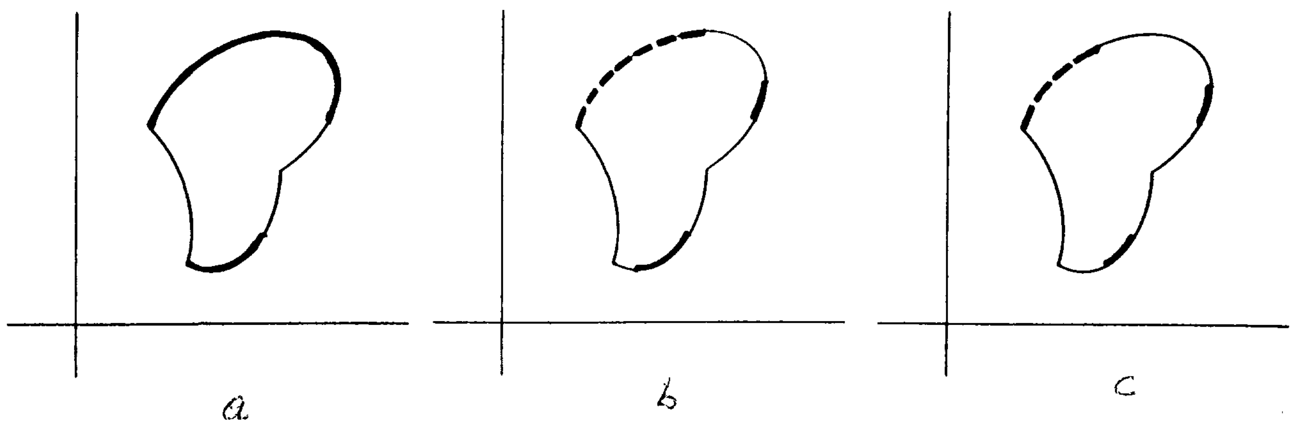

Figure 1: Reduction of set $\mathcal{E}$ using Propositions 1-3

As a first observation, note that if an element e of an $\mathcal{E}$-sufficient set $\mathcal{E}'$ is such that

$$\exists\ e^1,\ e^2 \neq e \in \mathcal{E}' \text{ s.t. } R(e^1,\omega) \leq R(e,\omega) \leq R(e^2,\omega) \quad \forall\ \omega \in \mathbb{R} \tag{2}$$

the set $\mathcal{E}'\backslash\{e\}$ is still $\mathcal{E}$-sufficient. Indeed, Relationships (1) trivially hold on that reduced subset. A particular case of (2) can be tested through a simple comparison between the components of e and those of the other elements in $\mathcal{E}'$. Thus the following holds.

Proposition 2. Let $\mathcal{E}'$ be $\mathcal{E}$-sufficient and let S be a subset of $\mathcal{E}'$ such that

$$\forall\ e \in S,\ \exists\ e^1,\ e^2 \in \mathcal{E}' \text{ s.t. } e^1 \neq e \neq e^2 \ \text{ and } (-1)^{j+1}e^1_j \leq\ (-1)^{j+1}e_j \leq (-1)^{j+1}e^2_j, \quad j = 1,\ldots,k.$$

Then, $\mathcal{E}'' \equiv \mathcal{E}'\backslash S$ is also $\mathcal{E}$-sufficient. □

Remark that when $\mathcal{E}$ is a hypercube with edges parallel to the coordinate axis, it is easy to obtain, using Proposition 2, an $\mathcal{E}$-sufficient set consisting of only two points. If $\mathcal{O}$ is also such a hypercube, it is therefore sufficient to check the stability of four polynomials (2×2). This is exactly Kharitonov's result. Also, note that, if $\mathcal{E}'$ have a very large number of uniformly distributed vertices, the cardinality of $\mathcal{E}''$ is roughly $\frac{1}{2^{k-1}}$ times that of $\mathcal{E}'$. To see this, consider the case when $\mathcal{E}$ is a sphere. Given any point $e \in \mathcal{E}$, let $\tilde{e} = e - c$ where c is the center of the sphere $\mathcal{E}$. Then,

$$(-1)^{j+1}((-1)^j|\tilde{e}_j|) = -|\tilde{e}_j| \leq (-1)^{j+1}\tilde{e}_j \leq |\tilde{e}_j| = (-1)^{j+1}((-1)^{j+1}|\tilde{e}_j|).$$

Thus, in view of Proposition 2, the set

$$\{e \in \mathcal{E} \text{ s.t. } \mathrm{Sgn}(\tilde{e}_j) = (-1)^j,\ j = 1,\ldots,k\} \cup \{e \in \mathcal{E} \text{ s.t. } \mathrm{Sgn}(\tilde{e}_j) = (-1)^{j+1},\ j = 1,\ldots,k\}$$

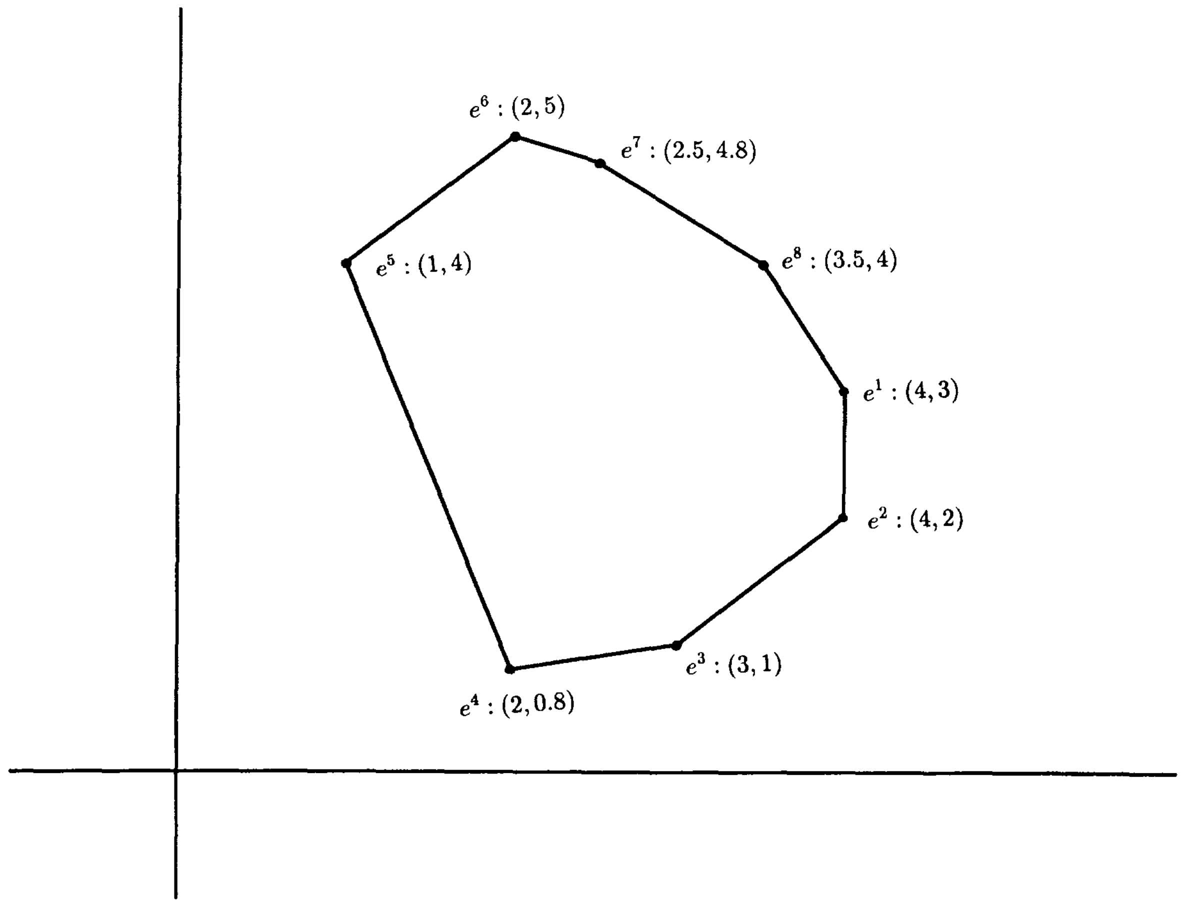

Figure 2: Reduction of set $\mathcal{E}$ in polytope case

is $\mathcal{E}$-sufficient. Similar considerations apply to $\mathcal{O}''$ compared to to $\mathcal{O}'$. Thus, if both $\mathcal{E}'$ and $\mathcal{O}'$ have a very large number of uniformly distributed vertices, the cardinality of $\mathcal{E}'' \times \mathcal{O}''$ is roughly $\frac{1}{2^{2k-2}}$ that of $\mathcal{E}' \times \mathcal{O}'$.

Making use of Proposition 2 and referring back to Figure 1, it is possible to reduce the $\mathcal{E}$-sufficient set $\mathcal{E}' = E(\mathcal{E})$ to the points on the thick line (corresponding to possible maxima in (1.a)) and the points on the dashed line (corresponding to possible minima in (1.b)) in Figure 1.b. A similar elimination on Figure 2 shows that only points e^2, e^3, e^4 (corresponding to possible maxima in (1.a)) and e^5, e^6 (corresponding to possible minima in (1.b) need be considered for the stability test.

Another simple consideration leads to eliminating more points from an $\mathcal{E}$-sufficient set $\mathcal{E}'$. If e is a point of $\mathcal{E}'$ corresponding to a maximizer in (1.a) (resp. a minimizer in (1.b)) for some $\omega' \in [0, \bar{\omega}_{\mathcal{O}}]$, then there exists a vector d in $\mathbb{R}^k$ of the form

$$d = [d_1 \; d_2 \; \dots \; d_k] \text{ s.t. } d_1 = 1, \; |d_j| \le \bar{\omega}_{\mathcal{O}}^{2(j-1)}, \text{ and } \mathrm{Sgn}(d_j) = -\mathrm{Sgn}(d_{j-1}), \; j = 2, \dots, k \quad (3)$$

satisfying

$$\langle d, e\rangle \ \geq \ \langle d, e'\rangle, \ \forall \ e' \in \mathcal{E}' \tag{4.a}$$

(resp.

$$\langle d, e\rangle \ \leq \ \langle d, e'\rangle, \ \forall \ e' \in \mathcal{E}') \tag{4.b}$$

where $\langle \cdot \ , \ \cdot\rangle$ represents the usual inner product in $\mathbb{R}^k$. In fact, such a vector is given by $d = (1, -\omega'^2, \omega'^4, \ldots, (-1)^{k-1}\omega'^{2k-2})$. Therefore, any point $e \in \mathcal{E}'$ for which there exists no vector d of the form (3) satisfying either (4.a) or (4.b) can be discarded. Thus the following holds.

Proposition 3. Let $\mathcal{E}'$ be $\mathcal{E}$-sufficient and let $S \subset \mathcal{E}'$ be such that, any vector $d \in \mathbb{R}^k$ satisfying

$$d_1 = 1$$

and

$$0 \leq (-1)^{j-1} d_j \leq \bar{\omega}_{\mathcal{O}}^{2(j-1)}, \quad j = 2, \ldots, k$$

also satisfies

$$\max_{e\in S} \min_{e'\in\mathcal{E}'} \langle d \ , \ e - e'\rangle \ < \ 0$$

and

$$\min_{e\in S} \max_{e'\in\mathcal{E}'} \langle d \ , \ e - e'\rangle \ > \ 0.$$

Then, $\mathcal{E}'' \equiv \mathcal{E}' \backslash S$ is also $\mathcal{E}$-sufficient. □

Suppose now that in the case of Figure 1, the value of $\bar{\omega}_{\mathcal{O}}$ is 2. After the second elimination process, only the points on the thick line (corresponding to possible maxima in (1.a)) and the points on the dashed line (corresponding to possible minima in (1.b)) in Figure 1.c will remain. If for the case of Figure 2, $\bar{\omega}_{\mathcal{O}} < \sqrt{5}$, the elimination process will eliminate the point e^4.

3 Examples

Example 1. As a first example, consider the family of polynomials of the form

$$s^3 + e_2 s^2 + o_1 s + e_1$$

where $[e_1 \ e_2]$ is an $\mathbb{R}^2$ vector varying in the shaded area depicted in Figure 3 and o_1 is a scalar varying in the interval [3.9,4.5]. We would like to determine whether the polynomials in that family are strictly Hurwitz. The bigger family of polynomials obtained by considering all the coefficients $[e_1 \ e_2]$ in the smallest hypercube with edges parallel to the axes, containing all these coefficients, is not stable. Indeed, one of the Kharitonov's polynomials, $s^3 + 0.8s^2 + 3.9s + 4$, is not strictly Hurwitz. We now consider the family obtained by drawing a small polytope containing all the

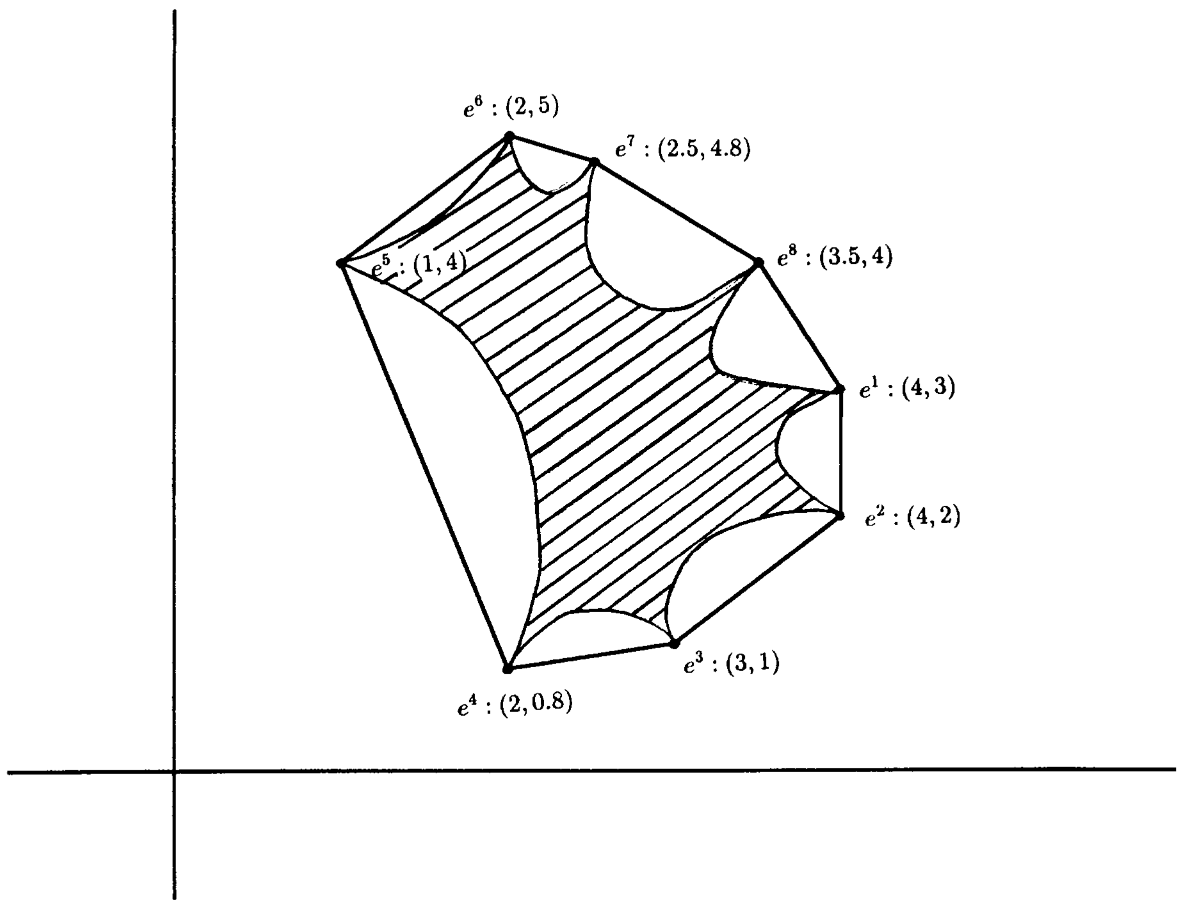

Figure 3: Set $\mathcal{E}$ for Example 1

coefficients $[e_1\ e_2]$ (see Figure 3). It is straightforward that $\bar{\omega}_{\mathcal{O}} = \sqrt{4.5}$ is an acceptable bound. Therefore, in view of the results of Section 2, it is sufficient to check the stability of the following extreme polynomials:

$$s^3 + 2s^2 + 3.9s + 4; \quad s^3 + s^2 + 3.9s + 3;$$

$$s^3 + 4s^2 + 3.9s + 1; \quad s^3 + 5s^2 + 3.9s + 2;$$

$$s^3 + 2s^2 + 4.5s + 4; \quad s^3 + s^2 + 4.5s + 3;$$

$$s^3 + 4s^2 + 4.5s + 1; \quad s^3 + 5s^2 + 4.5s + 2.$$

All these polynomials happen to be strictly Hurwitz. We can therefore conclude that the entire family of polynomials is strictly Hurwitz. □

When the coefficients of even and odd powers vary dependently, i.e., the coefficient set Ω is not the Cartesian product of even and odd coefficient sets, one may either apply Kharitonov's test on the smallest hypercube with edges parallel to the axes, containing Ω, or apply the results of Section 2 on the smallest polytope which contains Ω and can be expressed as the Cartesian product of some even and odd coefficient sets. For both cases, the stability test becomes only sufficient. However, it is obvious that the latter test is less conservative. The next example illustrates this fact.

Example 2. Consider the family of polynomials

$$p(z) = z^3 + a_1 z^2 + a_2 z + a_3$$

where a_1, a_2 and a_3 are only known to belong to the following intervals:

$$a_1 \in [0, 0.3], \quad a_2 \in [0, 0.4], \quad a_3 \in [0, 0.5]. \tag{5}$$

We would like to determine whether all the roots of $p(z)$ lie in the open unit disk of the z-plane for all vectors $[a_1\ a_2\ a_3]$ satisfying (5). It is easy to see that this is equivalent to checking whether $q(s)$ is strictly Hurwitz for all vectors $[a_1\ a_2\ a_3]$ satisfying (5), where $q(s)$ is defined by

$$q(s) = p(\frac{s+1}{s-1})(s-1)^3$$

$$= (1 + a_1 + a_2 + a_3)s^3 + (3 + a_1 - a_2 - 3a_3)s^2 + (3 - a_1 - a_2 + 3a_3)s + (1 - a_1 + a_2 - a_3)\ . \tag{6}$$

Since the polynomial

$$2.2s^3 + 1.1s^2 + 2.3s + 1.4$$

which is one of the polynomials corresponding to all vertices of the smallest hypercube containing the coefficient set, is not strictly Hurwitz, Kharitonov's test fails. We now conservatively assume that the coefficients of even and odd powers in (6) vary independently. It is clear that $\mathcal{E}$ and $\mathcal{O}$ (the set of all possible pairs of coefficients of even degree and the set of possible pairs of coefficients of odd degree) are both polytopes and that their vertices correspond to combinations of end points of the intervals in (5). Thus the subset of $\mathcal{E}'$ (resp. $\mathcal{O}'$) corresponding to all 8 such combinations is $\mathcal{E}$-sufficient (resp. $\mathcal{O}$-sufficient). Starting from these sets, an elimination process based on Proposition 2 reduces to the following 16 the set of polynomials to be checked for stability.

$$\begin{array}{llll}
1: & 1.7s^3+3.3s^2+2.3s+0.7; & 2: & 1.7s^3+1.8s^2+2.3s+0.2; \\
3: & 1.7s^3+2.6s^2+2.3s+1.4; & 4: & 1.7s^3+1.1s^2+2.3s+0.9; \\
5: & 2.2s^3+3.3s^2+3.8s+0.7; & 6: & 2.2s^3+1.8s^2+3.8s+0.2; \\
7: & 2.2s^3+2.6s^2+3.8s+1.4; & 8: & 2.2s^3+1.1s^2+3.8s+0.9; \\
9: & 1.0s^3+3.3s^2+3.0s+0.7; & 10: & 1.0s^3+1.8s^2+3.0s+0.2; \\
11: & 1.0s^3+2.6s^2+3.0s+1.4; & 12: & 1.0s^3+1.1s^2+3.0s+0.9; \\
13: & 1.5s^3+3.3s^2+4.5s+0.7; & 14: & 1.5s^3+1.8s^2+4.5s+0.2; \\
15: & 1.5s^3+2.6s^2+4.5s+1.4; & 16: & 1.5s^3+1.1s^2+4.5s+0.9;
\end{array}$$

A rough bound $\bar{\omega}_{\mathcal{E}} = 1.2$ can be obtained as the maximum of $(1 - a_1 + a_2 - a_3)$ over all possible coefficients divided by the minimum of $(3 + a_1 - a_2 - 3a_3)$ over these same coefficients. A bound $\overline{\omega}_{\mathcal{O}} = 2.2$ can be obtained in a similar fashion. In view of Proposition 3, it can be checked that only polynomials 1-4 and 13-16 are of interest for the stability test. Since they are all strictly Hurwitz, the roots of every polynomial $p(z)$ with coefficients $[a_1\ a_2\ a_3]$ satisfying (5) lie in the open unit disk of the z-plane. □

An application of practical importance in the context of dynamical systems is the question of determining whether all matrices in a given family are stable, i.e., have all their eigenvalues in the open left half plane. As the eigenvalues of a matrix are the roots of its characteristic polynomial, this question can be considered in the light of the results obtained in Section 2. This is the object of our third and last example.

Example 3. Consider the family of matrices

$$A = \begin{bmatrix} -\alpha_1 & 1 & 1 \\ 1 & -\alpha_2 & 1 \\ 1 & 1 & -\alpha_3 \end{bmatrix}$$

where α_1, α_2, α_3 are only known to lie in the interval $[m, M]$ of the positive real line. The characteristic polynomial is given by

$$\chi_A(s) = s^3 + e_2 s^2 + o_1 s + e_1$$

with

$$e_1 = \alpha_1\alpha_2\alpha_3 - (\alpha_1 + \alpha_2 + \alpha_3) - 2$$

$$e_2 = \alpha_1 + \alpha_2 + \alpha_3$$

$$o_1 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 - 3.$$

The set Ω of possible triples $[e_1\ e_2\ o_1]$ is contained in $\tilde{\Omega} = \mathcal{E} \times \mathcal{O}$, with

$$\mathcal{E} = \{[e_1\ e_2] \text{ s.t. } e_1 = \alpha_1\alpha_2\alpha_3 - (\alpha_1 + \alpha_2 + \alpha_3) - 2;\ e_2 = \alpha_1 + \alpha_2 + \alpha_3;\ \alpha_i \in [m, M],\ i = 1,2,3\}.$$

and

$$\mathcal{O} = \{[o_1\ 1] \text{ s.t. } \underline{o} \leq o_1 \leq \bar{o}\}$$

where $\underline{o} = 3m^2 - 3$ and $\bar{o} = 3M^2 - 3$. The set $\mathcal{E}$ is pictured on Figure 4. In view of Corollary 1, $\tilde{\Omega} \subset H$ if, and only if,

$$\{\text{A, B, C, D}\} \times \{[\underline{o}\ 1],\ [\bar{o}\ 1]\} \subset H, \tag{7}$$

a condition involving 8 polynomials. *Sufficient* conditions[1] for this to hold are that

$$\{\text{A, E, D}\} \times \{[\underline{o}\ 1],\ [\bar{o}\ 1]\} \subset H, \tag{8}$$

or

$$\{\text{A, F, D}\} \times \{[\underline{o}\ 1],\ [\bar{o}\ 1]\} \subset H. \tag{9}$$

Both of these conditions involve only 6 polynomials and (9) is particularly easy to check. Kharitonov's theorem applied to the set

$$\Omega_K = \{[e_1\ e_2\ o_1\ 1] \text{ s.t. } e_1 \in [m^3 - 3m - 2, M^3 - 3M - 2];\ e_2 \in [3m, 3M];\ o_1 \in [\underline{o}, \bar{o}]\}$$

gives the sufficient condition

$$\{\text{F, G}\} \times \{[\underline{o}\ 1],\ [\bar{o}\ 1]\} \subset H. \tag{10}$$

If, e.g., $m = 3$ and $M = 7$, (7) holds (as well as (8) and (9)) while (10) does not as

$$\{\text{G}\} \times \{[\underline{o}\ 1]\}$$

is unstable. Qualitatively similar considerations are likely to apply in more general situations, with matrices of higher dimensions. □

[1] In this particular case of polynomials of degree 3, using ideas similar to those in Anderson *et al.* [9], it is easy to see that these conditions are also necessary.

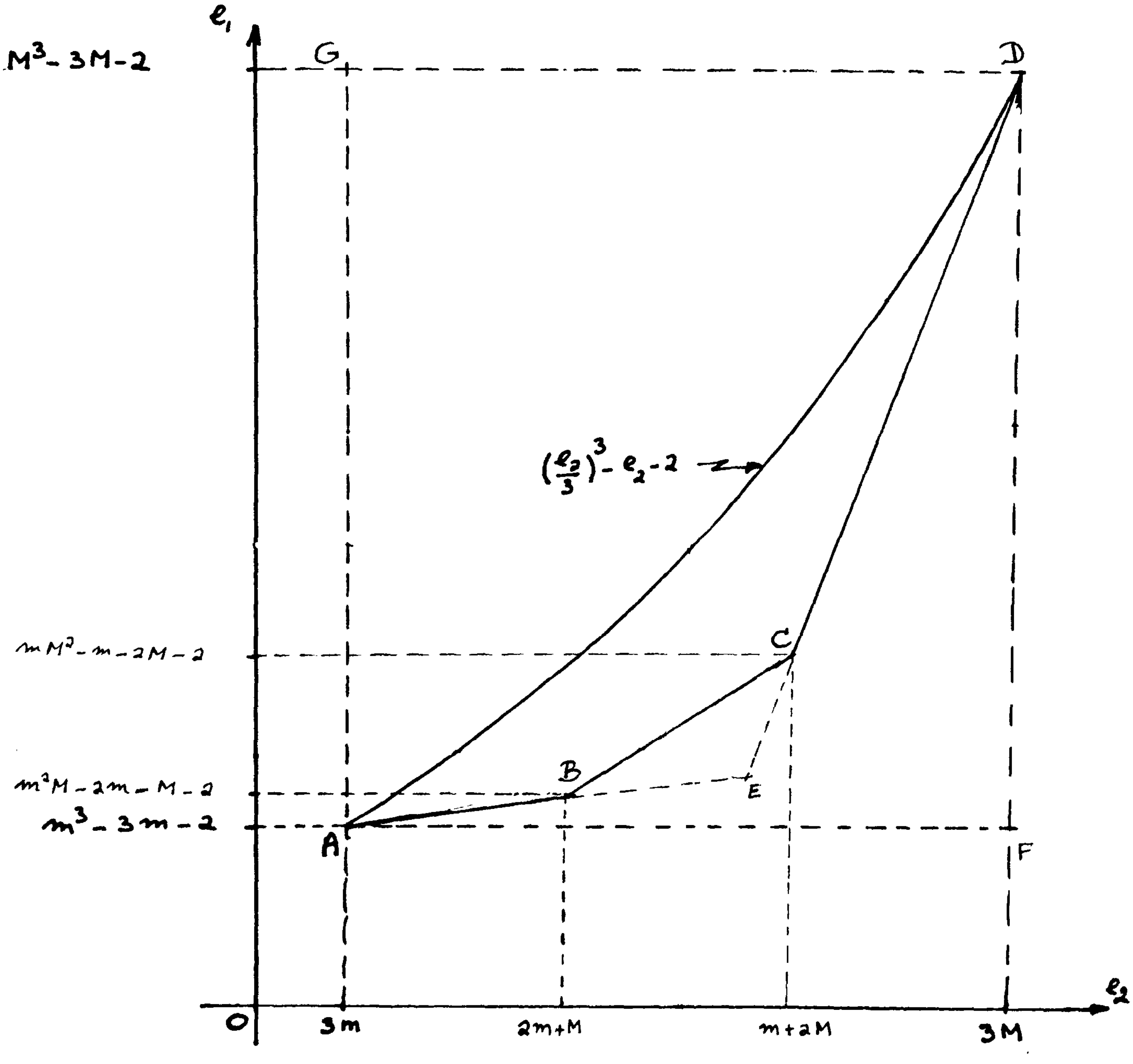

Figure 4: Set $\mathcal{E}$ for Example 3

4 Discussion

Suppose that $\mathcal{E}$ and $\mathcal{O}$ are polytopes. Then $\mathcal{E} \times \mathcal{O}$ is in H, provided a certain finite subset is in H, which can be assessed by performing finitely many Routh-Hurwitz type tests. In view of Corollaries 1 and 2, this in fact is the case whenever Co($\mathcal{E}$) and Co($\mathcal{O}$) are polytopes (see, e.g., Examples 1 and 3 in Section 3). The number of polynomials to be considered can then be reduced by making use of Propositions 2 and 3 as follows. First, it is readily checked whether selected vertices do belong to a set S as in Proposition 2, and thus can be discarded. Roughly, this will eliminate all but a $\frac{1}{2^{2k-2}}$ fraction of the vertices. Second, according to Proposition 3, one can sequentially discard vertices of $\mathcal{E}$ (resp. of $\mathcal{O}$) for which the corresponding system of linear equations and inequalities (3)-(4) is inconsistent. Note that it may not always be appropriate to perform the latter operation. Indeed, the cost of checking consistency of (3)-(4) with the hope of possibly discarding a vertex of $\mathcal{E}$ (resp. of $\mathcal{O}$) should first be weighted against the cost of directly performing Routh-Hurwitz type tests on the polynomials corresponding to that vertex (and all remaining vertices of $\mathcal{O}$ (resp. of $\mathcal{E}$)), with a possibility of immediate termination of the overall search (in case one of these polynomials is unstable). To be taken into account when assessing the relative complexity of these options are the degree of the polynomials involved, the current number of vertices, and the degree of dependency among the coefficients. Further investigation of this question is in order.

Suppose now that at least one of the sets Co($\mathcal{E}$) or Co($\mathcal{O}$) is not a polytope (or is not a *priori* known as such). In view of Corollary 2 and since convex sets can be 'outer-approximated' arbitrarily closely by polytopes, one could proceed as follows. First, replace Co($\mathcal{E}$) and Co($\mathcal{O}$) by polytopes $\mathcal{E}_0 \supset$ Co($\mathcal{E}$) and $\mathcal{O}_0 \supset$ Co($\mathcal{O}$). If $\mathcal{E}_0 \times \mathcal{O}_0$ is in H, Ω is also in H and the search is complete. Otherwise, reduce $\mathcal{E}_0$ to $\mathcal{E}_1$ and $\mathcal{O}_0$ to $\mathcal{O}_1$ by 'cutting' them with supporting hyperplanes to Co($\mathcal{E}$) and Co($\mathcal{O}$) respectively. If any pair of 'contact points' with Co($\mathcal{E}$) and Co($\mathcal{O}$) is not in H, then, in view of Corollary 2, Ω is not in H and, again, the search is complete. Otherwise, similarly generate $\mathcal{E}_2$, $\mathcal{E}_3$, ... and, $\mathcal{O}_2$, $\mathcal{O}_3$, ... When $\mathcal{E}_i \times \mathcal{O}_i$ is 'close enough' to Co($\mathcal{E}$) $\times$ Co($\mathcal{O}$), terminate the search, concluding that some of the polynomials of interest are at best 'marginally stable'.

Finally, as pointed out in our examples, given any compact set $\Omega \subset \mathbb{R}^{2k}$, *sufficient* conditions of stability can be obtained using either Kharitonov's theorem or the results presented in this paper. The former will provide a necessary and sufficient condition for stability of the set $\Omega_K \supset \Omega$ given by

$$\Omega_K = \prod_{i=1}^{k} E_i \times \prod_{i=1}^{k} O_i$$

with

$$E_i = \{s \in \mathbb{R} \text{ s.t. } \exists\ [e\ o] \in \Omega \text{ with } e^i = s\},\ i = 1, \ldots, k$$

$$O_i = \{s \in \mathbb{R} \text{ s.t. } \exists \, [e \; o] \in \Omega \text{ with } o^i = s\}, \; i = 1, \ldots, k.$$

The latter will provide a necessary and sufficient condition for stability of the set $\tilde{\Omega} = \mathcal{E} \times \mathcal{O} \supset \Omega$, with

$$\mathcal{E} = \{e \in \mathbb{R}^k \text{ s.t. } \exists \, o \in \mathbb{R}^k \text{ with } [e \; o] \in \Omega\}$$

$$\mathcal{O} = \{o \in \mathbb{R}^k \text{ s.t. } \exists \, e \in \mathbb{R}^k \text{ with } [e \; o] \in \Omega\}.$$

It is clear that, in most cases, $\tilde{\Omega}$ is a proper subset of Ω_K, so that (as was the case in all three examples of Section 3) conservativeness is likely to be reduced if one uses the results presented in this paper.

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