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**Gabor Representations and  
Wavelets**

**by**

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# Gabor representations and wavelets

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## Introduction

The first problem of wavelet theory is to construct best possible orthonormal bases  $\{\psi_n : n = 1, \dots\}$  of  $L^2(\mathbb{R})$ . "Best possible" means that each  $\psi_n$  should be as smooth as possible and should have controllable, preferably compact, support. Among other possibilities, it also means each basis  $\{\psi_n : n = 1, \dots\}$  should be sparse and "localizable," a notion which indicates that local changes or fine tuning in a signal can be made by adjustments to a small number of basis elements. The  $\psi_n$  are wavelets and the problem has been solved by the "Orchestre de Meyer," e.g., [4; 11; 12; 13; 18; 19; 20; 21; 22; 23].

Our modest goal in this paper is to define a generalization of the Fourier transform of  $L^1(\mathbb{R})$  and to prove the  $L^1(\mathbb{R})$  norm inversion theorem for such a transform (Theorem 1.5). Wiener's notion of deterministic autocorrelation (from the late 1920s) arises naturally in the proof of this result; and Gabor's representation of signals (from 1946), which is a fundamental example of wavelet decomposition, provides the setting.

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Besides Theorem 1.5, Section 1 reviews the relation between the Heisenberg group and Gabor representations, points out the development of modern wavelet theory from Gabor representations, and, notwithstanding some weaknesses of the Gabor theory [11; 15, Reference 7], shows that our generalized Fourier transform should give effective frequency resolution of signals.

The inversion theorem, Theorem 1.5, can be considered as a redundant (that is, not sparse) wavelet decomposition for  $L^1(\mathbb{R})$ . Sections 2 and 3 present the  $L^2(\mathbb{R})$  theory for writing discrete sparse wavelet decompositions  $f = \sum c_n \psi_n$ . Section 2 is devoted to the Zak transform  $Z$ ; and the main points, after proving  $Z$  is unitary on  $L^2(\mathbb{R})$  (Theorem 2.2 and Remark 2.3), are to emphasize how effectively orthonormal bases can be chosen and to observe that smoothness of wavelets and compactness of support cannot coexist in  $\{\psi_n : n = 1, \dots\}$  (Theorem 2.5 and Example 2.7). Section 3 discusses the Duffin-Schaeffer notion of a frame as a generalization of a bounded unconditional basis. An explicit wavelet decomposition formula is derived for frames (Theorem 3.5) and, even with the results of Meyer et al., the usefulness of frames for wavelet theory is established.

Almost everything in Sections 2 and 3 is known; and the main result of Section 1, although new and hopefully interesting, is not difficult. Criteria for a discrete  $L^1(\mathbb{R})$  wavelet decomposition in the  $L^1(\mathbb{R})$  norm are being sought.

Expanding on the material in Section 3 and studying most of the papers quoted above have been a shared venture with Christopher Heil and David Walnut. The material was challenging and the exper-

ience was exhilarating. I also acknowledge fruitful conversations with Hans Feichtinger and Karlheinz Gröchenig, as well as listening to Yves Meyer's stimulating Zygmund Lectures on wavelets.

## 0. Notation

$\mathbb{R}$  is the real line thought of as the time axis, and  $\hat{\mathbb{R}}$  is the real line, the dual group of  $\mathbb{R}$ , thought of as the frequency axis.  $L^1_{loc}(\mathbb{R})$  is the space of complex ( $\mathbb{C}$ )-valued locally Lebesgue integrable functions on  $\mathbb{R}$ . If  $f \in L^1_{loc}(\mathbb{R})$  then  $\tau_x f(u) = f(u-x)$ . Also,  $C_c^\infty(\mathbb{R}) \subseteq L^1_{loc}(\mathbb{R})$  is the space of infinitely differentiable functions having compact support.

We deal with the  $L^p$ -spaces  $L^1(\mathbb{R})$ ,  $L^2(\mathbb{R})$ , and  $L^\infty(\mathbb{R})$ , and denote the norm of  $f \in L^p(\mathbb{R})$  by  $\|f\|_p$ , e.g.,  $\|f\|_1 = \int |f(t)| dt$ , where " $\int$ " denotes integration over  $\mathbb{R}$ . The inner product of  $f, g \in L^2(\mathbb{R})$  is  $\langle f, g \rangle = \int f(t) \overline{g(t)} dt$ .

The Fourier transform of  $f \in L^1(\mathbb{R})$  is

$$\hat{f}(\gamma) = \int f(t) \exp\{-2\pi i t \gamma\} dt, \quad \gamma \in \hat{\mathbb{R}},$$

and the range of  $L^1(\mathbb{R})$  under the Fourier transform is  $A(\hat{\mathbb{R}})$ . If  $F \in A(\hat{\mathbb{R}})$  then  $F^\vee$  denotes the inverse Fourier transform. Convolution in  $L^1(\mathbb{R})$  is  $f * g(t) = \int f(t-x)g(x) dx$ .

If  $T : X \rightarrow Y$  is an operator then  $\mathcal{R}T \subseteq Y$  is the range; and an operator  $T : X \rightarrow X$  on an inner product space  $X$  is positive ( $T \geq 0$ ) if  $\langle Tf, f \rangle \geq 0$  for all  $f \in X$ .

Finally,  $\sum_k a_k$  or  $\sum_k a_k$  denotes  $\sum_{k=-\infty}^{\infty} a_k$ .

1. The Gabor representation of  $L^1(\mathbb{R})$

Definition 1.1. a. Given  $g \in L^1_{loc}(\mathbb{R})$ . The Gabor wavelet  $\psi = \psi_g$  is defined as

$$\psi_g(u; t, \omega) = g(u-t) \exp(2\pi i(u\omega - ct\omega)),$$

for fixed  $c \in \mathbb{R}$ .

b. For a given wavelet  $\psi = \psi_g$  the Gabor wavelet transform of  $f \in L^1_{loc}(\mathbb{R})$  is the function

$$F_\psi(f)(t, \omega) = F_\psi(t, \omega) = \int f(u) \overline{\psi(u; t, \omega)} du,$$

defined on  $\mathbb{R} \times \hat{\mathbb{R}}$ .

Gabor [10] introduced this transform for the Gaussian  $g$ . Also, Gabor wavelets fit naturally into the context of irreducible unitary representations of the Heisenberg group, e.g., Example 1.7.

If  $f, g \in L^2(\mathbb{R})$  or if  $f \in L^1(\mathbb{R})$  and  $g \in L^\infty(\mathbb{R})$  then  $F_\psi(f) \in A(\hat{\mathbb{R}})$  for almost all  $t \in \mathbb{R}$ , where Hölder's inequality is used for the  $L^2(\mathbb{R})$  case. Similarly, if  $f, g \in L^2(\mathbb{R})$  then  $F_\psi(f) \in A(\mathbb{R})$  for all  $\omega \in \hat{\mathbb{R}}$  since  $L^2(\mathbb{R}) * L^2(\mathbb{R}) = A(\mathbb{R})$ .

Example 1.2. The ambiguity function of  $f \in L^2(\mathbb{R})$  is the function,

$$A(f)(t, \omega) = \int \overline{f(u - \frac{t}{2})} f(u + \frac{t}{2}) \exp(-2\pi i u \omega) du,$$

defined on  $\mathbb{R} \times \hat{\mathbb{R}}$ , cf, the Wigner transform of  $f \in L^2(\mathbb{R})$  defined as the function,

$$W(f)(t, \omega) = \int \overline{f(t - \frac{u}{2})} f(t + \frac{u}{2}) \exp\{-2\pi i u \omega\} du.$$

Given the Gaussian,

$$g(u) = A \exp\{-Bu^2\}, \quad B > 0,$$

and corresponding Gabor wavelet  $\psi = \psi_g$ ,  $c = 1/2$ , and Gabor wavelet transform  $F_\psi$ . An easy calculation using Parseval's and Fubini's theorems yields the formula,

$$\begin{aligned} \iint \overline{F_\psi(f)(u-t, \gamma-\omega)} F_\psi(f)(u+t, \gamma+\omega) du d\gamma &= \\ &= A(f)(t, \omega) |A|^2 \sqrt{\frac{\pi}{2B}} \exp\{-\frac{1}{2} [Bt^2 + \frac{\pi^2 \omega^2}{B}]\}, \end{aligned}$$

e.g., [14]. The left hand side is a type of autocorrelation.

Our aim in Theorems 1.3 and 1.5 is to give integral representations of  $L^1(\mathbb{R})$  in terms of wavelets. To this end we let  $\{\rho_n\} \subseteq L^1(\hat{\mathbb{R}})$  have the property that  $\{\check{\rho}_n\} \subseteq L^1(\mathbb{R})$  is an  $L^1(\mathbb{R})$  approximate identity, i.e.,  $\int \check{\rho}_n(t) dt = 1$ ,  $\sup_n \|\check{\rho}_n\|_1 < \infty$ , and

$$\forall r > 0, \lim_{n \rightarrow \infty} \int_{|t| > r} |\check{\rho}_n(t)| dt = 0.$$

Theorem 1.3. Given  $g \in (L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})) \setminus \{0\}$  with corresponding Gabor wavelet  $\psi = \psi_g$  and Gabor wavelet transform  $F_\psi$ . For each  $f \in L^1(\mathbb{R})$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{\|g\|_2^2} \iint F_\psi(f)(t, \omega) \psi(u; t, \omega) \rho_n(\omega) dt d\omega = f(u)$$

in  $L^2(\mathbb{R})$ -norm.

We omit the proof of Theorem 1.3. It is analogous to the

proof of Theorem 1.5, which we shall give; and, as we'll see, this latter proof utilizes a more sophisticated form of autocorrelation than that used in Theorem 1.3 for  $g \in L^2(\mathbb{R})$ .

Definition 1.4. Given  $g \in L^\infty(\mathbb{R})$  and assume

$$\forall u \in \mathbb{R}, \exists \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \overline{g(t)} g(u+t) dt = G(u),$$

where  $G$  is continuous on  $\mathbb{R}$  and  $G(0) > 0$ .  $G$  is the autocorrelation of  $g$ .  $G$  is positive definite and the positive measure  $G^\vee$  is the power spectrum of  $g$ , e.g., [6; 26; 3, Sections I and IV].

In the case  $g \in L^2(\mathbb{R})$  we deal with the energy spectrum  $|g^\vee|^2$  and its Fourier transform plays a role in Theorem 1.3.

Theorem 1.5. Given  $g \in L^\infty(\mathbb{R})$  with corresponding Gabor wavelet  $\psi = \psi_g$  and Gabor wavelet transform  $F_\psi$ . Assume  $g$  has an autocorrelation  $G$ . For each  $f \in L^1(\mathbb{R})$ ,  $\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0$ , where

$$f_n(u) = \frac{1}{G(0)} \lim_{T \rightarrow \infty} \frac{1}{2T} \iint_{-T}^T F_\psi(f)(t, \omega) \psi(u; t, \omega) \rho_n(\omega) dt d\omega.$$

Proof. We first calculate

$$\begin{aligned} & \frac{1}{2T} \iint_{-T}^T F_\psi(f)(t, \omega) \psi_g(u; t, \omega) \rho_n(\omega) dt d\omega \\ (1.1) \quad & = \int f(u-x) \rho_n^\vee(x) G_T(u, x) dx, \end{aligned}$$

where



$$G_T(u, x) = \frac{1}{2T} \int_{-T+(u-x)}^{T+(u-x)} \overline{g(s)} g(s+x) ds$$

and where we can use Fubini's theorem in the verification of (1.1) since

$$\frac{1}{2T} \iiint_{-T}^T |f(v)| |\rho_n(\omega)| |\overline{g(v-t)} g(u-t)| dt d\omega dv \leq \|g\|_\infty^2 \|f\|_1 \|\rho_n\|_1 < \infty.$$

Next, we make the estimate

$$\begin{aligned} & \int |G(0)f(u) - \int f(u-x)\rho_n^\check{(x)}G_T(u, x)dx| du \\ & \leq \int |\rho_n^\check{(x)}| \left[ \int |G(0)f(u) - G_T(u, x)f(u-x)| du \right] dx \\ (1.2) \quad & \leq \int |\rho_n^\check{(x)}| \left[ \int |G_T(u, x)| |f(u) - f(u-x)| du \right] dx \\ & \quad + \int |\rho_n^\check{(x)}| \left[ \int |f(u)| |G(0) - G_T(u, x)| du \right] dx = I_1(n, T) + I_2(n, T). \end{aligned}$$

Note that

$$(1.3) \quad \lim_{n \rightarrow \infty} \sup_{T > 0} I_1(n, T) = 0;$$

in fact,

$$I_1(n, T) \leq \|g\|_\infty^2 \int |\rho_n^\check{(x)}| \|f - \tau_x f\|_1 dx$$

and

$$\lim_{n \rightarrow \infty} \int |\rho_n^\check{(x)}| \|f - \tau_x f\|_1 dx = 0$$

since  $\{\rho_n^\check{(x)}\}$  is an  $L^1(\mathbb{R})$  approximate identity (the property,  $\int \rho_n^\check{(x)} dx = 1$ , is not used) and  $\|f - \tau_x f\|_1$  is a bounded continuous function.

To estimate  $I_2(n,T)$  we first compute

$$\lim_{T \rightarrow \infty} \int |\rho_n^\vee(x)| |G(0) - G_T(u,x)| dx = \int |\rho_n^\vee(x)| |G(0) - G(x)| dx$$

by the dominated convergence theorem which is applicable since

$\lim_{T \rightarrow \infty} G_T(u,x) = G(x)$  for each  $x, u \in \mathbb{R}$  and since

$|\rho_n^\vee(x)| |G(0) - G_T(u,x)| \leq 2 \|g\|_\infty^2 |\rho_n^\vee(x)| \in L^1(\mathbb{R})$ . Consequently,

because

$$(1.4) \quad |f(u)| \int |\rho_n^\vee(x)| |G(0) - G_T(u,x)| dx \\ \leq 2 \|g\|_\infty^2 (\sup \|\rho_n^\vee(x)\|_1) |f(u)| \in L^1(\mathbb{R}),$$

we can apply the dominated convergence theorem again to compute

$$\lim_{T \rightarrow \infty} I_2(n,T) = \int |f(u)| \left[ \int |\rho_n^\vee(x)| |G(0) - G(x)| dx \right] du.$$

Now, we have

$$\lim_{n \rightarrow \infty} \int |\rho_n^\vee(x)| |G(0) - G(x)| dx = 0$$

since  $\{\rho_n^\vee\}$  is an  $L^1(\mathbb{R})$  approximate identity and since

$|G(0) - G(x)|$  is a bounded continuous function. Therefore, we can

once again use the dominated convergence theorem, invoking the

analogue of (1.4) with  $G$  replacing  $G_T$ , to state that

$$(1.5) \quad \lim_{n \rightarrow \infty} \lim_{T \rightarrow \infty} I_2(n,T) = 0.$$

Substituting (1.1) into the left hand side of (1.2) and substituting (1.3) and (1.5) into the right hand side of (1.2) we obtain our result.

q.e.d.

Remark 1.6. a. If  $g \equiv 1$  and  $c = 0$  in the definition of  $\psi_g$  then the Gabor wavelet transform is the ordinary Fourier transform; and Theorem 1.5 is the usual  $L^1(\mathbb{R})$  inversion theorem for the Fourier transform.

b. For peaked  $g \in L^2(\mathbb{R})$ , Gabor wavelet theory allows for good temporal (resp., frequency) localization at  $t$  and, correspondingly, adequate frequency (resp., temporal) localization at  $\omega$ ; for example, we see this by taking  $g(u) = \frac{\lambda}{\sqrt{\pi}} \exp\{-(\lambda u)^2\}$  and comparing it with its Fourier transform,  $\hat{g}(\gamma) = \exp\{-(\pi\gamma/\lambda)^2\}$ . Besides generalizing the Fourier transform, the  $L^\infty(\mathbb{R})$  Gabor wavelet theory, i.e.,  $g \in L^\infty(\mathbb{R})$ , allows for high resolution frequency localization.

c. An impetus for modern wavelet theory, i.e., post-Gabor, has been to modify the simultaneous  $(t, \omega)$  "localization" inherent in  $\psi_g$  in order to deal with "high frequency - high resolution" problems. To explain such problems, first note that the Gabor wavelet centers  $g \in L^2(\mathbb{R})$  at  $t$  and centers  $\hat{g}$  at  $\omega$ . However, in the case of large  $\omega$ , the wavelet  $\psi_g$ ,  $g \in L^2(\mathbb{R})$ , contains many cycles due to the  $\exp\{2\pi i u \omega\}$  term. Consequently, there is an intrinsic instability in representing, by means of formulas such as found in Theorem 1.3, high frequency low cycle signals in terms of Gabor wavelets. This particular representation problem has been solved by introducing a new class of wavelets [11; 12; 4]; and the problem has a natural formulation in terms of the non-unimodular "ax+b" group, cf, Example 1.7.

Example 1.7. a. Suppose  $u$  is an irreducible unitary representation of a locally compact group  $G$  onto  $L^2(\mathbb{R})$ . We require

irreducibility because, for such representations, the span of  $\{U(x)g : g \in L^2(\mathbb{R}) \setminus \{0\} \text{ fixed, } x \in G\}$  is dense in  $L^2(\mathbb{R})$ . We ask under what condition this density can be refined into an integral representation formula such as given in Theorem 1.3. The condition is square integrability, i.e., there exists  $g \in L^2(\mathbb{R}) \setminus \{0\}$  for which

$$\int_G |\langle U(x)g, g \rangle|^2 d_r x < \infty$$

( $d_r$  is right Haar measure). The theorem which proves the integral representation formula is well-known; and a readable, brief, self-contained exposition of it is found in [13, Theorem 3.1]. The constants  $\|g\|_2^2$  and  $G(0)$  in Theorem 1.3 and 1.5 are  $L^1(\mathbb{R})$  analogues of particular cases of the action of the positive operator  $C$  constructed in this  $L^2(\mathbb{R})$  theorem. In the  $L^2(\mathbb{R})$  case  $C$  is a multiple of the identity for unimodular groups.

b. The case of Gabor wavelets has the general formulation of Example 1.7a as follows. Define

$$(U(x)g)(u) = \exp(2\pi i a u) g(u+b) \exp(2\pi i c),$$

where  $x = (a, b, c) \in \mathbb{R}^3 = G$ . If we demand that  $U : G \rightarrow \mathcal{L}(L^2(\mathbb{R}))$  be a homomorphism, where  $U(xx')(g) = U(x)(U(x')g)$ , then we force the multiplication on  $G$  to be defined as

$$xx' = (a+a', b+b', c+c'+ba').$$

Consequently,  $G$  is essentially the Heisenberg group of matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

As such  $U$  is a square integrable irreducible unitary representation (and thus equivalent to a subrepresentation of the regular representation) of  $G$  onto  $L^2(\mathbb{R})$ ; and by the theorem mentioned in Example 1.7a we obtain the  $L^2(\mathbb{R})$ -vector-valued integral representation,

$$\forall f \in L^2(\mathbb{R}), f = c_g \int_G \langle f, U(x)g \rangle U(x)g d_r x,$$

where  $g \in L^2(\mathbb{R}) \setminus \{0\}$  is fixed, and otherwise arbitrary since the Heisenberg group is unimodular, cf, the latter part of Example 1.7a.

c. Integrable group representations have recently been used in a substantial and original way for the wavelet representation of functions by Feichtinger and Gröchenig [8; 9].

## 2. The Zak transform

Our goal is to prove discrete versions of results such as Theorems 1.3 and 1.5.

Definition 2.1. Given  $a > 0$  and  $\alpha > 0$ . The Zak transform of  $f \in L^1_{loc}(\mathbb{R})$  is the function

$$Z(h)(t, \omega) = a^{1/2} \sum_k h(t-ka) \exp\{2\pi i k \omega / \alpha\},$$

defined on the rectangle  $R_{a, \alpha} = I_a \times I_\alpha = [-a/2, a/2) \times [-\alpha/2, \alpha/2)$ , e.g., [28; 29].

Formally, if  $\omega = 0$  then  $Z(h)$  is an  $a$ -periodic function of  $t$ , and if  $t = 0$  then  $Z(h)$  is the Fourier series of an  $\alpha$ -periodic function. Lebesgue measure on  $R_{a, \alpha}$  is chosen so that

the integral of the function identically 1 on  $R_{a,\alpha}$  is 1. As such the Fourier coefficients of  $F \in L^1(R_{a,\alpha})$  are given by

$$\forall m,n \in \mathbb{Z}, \hat{F}(m,n) = \frac{1}{a\alpha} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} F(u,\gamma) \exp\{-2\pi i \left(\frac{mu}{a} + \frac{n\gamma}{\alpha}\right)\} d\gamma du.$$

In this case,  $F$  is viewed as a doubly periodic function on the  $(t,\omega)$ -plane. Of course,  $Z(h)$  is formally defined on the  $(t,\omega)$ -plane; but it only satisfies the quasi-double-periodicity condition,

$$(2.1) \quad \forall m,n \in \mathbb{Z}, Z(h)(t+ma, \omega+n\alpha) = Z(h)(t,\omega) \exp\{2\pi i m\omega/\alpha\}.$$

Theorem 2.2. The Zak transform is a unitary map,

$$Z : L^2(\mathbb{R}) \rightarrow L^2(R_{a,\alpha});$$

and, for all  $f \in L^2(\mathbb{R})$ ,  $Z(f)(t,\omega) = a^{1/2} \sum f(t-ka) \exp\{2\pi i k\omega/\alpha\}$ .

Proof. i. We begin by verifying that

$$(2.2) \quad \forall f \in C_c^\infty(\mathbb{R}), \|Z(f)\|_{L^2(R_{a,\alpha})} = \|f\|_2.$$

In fact, for  $f \in C_c^\infty(\mathbb{R})$ ,  $Z(f)$  is a finite sum and so the calculation,

$$\begin{aligned} \|Z(f)\|_{L^2(R_{a,\alpha})}^2 &= \frac{1}{a} \int_{R_{a,\alpha}} \int \sum_{m,n} f(t-ma) \overline{f(t-na)} \exp\{2\pi i(m-n)\omega/\alpha\} d\omega dt \\ &= \int_{-a/2}^{a/2} \sum_m |f(t-ma)|^2 dt = \|f\|_2^2, \end{aligned}$$

is valid.

Since  $C_c^\infty(\mathbb{R})$  is dense in  $L^2(\mathbb{R})$ ,  $Z$  extends, in the usual way, to a linear isometry  $Z : L^2(\mathbb{R}) \rightarrow L^2(R_{a,\alpha})$ . Further, if

$f \in L^2(\mathbb{R})$  we can use Fubini's theorem, applied to the measure spaces  $\mathbb{Z} \times \mathbb{Z}$  (with counting measure) and  $R_{a,\alpha}$ , to check that

$$\left\| \sum f(t-ka) \exp\{2\pi i k \omega / \alpha\} \right\|_{L^2(R_{a,\alpha})} < \infty;$$

and thus,  $Z(f)(t, \omega) = a^{1/2} \sum f(t-ka) \exp\{2\pi i k \omega / \alpha\}$ .

It remains to prove that  $Z$  is a surjection., cf, Remark 2.3.

ii. To this end we first note that  $L^2(I_a) \otimes L^2(I_\alpha)$  is dense in  $L^2(R_{a,\alpha})$ . In fact, the polynomials on  $R_{a,\alpha}$  are dense in  $L^2(R_{a,\alpha})$  (since the continuous functions are), and each polynomial, as a sum of products, is an element of  $L^2(I_a) \otimes L^2(I_\alpha)$ .

If  $F \in L^2(R_{a,\alpha})$  then the definition of the Zak transform and our comment after Definition 2.1 about Fourier series give us a natural candidate  $f$  for which to expect that  $Z(f) = F$ . In fact, each  $t \in \mathbb{R}$  is written uniquely as  $t = u - ka$ ,  $u \in [-a/2, a/2)$  and  $k \in \mathbb{Z}$ , and so we formally define

$$(2.3) \quad f(t) = f(u-ka) = \frac{1}{a^{1/2}} \int_{-a/2}^{a/2} F(u, \gamma) \exp\{-2\pi i k \gamma / \alpha\} d\gamma.$$

iii. For  $F \in L^2(I_a) \otimes L^2(I_\alpha)$  we shall show that  $\|f\|_2 < \infty$  where  $f$  is defined by (2.3). To this end, write  $F$  as the finite sum  $\sum_{m=1}^M U_m(u) \Omega_m(\omega)$  where each  $U_m \in L^2[-a/2, a/2)$  and each  $\Omega_m \in L^2[-\alpha/2, \alpha/2)$ . Using (2.3) and Minkowski's and Hölder's inequalities, we compute

$$\int |f(s)|^2 ds = \sum_k \int_{-a(k+1/2)}^{-a(k-1/2)} |f(t)|^2 dt$$

$$\begin{aligned}
&= \frac{1}{a\alpha^2} \sum_k \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \left[ \sum_{m,n=1}^M U_m(u) \overline{U_n(u)} \Omega_m(\omega) \overline{\Omega_n(\lambda)} \right] \times \\
&\quad \exp\{-2\pi i k(\omega-\lambda)/\alpha\} d\omega d\lambda du \\
&= \frac{1}{a} \sum_k \int_{-a/2}^{a/2} |U_m(u) \hat{\Omega}_m(k)|^2 du \\
&\leq \frac{1}{a} \sum_k \left[ \sum_{m=1}^M |\hat{\Omega}_m(k)| \|U_m\|_{L^2(I_a)} \right]^2 \leq C \sum_k \sum_{m,n=1}^M |\hat{\Omega}_m(k) \hat{\Omega}_n(k)| \\
&\leq C \sum_{m,n=1}^M \left[ \sum_k |\hat{\Omega}_m(k)|^2 \right]^{1/2} \left[ \sum_k |\hat{\Omega}_n(k)|^2 \right]^{1/2},
\end{aligned}$$

and this last term is finite since  $\Omega_m \in L^2(I_a)$ .

For these functions  $f$  and  $F$  we now verify that  $Z(f) = F$ .

$Z(f)(u, \omega)$  is formally defined as

$$a^{1/2} \sum_k \sum_{m=1}^M U_m(u) \left[ \frac{1}{a^{1/2}\alpha} \int_{-a/2}^{a/2} \Omega_m(\gamma) \exp\{-2\pi i k\gamma/\alpha\} d\gamma \right] \exp\{2\pi i k\omega/\alpha\}$$

which in turn equals

$$\sum_{m=1}^M U_m(u) \left[ \sum_k \hat{\Omega}_m(k) \exp\{2\pi i k\omega/\alpha\} \right] = F(u, \omega) \text{ a.e.}$$

Therefore,  $Z(f) = F$ . (Of course, we could adjust our dense tensor product to avoid using the power of Carleson's theorem in the last step.)

Because of parts ii and iii we established that  $\mathcal{R}Z$  is dense in  $L^2(R_{a,\alpha})$ .

iv. Let  $Z^*$  be the transpose of  $Z$ . From part i we know  $Z^{-1}$  exists and is continuous on its domain. We now show this



implies  $\mathcal{R}Z^* = L^2(\mathbb{R})$ .

Fix  $f^* \in L^2(\mathbb{R})$ . Then a continuous linear functional is defined on  $\mathcal{R}Z$  by the expression,

$$\forall H \in \mathcal{R}Z, \langle f^*, Z^{-1}H \rangle;$$

and so, since  $\overline{\mathcal{R}Z} = L^2(\mathbb{R}_{a,\alpha})$ , there is  $H^* \in L^2(\mathbb{R}_{a,\alpha})$  such that

$$\forall H \in \mathcal{R}Z, \langle H^*, H \rangle = \langle f^*, Z^{-1}H \rangle.$$

Therefore,  $\langle H^*, Z(f) \rangle = \langle f^*, Z^{-1}(Z(f)) \rangle = \langle f^*, f \rangle$  for all  $f \in L^2(\mathbb{R})$ ; but  $\langle H^*, Z(f) \rangle = \langle Z^*(H^*), f \rangle$  by definition of  $Z^*$ , and so  $Z^*(H^*) = f^*$ , i.e.,  $\mathcal{R}Z^* = L^2(\mathbb{R})$ . (A Hahn-Banach argument could replace the fact,  $\overline{\mathcal{R}Z} = L^2(\mathbb{R}_{a,\alpha})$ , in this part of our proof.)

v. It is a standard fact from functional analysis (e.g., Rudin's Functional analysis, Theorem 4.14) that, for an operator  $T$ ,  $\mathcal{R}T$  is closed if  $\mathcal{R}T^*$  is closed. In our case this implies  $\mathcal{R}Z$  is closed. This coupled with the density of  $\mathcal{R}Z$  yields the desired surjectivity.

q.e.d.

Remark 2.3. The surjectivity in Theorem 2.2 has the following quick formal proof. Let  $\{c_{m,n}\}$  be the Fourier coefficients of  $F \in L^2(\mathbb{R}_{a,\alpha})$ . Define  $F_m$  by the  $L^2$ -Fourier series with coefficients  $\{c_{m,n} : n \in \mathbb{Z}, m \text{ fixed}\}$ , and set  $f(u-ma) = F_m(u)$ ,  $u \in I_a$  and  $m \in \mathbb{Z}$ . Then  $f \in L^2(\mathbb{R})$  and, formally, the Fourier coefficients of  $Z(f)$  equal to those of  $F$ . The interchange of integration (over  $\mathbb{R}_{a,\alpha}$ ) and summation in this formal calculation

is valid if  $f \in L^1(\mathbb{R})$ .

The following fact was observed by von Neumann [25, pp. 405 ff.]; and the array  $\{(ma, n/a) : m, n \in \mathbb{Z}\}$  we use is the von Neumann lattice. There are several treatments of this fact in the physics literature, e.g., [1]. The functions  $g$  and  $\psi_{m,n}$  of Theorem 2.4 are the analyzing wavelet and discrete Gabor wavelets, respectively.

Theorem 2.4. Given  $g \in L^2(\mathbb{R})$  and  $a > 0$ , and let  $\alpha = 1/a$  in the definition of the Zak transform. Define

$$X = \{\psi_{m,n}(u) = \psi_g(u; ma, n/a) : m, n \in \mathbb{Z}\} \subseteq L^2(\mathbb{R}),$$

where  $c = 1$  in the definition of  $\psi_g$ . Then the linear span of  $X$  is dense in  $L^2(\mathbb{R})$  if and only if  $|N(Z(g))| = 0$ , where  $|\dots|$  designates two-dimensional Lebesgue measure on  $R_{a,\alpha}$  and  $N(H) = \{(t, \omega) : H(t, \omega) = 0\}$ .

Proof. By Theorem 2.2 the span of  $X$  is dense in  $L^2(\mathbb{R})$  if and only if the span of  $Z(X)$  is dense in  $L^2(R_{a,1/a})$ , and this occurs if and only if  $H \in L^2(R_{a,1/a})$  is the zero function whenever  $H$  annihilates  $Z(X)$ .

Since  $\alpha = 1/a$  we see that

$$(2.4) \quad Z(\psi_{m,n})(t, \omega) = \exp\{-2\pi i(\omega ma - tn/a)\} Z(g)(t, \omega).$$

Thus, the condition that  $H$  annihilates  $Z(X)$  is equivalent to the system of equations,

$$(2.5) \quad \forall m, n \in \mathbb{Z}, \int_{R_{a,1/a}} \overline{H(t, \omega)} Z(g)(t, \omega) \exp\{-2\pi i(\omega ma - tn/a)\} dt d\omega = 0.$$

Viewing  $R_{a,1/a}$  as a compact group and noting that  $\bar{H}Z(g) \in L^1(R_{a,1/a})$ , the uniqueness theorem for Fourier series thereby shows that the equation,  $\bar{H}Z(g) = 0$  a.e., is equivalent to the condition that  $H$  annihilates  $Z(X)$ .

The result follows by the equivalences for density mentioned at the start of the proof.

q.e.d.

Theorem 2.5. Define the sequence  $\{\psi_{m,n}\}$  of wavelets as in Theorem 2.4 for  $g \in L^2(\mathbb{R})$ ,  $a = 1/a$ , and  $c = 1$ .  $\{\psi_{m,n} : m,n \in \mathbb{Z}\}$  is an orthonormal basis of  $L^2(\mathbb{R})$  if and only if  $|Z(g)| = 1$  a.e. on  $R_{a,1/a}$ .

Proof. We calculate

$$(2.5) \quad \langle \psi_{m,n}, \psi_{p,q} \rangle = \langle Z(\psi_{m,n}), Z(\psi_{p,q}) \rangle \\ = \int_{R_{a,1/a}} \int |Z(g)(t,\omega)|^2 \exp\{-2\pi i[\omega(m-p)a - t(n-q)/a]\} dt d\omega.$$

The first equation is a consequence of the isometry  $\|f\|_2 = \|Z(f)\|_{L^2(R_{a,1/a})}$  and the fact that  $4f\bar{g} = |f+g|^2 - |f-g|^2 + i|f+ig|^2 - i|f-ig|^2$ . The second equation follows from (2.4).

If  $|Z(g)| = 1$  then (2.5) provides the orthonormality of  $\{\psi_{m,n}\}$  and Theorem 2.4 gives the density of the span of  $\{\psi_{m,n}\}$ . Consequently, by standard Hilbert space properties,  $\{\psi_{m,n}\}$  is an orthonormal basis of  $L^2(\mathbb{R})$ .

For the converse, the orthonormality and (2.5) show that the Fourier coefficients  $\hat{F}(m,n) = 0$  if  $(m,n) \neq (0,0)$  and  $\hat{F}(0,0) =$

1, where  $F = |Z(g)|^2$ . Thus the Fourier coefficients of  $F$  and of the function which is identically 1 on  $R_{a,1/a}$  are the same; and so, since  $F \in L^1(R_{a,1/a})$ ,  $F \equiv 1$  by the uniqueness theorem. We did not need the hypothesis that  $\{\psi_{m,n}\}$  is a basis.

q.e.d.

For a given  $g$  and corresponding sequence  $\{\psi_{m,n}\}$  of wavelets, there are several formal ways of writing  $f$  as a sum,  $\sum c_{m,n} \psi_{m,n}$ , e.g., [2; 15; 17; 7; 9]. In case  $|Z(g)| \equiv 1$  we've seen there is a unique decomposition in terms of a computable sequence  $\{c_{m,n}\}$  (Theorem 2.5) and in case  $|N(Z(g))| > 0$  we've seen that there are  $f \in L^2(\mathbb{R})$  which aren't even in the closed span of  $\{\psi_{m,n}\}$ . If  $|N(Z(g))| = 0$  and  $N(Z(g)) \neq \emptyset$  then the situation is generally complicated, c.f., Example 3.7. We shall prove a theorem for wavelet decomposition (Theorem 3.5), but the following algorithm, e.g., [2; 16], is also potentially effective.

Algorithm 2.6. For a given  $f$  we define the coefficients

$c_{m,n}(f)$  as

$$c_{m,n} = \int_{-a/2}^{a/2} \int_{-1/2a}^{1/2a} \frac{Z(f)(t,\omega)}{Z(g)(t,\omega)} \exp\{2\pi i(m\omega a - nt/a)\} d\omega dt.$$

Then, formally, we compute  $\sum_{m,n} c_{m,n} \psi_{m,n}(u) =$

$$a^{1/2} \sum_{m,n} g(u - ma) \exp\{2\pi i un/a\} \int_{-1/2a}^{1/2a} S(\omega) \exp\{2\pi i m\omega a\} d\omega,$$

where

$$\begin{aligned}
S(\omega) &= \sum_k \int_{-1/2a}^{1/2a} \frac{f(t-ka)}{Z(g)(t,\omega)} \exp\{2\pi i(k\omega a - nt/a)\} dt \\
&= \int \frac{f(x)}{Z(g)(x,\omega)} \exp\{-2\pi i n x/a\} dx,
\end{aligned}$$

and where this last equality is a consequence of (2.1). Substituting this expression for  $S(\omega)$  into the original sum, we have

$$\begin{aligned}
\sum_{m,n} c_{m,n} \psi_{m,n}(u) &= \int f(x) \left[ \sum_n \exp\{2\pi i n(u-x)/a\} \int_{-1/2a}^{1/2a} \frac{Z(g)(u,a)}{Z(g)(x,\omega)} d\omega \right] dx \\
&= a \int \left[ f(u-xa) \int_{-1/2a}^{1/2a} \frac{Z(g)(u,\omega)}{Z(u-xa,\omega)} d\omega \right] \left[ \sum \exp\{2\pi i n x\} \right] dx \\
&= f(u),
\end{aligned}$$

where the last equation is a consequence of the fact that  $\sum \exp\{2\pi i n x\}$  is the Fourier series of the Dirac measure  $\delta(x)$ .

**Example 2.7.** a. Given  $a > 0$  and let  $g(u) = a^{-1/2} \chi_{[r, a+r)}(u) \exp\{2\pi i f(u)\}$ , where  $f$  is real valued and  $r \in \mathbb{R}$  is fixed. Then

$$Z(g)(t,\omega) = \exp\{2\pi i(f(t-ka) + k\omega a)\},$$

where  $(t,\omega) \in R_{a,1/a}$  and  $k$  is chosen so that  $t \in [ka+r, (k+1)a+r)$ . Thus, Theorem 2.5 is applicable and the associated set  $\{\psi_{m,n}\}$  of Gabor wavelets is an orthonormal basis of  $L^2(\mathbb{R})$ . The special case  $Z(g) = 1$  on  $R_{a,1/a}$  occurs if  $r = -a/2$  and  $f = 0$  on  $\mathbb{R}$ .

b. In order to construct orthonormal bases of Gabor wavelets we need only take  $G \in L^2(R_{a,1/a})$  for which  $|G| = 1$  a.e. on  $R_{a,1/a}$  and use (2.3) to construct the corresponding analyzing

wavelet  $g$ . For example, if  $G(u, \gamma) = \exp\{2\pi i(U(u) + \Gamma(\gamma))\}$  then

$$g(t) = a^{1/2} \exp\{2\pi i U(u)\} \int_{-1/2a}^{1/2a} \exp\{2\pi i(\Gamma(\gamma) - k\gamma a)\} d\gamma,$$

where  $t \in \mathbb{R}$  is given and  $t = u - ka$ ,  $u \in [-a/2, a/2)$ .

If  $\Gamma(\gamma) = \gamma$  then

$$g(t) = a^{-1/2} \exp\{2\pi i U(u)\} \left[ \frac{a}{\pi(1-ka)} \sin \left[ \frac{\pi(1-ka)}{a} \right] \right]$$

and

$$\|g\|_2^2 = a \sum_k \left[ \frac{\sin(\pi(1-ka)/a)}{\pi(1-ka)} \right]^2.$$

If  $U = \Gamma = 0$  then  $G = 1$  on  $R_{a, 1/a}$  and  $g = a^{-1/2} \chi_{[-a/2, a/2)}$ , cf, part a. Note that in this case  $Z(g) = 1$  on  $R_{a, 1/a}$  and, by (2.1),  $Z(g)(t, \omega) = \exp\{2\pi i \omega a\}$  on  $\tau_a R_{a, 1/a}$  (translation in the  $t$ -direction). Thus,  $Zg$  is discontinuous on  $\{(a/2, \omega) : \omega \in (-\frac{1}{2a}, \frac{1}{2a}) \setminus \{0\}\}$  (as well as at many other points).

c. It is natural to ask if there is an analyzing wavelet  $g \in C_c^\infty(\mathbb{R})$  — or even just compactly supported and continuous  $g$  — for which  $\{\psi_{m,n}\} \subseteq L^2(\mathbb{R})$  is an orthonormal basis. The answer is "no" because if  $Z(f)$  is continuous on  $\mathbb{R} \times \mathbb{R}$  then  $Z(f)$  has at least one zero in  $R_{a, 1/a}$ . This last fact is a simple observation, due to Janssen and Zak, e.g., [16], depending on (2.1). The answer "no" becomes "yes" if a slightly more general notion than orthonormal basis is considered, e.g., Section 3 and [4, Theorem 2, Section IIE].

This leaves open the question of finding continuous analyzing vectors for which  $\{\psi_{m,n}\}$  is an orthonormal basis.

d. The Haar system is an orthonormal basis for  $L^2(\mathbb{R})$  gen-

erated by characteristic functions as in part a but with a different coherence property than the discrete Gabor wavelets.

### 3. Discrete wavelet decomposition

Let  $H$  be a separable Hilbert space.

Definition 3.1.  $\{\psi_n : n = 1, \dots\} \subseteq H$  is a Schauder basis or basis of  $H$  if each  $f \in H$  has a unique decomposition  $f = \sum_1^{\infty} c_n(f) \psi_n$ .

A basis  $\{\psi_n\}$  is an unconditional basis if

$\exists C$  such that  $\forall N \geq 1$  and  $\forall b_1, \dots, b_N, c_1, \dots, c_N \in \mathbb{C}$ ,  
where  $|b_j| \leq |c_j|$ ,

$$\left\| \sum_1^N b_n \psi_n \right\| \leq C \left\| \sum_1^N c_n \psi_n \right\|.$$

An unconditional basis  $\{\psi_n\}$  is bounded if

$\exists A, B > 0$  such that  $\forall n, A \leq \|\psi_n\| \leq B$ .

Definition 3.2.  $\{\psi_n : n = 1, \dots\} \subseteq H$  is a frame if

$\exists A, B > 0$  such that  $\forall f \in H$ ,

$$A \|f\|^2 \leq \sum_1^{\infty} |\langle f, \psi_n \rangle|^2 \leq B \|f\|^2,$$

e.g., [5; 27; 4].

Separable Hilbert spaces have orthonormal bases. In the following result the first simplification is trivial, the second is well known, and we shall expand upon the third in Theorem 3.5.

Theorem 3.3. Orthonormal bases are bounded unconditional bases,

bounded unconditional bases are frames, and if  $\{\psi_n\}$  is a frame then

$$(3.1) \quad \forall f \in H, \exists \{c_n(f)\} \text{ such that } f = \sum_1^{\infty} c_n(f) \psi_n.$$

Example 3.4. a. (3.1) does not assert  $c_n(f) = \langle f, \psi_n \rangle$ . If, however,  $\{\psi_n\}$  is a frame it is easy to see (using Cauchy sequences, the definition of norm, and Hölder's inequality) that  $\sum \langle f, \psi_n \rangle \psi_n$  converges in  $H$  for each  $f \in H$ .

b. None of the implications in Theorem 3.3 goes in the other direction. In particular, there exists a non-frame  $\{\psi_n\}$  for which (3.1) holds, where  $\{\psi_n\}$  can even be taken to be an orthogonal sequence and the representations in (3.1) are unique. To see this, let  $\{\varphi_n\} \subseteq H$  be an orthonormal basis and define  $\psi_n = n\varphi_n$ .  $\{\psi_n\}$  is not a frame since  $\sum_1^{\infty} |\langle f, \psi_n \rangle|^2 = \infty$  for  $f \in \sum_1^{\infty} (1/n)\varphi_n$ . On the other hand, each  $h \in H$  has the unique decomposition  $\sum_1^{\infty} (\langle h, \varphi_n \rangle / n) \psi_n$ .

Theorem 3.5. Given a frame  $\{\psi_n : n = 1, \dots\} \subseteq H$  and define the map  $T : H \rightarrow H, f \mapsto \sum_1^{\infty} \langle f, \psi_n \rangle \psi_n$ .

a.  $T$  is an isomorphism, i.e.,  $T$  is a linear bijective topological isomorphism on  $H$ .

b.  $((1/B)T)^{-1}$  exists and equals  $\sum_0^{\infty} (I - \frac{1}{B}T)^k$ ; and for all  $f \in H$ ,

$$f = \frac{1}{B} \sum \langle f, \psi_n \rangle (\frac{1}{B}T)^{-1} \psi_n$$

( $B$  as in Definition 3.2 and  $I$  the identity operator).



c. For all  $f \in H$ ,  $f = \sum_1^{\infty} \langle h, \psi_n \rangle \psi_n$  where  $h = T^{-1}f$ .

Proof. a.i.  $T$  is well defined and continuous since

$$\begin{aligned} \|Tf\|^2 &\leq \|T\|^2 \|f\|^2 = (\sup |\sum \langle h, \psi_n \rangle \langle \psi_n, g \rangle|)^2 \|f\|^2 \\ &\leq \|f\|^2 \sup \sum |\langle h, \psi_n \rangle|^2 \sum |\langle g, \psi_n \rangle|^2 \\ &\leq B^2 \|f\|^2 < \infty, \end{aligned}$$

where the sup is taken over  $\|h\| \leq 1$ ,  $\|g\| \leq 1$ .

$T$  is clearly linear. If  $Tf = 0$  then  $f = 0$  since  $0 = \sum |\langle f, \psi_n \rangle|^2 \geq A\|f\|^2$ , and, thus,  $T$  is injective.

It remains to prove that  $T$  is a surjection.

a.ii.  $T$  is self-adjoint since  $T$  is positive (or by direct calculation). Also  $T^{-1}$  is continuous on its domain since  $A\|f\| \leq \|Tf\|$  (by definition of frame), and so  $\|T^{-1}g\| \leq (1/A)\|g\|$ ,  $Tf = g$ .

a.iii. If we knew  $\overline{T(H)} = H$  then part a.ii coupled with the argument in part iv of Theorem 2.2 would give the surjectivity. Instead we appeal to part a.ii and the standard fact from functional analysis (e.g., Taylor's Functional analysis, Theorem 4.7C) that  $\mathcal{R}T = H$  if the transpose  $T^*$  has a continuous inverse. This fact is based on the Hahn-Banach theorem and an argument similar to the proof of the open mapping theorem.

b. Since  $\{\psi_n\}$  is a frame we have  $(A/B)I \leq (1/B)I \leq I$ . Therefore  $\|I - (1/B)T\| \leq \|I - (A/B)I\| \leq (B-A)/B < 1$ , and so  $((1/B)T)^{-1}$  exists and equals  $\sum_0^{\infty} (I - (1/B)T)^k$ . We write out the expression for  $((1/B)T)f$ , and obtain the decomposition in part b by inverting both sides.

c. Given  $f \in H$  and choose  $g \in H$  such that  $Tg = f$  (by

part a). Then part b gives  $g = (1/B)\sum \langle g, \psi_n \rangle ((1/B)T)^{-1} \psi_n$ ; and so  $f = TT^{-1}f = \sum \langle g, \psi_n \rangle ((1/B)T)((1/B)T)^{-1} \psi_n$ .

q.e.d.

Remark 3.6. In light of the results of Meyer et al., which we referenced in the Introduction and which established smooth compactly supported localizable (or coherent) orthonormal bases  $\{\psi_n\} \subseteq L^2(\mathbb{R})$  (but  $\psi_n \notin C_c^\infty(\mathbb{R})$ ), it is natural to question the introduction of frames. First, there are analyzing wavelets  $g \in C_c^\infty(\mathbb{R})$  for which the sequence  $\{\psi_{m,n}\}$  of discrete Gabor wavelets is a frame (in which  $A = B$ ), e.g., [4, Theorem 2, Section IIE], cf, Example 2.7c. Second, although the construction of Daubechies, Mallat, and Meyer in terms of multiscale analysis is elegant and deep, some of the constructions of frames might be essentially easier to implement. Third, in light of our desire to discretize the results of Section 1, the methodology of Theorem 3.5 may be more adaptable to spaces, such as  $L^1(\mathbb{R})$ , which not only don't have orthonormal bases but which don't have unconditional bases. On this matter, we point out that the Zak transform, although a norm decreasing injection  $L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R}_{a,1/a})$ , is not a surjection. Fourth, if  $\{\psi_n : n = 1, \dots\} \subseteq H$  is an unconditional basis and  $T : H \rightarrow H$  is linear on the span of  $\{\psi_n\}$  then we can conclude that  $T$  is bounded when

$$\forall n, T\psi_n = \lambda_n \psi_n \quad \text{and} \quad \sup |\lambda_n| < \infty.$$

The point is, as noted by Meyer, that there is a workable condition to check the continuity of operators in the case orthonormal bases are not available or too complex.

Example 3.7. Let  $g$  be the Gaussian and define the wavelets  $\{\psi_{m,n}\}$  as in Section 2. Then, not only is  $\{\psi_{m,n}\}$  not an orthonormal basis of  $L^2(\mathbb{R})$  (in fact,  $|Zg| \neq 1$  on  $\mathbb{R}_{a,1/a}$ ) but it is not a frame. (This latter fact is a consequence of properties of zeros of Jacobi's theta functions, e.g., [24; 17; 4, Section IIB].) On the other hand, each  $f \in L^2(\mathbb{R})$  (resp.,  $L^1(\mathbb{R})$ ) has a wavelet decomposition  $\sum c_{m,n} \psi_{m,n}$  for a computable set  $\{c_{m,n}\}$ . The weaknesses of this decomposition are that  $\{c_{m,n}\}$  can be an unbounded sequence and the convergence is in a much weaker than  $L^2(\mathbb{R})$  (resp.,  $L^1(\mathbb{R})$ ) norm topology [15, Theorems 4.6 and 4.7; 16, Section 4.4], cf, [2; 7]. There are also decompositions in the non-Gaussian case but with even less control on the coefficients [16, Section 4.3].

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