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**Graph Bipartization and Via  
Minimization**

**by**

**H. Choi, K. Nakajima, and C.S. Rim**

# Graph Bipartization and Via Minimization

**Hyeong-Ah Choi**<sup>1</sup>

Department of Computer Science  
Michigan State University,  
East Lansing, MI 48824

**Kazuo Nakajima**<sup>2</sup>

Electrical Engineering Department,  
Institute for Advanced Computer Studies,  
and Systems Research Center  
University of Maryland  
College Park, MD 20742

**Chong S. Rim**<sup>2</sup>

Electrical Engineering Department,  
and Systems Research Center  
University of Maryland  
College Park, MD 20742

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## Abstract

The vertex- (resp., edge-) deletion graph bipartization problem is the problem of deleting a set of vertices (resp., edges) from a graph so as to make the remaining graph bipartite. In this paper, we first show that the vertex-deletion graph bipartization problem has a solution of size  $k$  or less if and only if the edge-deletion graph bipartization problem has a solution of size  $k$  or less, when the maximum vertex degree is limited to three. This immediately implies that (1) the vertex-deletion graph bipartization problem is NP-complete for cubic graphs, and (2) the minimum vertex-deletion graph bipartization problem is solvable in polynomial time for planar graphs when the maximum vertex degree is limited to three. We then prove that the vertex-deletion graph bipartization problem is NP-complete for planar graphs when the maximum vertex degree exceeds three. Using this result, we finally show that the via minimization problem, which arises in the design of integrated circuits and printed circuit boards, is NP-complete even when the maximum “junction” degree is limited to four.

**Key words.** computational complexity, maximum bipartite subgraphs, NP-completeness, via minimization

**AMS(MOS) subject classifications.** 68C25, 68E10

# 1. Introduction

A *graph*  $G$  is an ordered pair  $(V, E)$  where  $V$  is a set of elements, called *vertices* and  $E$  is a set of unordered pairs of distinct vertices in  $V$ , called *edges*. If  $(u, v) \in E$ , vertices  $u$  and  $v$  are said to be *adjacent* to each other, and the edge  $(u, v)$  is said to be *incident* upon each of  $u$  and  $v$ . A sequence of vertices  $[v_{i_1}, v_{i_2}, \dots, v_{i_k}]$  is called a *path* from  $v_{i_1}$  to  $v_{i_k}$  in  $G$  if the vertices are all distinct and  $(v_{i_j}, v_{i_{j+1}}) \in E$  for  $j = 1, 2, \dots, k - 1$ . Furthermore, if  $v_{i_1} = v_{i_k}$ , the path is called a *cycle*. The length of a path (or cycle) is the number of edges on the path (resp., cycle). If the length of a cycle is odd (resp., even), it is called an *odd* (resp., *even*) *cycle*. The degree of vertex  $v$ , denoted by  $d_G(v)$ , is the number of edges incident upon  $v$  in  $G$ . Let  $\Delta(G) = \max_{v \in V} \{d_G(v)\}$ .

Given a graph  $G = (V, E)$ , we can partition  $V$  into subsets  $V_1, V_2, \dots, V_r$  such that vertices  $u$  and  $v$  belong to  $V_i$  if and only if there is a path from  $u$  to  $v$  in  $G$ . Let  $E(V_i) = \{(u, v) \in E \mid u, v \in V_i\}$  for  $i = 1, 2, \dots, r$ . The graphs  $G_i = (V_i, E(V_i))$  are called *connected components* of  $G$ .

A graph  $G = (V, E)$  is called a *planar graph* if it can be drawn in the plane in such a way that each vertex in  $V$  is represented by a point; each edge  $(u, v) \in E$  is represented by a continuous line connecting the two points which represent  $u$  and  $v$ ; and no two lines, which represent edges, share any points, except in their ends. Such a drawing is called a *planar embedding* of  $G$ . A graph  $G = (V, E)$  is called a *bipartite graph* if  $V$  can be partitioned into two nonempty subsets  $V_1$  and  $V_2$  such that  $V_1 \cap V_2 = \phi$  and no two vertices

in the same subset are adjacent. It is well known [4] that a graph  $G$  is bipartite if and only if there is no odd cycle in  $G$ .

For a subset  $V' \subset V$  of  $G = (V, E)$ , the graph obtained by deleting from  $G$  all vertices in  $V'$  and all the edges incident upon them is called a *vertex-deleted subgraph* of  $G$  and is denoted by  $G^v(V - V')$ . Namely,  $G^v(V - V') = (V - V', E(V - V'))$  where  $E(V - V') = \{(u, v) \in E \mid u, v \in V - V'\}$ . Likewise, for a subset  $E' \subset E$  of  $G = (V, E)$ , an *edge-deleted subgraph* of  $G$ , denoted by  $G^e(E - E')$ , is obtained by deleting all edges in  $E'$  from  $G$ ; that is,  $G^e(E - E') = (V, E - E')$ .

Given a graph  $G = (V, E)$  and an integer  $k \geq 0$ , the *vertex-* (resp., *edge-*) *deletion graph bipartization problem*, abbreviated to VDB (resp., EDB), is the problem of finding a set of  $k$  or fewer vertices  $V' \subset V$  (resp., edges  $E' \subset E$ ) in  $G$  such that the subgraph  $G^v(V - V')$  (resp.,  $G^e(E - E')$ ) is bipartite, or equivalently, free of odd cycles. The *minimum vertex-* (resp., *edge-*) *deletion graph bipartization problem*, abbreviated to MVDB (resp., MEDB), is the problem of finding such a vertex (resp., edge) set of minimum cardinality.

Garey, Johnson and Stockmeyer [7] and Yannakakis [17] proved that the EDB problem is NP-complete even if  $G = (V, E)$  is a cubic graph, i.e.,  $d_G(v) = 3$  for every  $v \in V$ . On the other hand, Hadlock [8,1] showed that the MEDB problem is solvable in  $O(|V|^3)$  time if the graph is planar. As for the VDB problem, combined results of Garey and Johnson [6] and Krishnamoorthy and Deo [11] imply that it is NP-complete even if  $G$  is planar and  $\Delta(G) = 6$ . Given a graph  $G = (V, E)$ , the *line graph*, denoted by  $G_l = (V_l, E_l)$ , of  $G$  is defined as follows: There is a one-to-one correspondence between  $V_l$  and  $E$ . If two edges in

$E$  are incident upon the same vertex in  $G$ , then there is an edge in  $E_l$  which connects the two corresponding vertices in  $V_l$ . One can easily show by constructing the line graph of a cubic graph that the VDB problem is NP-complete for a general graph  $G$  with  $\Delta(G) = 4$ .

The first goal of this paper is to present complete complexity results for the VDB problem. We arrive at this by first establishing a relationship between the VDB and EDB problems for a graph  $G = (V, E)$  with  $\Delta(G) \leq 3$ . Namely, we prove that the VDB problem has a solution  $V' \subset V$  with  $|V'| \leq k$  if and only if the EDB problem has a solution  $E' \subset E$  with  $|E'| \leq k$ . Since the EDB problem is NP-complete for cubic graphs [7,17], this immediately implies that the VDB problem is NP-complete for cubic graphs. Furthermore, since the MEDB problem is solvable in  $O(|V|^3)$  time if  $G = (V, E)$  is planar [8,1], this relationship also implies that the MVDB problem is solvable in  $O(|V|^3)$  time for the case when  $G$  is planar and  $\Delta(G) \leq 3$ . It should further be noted that if  $\Delta(G) \leq 2$ , the graph  $G$  is always planar and hence the VDB problem is solvable in polynomial time, and in fact, in  $O(|V|)$  time. Finally, we prove that the VDB problem becomes NP-complete for a planar graph  $G$  if  $\Delta(G) = 4$ . We give a polynomial transformation from the Planar 3-Satisfiability problem [12] to this problem. The complete complexity results for the VDB problem are summarized in Table 1.

The second goal of this paper is to show the NP-completeness of the *via minimization problem* which arises in the design of integrated circuits and printed circuit boards. Given a set of wire segments and two layers for routing, the problem is to assign the segments to one of the layers so that the number of vias required to electrically connect the segments

$G$	$\Delta(G)$		
	$\leq 2$	$= 3$	$\geq 4$
General	$O( V )$	NPC	NPC
Planar	$O( V )$	$O( V ^3)$	NPC

Table 1: Complexity Results for the VDB Problem

which are assigned to different layers is minimized.

In 1971, Hashimoto and Stevens [9] first considered this problem for the case of grid based layouts with the maximum “junction” degree being limited to two, where a *grid based layout* is a layout in which all wire segments are placed in parallel to one of the two perpendicular axes, and “*junction degree*” is defined to be the number of wire segments which meet at a single point and which are to be electrically connected. In 1980, Kajitani [10] showed that the problem is solvable in polynomial time for this particular case. Later Chen, Kajitani and Chan [2] and Pinter [16] extended its polynomial solvability for grid-based layouts to the case of the maximum junction degree three. Quite recently, Molitor [13] and Naclerio, Masuda, and Nakajima [14] showed that the via minimization problem is solvable in polynomial time even for general layouts if the maximum junction degree is limited to three.

On the other hand, Naclerio, Masuda, and Nakajima [15] showed that the via minimization decision problem is NP-complete if the maximum junction degree exceeds five.

Thus, the cases of the maximum junction degrees 4 and 5 have been left as open problems. In this paper, using the NP-completeness result for the VDB problem for planar graphs, we prove that the via minimization decision problem is NP-complete even when the maximum junction degree is limited to four.

In Section 2, we first show that the MVDB problem for a general graph  $G = (V, E)$  is easily solvable in  $O(|V|)$  time if  $\Delta(G) \leq 2$ . We then prove that if  $\Delta(G) = 3$ , the VDB problem has a solution of size  $k$  or less if and only if the EDB problem has a solution of size  $k$  or less. As corollaries of this result, we show that (1) the VDB problem is NP-complete even for cubic graphs, and (2) the MVDB problem is solvable in  $O(|V|^3)$  time when  $G$  is planar and  $\Delta(G) = 3$ . Finally, using a transformation from the Planar 3-Satisfiability problem [12], we prove that the VDB problem becomes NP-complete for a planar graph  $G$  if  $\Delta(G) = 4$ . In Section 3, we consider the via minimization problem for two layers. We show how the VDB problem for planar graphs can be transformed to this problem. This leads us to prove that the via minimization decision problem is NP-complete even when the maximum junction degree is limited to four. Section 4 concludes the paper with some further comments on the via minimization problem.

## 2. Graph Bipartization

In this section, we investigate the computational complexity of the VDB problem from the point of view of vertex degree constraints. We first show that there is a close relationship between the VDB and EDB problems for a graph  $G = (V, E)$  with  $\Delta(G) \leq 3$ . If



$\Delta(G) \leq 2$ , each connected component of the graph  $G$  is either a single vertex, a path or a cycle. Therefore, the MVDB and MEDB problems can be solved by first finding all odd cycles and then selecting an arbitrary vertex and edge, respectively, from each such cycle. This can easily be done in  $O(|E|)$  time, which is in fact,  $O(|V|)$  time, since  $G$  is planar in this case. We now consider the case of  $\Delta(G) = 3$ , and establish a close relationship between the VDB and EDB problems.

**Theorem 1.** *For any graph  $G = (V, E)$  with  $\Delta(G) = 3$ , there exists a subset  $E' \subset E$  such that  $|E'| \leq k$  and  $G^e(E - E')$  is bipartite if and only if there exists a subset  $V' \subset V$  such that  $|V'| \leq k$  and  $G^v(V - V')$  is bipartite.*

**Proof.** Suppose that there exists a subset  $E' \subset E$  with  $|E'| \leq k$  such that  $G^e(E - E')$  is bipartite. Let  $V' = \{v \in V \mid \text{for each edge } (u, w) \in E', v \text{ is either } u \text{ or } w\}$ . It is clear that  $|V'| \leq |E'| \leq k$  and  $G^v(V - V')$  is bipartite.

Conversely, suppose that there exists a subset  $V' \subset V$  with  $|V'| \leq k$  such that  $G^v(V - V')$  is bipartite. First, we construct, if necessary, a new graph  $\hat{G} = (\hat{V}, \hat{E})$  from  $G = (V, E)$  such that (1) for any pair of vertices  $u, w \in V'$ ,  $(u, w) \notin \hat{E}$  and (2)  $\hat{G}^v(\hat{V} - V')$  is bipartite. Let  $E_p = \{(u, w) \in E \mid u, w \in V'\}$ . If  $E_p = \phi$ , we do nothing, *i.e.*,  $\hat{G} = G$ . Otherwise, we replace each edge  $(u, w) \in E_p$  by a path of length three  $\{u, u', w', w\}$  in  $\hat{G}$ . More precisely,  $\hat{V} = V \cup \{u', w' \mid (u, w) \in E_p\}$  and  $\hat{E} = E - E_p \cup \{(u, u'), (u', w'), (w', w) \mid (u, w) \in E_p\}$ . Note that any odd (resp., even) cycle containing  $(u, w) \in E_p$  in  $G$  remains odd (resp., even) in  $\hat{G}$ , and hence  $\hat{G}^v(\hat{V} - V')$  remains bipartite. Furthermore,  $\Delta(\hat{G}) = 3$ .

Next, we obtain, from  $V'$ , a subset  $\hat{E}' \subset \hat{E}$  such that  $|\hat{E}'| \leq k$  and  $\hat{G}^e(\hat{E} - \hat{E}')$  is bipartite. Since  $\hat{G}^v(\hat{V} - V')$  is bipartite and for any two vertices  $u, w \in V'$ ,  $(u, w) \notin \hat{E}$ , the vertex set  $\hat{V}$  can be partitioned into three mutually disjoint subsets,  $V'$ ,  $X$ , and  $Y$  such that no two vertices in the same subset are adjacent.

We first remove from  $V'$  those vertices  $v$  with  $d_{\hat{G}}(v) = 1$  because the edges incident upon  $v$  are not in any cycle in  $\hat{G}$ . Let  $V''$  denote the set of the remaining vertices. It is not difficult to see, as illustrated in Fig. 1 (a), that an odd cycle is formed only if there exists a vertex in  $V''$  which is adjacent to a vertex in  $X$  and a vertex in  $Y$  simultaneously. Otherwise, as can be seen from Fig. 1 (b), all cycles are of even length since  $\hat{G}^v(V - V') = \hat{G}^v(X \cup Y)$  is bipartite. Note that if not all of the vertices adjacent to a vertex  $v$  in  $V''$  are in one of the two sets  $X$  and  $Y$ , then either  $X$  or  $Y$  contains exactly one vertex, say  $w$ , since  $d_{\hat{G}}(v) \leq 3$ . It is clear that if we delete this edge  $(v, w)$ , all odd cycles passing through  $v$  will be removed. Therefore, we select all such edges  $(v, w)$  as members of  $\hat{E}'$ . Then  $\hat{G}^e(\hat{E} - \hat{E}')$  is bipartite and  $|\hat{E}'| \leq |V''| \leq |V'| \leq k$ .

Finally, from  $\hat{E}'$  we construct a subset  $E' \subset E$  such that  $|E'| \leq k$  and  $G^e(E - E')$  is bipartite. Let  $e \in \hat{E}'$ . If  $e \in E$ , we put  $e$  into  $E'$ . If  $e \notin E$ , then  $e$  must be of the form  $(u, u')$  such that  $u \in V' \subset V$  and  $u' \in \hat{V} - V$ . In this case, there is a vertex  $w \in V' \subset V$  such that  $(u, w) \in E$ , and we put this edge  $(u, w)$  into  $E'$ . The deletion of the edge  $(u, u')$  from  $\hat{G}$  has the same effect on the elimination of cycles as that of its corresponding unique path of length three from  $u$  to  $w$  in  $\hat{G}$ . Thus, the deletion of the edge  $(u, w)$  from  $G$  eliminates all cycles passing through  $u$ . Therefore,  $G^e(E - E')$  is bipartite,

and furthermore,  $|E'| \leq |\hat{E}'| \leq k$ . This completes the proof of the theorem.  $\square$

Garey, *et. al* [7] and Yannakakis [17] showed that the EDB problem is NP-complete even if a graph  $G = (V, E)$  is a cubic graph, *i.e.*,  $d_G(v) = 3$  for every  $v \in V$ . Thus, we can derive the following result from Theorem 1.

**Corollary 1.** *The VDB problem is NP-complete even for a cubic graph.*  $\square$

On the other hand, if the graph  $G = (V, E)$  is planar, Hadlock [8,1] showed that the MEDB problem can be solved in  $O(|V|^3)$  time. Therefore, we obtain the following result from Theorem 1.

**Corollary 2.** *For a planar graph  $G = (V, E)$  with  $\Delta(G) = 3$ , the MVDB problem is solvable in  $O(|V|^3)$  time.*  $\square$

**Remark 1.** The above discussion suggests a way to solve the MVDB problem for a planar graph  $G$  with  $\Delta(G) = 3$ . That is, we first solve the MEDB problem using Hadlock's approach [8], and then convert its solution to a solution to the MVDB problem. We present a more direct approach to the MVDB problem. This approach can also be used to obtain an approximate solution for the case of  $\Delta(G) \geq 4$ .

Let  $G = (V, E)$  be a given planar graph. Without loss of generality, we can assume that  $G$  is *biconnected*, that is, for every pair of vertices  $u$  and  $v$  in  $V$ , there are at least two vertex disjoint paths from  $u$  to  $v$ .

Let  $\tilde{G}$  be a planar embedding of  $G$ . Since  $G$  is biconnected,  $\tilde{G}$  partitions the rest of

the plane into a number of connected regions. The closures of those regions are called the *faces* of  $\tilde{G}$ . Let  $F$  be a set of such faces.  $F$  includes a special face called an *exterior face* which represents the infinite region outside the embedding  $\tilde{G}$ . Note that each face in  $F$  corresponds to a cycle in  $G$ , which is called a *fundamental cycle*. We create a new graph  $G^n = (V^n, E^n)$ , where  $V^n = V \cup F$  and  $E^n = \{(v, f) \mid v \in V \text{ is on the fundamental cycle corresponding to } f \in F\}$ . Then we follow Hadlock [8] and find a pairing of odd degree vertices (or faces) in  $F$  such that the total sum of lengths of shortest paths between such pairs is a minimum.  $\square$

We now consider the case in which  $G = (V, E)$  is planar and  $\Delta(G) = 4$ . We prove that the VDB problem becomes NP-complete for this case.

**Theorem 2.** *The VDB problem is NP-complete for a planar graph even if  $\Delta(G) = 4$ .*

**Proof.** Since the VDB problem belongs to the class NP, it suffices to show that a known NP-complete problem is transformable in polynomial time to this problem. We start with the following problem which was shown to be NP-complete by Licktenstein [12].

Planar 3-Satisfiability (P3SAT)

Instance: A set  $U = \{v_i \mid 1 \leq i \leq n\}$  of  $n$  Boolean variables and a set  $C = \{c_j \mid 1 \leq j \leq m\}$  of  $m$  clauses over  $U$  such that each clause  $c_j$  contains exactly three variables or their complements. Furthermore, the following graph is planar:

$$G_C = (V_C, E_C), \text{ where}$$

$$V_C = \{c_j \mid 1 \leq j \leq m\} \cup \{v_i \mid 1 \leq i \leq n\} \quad \text{and}$$

$$E_C = \{(c_j, v_i) \mid v_i \in c_j \text{ or } \bar{v}_i \in c_j\} \cup \{(v_i, v_{i+1}) \mid 1 \leq i < n\} \cup \{(v_n, v_1)\}.$$

Question: Is  $C$  satisfiable? Namely, is there a truth assignment for  $U$  such that each clause in  $C$  is true?

Given  $U = \{v_i \mid 1 \leq i \leq n\}$  and  $C = \{c_j \mid 1 \leq j \leq m\}$  together with a planar embedding of  $G_C = (V_C, E_C)$ , we construct a planar graph  $G = (V, E)$  in the following way. The graph contains two kinds of components: clause components and variable components. And each of them is placed in the region in which the corresponding vertex  $c_j \in V_C$  or  $v_i \in V_C$  is drawn. For each clause  $c_j \in C$  we create a clause component of four vertices  $w_j^1, w_j^2, w_j^3, w_j^4$  and one edge  $(w_j^1, w_j^4)$  as shown in Fig. 2 (a). As will be explained later, each pair of vertices  $w_j^l$  and  $w_j^{l+1}$  for  $l = 1, 2, 3$ , will be connected to a pair of vertices of an appropriate variable component. For each variable  $v_i \in U$  we construct a variable component of  $8n_i$  vertices and  $12n_i$  edges, where  $n_i$  is the number of times variable  $v_i$  or its complement  $\bar{v}_i$  appears in  $C$ . As illustrated in Fig. 2(b), each variable component is made up of  $4n_i$  triangles whose bottom edges form a cycle of length  $4n_i$ . The vertices on this cycle correspond to  $v_i$  and  $\bar{v}_i$ , alternately. Namely,  $b_i^{k1}$  and  $b_i^{k3}$  correspond to  $v_i$  and  $b_i^{k2}$  and  $b_i^{k4}$  correspond to  $\bar{v}_i$  for  $k = 1, 2, \dots, n_i$ . Note that a group of four consecutive triangles, more precisely, eight vertices  $a_i^{kl}$  and  $b_i^{kl}$  for  $l = 1, 2, 3, 4$  correspond to variable  $v_i$ . If  $\bar{v}_i \in c_j$  (resp.,  $v_i \in c_j$ ), then the two top vertices  $a_i^{k1}$  and  $a_i^{k2}$  (resp.,  $a_i^{k2}$  and  $a_i^{k3}$ ) for some  $k \in \{1, 2, \dots, n_i\}$  are connected to a pair of vertices  $w_j^l$  and  $w_j^{l+1}$  for some  $l \in \{1, 2, 3\}$  of the clause component corresponding to  $c_j$ . For example, see Fig. 3. It is easy to see

that  $|V| = 4m + \sum_{i=1}^n 8n_i = 28m$  and  $|E| = 7m + \sum_{i=1}^n 12n_i = 43m$ . Therefore, this transformation is done in polynomial time. Furthermore, it is clear that  $G$  is planar and  $\Delta(G) = 4$ .

We now prove that there is a truth assignment for  $U$  such that each clause  $c_j \in C$  is true if and only if there is a subset  $V' \subset V$  of  $G = (V, E)$  such that  $|V'| = 6m$  and  $G^v(V - V')$  is bipartite.

Suppose that there is a subset  $V' \subset V$  such that  $|V'| = 6m$  and  $G^v(V - V')$  is bipartite. Since each variable component contains  $4n_i$  triangles, which are odd cycles, we must delete at least  $2n_i$  vertices to break these triangles. This can be done only if every other vertex on the cycle of length  $4n_i$  is so chosen. More precisely, we must delete  $n_i$  pairs of vertices either  $b_i^{k1}$  and  $b_i^{k3}$  or  $b_i^{k2}$  and  $b_i^{k4}$  for  $k = 1, 2, \dots, n_i$ , for each  $i = 1, 2, \dots, n$ . Since  $\sum_{i=1}^n 2n_i = 6m$ , no other vertices are deleted. Therefore, we can make a consistent assignment of value *true* or *false* to each variable  $v_i$  in the following manner: If vertex  $b_i^{12}$  (resp.,  $b_i^{13}$ ) is deleted from the variable component corresponding to  $v_i$ , assign *false* (resp., *true*) to  $v_i$ , for  $i = 1, 2, \dots, n$ .

Note that all four vertices in each clause component belong to a cycle of length 13 which connects those four vertices and three vertices, one from each of the three corresponding variable components. We call such a cycle a *clause cycle*. For example, a cycle  $[w_2^1, a_4^{13}, b_4^{13}, a_4^{12}, w_2^2, a_2^{22}, b_2^{22}, a_2^{21}, w_2^3, a_3^{23}, b_3^{23}, a_3^{22}, w_4^2, w_2^1]$  is a clause cycle for clause  $c_2$  in Fig. 3. Since by assumption, each such cycle is broken, at least one vertex labeled  $b$  in the three variable components must be deleted. Since the variable or its complement corresponding

to such a vertex is assigned *true*, the clause corresponding to this cycle of length 13 is *true*.

Therefore, there is a truth assignment for  $U$  such that each clause in  $C$  is true.

On the other hand, suppose that there is a truth assignment for  $U$  such that each clause  $c_j$  in  $C$  is true. If  $v_i$  is assigned *true* (resp., *false*), then delete  $n_i$  pairs of vertices  $b_i^{k2}$  and  $b_i^{k4}$  (resp.,  $b_i^{k1}$  and  $b_i^{k3}$ ) for  $k = 1, 2, \dots, n_i$ , for each  $i = 1, 2, \dots, n$ . As mentioned before, the removal of these vertices breaks all  $4n_i$  triangles and leaves  $2n_i$  paths of length 2 in the variable component corresponding to  $v_i$ . It also breaks all clause cycles. Furthermore, it is clear that all the other (possibly odd) cycles are eliminated because each variable component is chopped into  $2n_i$  paths of length 2. Therefore, the remaining graph does not contain any odd cycles and hence it is bipartite. This completes the proof.  $\square$

### 3. Via Minimization

In this section, using Theorem 2, we prove that the via minimization decision problem for *two* layers is NP-complete even if the maximum junction degree is limited to four. We start with some definitions.

A *circuit* is specified by a set of modules  $M$ , a set of terminals  $T$  and a set of nets  $N$ . The *terminals* in  $T$  are located on the boundary of the *modules* in  $M$  and each *net* specifies which terminals are to be electrically connected. Such connections are made by patterning conductive paths on one of two *layers*. Such paths are made up of straight line segments, called *wire segments*. We assume that the terminals are available on both layers and each wire segment can be assigned to either layer. Note that the vertical projection of each wire

segment is fixed in the plane but its layer assignment is not specified. A point other than a terminal at which two or more wire segments meet and are electrically connected is called a *junction*. A wire segment is said to be *incident* upon a junction at which it meets. Wire segments which are incident upon the same junction are said to be *adjacent* to each other, and the number of such segments is called the *junction degree*. If wire segments incident upon the same junction are assigned to different layers, a *via* is placed at the junction to electrically connect them. If the vertical projections of two wire segments that are not electrically connected intersect, they are said to *cross* each other. A layer assignment is said to be *valid* if no two wire segments that cross each other are assigned to the same layer and no two adjacent wire segments are assigned to different layers without a via.

The *via minimization decision problem* (abbreviated to VM) for two layers is defined as follows:

Via Minimization (VM)

Instance: A set  $M = \{m_i \mid 1 \leq i \leq p\}$  of modules, a set  $T = \{t_i \mid 1 \leq i \leq q\}$  of terminals, a set  $W = \{w_i \mid 1 \leq i \leq r\}$  of wire segments whose vertical projections are fixed in the plane, and an integer  $k \geq 0$ .

Question: Is there a valid layer assignment for  $W$  which requires  $k$  or fewer vias using two layers?

In order to show a transformation from the VDB problem for planar graphs to the VM problem, we will use the sublayout  $L$  shown in Fig. 4.  $L$  has a single module  $m$ , two



terminals  $t$  and  $t'$  on its boundary, and wire segments  $w$  and  $w'$  which connect terminals  $t$  and  $t'$  to junctions  $j$  and  $j'$ , respectively. Note that the terminals  $t$  and  $t'$  belong to different nets and thus the wires  $w$  and  $w'$  must be assigned to different layers.

Let  $G = (V, E)$  be a planar graph. A *straight line planar embedding* of  $G$  is a planer embedding of  $G$  in which every edge in  $G$  is represented by a straight line segment. It is known [3,5] that such an embedding of  $G$  can be obtained in polynomial time, and we will denote it by  $\bar{G}$ . For simplicity, we call a straight line segment in  $\bar{G}$  which represents an edge  $(x, y)$  in  $G$ , a *segment*  $(x, y)$ , and the endpoints of the line segment which represent vertices  $x$  and  $y$ , *points*  $x$  and  $y$ , respectively.

We first create a small region surrounding each segment  $(x, y)$  in  $\bar{G}$  so that no two such regions overlap. We then replace each segment  $(x, y)$  by a sublayout  $L$  in such a manner that  $L$  is completely within the region surrounding  $(x, y)$  and junctions  $j$  and  $j'$  coincide with points  $x$  and  $y$ , respectively.

We denote the resultant layout by  $L(G)$ . Fig. 5 shows an example graph  $G_1$  and its corresponding layout  $L(G_1)$ . Note that for each cycle in  $G$ , there is a cycle of sublayouts or modules in  $L(G)$ . We call such a cycle of modules an *m-cycle*. If the number of modules in an m-cycle is odd (resp., even), it is called an *odd* (resp., *even*) m-cycle. Consider the m-cycle consisting of modules  $m_1, m_2$  and  $m_3$  shown in Fig. 5(b). Suppose that wire segment  $w_1$  is assigned to layer 1. Then  $w'_1$  must be assigned to layer 2. To avoid a via at junction  $j_2$ ,  $w_2$  should be assigned to layer 2. Consequently,  $w'_2$  must be assigned to layer 1 and hence  $w'_3$  should be assigned to layer 1. This implies that  $w_3$  must be assigned to layer

2. In order to electrically connect the wire segments  $w_3$  and  $w_1$ , a via needs to be placed at junction  $j_1$ . It is not difficult to see that a via is always required to have a valid layer assignment if there is an odd  $m$ -cycle in  $L(G)$ .

On the other hand, suppose that there is no odd  $m$ -cycle in  $L(G)$ . Then, a valid layer assignment for  $L(G)$  which requires no via will be obtained in the following manner. Assign an arbitrary wire segment and all of its adjacent wire segments to layer 1. Find wire segments which cross the wire segments just assigned to layer 1. Assign those segments to layer 2. Then find unassigned wire segments which cross the segments just assigned to layer 2 and assign them to layer 1. Repeat this process until all segments are assigned to one of the layers. Since there is no odd  $m$ -cycle, no conflict on layer assignment would occur. Therefore, we have the following lemma.

**Lemma 1.** *There exists a valid layer assignment for layout  $L(G)$  which requires no via if and only if it is free of odd  $m$ -cycles.  $\square$*

We are now ready to show the NP-completeness of the VM problem even when the maximum junction degree is limited to four.

**Theorem 4.** *The VM problem is NP-complete even when the maximum junction degree is limited to four.*

**Proof.** It is easy to see that the VM problem belongs to the class NP. Thus, it suffices to show that the VDB problem for a planar graph  $G$  with  $\Delta(G) = 4$  is transformable in polynomial time to the VM problem.

Let  $G = (V, E)$  be a planer graph such that  $\Delta(G) = 4$  and  $k$  be a nonnegative integer.

We construct a layout  $L(G)$  in the manner described above.

Suppose that there is a set of vertices  $V' \subset V$  such that  $|V'| \leq k$  and  $G^v(V - V')$  is bipartite. Let  $J$  be a set of junctions in  $L(G)$  which correspond to the vertices in  $V'$ . We first delete from  $L(G)$  all junctions in  $J$  and the wire segments incident upon them. Let  $L^d(G)$  denote the resultant layout. Fig. 6 (a) depicts such a layout which is obtained by deleting junction  $j_1$  from  $L(G_1)$  of Fig. 5 (b). Note that there are two types of “*degenerated*” sublayouts in  $L^d(G)$ . Sublayouts of Type 1 (resp., 0) are those that consist of a module and one (resp., no) wire segment. In Fig. 6 (a), sublayouts containing modules  $m_1, m_3, m_6$ , and  $m_8$  are of Type 1. If junction  $j_2$  were also deleted, the sublayout containing module  $m_1$  would be of Type 0. We then delete those degenerated sublayouts from  $L^d(G)$ . Let  $L'(G)$  be the resultant layout. Since  $L'(G)$  does not contain any odd m-cycle, by Lemma 1 there exists a valid layer assignment for  $L'(G)$  which requires no via. Fig. 6 (b) illustrates such a layer assignment for  $L'(G_1)$ . Then, based on this layer assignment, we can obtain a valid layer assignment for  $L^d(G)$  as follows: For each degenerated sublayout of Type 1, assign its only wire segment to the same layer as all of its adjacent wire segments in  $L'(G)$ . Such a layer assignment for  $L'(G_1)$  is shown in Fig. 6 (c).

We now find layer assignments for those wire segments which are incident upon the junctions in  $J$ . For each wire segment which is missing from some degenerated sublayout of Type 1, if the remaining wire segment is assigned to layer 1, we assign the missing segment to layer 2 and vice versa. We assign a pair of wire segments in each degenerated sublayout

of Type 0 to different layers arbitrarily. Because vias are placed at those junctions in  $J$ , necessary electrical connections will be made. Therefore, a valid layer assignment exists for  $L(G)$  which requires  $k$  or fewer vias, since  $|J| \leq k$ . A final valid layer assignment for  $L(G_1)$  is shown in Fig. 6 (d).

Conversely, suppose that there is a set of  $k$  or fewer vias for which a valid layer assignment for  $L(G)$  exists. Let  $J$  be the set of junctions at which the vias are placed. Let  $V'$  be the set of vertices which correspond to the junctions in  $J$ .

Consider the vertex-deleted subgraph  $G^v(V - V')$  and its corresponding layout  $L(G^v(V - V'))$ . It is not difficult to see that  $L(G^v(V - V'))$  is obtained by deleting all such sublayouts that contain at least one wire segment which is incident upon some junction in  $J$ . Since the layer assignment for  $L(G^v(V - V'))$  is valid and requires no via, by Lemma 1 there is no odd  $m$ -cycle in  $L(G^v(V - V'))$ . This implies that there is no odd cycle in  $G^v(V - V')$  and hence  $G^v(V - V')$  is bipartite. Therefore, there exists a set of vertices  $V'$  such that  $|V'| = |J| \leq k$  and  $G^v(V - V')$  is bipartite.

We have proved that there is a set of vertices  $V' \subset V$  such that  $|V'| \leq k$  and  $G^v(V - V')$  is bipartite if and only if a valid layer assignment exists for  $L(G)$  which requires  $k$  or fewer vias. Since the construction of  $L(G)$  only requires replacing each segment in  $\bar{G}$  with a sublayout  $L$ , it can easily be accomplished in polynomial time. Furthermore, the number of wire segments incident upon a junction in  $L(G)$  is the same as the number of segments incident upon the corresponding point in  $\bar{G}$ . Since  $\Delta(G) = 4$ , the maximum junction degree in  $L(G)$  is limited to four. This completes the proof of the theorem.  $\square$

**Remark 2.** Using the same arguments as in Naclario, Masuda, and Nakajima [15], one can establish the NP-completeness of the VM problem for two layers under any combination of the following two constraints as long as the maximum junction degree is limited to four or more:

1. The layout is restricted to be grid based.
2. Vias are placed only at the junctions which existed in the input layout.

## 4. Conclusion

We have presented complexity results for the vertex-deletion graph bipartization (VDB) problem. These results completely close the gap between the polynomially solvable cases and NP-complete cases for general and planar graphs as shown in Table 1. We have also shown that the via minimization decision problem is NP-complete for two layers when the maximum junction degree is limited to four. Quite recently, Molitor [13] showed that the problem of assigning wire segments to three or more layers without using vias is NP-complete when the maximum junction degree is limited to four. Since the via minimization problem for two layers is solvable in polynomial time when the maximum junction degree is limited to three [2,13,14,16], our result has completely settled the complexity issues on via minimization.

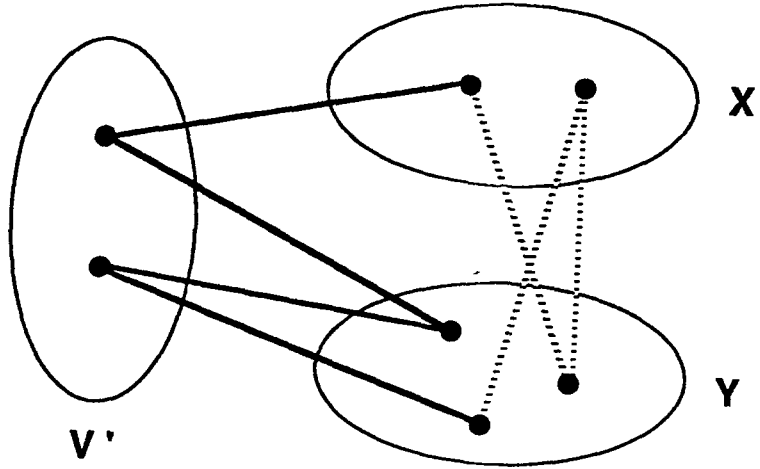
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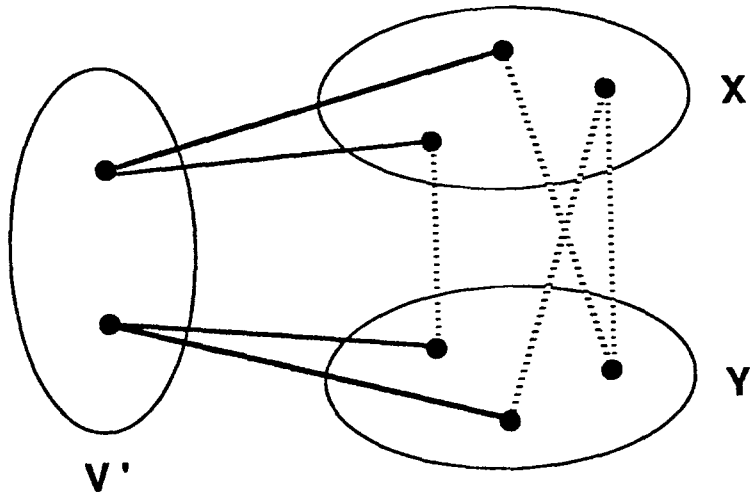
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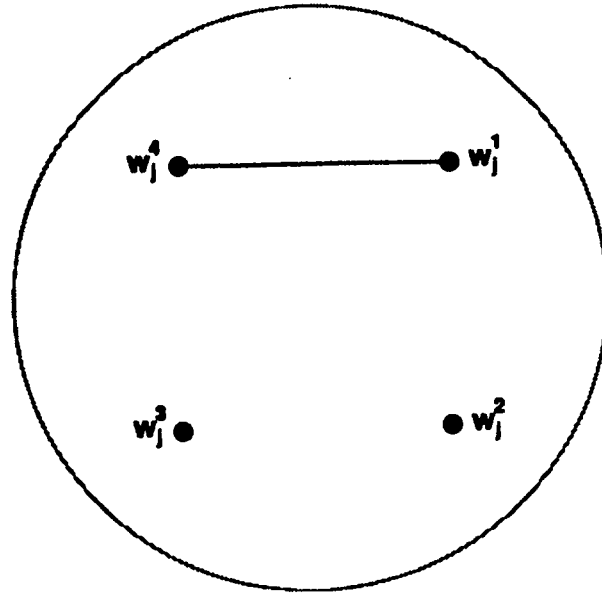


(a) A case in which an odd cycle is formed.

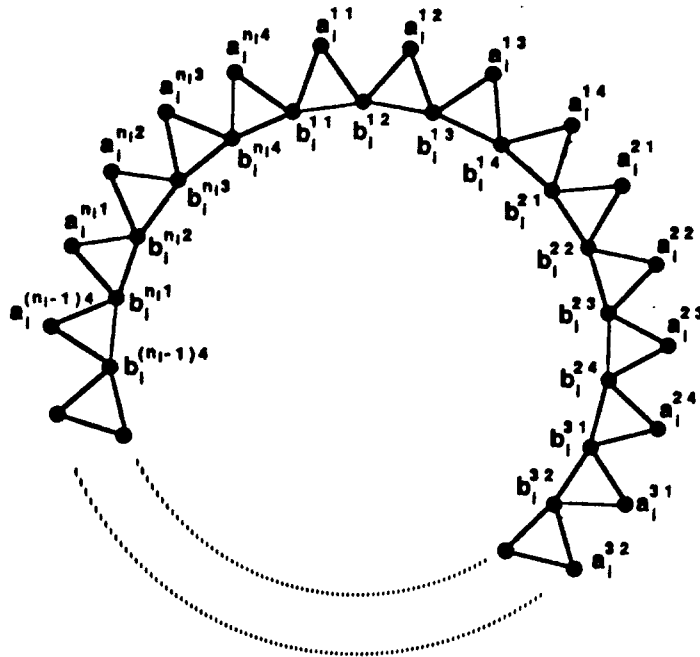


(b) A case in which an even cycle is formed.

Fig. 1. Formation of cycles.



(a) Clause component.



(b) Variable component.

Fig. 2. Clause and variable components.

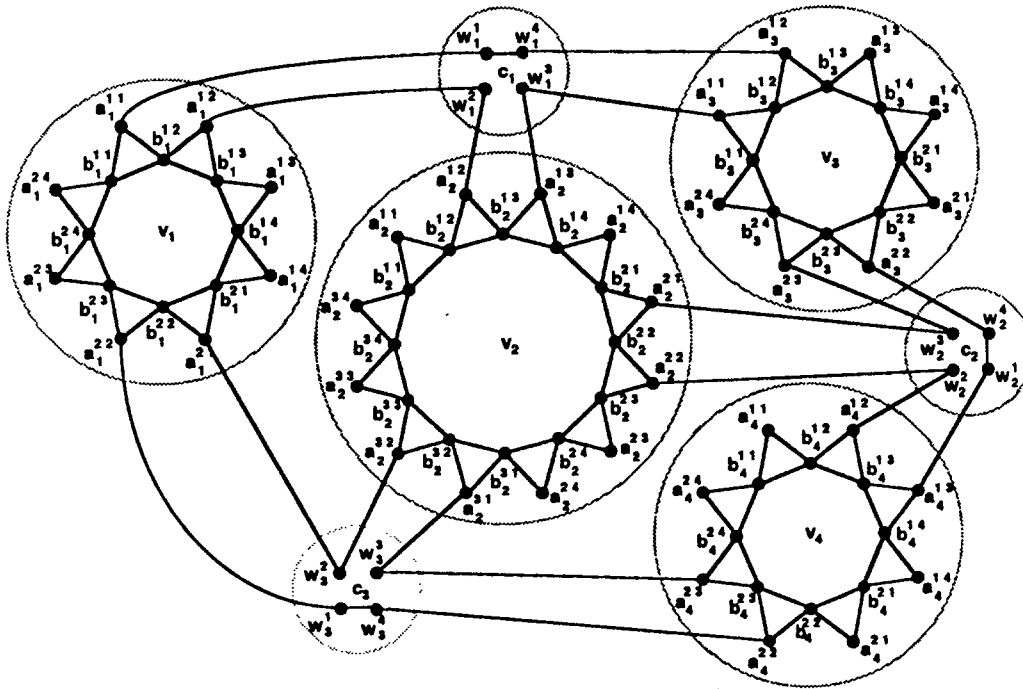


Fig. 3. Example.  $C = \{c_1, c_2, c_3\}$ ;  $c_1 = \{\bar{v}_1, v_2, \bar{v}_3\}$ ,  $c_2 = \{\bar{v}_2, v_3, v_4\}$ ,  $c_3 = \{\bar{v}_1, \bar{v}_2, v_4\}$ .

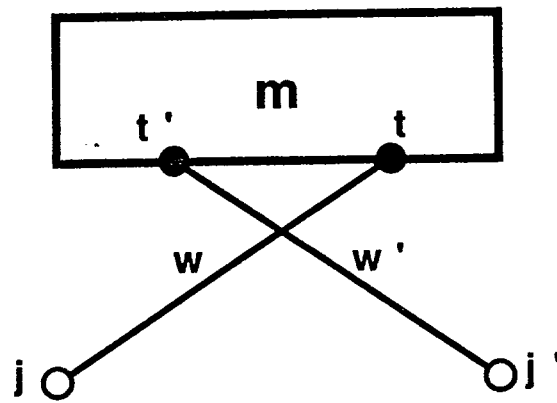
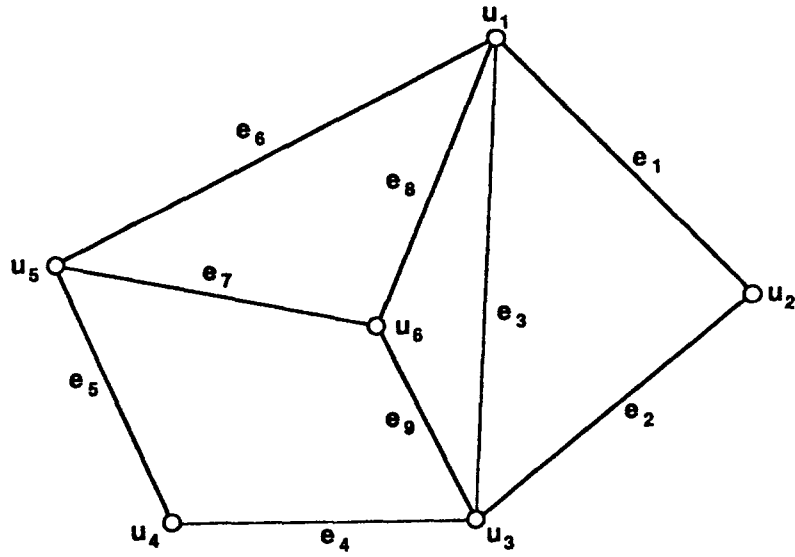
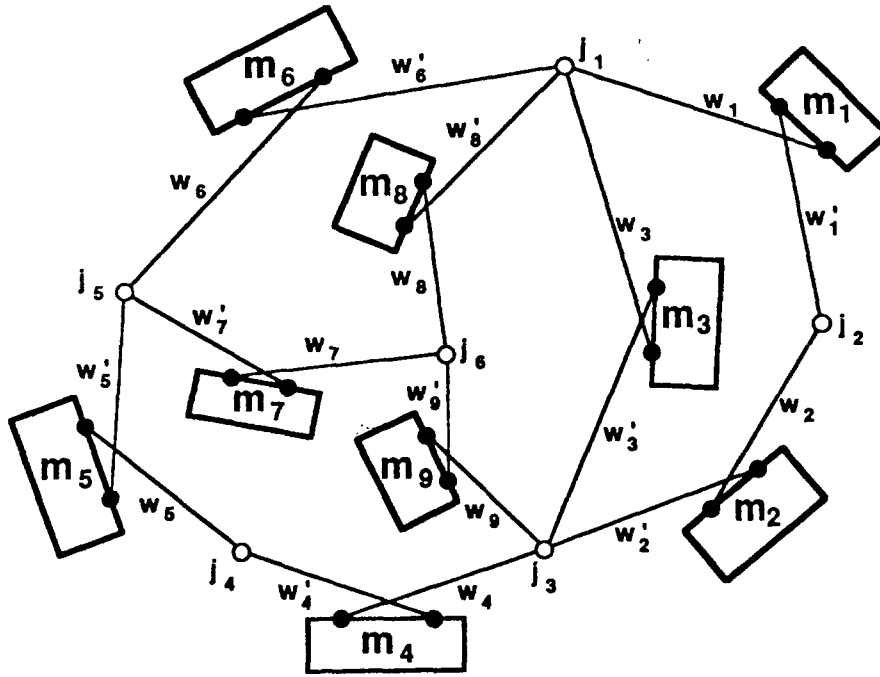


Fig. 4. Sublayout  $L$ .

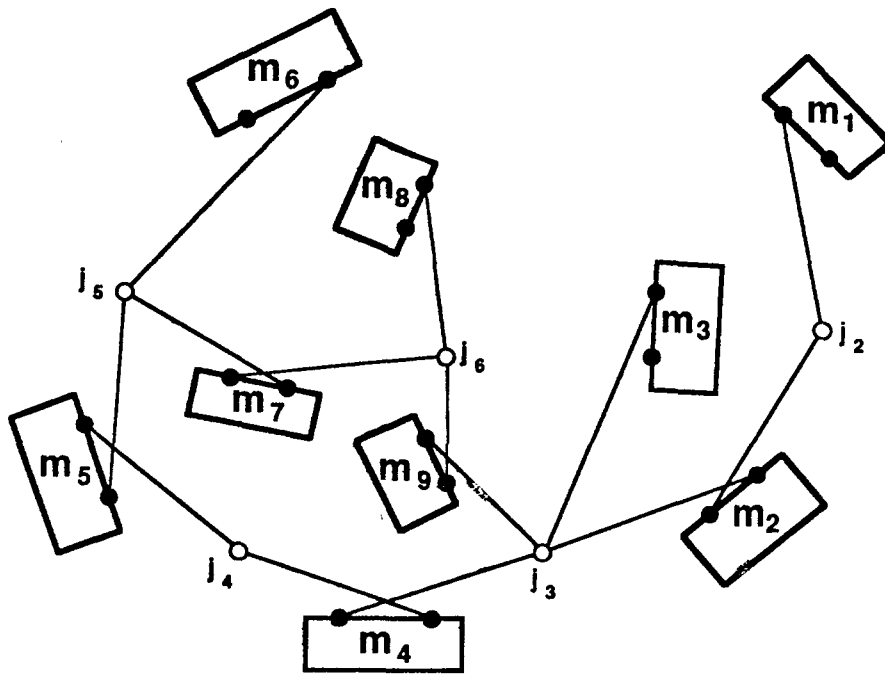


(a)  $G_1$ .

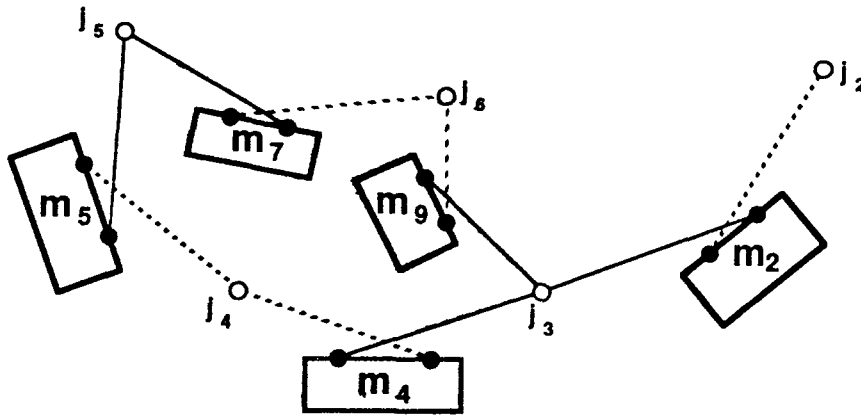


(b)  $L(G_1)$ .

Fig. 5. An example graph  $G_1$  and its corresponding layout  $L(G_1)$ .

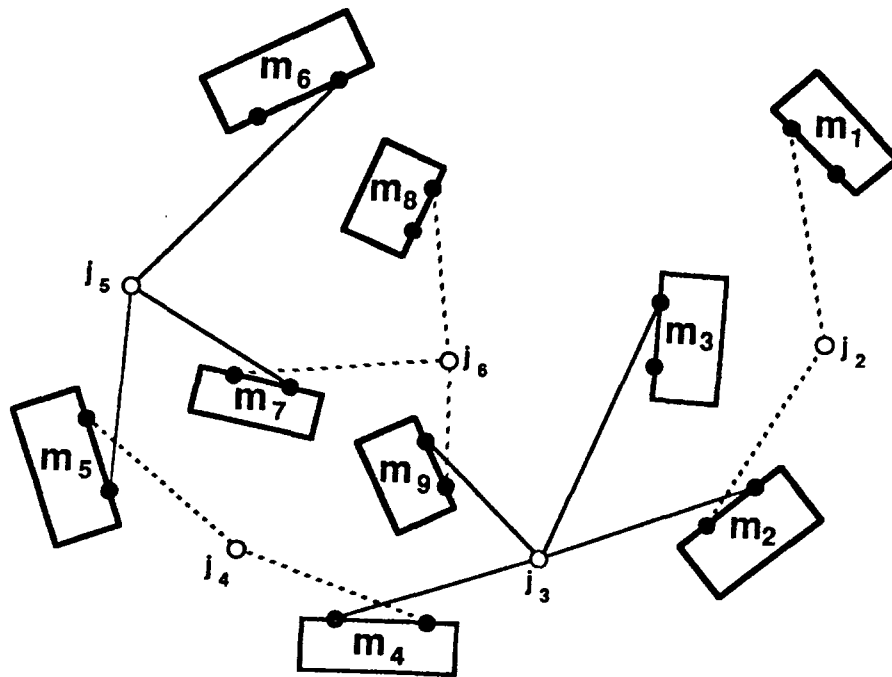


(a)  $L(G_1^d)$ .

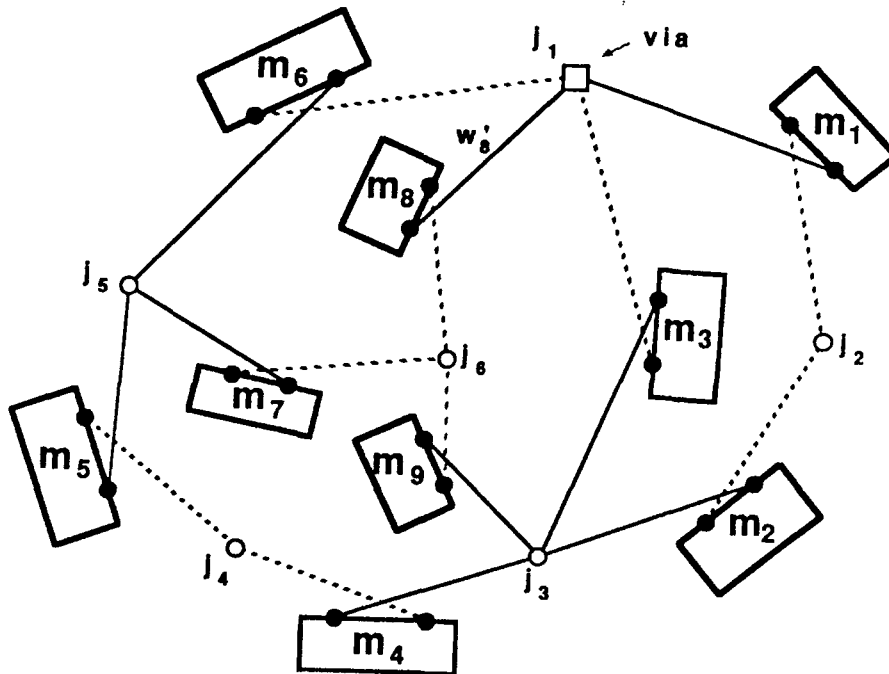


(b) A valid layer assignment for  $L(G_1^d)$ .

Fig. 6. Illustrations for the proof of Theorem 3.



(c) A valid layer assignment for  $L(G_1^d)$ .



(d) A valid layer assignment for  $L(G_1)$ .

Fig. 6 (cont'd). Illustrations for the proof of Theorem 3.