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**Homogeneous Indices, Feedback  
Invariants and Control Structure  
Theorem for Generalized Linear  
Systems**

**by**

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HOMOGENEOUS INDICES, FEEDBACK INVARIANTS AND CONTROL STRUCTURE THEOREM  
FOR GENERALIZED LINEAR SYSTEMS\*

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## ABSTRACT

We define a new set of indices for a generalized linear system. These indices, referred to as the homogeneous indices, are a natural generalization of the minimal column indices (Kronecker indices) of an ordinary state-space system. We prove that the homogeneous indices are a complete set of invariants for the action of a natural group of feedback transformations on generalized linear systems. We also show that the homogeneous indices determine exactly which closed loop invariant polynomials can be assigned by feedback, thereby generalizing the Control Structure Theorem of Rosenbrock.

Keywords: generalized linear system, proportional and derivative feedback, feedback invariants, homogeneous indices, singular pencils.

## 1. Introduction.

In the past several years, there has been considerable interest in generalized linear systems (also called "descriptor systems") -- i.e., generalized state-space models of the form

$$E \dot{x}(t) = Ax(t) + Bu(t) \quad (1.1)$$

with the matrix  $E$  possibly singular. (See, e.g., [1][16][17].) We represent this system by the matrix triple  $(E,A,B)$  and refer to it as a regular system if  $E$  is nonsingular and as a singular system if  $E$  is singular.

Recently, Shayman and Zhou [2] have presented a unified theory of control synthesis for generalized linear systems using constant-ratio proportional and derivative (CRPD) feedback. The framework includes the theory of static state feedback and output feedback for regular systems as a special case. The main elements of this theory include (1) a covering of the space of all systems, both regular and singular, by a family of open and dense subsets indexed by the unit circle; (2) a group of transformations which may be viewed as symmetries of the cover; (3) an admissible class of feedback transformations on each subset which is specifically adapted to that subset. A general procedure of control synthesis of CRPD feedback for generalized linear systems is obtained which uses the symmetry transformations to systematically reduce each synthesis problem to an ordinary static state feedback (or output feedback) synthesis problem for a corresponding regular system. This procedure was used to obtain natural generalizations of the Disturbance Decoupling Theorem, the Pole Assignment Theorem, and Brunovsky's canonical form.

In order to give a precise statement of the problems to be addressed in the present paper, we review the three main elements of the theory presented in [2]. We begin with the covering of the space of generalized systems. Let  $\hat{\Sigma}(n,m)$

denote the space of all matrix triples  $(E,A,B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ . Let  $\Sigma(n,m)$  denote the open and dense subset of  $\hat{\Sigma}(n,m)$  characterized by the requirement that  $\det(sE-A)$  does not vanish identically. This condition guarantees uniqueness for the solutions of (1.1). In the literature, the systems belonging to  $\Sigma(n,m)$  are generally referred to as "regular systems." However, we will reserve the word "regular" to refer to a generalized linear system  $(E,A,B)$  for which  $E$  is nonsingular. We refer to the systems in  $\Sigma(n,m)$  as the admissible systems, and to the condition  $\det(sE-A) \neq 0$  as the admissibility assumption.

We now define a covering of the space  $\Sigma(n,m)$  of admissible systems. For each  $\theta \in \mathbb{R}$ , let  $\Sigma_\theta(n,m)$  denote the subset of  $\Sigma(n,m)$  given by

$$\Sigma_\theta(n,m) = \{(E,A,B) \in \Sigma(n,m) : \det(\cos\theta E - \sin\theta A) \neq 0\}. \quad (1.2)$$

It is easy to show that  $\Sigma_{\theta+\pi}(n,m) = \Sigma_\theta(n,m)$ , and that  $\{\Sigma_\theta(n,m) : \theta \in [0,\pi)\}$  is a covering of  $\Sigma(n,m)$  by open and dense subsets. By virtue of the periodicity, it is natural to regard the parameter  $\theta$  as a point on the unit circle. Note that in the special case where  $\theta = 0$ ,  $\Sigma_0(n,m)$  consists of those triples  $(E,A,B)$  for which  $E$  is nonsingular -- i.e., the regular systems.

Next, we define a group of symmetries of the cover  $\{\Sigma_\theta(n,m) : \theta \in [0,\pi)\}$  -- transformations which map these subsets into each other. For each  $\phi \in \mathbb{R}$ , define a mapping  $R_\phi : \hat{\Sigma}(n,m) \rightarrow \hat{\Sigma}(n,m)$  by

$$R_\phi(E,A,B) = (\cos\phi E + \sin\phi A, -\sin\phi E + \cos\phi A, B). \quad (1.3)$$

It is straightforward to show that  $R_\phi$  maps  $\Sigma_\theta(n,m)$  isomorphically onto  $\Sigma_{\theta+\phi}(n,m)$ . In particular, this implies that each subset  $\Sigma_\theta(n,m)$  in the covering is isomorphic to the set  $\Sigma_0(n,m)$  of regular systems.

We now define a class of admissible feedback transformations for each subset  $\Sigma_\theta(n,m)$ . Specifically, we allow feedback of the form

$$u = F(\cos\theta x - \sin\theta \dot{x}) + v \quad (1.4)$$

to be applied to the systems belonging to the subset  $\sum_{\theta}(n,m)$ . In (1.4),  $\theta$  is fixed while the  $m \times n$  gain matrix  $F$  is arbitrary, and  $v$  represents a new external input. The fixed parameter  $\theta$  specifies the ratio of state to derivative in the feedback law. Consequently, we refer to (1.4) as constant-ratio proportional and derivative (CRPD) state feedback. This specialized form of proportional and derivative feedback was suggested as a design tool for singular systems in the doctoral thesis of Zhou [3] (also Zhou-Shayman-Tarn [4]), and independently by Christodoulou [5].

Remark 1.1: As mentioned previously, in the special case where  $\theta = 0$ ,  $\sum_0(n,m)$  is the set of all regular systems. In this case, (1.4) is ordinary state feedback. Thus, the theory outlined above includes the theory of state feedback for regular systems as a special case.

There are three main contributions in the present paper. The first is the introduction of a new set of indices for a generalized linear system, which we refer to as the homogeneous indices of  $(E,A,B)$ . These indices are a natural generalization of the minimal column indices ("Kronecker indices") of a regular system. In fact, we will show that if  $(E,A,B)$  is a controllable regular system, its homogeneous indices and its minimal column indices coincide.

The second contribution is a solution to the CRPD feedback equivalence problem for generalized linear systems. We determine necessary and sufficient conditions for two controllable systems in  $\sum_{\theta}(n,m)$  to be transformable to each other via the CRPD feedback (1.4) together with change of basis in the state-space, change of basis in the input space, and left-multiplication of (1.1) by a nonsingular matrix. We show that two such systems are feedback equivalent if and only if they have the same homogeneous indices. This generalizes the well-

known result that two controllable regular systems are equivalent under the state feedback group if and only if they have identical Kronecker indices.

The third contribution in this paper is a generalization of Rosenbrock's Control Structure Theorem [6]. Rosenbrock's Theorem describes precisely which closed-loop invariant polynomials are attainable by applying state feedback to a given controllable regular system. Using the concept of homogeneous indices, we are able to describe exactly which closed-loop invariant polynomials are attainable by applying CRPD feedback (1.4) to a controllable system in  $\sum_{\theta}(n,m)$ . Rosenbrock's Theorem is recovered as a special case of our result by setting  $\theta = 0$ .

## 2. Homogeneous Indices.

We begin by reviewing Kronecker's definition of the minimal column indices of a singular pencil of matrices [7]. (See also [8, p. 37] and [18, p. 55].) Let  $M$  and  $N$  be real  $m \times n$  matrices. The matrix pencil  $\lambda M + N$  is called a regular pencil if  $m = n$  and  $\det(\lambda M + N)$  does not vanish identically. Otherwise, it is called a singular pencil.

The minimal column indices of a singular pencil are defined as follows: Let  $v_1$  be a minimal degree nonzero polynomial solution to the equation

$$(\lambda M + N)v = 0 . \tag{2.1}$$

Let  $v_2$  be a minimal degree solution which is linearly independent (over the polynomial ring  $\mathbb{R}[\lambda]$ ) of  $v_1$ . Let  $v_3$  be a minimal degree solution which is linearly independent of  $\{v_1, v_2\}$ . Proceeding in this way, one obtains a sequence  $v_1, \dots, v_p$  of solutions. Such a sequence is called a fundamental series of solutions of (2.1). Let  $\epsilon_1 < \epsilon_2 < \dots < \epsilon_p$  denote the degrees of  $v_1, \dots, v_p$ , respectively. Using the fact that column vectors over a polynomial ring are linearly independent if and only if they are linearly independent over the

corresponding field of fractions, it is easy to show [8, p. 38] that these non-negative integers are independent of the choice of fundamental series.

$(\epsilon_1, \dots, \epsilon_p)$  are called the minimal column indices of the singular pencil  $\lambda M + N$ .

Recall that two  $m \times n$  pencils,  $\lambda M + N$  and  $\lambda \bar{M} + \bar{N}$  are said to be strictly equivalent [8, p. 24] if there exist nonsingular constant matrices  $P$  and  $Q$  of dimensions  $m \times m$  and  $n \times n$  such that

$$P(\lambda M + N)Q = \lambda \bar{M} + \bar{N} . \quad (2.2)$$

It is well-known that strictly equivalent singular pencils have identical minimal column indices.

Let  $(E,A,B)$  be a regular system -- i.e.,  $(E,A,B) \in \sum_0(n,m)$ , and assume  $(E,A,B)$  is controllable. Let  $(\epsilon_1, \dots, \epsilon_p)$  denote the minimal column indices of the singular pencil  $[\lambda E - A, B]$ . Let  $r$  denote the rank of  $B$ , and let  $M_i$  denote the matrix  $[E^{-1}B, (E^{-1}A)(E^{-1}B), \dots, (E^{-1}A)^{i-1}(E^{-1}B)]$ . Let  $\ell_1 = \text{rank } M_1$  and let  $\ell_i = \text{rank } M_i - \text{rank } M_{i-1}$ ,  $i = 2, \dots, n$ . Then  $\ell_1 > \dots > \ell_n > 0$ . The following facts are well-known. (See, e.g., [9].)

Proposition 2.1: Let  $(E,A,B)$  be a controllable regular system. Then,

- (a)  $(\epsilon_1, \dots, \epsilon_p)$  is a partition of  $n$  into  $m$  parts, with  $\epsilon_1, \dots, \epsilon_{m-r}$  zero and  $\epsilon_{m-r+1}, \dots, \epsilon_m$  strictly positive. (Thus,  $p = m$ .)
- (b)  $\epsilon_j = \text{Card}\{i: \ell_i > m - j + 1\}$  ( $j = 1, \dots, m$ ).

Remark 2.1: Proposition 2.1 is no longer true if the assumption that  $(E,A,B)$  be a regular system is dropped. Rosenbrock has shown [10] that if the system  $(E,A,B) \in \sum_0(n,m)$  has no finite or infinite input decoupling zero (i.e., is controllable), then the pencil  $[\lambda E - A, B]$  has no finite elementary divisor and no minimal index for the rows. It has infinite elementary divisors, each of degree 1, equal in number to the rank defect of  $E$ . It has  $m$  minimal indices for



the columns. Thus,  $p = m$ , and  $(\epsilon_1, \dots, \epsilon_p)$  is a partition of rank  $E$ , rather than a partition of  $n$  as it is in the case of a regular system.

We now define the homogeneous indices of a generalized linear system. Let  $(E, A, B) \in \hat{\Sigma}(n, m)$ . We associate to  $(E, A, B)$  the degree one matrix polynomial in two variables given by  $[\lambda E - \mu A, B]$ . Abusing terminology slightly, we refer to  $[\lambda E - \mu A, B]$  as a matrix pencil. Let  $z_1$  be a column vector with entries in the ring  $\mathbb{R}[\lambda, \mu]$  of polynomials in two variables which is a minimal degree nonzero solution to the equation

$$[\lambda E - \mu A, B]z = 0 . \tag{2.3}$$

(For a polynomial in two variables, "degree" refers to the total degree, and the degree of a solution  $z$  is the highest degree of its components.) Let  $z_2$  be a minimal degree solution which is linearly independent over  $\mathbb{R}[\lambda, \mu]$  of  $z_1$ . Let  $z_3$  be a minimal degree solution which is linearly independent of  $\{z_1, z_2\}$ . Proceeding in this way, we obtain a sequence  $z_1, \dots, z_q$  of solutions, which we refer to as a fundamental series of solutions of (2.3). Since linear independence over  $\mathbb{R}[\lambda, \mu]$  is equivalent to linear independence over the fraction field  $\mathbb{R}(\lambda, \mu)$  of rational functions in two variables, it follows that  $q$  is at most equal to  $n+m$ . Let  $\delta_1 < \dots < \delta_q$  denote the degrees of  $z_1, \dots, z_q$ , respectively. Using the fraction field  $\mathbb{R}(\lambda, \mu)$ , it follows by an argument which is analogous to the one given in [8, p. 38] for the minimal column indices that  $\delta_1, \dots, \delta_q$  are well-defined -- i.e., independent of the choice of fundamental series. We will refer to  $(\delta_1, \dots, \delta_q)$  as the homogeneous indices of the system  $(E, A, B)$ .

Remark 2.2: It should be noted that the pencil  $[\lambda E - \mu A, B]$  is not the homogenization of the pencil  $[\lambda E - A, B]$ . Since  $[\lambda E - A, B] = \lambda[E, 0] + [-A, B]$ , the homogenization of  $[\lambda E - A, B]$  would be  $\lambda[E, 0] + \mu[-A, B] = [\lambda E - \mu A, \mu B]$ . However, the term "homogeneous indices" seems appropriate since the submatrix  $\lambda E - \mu A$  of

$[\lambda E - \mu A, B]$  is the homogenization of the submatrix  $\lambda E - A$  of  $[\lambda E - A, B]$ . The homogenization of  $\lambda E - A$  to  $\lambda E - \mu A$  plays a crucial role in the Weierstrass theory of regular matrix pencils [8, p. 26].

We now establish important properties of the homogeneous indices which will be needed later. Given a triple  $(E, A, B)$ , let  $HI(E, A, B)$  denote its set of homogeneous indices, and let  $CI(E, A, B)$  denote its set of minimal column indices -- i.e., the minimal column indices of the singular pencil  $[\lambda E - A, B]$ .

The following result shows that the homogeneous indices of  $(E, A, B)$  are invariant under system rotation.

Proposition 2.2: If  $(E, A, B), (\hat{E}, \hat{A}, B) \in \hat{\Sigma}(n, m)$  with  $(\hat{E}, \hat{A}, B) = R_\phi(E, A, B)$ , then  $HI(E, A, B) = HI(\hat{E}, \hat{A}, B)$ .

Proof: Let  $z_1, \dots, z_p$  be a fundamental series of solutions of the equation

$$[\lambda E - \mu A, B]z = 0,$$

and let  $\hat{z}_i(\hat{\lambda}, \hat{\mu}) = z_i((\cos\phi)\hat{\lambda} + (\sin\phi)\hat{\mu}, (-\sin\phi)\hat{\lambda} + (\cos\phi)\hat{\mu})$ . It is easy to verify that  $\hat{z}_1, \dots, \hat{z}_p$  is a fundamental series of solutions of the equation

$$[\hat{\lambda}\hat{E} - \hat{\mu}\hat{A}, B]\hat{z} = 0.$$

Since  $\deg \hat{z}_i = \deg z_i$ , the result follows immediately.  $\square$

Remark 2.3: Proposition 2.2 describes a crucial difference between the homogeneous indices and the minimal column indices. In contrast to the homogeneous indices, the minimal column indices are not invariant under system rotation.

For example, consider the system  $(E, A, B)$  with

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let  $\phi = \pi/2$ , and let  $(\hat{E}, \hat{A}, B) = R_\phi(E, A, B)$ . Then  $\hat{E} = A$  and  $\hat{A} = -E$ . It is easy to check that  $CI(E, A, B) = (1)$  whereas  $CI(\hat{E}, \hat{A}, B) = (2)$ . On the other hand, we have  $HI(E, A, B) = HI(\hat{E}, \hat{A}, B) = (2)$ .

The next result shows that for a controllable regular system, the homogeneous indices coincide with the minimal column indices. The proof is deferred to the next section.

Proposition 2.3: Let  $(E, A, B)$  be a controllable regular system. Then,

$$HI(E, A, B) = CI(E, A, B) .$$

Remark 2.4: Using Propositions 2.2, 2.3, and 2.1, we can obtain a simple procedure for computing the homogeneous indices of a controllable generalized system. Let  $C_\theta(n, m)$  denote the subset of  $\sum_\theta(n, m)$  consisting of those systems which are controllable according to the definition of Yip and Sincovec [11]. (I.e., there are no finite or infinite input decoupling zeroes.) It is proven in [2] that controllability is invariant under system rotation. Thus,  $R_\phi(C_\theta(n, m)) = C_{\theta+\phi}(n, m)$ . Let  $(E, A, B) \in C_\theta(n, m)$ , and let  $(\hat{E}, \hat{A}, B) = R_{-\theta}(E, A, B) \in C_0(n, m)$ . By Propositions 2.2 and 2.3, we have  $HI(E, A, B) = HI(\hat{E}, \hat{A}, B) = CI(\hat{E}, \hat{A}, B)$ . Thus, the homogeneous indices of  $(E, A, B)$  can be determined by computing the  $\ell_1$ 's for the controllable regular system  $(\hat{E}, \hat{A}, B)$ , and then using Proposition 2.1(b) to obtain  $CI(\hat{E}, \hat{A}, B)$ . For example, let  $(E, A, B)$  be as in Remark 2.3, and let  $\theta = -\pi/2$ . Then,  $(E, A, B) \in C_\theta(n, m)$  and  $(\hat{E}, \hat{A}, B) = R_{-\theta}(E, A, B) = (A, -E, B)$ . Since  $\text{rank } B = 1$  and  $\text{rank } [B, -EB] = 2$ , we get  $\ell_1 = 1$ ,  $\ell_2 = 1$ . Applying Proposition 2.1(b), we obtain  $HI(E, A, B) = (2)$ .

Proposition 2.4: If  $(E, A, B)$  is a controllable admissible system, then  $HI(E, A, B)$  is a partition of  $n$  into  $m$  parts, of which  $\text{rank } B$  parts are strictly positive.

Proof: By assumption,  $(E,A,B) \in C_\theta(n,m)$  for some  $\theta$ . Using the notation of Remark 2.4, we have  $HI(E,A,B) = CI(\hat{E},\hat{A},B)$ . The result follows from this together with Proposition 2.1(a).  $\square$

Remark 2.5: Proposition 2.4 describes an important difference between the homogeneous indices and the minimal column indices of a controllable system. The homogeneous indices sum to  $n$  regardless of whether the system is regular or singular. In contrast, the minimal column indices sum to  $\text{rank } E$  (Remark 2.1), which is equal to  $n$  only if the system is regular.

Remark 2.3 shows that in contrast to the homogeneous indices, the minimal column indices are not invariant under system rotation. However, what is true is that if two controllable regular systems are related by a system rotation, then they have identical minimal column indices.

Proposition 2.5: If  $(E,A,B)$  and  $(\hat{E},\hat{A},B)$  are controllable regular systems with  $(\hat{E},\hat{A},B) = R_\phi(E,A,B)$ , then  $CI(E,A,B) = CI(\hat{E},\hat{A},B)$ .

Proof: Using Propositions 2.2 and 2.3, we have  $CI(E,A,B) = HI(E,A,B) = HI(\hat{E},\hat{A},B) = CI(\hat{E},\hat{A},B)$ .  $\square$

### 3. Feedback Invariants.

We begin by reviewing the definition of the state feedback group. (See e.g. [12],[9],[13],[14].) Consider the ordinary state-space model

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.1}$$

where  $(A,B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ . We consider three types of elementary transformations on the system (3.1). They are (1) change of basis in the state-space,  $x = Pz$  with  $P$  a nonsingular  $n \times n$  matrix; (2) change of basis in the input space,  $u = Qv$  with  $Q$  a nonsingular  $m \times m$  matrix; (3) state feedback  $u = Fx + v$ .

These operations transform the matrix pair (A,B) as follows:

$$(A,B) \rightarrow (P^{-1}AP, P^{-1}B) \quad (3.2)$$

$$(A,B) \rightarrow (A, BQ) \quad (3.3)$$

$$(A,B) \rightarrow (A + BF, B) \quad (3.4)$$

The transformation group generated by (3.2), (3.3), (3.4) can be conveniently represented in the following way. Recall that a right group action of a group G on a set X is a mapping  $\eta: X \times G \rightarrow X$  satisfying the conditions

$$\eta(x, e) = x \text{ and } \eta(x, g_1 g_2) = \eta(\eta(x, g_1), g_2) \text{ where } e \text{ denotes the identity element}$$

of G. If  $x \in X$ , the orbit of x, denoted  $xG$ , consists of the subset

$$\{\eta(x, g): g \in G\} \text{ of } X.$$

Let  $C(n,m)$  denote the space of all matrix pairs  $(A,B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$  which are controllable. Let  $H(n,m)$  denote the group consisting of all nonsingular  $(n+m) \times (n+m)$  matrices of the form

$$\begin{bmatrix} P & 0 \\ F & Q \end{bmatrix}$$

with  $P \ n \times n$ ,  $F \ m \times n$ ,  $Q \ m \times m$ . We refer to  $C(n,m)$  as the space of controllable pairs and to  $H(n,m)$  as the state feedback group. Define a right group action of  $H(n,m)$  on  $C(n,m)$  by

$$\eta((A,B), \begin{bmatrix} P & 0 \\ F & Q \end{bmatrix}) = (P^{-1}AP + P^{-1}BF, P^{-1}BQ). \quad (3.5)$$

The transformations (3.2), (3.3), (3.4) correspond to the special cases of (3.5) where  $F = 0$  and  $Q = I$ ,  $P = I$  and  $F = 0$ ,  $P = I$  and  $Q = I$  respectively.

It is of interest to know when two systems  $(A_1, B_1)$  and  $(A_2, B_2)$  are related by a transformation in the state feedback group -- i.e. belong to the same  $H(n,m)$ -orbit. It is also useful to have a canonical form for this group

action -- to identify the "simplest" element on each orbit. This is provided by the following result of Brunovsky [12]. (See also [9],[6],[15].)

Theorem 3.1 [12]:

(a)  $(A_1, B_1), (A_2, B_2) \in C(n, m)$  belong to the same  $H(n, m)$ -orbit if and only if  $CI(I, A_1, B_1) = CI(I, A_2, B_2)$ .

(b) Let  $r$  be a positive integer with  $r < \min(n, m)$ , and let

$n_1 > n_2 > \dots > n_r$  be a partition of  $n$  into  $r$  positive parts. The

$H(n, m)$ -orbit consisting of those pairs  $(A, B) \in C(n, m)$  for which

$CI(I, A, B) = (0, \dots, 0, n_r, n_{r-1}, \dots, n_1)$  contains the canonical pair

$(A_c, B_c)$  given by

$$A_c = \begin{bmatrix} J_{n_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2} & 0 & \dots & 0 \\ 0 & 0 & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot \\ & & & & J_{n_r} \end{bmatrix}$$

$$B_c = \begin{bmatrix} e_{n_1} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & e_{n_2} & 0 & \dots & 0 & \cdot & & \cdot \\ 0 & 0 & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & e_{n_r} & 0 & \dots & 0 \end{bmatrix}$$

where  $J_k$  is a  $k \times k$  matrix of the form

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot \\ 0 & 0 & \cdot & \cdot & 0 \end{bmatrix}$$

and  $e_k$  is a  $k$ -dimensional column vector in which the only nonzero component is the last, which is 1.

We will refer to the pair  $(A_c, B_c)$  in Theorem 3.1 as the Brunovsky canonical form associated with the set of minimal column indices  $(0, \dots, 0, n_r, \dots, n_1)$ .

The following result is needed for the proof of Proposition 2.3.

Lemma 3.1: If  $(A, B), (\hat{A}, \hat{B}) \in C(n, m)$  belong to the same  $H(n, m)$ -orbit, then

$$HI(I, A, B) = HI(I, \hat{A}, \hat{B}).$$

Proof: It suffices to consider the following two special cases:

- (a)  $(\hat{A}, \hat{B}) = (P^{-1}AP, P^{-1}BQ)$
- (b)  $(\hat{A}, \hat{B}) = (A + BF, B)$ .

First consider (a). We have

$$[\lambda I - \mu \hat{A}, \hat{B}] = P^{-1}[\lambda I - \mu A, B] \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}. \quad (3.6)$$

Let  $z_1, \dots, z_p$  be a fundamental series of solutions of the equation

$[\lambda I - \mu A, B]z = 0$ . Let  $\hat{z}_i = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}^{-1} z_i$ . Then it is clear that  $\hat{z}_1, \dots, \hat{z}_p$  is a

fundamental series of solutions of the equation  $[\lambda I - \mu \hat{A}, \hat{B}]\hat{z} = 0$ . Since

$\deg \hat{z}_i = \deg z_i$ , it follows that  $HI(I, A, B) = HI(I, \hat{A}, \hat{B})$ .

Now consider (b). We have

$$[\lambda I - \mu \hat{A}, \hat{B}] = [\lambda I - \mu A, B] \begin{bmatrix} I & 0 \\ -\mu F & I \end{bmatrix}. \quad (3.7)$$

Let  $z_1, \dots, z_p$  be a fundamental series of solutions of  $[\lambda I - \mu A, B]z = 0$ , and let

$\hat{z}_i = \begin{bmatrix} I & 0 \\ \mu F & I \end{bmatrix} z_i$ . Then  $\hat{z}_1, \dots, \hat{z}_p$  are solutions of  $[\lambda I - \mu \hat{A}, \hat{B}]\hat{z} = 0$  which are

linearly independent (over  $\mathbb{R}[\lambda, \mu]$ ).

We claim that  $\deg \hat{z}_i = \deg z_i$ . Let  $z_i = \begin{bmatrix} x_i \\ u_i \end{bmatrix}$  and let  $\hat{z}_i = \begin{bmatrix} \hat{x}_i \\ \hat{u}_i \end{bmatrix}$ . Then

$\hat{x}_i = x_i$  and  $\hat{u}_i = \mu F x_i + u_i$ . Thus, it suffices to show that  $\deg \hat{u}_i = \deg u_i$ .

Let  $d = \deg x_i$ , and write

$$x_i = x_{i0} \lambda^d + x_{i1} \lambda^{d-1} \mu + \dots + x_{id} \mu^d + \bar{x}_i$$

with  $\deg \bar{x}_i < d$ . Suppose that  $x_{i0}, \dots, x_{i,k-1}$  are zero, but  $x_{ik}$  is nonzero. Since  $Bu_i = \mu Ax_i - \lambda x_i$ , it follows that the coefficient vector of  $\lambda^{d-k+1} \mu^k$  in  $u_i$  is nonzero. In particular,  $\deg u_i > d + 1$ . If  $\deg u_i > d + 1$ , then since  $\deg \mu F x_i \leq d + 1$ , it follows that  $\deg \hat{u}_i = \deg u_i$ . Therefore, we may assume  $\deg u_i = d + 1$ . Since  $u_i$  contains  $\lambda^{d-k+1} \mu^k$  but  $\mu F x_i$  does not contain  $\lambda^{d-k+1} \mu^k$ , it follows that  $\deg \hat{u}_i = d + 1 = \deg u_i$ . Thus,  $\deg \hat{z}_i = \deg z_i$  as claimed.

We claim that  $\hat{z}_1, \dots, \hat{z}_p$  is a fundamental series of solutions of  $[\lambda I - \mu \hat{A}, \hat{B}] \hat{z} = 0$ . Suppose not. Then there is a smallest positive integer  $k$  for which  $\deg \hat{z}_k$  is not minimal. Thus, there exists a solution  $\hat{y}_k$  with  $\deg \hat{y}_k <$

$\deg \hat{z}_k$  such that  $\hat{z}_1, \dots, \hat{z}_{k-1}, \hat{y}_k$  are linearly independent. Let  $y_k = \begin{bmatrix} I & 0 \\ -\mu F & I \end{bmatrix} \hat{y}_k$ .

Then,  $z_1, \dots, z_{k-1}, y_k$  is a linearly independent set of solutions of  $[\lambda I - \mu A, B] z = 0$ .

Essentially the same argument as given in the preceding paragraph shows that

$\deg y_k = \deg \hat{y}_k$ , so  $\deg y_k < \deg z_k$ . This contradicts the assumption that

$z_1, \dots, z_p$  is a fundamental series. Thus,  $\hat{z}_1, \dots, \hat{z}_p$  is a fundamental series.

Since  $\deg \hat{z}_i = \deg z_i$ , this proves that  $HI(I, A, B) = HI(I, \hat{A}, \hat{B})$ .  $\square$

**Remark 3.1:** We can use (3.5) to define the action of  $H(n, m)$  on all matrix pairs  $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ , rather than on the set of controllable pairs  $C(n, m)$ .

Since the proof of Lemma 3.1 does not use controllability, it follows that the

statement of Lemma 3.1 remains true with  $C(n, m)$  replaced by  $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ .



Proof of Proposition 2.3: Let  $(E,A,B)$  be a controllable regular system. Since left-multiplication is a strict equivalence transformation, it does not change either the minimal column indices or the homogeneous indices. Consequently, it suffices to show that  $HI(I,E^{-1}A,E^{-1}B) = CI(I,E^{-1}A,E^{-1}B)$ . Let  $(A_c,B_c)$  denote the Brunovsky canonical form of  $(E^{-1}A,E^{-1}B)$ . From Theorem 3.1(a) and Lemma 3.1, we have  $CI(I,A_c,B_c) = CI(I,E^{-1}A,E^{-1}B)$  and  $HI(I,A_c,B_c) = HI(I,E^{-1}A,E^{-1}B)$ . Thus, it suffices to show that  $HI(I,A_c,B_c) = CI(I,A_c,B_c)$ .

Let  $(0, \dots, 0, n_r, \dots, n_1)$  denote the minimal column indices of  $(I,A_c,B_c)$ . We can choose an  $(n+m) \times (n+m)$  permutation matrix  $N$  such that

$$[\lambda I - \mu_{A_c, B_c}]N = [\text{diag}\{K(n_1), \dots, K(n_r)\}, 0_{n \times (m-r)}],$$

where  $K(n_i)$  denotes the  $n_i \times (n_i + 1)$  pencil  $[\lambda I_{n_i} - \mu J_{n_i}, e_{n_i}]$ . Although  $[\lambda I - \mu_{A_c, B_c}]N$  is not of the form  $[\lambda E - \mu A, B]$ , we can define a fundamental series of solutions for the equation  $[\lambda I - \mu_{A_c, B_c}]N\hat{z} = 0$  in the obvious way. Hence, homogeneous indices are well-defined for the pencil  $[\lambda I - \mu_{A_c, B_c}]N$ . If  $\hat{z}_1, \dots, \hat{z}_p$  denotes such a fundamental series, then it is clear that  $N\hat{z}_1, \dots, N\hat{z}_p$  is a fundamental series for the equation  $[\lambda I - \mu_{A_c, B_c}]z = 0$ . Since  $\deg N\hat{z}_i = \deg \hat{z}_i$ , it follows that the homogeneous indices of  $[\lambda I - \mu_{A_c, B_c}]$  -- i.e.,  $HI(I,A_c,B_c)$  -- are equal to the homogeneous indices of  $[\lambda I - \mu_{A_c, B_c}]N$ . Thus, it suffices to show that the homogeneous indices of  $[\lambda I - \mu_{A_c, B_c}]N$  are  $(0, \dots, 0, n_r, \dots, n_1)$ .

It is easy to see that for a block-diagonal pencil, the complete set of homogeneous indices is obtained as the union of the corresponding systems of homogeneous indices of the individual diagonal blocks. (This is analogous to the situation for the minimal column indices noted in [8, p. 39].) Consequently, the homogeneous indices of  $[\lambda I - \mu_{A_c, B_c}]N$  consist of  $0, \dots, 0$  (multiplicity  $m-r$ ) together with the homogeneous indices of the  $r$  pencils  $[\lambda I_{n_i} - \mu J_{n_i}, e_{n_i}]$ . Let  $z_{n_i}^*$  denote the  $(n_i + 1)$  - component vector with  $j^{\text{th}}$  component  $\lambda^{j-1} \mu^{n_i-j}$

( $j=1, \dots, n_i$ ) and ( $n_i + 1$ )<sup>th</sup> component  $-\lambda^{n_i}$ . It is easy to verify that every nonzero solution of  $[\lambda I_{n_i} - \mu J_{n_i}, e_{n_i}] z_i = 0$  is an  $\mathbb{R}[\lambda, \mu]$ -multiple of  $z_{n_i}^*$ . Consequently,  $[\lambda I_{n_i} - \mu J_{n_i}, e_{n_i}]$  has the single homogeneous index ( $n_i$ ). Thus, the homogeneous indices of  $[\lambda I - \mu A_c, B_c] N$  are  $(0, \dots, 0, n_r, \dots, n_1)$ , completing the proof.  $\square$

We now review the definition of the CRPD state feedback groups given in [2]. We consider four types of elementary transformations on the system (1.1). They are (1) change of basis in the state-space,  $x = Pz$  with  $P$  a nonsingular  $n \times n$  matrix; (2) change of basis in the input space,  $u = Qv$  with  $Q$  a nonsingular  $m \times m$  matrix; (3) CRPD feedback  $u = F(\cos\theta x - \sin\theta \dot{x}) + v$  with  $\theta$  a fixed number in  $[0, \pi)$  and  $F$  an arbitrary  $m \times n$  matrix; (4) left-multiplication by a nonsingular  $n \times n$  matrix  $R^{-1}$ . These operations transform the matrix triple  $(E, A, B)$  as follows:

$$(E, A, B) \rightarrow (P^{-1}EP, P^{-1}AP, P^{-1}B) \quad (3.8)$$

$$(E, A, B) \rightarrow (E, A, BQ) \quad (3.9)$$

$$(E, A, B) \rightarrow (E + \sin\theta BF, A + \cos\theta BF, B) \quad (3.10)$$

$$(E, A, B) \rightarrow (R^{-1}E, R^{-1}A, R^{-1}B) \quad (3.11)$$

The transformation group generated by (3.8), (3.9), (3.10), (3.11) can be conveniently represented in the following way. For each  $\theta$ , let  $G_\theta(n, m)$  denote the group consisting of all nonsingular  $(3n+m) \times (3n+m)$  matrices of the form

$$\begin{bmatrix} R & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & \sin\theta F & \cos\theta F & Q \end{bmatrix} . \quad (3.12)$$

We refer to the family of groups  $\{G_\theta(n, m) : \theta \in [0, \pi)\}$  as the CRPD state feedback groups.

Remark 3.2: If  $\cos\theta F$  and  $\sin\theta F$  in (3.12) are replaced with arbitrary  $m \times n$  matrices  $F_1, F_2$ , then one obtains a larger group which we denote by  $G(n,m)$  and refer to as the proportional and derivative state feedback group. This corresponds to replacing the CRPD feedback (1.4) with the more general proportional and derivative feedback  $u = F_1 x - F_2 \dot{x} + v$ . Each of the CRPD feedback groups,  $G_\theta(n,m)$ , is a subgroup of  $G(n,m)$ .

Define a right group action of  $G_\theta(n,m)$  on  $\hat{\Sigma}(n,m)$  by

$$\left( (E,A,B), \begin{bmatrix} R & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & \sin\theta F & \cos\theta F & Q \end{bmatrix} \right) \quad (3.13)$$

$$\rightarrow (R^{-1}EP + \sin\theta R^{-1}BF, R^{-1}AP + \cos\theta R^{-1}BF, R^{-1}BQ).$$

Each of the transformations (3.8), (3.9), (3.10), (3.11) is a special case of (3.13). Let  $g_\theta(R,P,Q,F)$  denote the transformation on  $\hat{\Sigma}(n,m)$  induced by the matrix (3.12) in  $G_\theta(n,m)$ . In other words,  $g_\theta(R,P,Q,F)(E,A,B)$  denotes the right-hand side of (3.13).

Recall from Remark 2.4 that  $C_\theta(n,m)$  denotes the controllable systems in the open and dense subset  $\Sigma_\theta(n,m)$ , and that  $R_\phi(C_\theta(n,m)) = C_{\theta+\phi}(n,m)$ . The following three results are proven in [2].

Proposition 3.1 [2]:  $\Sigma_\theta(n,m)$  is invariant under the action of  $G_\theta(n,m)$ .

Proposition 3.2 [2]: The following is a commutative diagram:

$$\begin{array}{ccc}
& g_{\theta}(R,P,Q,F) & \\
\sum_{\theta}(n,m) & \xrightarrow{\quad} & \sum_{\theta}(n,m) \\
\downarrow R_{\phi} & & \downarrow R_{\phi} \\
\sum_{\theta+\phi}(n,m) & \xrightarrow{\quad} & \sum_{\theta+\phi}(n,m) \\
& g_{\theta+\phi}(R,P,Q,F) &
\end{array}$$

I.e.,  $R_{\phi} \circ g_{\theta}(R,P,Q,F) = g_{\theta+\phi}(R,P,Q,F) \circ R_{\phi}$ .

Proposition 3.3 [2]:  $C_{\theta}(n,m)$  is invariant under the action of  $G_{\theta}(n,m)$ .

By virtue of Proposition 3.3, we can restrict the action of  $G_{\theta}(n,m)$  on  $\sum_{\theta}(n,m)$  to the invariant subset  $C_{\theta}(n,m)$ . The problem which we consider is the one of determining a complete set of invariants for the action of  $G_{\theta}(n,m)$  on  $C_{\theta}(n,m)$ . Roughly speaking, this means finding a set of functions of  $(E,A,B)$  with the property that these functions have the same values on  $(E_1,A_1,B_1)$  as on  $(E_2,A_2,B_2)$  if and only if  $(E_1,A_1,B_1)$  and  $(E_2,A_2,B_2)$  are related by a transformation in the CRPD feedback group  $G_{\theta}(n,m)$ .

Remark 3.3: In the special case where  $\theta = 0$ ,  $C_0(n,m)$  consists of the controllable regular systems, and  $G_0(n,m)$  can be regarded as the state feedback group  $H(n,m)$  augmented by left-multiplication. In this case, it follows from Theorem 3.1(a) that the minimal column indices are a complete set of invariants. In other words,  $(E_1,A_1,B_1), (E_2,A_2,B_2) \in C_0(n,m)$  are equivalent under the action of  $G_0(n,m)$  if and only if  $CI(E_1,A_1,B_1) = CI(E_2,A_2,B_2)$ .

The following result is the main result of this section. It presents the solution to the problem posed above, and represents a natural generalization of Theorem 3.1.

Theorem 3.2:

- (a)  $(E_1, A_1, B_1), (E_2, A_2, B_2) \in C_\theta(n, m)$  belong to the same  $G_\theta(n, m)$ -orbit if and only if

$$HI(E_1, A_1, B_1) = HI(E_2, A_2, B_2) .$$

- (b) Let  $n_1 \geq \dots \geq n_m$  be a partition of  $n$  into  $m$  nonnegative parts. The  $G_\theta(n, m)$ -orbit consisting of those triples  $(E, A, B) \in C_\theta(n, m)$  for which  $HI(E, A, B) = (n_m, \dots, n_1)$  contains the canonical triple  $(\cos\theta I + \sin\theta A_c, -\sin\theta I + \cos\theta A_c, B_c)$  where  $(A_c, B_c)$  is the Brunovsky canonical form associated with the minimal column indices  $(n_m, \dots, n_1)$ .

Proof: (a) Let  $(E_1, A_1, B_1), (E_2, A_2, B_2) \in C_\theta(n, m)$ , and suppose there exist  $R, P, Q, F$  such that  $(E_2, A_2, B_2) = g_\theta(R, P, Q, F)(E_1, A_1, B_1)$ . Applying successively Proposition 2.2, Proposition 2.3, Theorem 3.1(a) together with Remark 3.3, Proposition 2.3, Proposition 2.2, and Proposition 3.2, we have

$$\begin{aligned} HI(E_1, A_1, B_1) &= HI(R_{-\theta}(E_1, A_1, B_1)) \\ &= CI(R_{-\theta}(E_1, A_1, B_1)) = CI(g_0(R, P, Q, F) \circ R_{-\theta}(E_1, A_1, B_1)) \\ &= HI(g_0(R, P, Q, F) \circ R_{-\theta}(E_1, A_1, B_1)) \\ &= HI(R_\theta \circ g_0(R, P, Q, F) \circ R_{-\theta}(E_1, A_1, B_1)) \\ &= HI(g_\theta(R, P, Q, F)(E_1, A_1, B_1)) = HI(E_2, A_2, B_2). \end{aligned}$$

Conversely, suppose  $(E_1, A_1, B_1), (E_2, A_2, B_2) \in C_\theta(n, m)$  with  $HI(E_1, A_1, B_1) = HI(E_2, A_2, B_2)$ . By Proposition 2.2, we have  $HI(R_{-\theta}(E_1, A_1, B_1)) = HI(R_{-\theta}(E_2, A_2, B_2))$ . By Proposition 2.3, it follows that  $CI(R_{-\theta}(E_1, A_1, B_1)) = CI(R_{-\theta}(E_2, A_2, B_2))$ . Consequently, by Theorem 3.1(a) together with Remark 3.3, there exist  $R, P, Q, F$  such that  $R_{-\theta}(E_2, A_2, B_2) = g_0(R, P, Q, F) \circ R_{-\theta}(E_1, A_1, B_1)$ . Thus, it follows from Proposition 3.2 that  $(E_2, A_2, B_2) = g_\theta(R, P, Q, F)(E_1, A_1, B_1)$ .

- (b) Let  $(E, A, B) \in C_\theta(n, m)$ . It is proven in [2] that there is some partition of  $n$  into  $m$  nonnegative parts,  $n_1 \geq n_2 \geq \dots \geq n_m$ , such that the  $G_\theta(n, m)$ -orbit of  $(E, A, B)$  contains the triple  $(\cos\theta I + \sin\theta A_c, -\sin\theta I + \cos\theta A_c, B_c)$ , where

$(A_c, B_c)$  is the Brunovsky canonical form associated with the minimal column indices  $(n_m, \dots, n_1)$ . It remains to show that  $(n_m, \dots, n_1)$  are the homogeneous indices of  $(E, A, B)$ . By successively using part (a), Proposition 2.2, and Proposition 2.3, we have  $HI(E, A, B) = HI(\cos\theta I + \sin\theta A_c, -\sin\theta I + \cos\theta A_c, B_c) = HI(I, A_c, B_c) = CI(I, A_c, B_c) = (n_m, \dots, n_1)$ .  $\square$

Remark 3.4: Kalman has noted [9] that the action (3.5) of the state feedback group can be regarded as a special case of the strict equivalence action on matrix pencils. To see this, let  $(\hat{A}, \hat{B})$  denote the righthand side of (3.5).

Then,

$$[\lambda I - \hat{A}, \hat{B}] = P^{-1}[\lambda I - A, B] \begin{bmatrix} P & 0 \\ -F & Q \end{bmatrix}. \quad (3.14)$$

Thus, the pencil  $[\lambda I - \hat{A}, \hat{B}]$  is obtained from the pencil  $[\lambda I - A, B]$  by a strict equivalence transformation. Consequently, Brunovsky canonical form is a special case of the Kronecker canonical form for matrix pencils.

An obvious question to ask is whether the action (3.13) of the CRPD feedback group  $G_\theta(n, m)$  can somehow be regarded as a special case of the strict equivalence action on matrix pencils. It is easy to show that the answer to this question is negative. One way to show this is to note that if (3.13) corresponded to a strict equivalence transformation, then the minimal column indices would be invariant under the action of  $G_\theta(n, m)$ . This is not the case. For example, let  $(E, A, B)$  be as in Remark 2.3, and let  $\theta = \pi/2$ . Then,  $R_{-\theta}(E, A, B) = (-A, E, B)$ , which is a controllable regular system -- i.e., an element in  $C_\theta(n, m)$ . Thus,  $(E, A, B) \in C_\theta(n, m)$ . Let  $F = [1 \ 0]$ . Then  $g_\theta(I, I, I, F)(E, A, B) = (E + BF, A, B)$  with

$$E + BF = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus,  $(E + BF, A, B)$  is a controllable regular single-input system. Consequently,  $CI(E + BF, A, B) = (2)$ . On the other hand, it was noted in Remark 2.3 that  $CI(E, A, B) = (1)$ .

Another way to appreciate the distinction between the action of the CRPD feedback groups and the strict equivalence action is to express (3.13) as a transformation on matrix pencils. Let  $(\tilde{E}, \tilde{A}, \tilde{B})$  denote the righthand side of (3.13). Then,

$$[\lambda \tilde{E} - \mu \tilde{A}, \tilde{B}] = R^{-1}[\lambda E - \mu A, B] \begin{bmatrix} P & 0 \\ (\lambda \sin \theta - \mu \cos \theta)F & Q \end{bmatrix}. \quad (3.15)$$

In contrast to (3.14), the transformation (3.15) is not of the strict equivalence type since the right-multiplication is not by a constant matrix.

In the literature, other types of feedback groups have been considered in conjunction with generalized linear systems. Hayton [19] studies the action of the state feedback group  $H(n, m)$  on generalized linear systems. Pandolfi [20] considers the transformation group generated by exponential rescaling, left-multiplication, change of basis in the state-space, change of basis in the input space, and static state feedback. A complete set of invariants for the action of this group on the set of controllable admissible generalized linear systems is determined.

The transformation groups  $\{G_\theta(n, m)\}$  which we study differ considerably from those in [19] and [20]. The feedback in [19] and [20] is static state feedback, which implies that the rank of  $E$  is invariant. In contrast, the feedback in  $G_\theta(n, m)$  is of the CRPD-type, and can modify the rank of  $E$ . Only when  $\theta = 0$  does the CRPD feedback coincide with pure state feedback. If  $(E, A, B)$  is a singular system, then  $(E, A, B) \notin \sum_0(n, m)$ . Since the transformations in  $G_\theta(n, m)$  are applied only to the systems in  $\sum_0(n, m)$ , pure state feedback (i.e., no derivative contribution) is never applied to a singular system.

#### 4. Control Structure Theorem.

The problem of pole-assignment by state feedback for singular systems has been studied by Cobb [21] and Pandolfi [22]. Armentano [23] and Lewis and Ozcaldiran [24] have investigated eigenvector-assignment by state feedback. Mukundan and Dayawansa [25] have studied pole-assignment by proportional and derivative state feedback. In this section, we consider a different problem, namely, the determination of which closed-loop invariant polynomials can be obtained using constant-ratio proportional and derivative feedback.

The Control Structure Theorem of Rosenbrock [6] is an important result which describes precisely which invariant polynomials can be assigned by application of state feedback to a controllable regular system:

Theorem 4.1 [6]: Let  $(A,B)$  be controllable. Let  $r = \text{rank } B$ , and let  $n_1 > \dots > n_r$  be the nonzero minimal column indices of  $(I,A,B)$ . Let  $q \leq r$ , and let  $\tilde{\psi}_1(\lambda), \dots, \tilde{\psi}_q(\lambda)$  be any set of nonunity monic polynomials such that  $\tilde{\psi}_{i+1} | \tilde{\psi}_i$  ( $i = 1, \dots, q-1$ ) and  $\sum_{i=1}^q \deg \tilde{\psi}_i = n$ . Then there is a state feedback gain  $F$  such that the given polynomials are the nonunity invariant polynomials of the closed loop system  $A + BF$  if and only if

$$\sum_{i=1}^p \deg \tilde{\psi}_i \geq \sum_{i=1}^p n_i, \quad p = 1, \dots, q.$$

Let  $(E,A,B)$  be an admissible generalized linear system -- i.e.,  $(E,A,B) \in \Sigma(n,m)$ . By the invariant polynomials of  $(E,A,B)$ , we mean the invariant polynomials of the pencil  $\lambda E - A$ . In order to generalize Rosenbrock's Theorem, it is necessary to define the homogeneous invariant polynomials for  $(E,A,B)$ . In the Weierstrass treatment of infinite elementary divisors for a regular pencil [8, p. 26], (homogeneous) invariant polynomials are defined for the homogeneous pencil  $\lambda E - \mu A$ . Let  $D_k(\lambda, \mu)$  be the greatest common divisor of the minors of



order  $k$  ( $k=1, \dots, n$ ). The invariant polynomials of  $\lambda E - \mu A$  are the quotients  $i_1 = D_n/D_{n-1}$ ,  $i_2 = D_{n-1}/D_{n-2}$ , etc. Each  $D_k, i_j$  is homogeneous. We define the homogeneous invariant polynomials of  $(E, A, B)$  to be the invariant polynomials of  $\lambda E - \mu A$ . Note that these polynomials are defined modulo multiplication by non-zero real numbers.

Let  $r_\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with

$$r_\phi(\lambda, \mu) = ((\cos\phi)\lambda + (\sin\phi)\mu, (-\sin\phi)\lambda + (\cos\phi)\mu). \quad (4.1)$$

The following proposition describes how the homogeneous invariant polynomials transform under system rotation.

Proposition 4.1: Let  $\psi_1, \dots, \psi_n$  denote the homogeneous invariant polynomials of  $(E, A, B) \in \tilde{\Sigma}(n, m)$ . Then the homogeneous invariant polynomials of  $R_\phi(E, A, B)$  are  $\psi_1 \circ r_\phi, \dots, \psi_n \circ r_\phi$ .

Proof: Let  $(\hat{E}, \hat{A}, B) = R_\phi(E, A, B)$ , and let  $(\hat{\lambda}, \hat{\mu}) = r_{-\phi}(\lambda, \mu)$ . Then,

$$\lambda E - \mu A = \hat{\lambda} \hat{E} - \hat{\mu} \hat{A}. \quad (4.2)$$

Since  $(\lambda, \mu) = r_\phi(\hat{\lambda}, \hat{\mu})$ , it follows immediately from (4.2) that the invariant polynomials of  $\lambda E - \mu A$  are  $\psi_1 \circ r_\phi, \dots, \psi_n \circ r_\phi$ .  $\square$

Remark 4.1: It follows from Proposition 4.1 that the degrees of the homogeneous invariant polynomials of  $R_\phi(E, A, B)$  are equal to the degrees of the homogeneous invariant polynomials of  $(E, A, B)$ . However, the corresponding statement for the ordinary invariant polynomials is definitely not true. For example, let  $(E, A, B)$  be as in Remark 2.3, and let  $\phi = \pi/2$ . Then  $R_\phi(E, A, B) = (A, -E, B)$ . The homogeneous invariant polynomials of  $(E, A, B)$  are  $\mu^2, 1$ , which have the same degrees as  $\lambda^2, 1$ , the homogeneous invariant polynomials of  $R_\phi(E, A, B)$ . On the other hand, the (ordinary) invariant polynomials of  $(E, A, B)$  are  $1, 1$ , whereas those of

$R_\phi(E,A,B)$  are  $\lambda^2, 1$ .

Proposition 4.2:  $(E,A,B) \in \Sigma(n,m)$  is a regular system if and only if no homogeneous invariant polynomial is divisible by  $\mu$ . In this case, there is a degree-preserving one-to-one correspondence between the homogeneous invariant polynomials and the invariant polynomials given by  $\psi_i(\lambda,\mu) \leftrightarrow \psi_i(\lambda,1)$ .

Proof: See [8, p. 27].  $\square$

Remark 4.2: The condition in Proposition 4.2 that no homogeneous invariant polynomial is divisible by  $\mu$  is equivalent to the condition that  $\det(\lambda E - \mu A)$  is not divisible by  $\mu$ .

Remark 4.3: Roughly speaking, Propositions 4.1 and 4.2 are the analogs for the homogeneous invariant polynomials of Propositions 2.2 and 2.3 for the homogeneous indices.

We are now ready to state and prove the generalization of Rosenbrock's Theorem.

Theorem 4.2 (Control Structure Theorem for Generalized Linear Systems):

Let  $(E,A,B) \in C_\theta(n,m)$ . Let  $r = \text{rank } B$ , and let  $n_1 > \dots > n_r$  be the nonzero homogeneous indices of  $(E,A,B)$ . Let  $q < r$ , and let  $\psi_1(\lambda,\mu), \dots, \psi_q(\lambda,\mu)$  be any set of nonconstant homogeneous polynomials such that  $\psi_{i+1} | \psi_i$  ( $i=1, \dots, q-1$ ) and  $\sum_{i=1}^q \deg \psi_i = n$ . Then there is a CRPD state feedback gain  $F$  such that the given polynomials are the nonconstant homogeneous invariant polynomials of the closed loop system  $g_\theta(F)(E,A,B)$  if and only if the following two conditions are satisfied:

- (i)  $\psi_i$  is not divisible by  $(-\sin\theta)\lambda + (\cos\theta)\mu$ ,  $i=1, \dots, q$ .
- (ii)  $\sum_{i=1}^p \deg \psi_i > \sum_{i=1}^p n_i$ ,  $p=1, \dots, q$ .

Proof: Let  $(\hat{E}, \hat{A}, B) = R_{-\theta}(E, A, B) \in C_0(n, m)$ . By Propositions 2.2 and 2.3, we have  $HI(E, A, B) = HI(\hat{E}, \hat{A}, B) = CI(\hat{E}, \hat{A}, B)$ . Thus,  $n_1 > \dots > n_r$  are the nonzero minimal column indices of  $(\hat{E}, \hat{A}, B)$ . Let  $\hat{\psi}_i = \psi_i \circ r_{-\theta}$ , and let  $\tilde{\psi}_i(\lambda) = \hat{\psi}_i(\lambda, 1)$ .

Clearly,  $\deg \hat{\psi}_i = \deg \psi_i$ .

Suppose that  $\psi_1, \dots, \psi_q$  satisfy (i) and (ii). Since  $\psi_i$  is not divisible by  $(-\sin\theta)\lambda + (\cos\theta)\mu$ ,  $\hat{\psi}_i$  is not divisible by  $\mu$ . Consequently,  $\deg \tilde{\psi}_i = \deg \hat{\psi}_i$ .

Thus,

$$\sum_{i=1}^p \deg \tilde{\psi}_i \geq \sum_{i=1}^p n_i, \quad p = 1, \dots, q.$$

Replacing  $\tilde{\psi}_1, \dots, \tilde{\psi}_q$  with nonzero scalar multiples if necessary, we may assume  $\tilde{\psi}_1, \dots, \tilde{\psi}_q$  are monic. Applying Rosenbrock's Theorem (Theorem 4.1) to the controllable pair  $(\hat{E}^{-1}\hat{A}, \hat{E}^{-1}B)$ , we conclude that there is a feedback gain  $F$  such that  $\tilde{\psi}_1, \dots, \tilde{\psi}_q$  are the nonunity invariant polynomials of  $\hat{E}^{-1}\hat{A} + \hat{E}^{-1}BF$  - i.e. of the pencil  $\lambda I - (\hat{E}^{-1}\hat{A} + \hat{E}^{-1}BF)$ . Thus,  $\tilde{\psi}_1, \dots, \tilde{\psi}_q$  are the nonunity invariant polynomials of the pencil  $\lambda\hat{E} - (\hat{A} + BF)$ , and hence of the regular system  $(\hat{E}, \hat{A} + BF, B)$ . By Proposition 4.2,  $\hat{\psi}_1, \dots, \hat{\psi}_q$  are the nonconstant homogeneous invariant polynomials of  $(\hat{E}, \hat{A} + BF, B)$ . By Proposition 4.1,  $\psi_1, \dots, \psi_q$  are the nonconstant homogeneous invariant polynomials of  $R_{\theta}(\hat{E}, \hat{A} + BF, B) = R_{\theta} \circ g_0(F) \circ R_{-\theta}(E, A, B) = g_{\theta}(F)(E, A, B)$ , as required.

Conversely, suppose that there exists a feedback gain  $F$  for which the closed loop system  $g_{\theta}(F)(E, A, B)$  has  $\psi_1, \dots, \psi_q$  as its nonconstant homogeneous polynomials. By Proposition 4.1,  $\hat{\psi}_1, \dots, \hat{\psi}_q$  are the nonconstant homogeneous invariant polynomials of  $R_{-\theta} \circ g_{\theta}(F)(E, A, B) = g_0(F)(\hat{E}, \hat{A}, B)$ . By Proposition 4.2,  $\hat{\psi}_1, \dots, \hat{\psi}_q$  are not divisible by  $\mu$ ,  $\tilde{\psi}_1, \dots, \tilde{\psi}_q$  are the nonconstant invariant polynomials of  $g_0(F)(\hat{E}, \hat{A}, B)$ , and  $\deg \tilde{\psi}_i = \deg \hat{\psi}_i$ . Since  $\hat{\psi}_i$  is not divisible by  $\mu$  and  $\psi_i = \hat{\psi}_i \circ r_{\theta}$ , it follows that  $\psi_i$  is not divisible by  $(-\sin\theta)\lambda + (\cos\theta)\mu$ .

Thus, condition (i) is satisfied. Applying Theorem 4.1, we conclude that

$\sum_{i=1}^p \deg \tilde{\psi}_i > \sum_{i=1}^p n_i$ ,  $p = 1, \dots, q$ . Since  $\deg \psi_i = \deg \hat{\psi}_i = \deg \tilde{\psi}_i$ , it follows immediately that condition (ii) is satisfied.  $\square$

Remark 4.4: Theorem 4.2 says that for a generalized linear system

$(E, A, B) \in C_\theta(n, m)$ , the closed loop homogeneous invariant polynomials can be assigned arbitrarily by CRPD feedback  $g_\theta(F)$  subject to two restrictions. Let  $(E_F, A_F, B) = g_\theta(F)(E, A, B)$ . The first restriction, that the homogeneous invariant polynomials of  $(E_F, A_F, B)$  cannot be divisible by  $(-\sin\theta)\lambda + (\cos\theta)\mu$ , is equivalent to the restriction that  $\det(\lambda E_F - \mu A_F)$  is not divisible by  $(-\sin\theta)\lambda + (\cos\theta)\mu$ . This corresponds to the fact that  $\Sigma_\theta(n, m)$  is invariant under the CRPD feedback  $g_\theta(F)$ , so we must have  $\det(\cos\theta E_F - \sin\theta A_F) \neq 0$ .

Remark 4.5: Rosenbrock's Theorem (Theorem 4.1) can be easily recovered by

setting  $\theta=0$  in Theorem 4.2. Let  $(A, B)$ ,  $r$ ,  $n_1 > \dots > n_r$ ,  $q$ , and  $\tilde{\psi}_1(\lambda), \dots, \tilde{\psi}_q(\lambda)$  be as in the hypotheses of Theorem 4.1. Then  $(I, A, B) \in C_0(n, m)$ , and by

Proposition 2.3,  $n_1 > \dots > n_r$  are the nonzero homogeneous indices of  $(I, A, B)$ .

Let  $\psi_i(\lambda, \mu)$  be the homogenization of  $\tilde{\psi}_i(\lambda)$ . I.e.  $\psi_i(\lambda, \mu) = \mu^{d_i} \tilde{\psi}_i(\lambda/\mu)$ , where  $d_i$  denotes the degree of  $\tilde{\psi}_i$ . Since  $\psi_i$  is automatically not divisible by  $\mu$ , it

follows from setting  $\theta=0$  in Theorem 4.2 that there exists  $F$  such that  $(I, A+BF, B)$

has nonconstant homogeneous invariant polynomials  $\psi_1, \dots, \psi_q$  if and only if

$\sum_{i=1}^p \deg \psi_i > \sum_{i=1}^p n_i$ ,  $p=1, \dots, q$ . By Proposition 4.2,  $(I, A+BF, B)$  has nonconstant invariant polynomials  $\tilde{\psi}_1, \dots, \tilde{\psi}_q$  if and only if it has nonconstant homogeneous

invariant polynomials  $\psi_1, \dots, \psi_q$ . Since  $\deg \tilde{\psi}_i = \deg \psi_i$ , we conclude that there

exists  $F$  such that  $A+BF$  has nonconstant invariant polynomials  $\tilde{\psi}_1, \dots, \tilde{\psi}_q$  if and

only if  $\sum_{i=1}^p \deg \tilde{\psi}_i > \sum_{i=1}^p n_i$ ,  $p=1, \dots, q$ .

Remark 4.6: The analogue of condition (i) in Theorem 4.2 is absent from Theorem

4.1 only because Theorem 4.1 is stated in terms of the invariant polynomials

rather than the homogeneous invariant polynomials. If Theorem 4.1 were restated in terms of the homogeneous invariant polynomials, it would be necessary to include the requirement that they not be divisible by  $\mu$ .

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