Robust Matched Filters for Optical
Receivers

by

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ABSTRACT

The problem of designing optical receivers that are robust against uncertainty in the statistics of the observation process in photodetection is investigated. In particular a modification in the design of the post-detection matched filter is proposed to account for possible uncertainty in the rate function of the incident light, the rate of the dark current, and in the statistics of the additive noise present at the input to the optical receiver. This design is based on a game-theoretic approach in which a filter is sought that has the maximum worst-case output signal-to-noise ratio possible over the class of allowable statistics; that is, the design criterion is maximin signal-to-ratio. A general characterization of maximin robust matched filters for observed Poisson processes is presented in this context, and specific solutions for several useful uncertainty models are obtained. Numerical results are presented for a specific example to illustrate the performance of the proposed technique.

I. INTRODUCTION

Most techniques for signal processing in communication receivers are based on statistical models for various signals and noises. Since these models are rarely accurate, it is of interest to design procedures that not only perform well for the chosen or nominal statistical model but also exhibit a degree of robustness in the face of deviations from this model. There has been considerable recent interest in the development of such methods, and many of these are described in a recent survey paper by Kassam and Poor [1]. One class of applications that have not been treated extensively in this context are those in which the observations are derived from point processes, such as is in the case in optical communications. This paper together with a companion paper [2] is concerned with the study of robust detection from observed point processes with uncertain statistics. In [2] we have examined robust decision designs based on the error probabilities for general discrimination problems between point processes. Here we develop a robust matched filter design based on a game-theoretic approach in which we seek a receiving filter to maximize the minimum output signal-to-noise ratio over classes that model uncertainty in the rate functions of the incident light and dark current of a photodetector, and in the statistics of the noise at the input to the receiver.

Our system model consists of a *continuous-time* system with *discontinuous observations;* i.e. a systems observed through a point process. We assume that a deterministic function modulates the photon rate of a transmitting light source and that dark current and additive thermal noise corrupt the received signal as observed at the output of a photodetector. The objective of an optical detection scheme is to decide the presence or absence of the modulated light at the photodetector by comparing the output of a matched filter following the photodetector to an appropriate threshold. The transmitted light signal passes through a channel and can be distorted in ways that are difficult to model accurately, whereas the dark current and the thermal noise are receiver-generated and thus are more easily modeled. Hence, although we will consider uncertainty in both the incident statistics and in the dark current/thermal noise, our emphasis will be placed on the former. To model uncertainty in the statistics of the incident light and the dark current we adopt inhomogeneous Poisson models for these phenomena in which the corresponding rate functions are assumed to be in some *classes* of rate functions, but otherwise are unknown. A similar model is used for the additive thermal noise by placing its auto-correlation function in an uncertainty class. A number of specific uncertainty classes of practical interest are

discussed in detail in the sequel.

As noted above, in order to design matched filters that exhibit performance robustness over such classes of input statistics, we adopt a maximin formulation similar to that used in related robust matched filter design problems for *continuous observations* (e.g., [3-5]). For the case of *discontinuous observations*, the matched filter for *known* incident-light signal and dark current rate functions has not been discussed extensively in the literature, and thus an appropriate signal-to-noise ratio must be defined carefully and put into a form analogous to that available for the continuous observations case. As is usual in optical signal processing problems, the appearance of *self-noise* terms distinguishes this criterion from its continuous observations counterpart. Having chosen an SNR criterion, our objective is then to select an appropriate receiving filter such that the minimum value of the output signal-to-noise ratio over the class of possible signal rate functions, dark current rate functions, and (thermal) noise autocorrelation functions is maximized.

This paper is organized as follows. In Section II we describe the system model that we consider throughout this paper, introduce the necessary preliminaries and notation, and formulate the problems of signal-to-noise ratio maximization and design of matched filters for discontinuous observations with known signal and dark current rates and thermal noise autocorrelation. In Section III we formulate the robust optical matched filter design problem and characterize its solutions for general uncertainty in the statistics of the incident light and the dark current/thermal noise processes. We then give explicit solutions in Section IV for the robust optical matched filter in the presence of uncertainty in the statistics of the incident light and the noise (dark current/thermal noise) when the uncertainty is described by several general types of uncertainty classes. To illustrate these results, in Section V, we carry out the robust optical matched filter design for three special cases of the general uncertainty models of Section IV: an ε-contamination model, a band model, and a mean-absolute-distortion model. In Section V we present some numerical results to illustrate the effectiveness of our approach. Finally, in Section VI, we give a summary of the reults of this paper.

II. PRELIMINARIES

A. *System Model*

Consider the photodetector model depicted in Figure 1 which is valid for both fiber and free-space optical communication systems (see [6]) and which allows for the treatment of various noise sources. The output of the photodetector in this model is given by the sum of a filtered Poisson process $i_s(t)$ plus an independent zero-mean thermal noise process $i_{th}(t)$. The current $i_s(t)$ can be expressed, for $t \geq 0$, as

$$i_s(t) = \sum_{k=1}^{N_t} eg_k \delta(t-\tau_k) \tag{1}$$

where $\{N_t, t{\geq}0\}$ is an inhomogeneous Poisson counting processes such that N_t is the number of photoelectrons generated during $[0,t]$, τ_k is the emission time of the k-th electron, and the g_k's are independent and identically distributed random variables which, in avalanche photodiode (APD's), model the number of secondary electrons generated for each primary photoelectron. Here e is the electronic charge and the detector impulse response is assumed to be $g_k \delta$ (t) where $\delta(\cdot)$ is the Dirac delta function (i.e., the photodetector is assumed to be ideal). This latter assumption is not a major restriction, and the results that follow may be modified straightforwardly for the general case in which the photodetector is not ideal.

The intensity $\lambda(t)$ of $\{N_t, t{\geq}0\}$ is related to the incident optical power p(t) by the expression

$$\lambda(t) = \frac{\eta}{h_p v} p(t){+}\lambda_d = \lambda_s(t){+}\lambda_d \tag{2}$$

where η is the quantum efficiency of the photodetector, h_p is Plank's constant and v is the unmodulated optical carrier frequency. We assume that p(t) is deterministic, and thus so is $\lambda_s(t)$ defined in (2). The rate λ_d accounts for the dark current rate at which spontaneous but extraneous electrons are generated in the photodetector. (Background radiation can also be lumped into this term.)

The filtered Poisson process y(t) at the output of the receiver filter (see Fig. 1) can be written as

$$y(t) = \sum_{k=1}^{N_t} eg_k h(t-\tau_k){+}\int_0^t i_{th}(\tau)h(t-\tau)d\tau, \; 0{\leq}t{\leq}T. \tag{3}$$

where h is the impulse response of the receiving filter. In [7] the characteristic function and the moments of y(t)
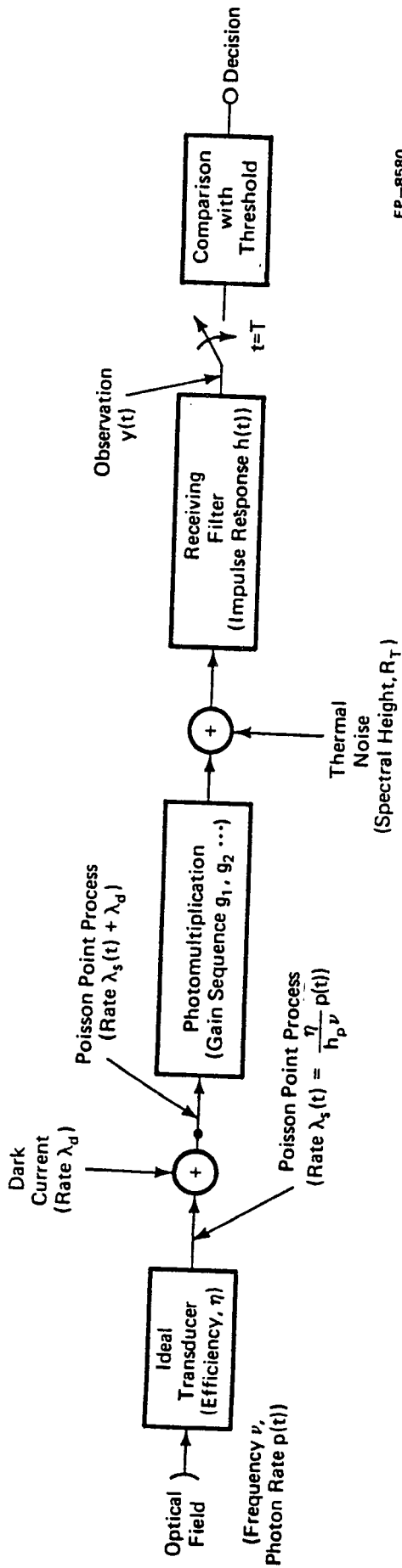
Figure 1: Optical Receiver Model

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are evaluated. From these quantities the signal-to-noise ratio at the output of the receiver filter at the end of the observation interval (t=T) is given by

$$SNR = \frac{E^2\{y(T)|\lambda_d = 0, i_{th}(t) = 0, 0 \le t \le T\}}{Var\{y(T)\}} \tag{4a}$$

$$= \frac{E^2\{g\}}{E\{g^2\}} \cdot \frac{[\int_0^T h(T-t)\lambda_s(t)dt]^2}{[\int_0^T h^2(T-t)\{\lambda_s(t)+[N_0/(2e^2E\{g^2\})+\lambda_d]\}dt} \tag{4b}$$

where $N_0/2$ is the two-sided spectral density of the thermal noise process which for now is assumed to be white and Gaussian, and the moments $E\{g\}$ and $E\{g^2\}$ are the common first and second moments of the gain sequence $g_1, g_2, \ldots$, which for our problem are given constants (see also [8]).

The assumptions leading to (4) about the thermal noise being a white Gaussian process and the dark current rate being constant can be relaxed easily. In particular, we can treat the general situation where the dark current rate is time-varying (i.e., we have $\lambda_d(t)$ instead of λ_d) and the autocorrelation function of the thermal noise is $R_T(t)\delta(t-\tau)$ instead of $\frac{N_0}{2}\delta(t-\tau)$ (i.e., the noise intensity is time varying). In this case (4b) becomes

$$SNR = \frac{E^2\{g\}}{E\{g^2\}} \cdot \frac{[\int_0^T h(T-t)\lambda_s(t)dt]^2}{\int_0^T h^2(T-t)[\lambda_s(t)+R_n(t)]dt} \tag{5}$$

where the term $R_n(t) \triangleq [R_T(t)/(e^2E\{g^2\})+\lambda_d(t)]$ incorporates the effects of dark current and thermal noise. Note that the rate functions $\lambda_s(t)$ and $\lambda_d(t)$ and the thermal noise intensity function $R_T(t)$ must all be *nonnegative* functions.

B. *The Matched Filter for Optical Reception*

As is proved in [7], as certain parameters tend to prescribed limits, the process y(t) defined in (3) tends to a Gaussian process on [0,T] with mean the unsquared numerator of (4b) [or (5)] and variance the denominator of (4b) [or (5)]. Therefore the probability of error reduces to $Q(SNR^{\frac{1}{2}})$ (where Q is the tail of the standard normal distribution) and the quantity SNR becomes a useful performance measure. Thus, maximizing the SNR over all possible filter impulse responses is desirable. The resulting optimal filter is, of course, the *matched filter*. In order to simplify notation and also to make our analysis analogous to that of robust matched filtering with continuous observations developed in [3], we adopt the following formulation for this problem.

Assume that each of the functions λ_s, h, and R_n lies in L_2 [0,T] (the space of square-integrable functions on [0,T]) which we denote by H, and that λ_s and R_n are in H^+, the subset of H consisting of (a.e.) positive functions. If for a, b $\in H$ we define $<a,b> = \int_0^T a(t)b(t)dt$, then (5) can be rewritten as $SNR = E^2\{g\}\rho(h; \lambda_s, R_n)/E\{g^2\}$ where

$$\rho(h;\lambda_s, R_n) \triangleq \frac{<\overline{h}, \lambda_s>^2}{<\overline{h}, (\lambda_s+R_n)h>} , \qquad (6)$$

and where the overbar denotes time reversal; i.e. $\overline{h}(t) = h(T-t)$. This quantity differs from the analogous quantity for the continuous-observations case (see [3]) in that the signal λ_s appears in (6) both in the numerator and denominator, whereas for continuous observations it appears only in the numerator. Of course, it is this *self-noise* phenomenon that distinguishes these two problems analytically.

Within this formulation, the optical matched filtering problem for fixed λ_s and R_n is now described by (compare with [3, section II]) the maximization problem:

$$\max_{h \in H} \rho(h; \lambda_s, R_n). \qquad (7)$$

The solution to this problem is given by

Property 1: (Optical Matched Filter) Define a filter h_o by

$$h_o(t) = \frac{\lambda_s(T-t)}{\lambda_s(T-t)+R_n(T-t)}, \quad 0\leq t\leq T. \qquad (8)$$

Then

$$\max_{h \in H} \rho(h; \lambda_s, R_n) = \rho(h_0; \lambda_s, R_n) = <\lambda_s, (\lambda_s + R_n)^{-1}\lambda_s>. \tag{9}$$

Proof: Since $\lambda_s > 0$ and $R_n > 0$ by assumption, $(\lambda_s + R_n)^{-1}$ is well-defined. Then (9) follows from an application to (6) of the Schwarz Inequality for the inner product $[a, b] = <a, (\lambda_s + R_n) b>$.

Note that the impulse response (8) is reminiscent of, but is quite different from, the conventional matched filter impulse response for this model, which does not have the λ_s term in the denominator.[1] Note that h_0 of (8) satisfies the condition $0 \le h_0(t) \le 1$ for all $t \in [0,T]$.

In the sequel, we will need the following properties of the functional ρ:

Property 2: For fixed $h \in H$, $\rho(h; \lambda_s, R_n)$ is convex in $(\lambda_s, R_n) \in H^+ \times H^+$.

Proof: The proof is similar to that of Property 2 of [3] and will be omitted.

Property 3: The functional $\max_h \rho(h; \lambda_s, R_n) = <\lambda_s, (\lambda_s + R_n)^{-1}\lambda_s>$ is convex in (λ_s, R_n) on $H^+ \times H^+$.

Proof: Property 3 follows straightforwardly from Property 2.

Remark 1: In comparing $<\lambda_s, (\lambda_s + R_n)^{-1}\lambda_s>$ of (8) with the analogous quantity $<\lambda_s, R_n^{-1}\lambda_s>$ for the conventional problem of [3] notice that the discontinuous-observations case is equivalent to the continuous observations case with autocorrelation function $[R_n(t) + \lambda_s(t)]\delta(t-\tau)$ (i.e., the useful signal also plays the role of additive time-wise-uncorrelated noise).

Remark 2: Let us assume that $\lambda_d = 0$ (no dark current is present) and that no thermal noise disturbs the system of Fig. 1. Then via (4b) [or (5)] (6) reduces to

$$\rho(h; \lambda_s, 0) = <\overline{h}, \lambda_s>^2/<\overline{h}, \lambda_s h>$$

and (9) to

$$\max_{h \in H} \rho(h; \lambda_s, 0) = \rho(1; \lambda_s, 0) = <\lambda_s, 1> .$$

Thus the optimal filter in this case is one with impulse response identically equal to unity on $[0,T]$ which corresponds to a pure integrator (i.e., an unweighted photon counter).

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[1] It should be noted that the model under consideration could be further refined to include completely general thermal noise autocorrelation functions (as in [3]), in which case (8) would be replaced by an integral equation. However, our model of time-wise-uncorrelated thermal noise should be adequate for most optical applications.

III. DESIGN OF ROBUST OPTICAL MATCHED FILTERS - GENERAL CHARACTERIZATION

Property 1 indicates that for known statistical characteristics λ_s and R_n, the optical matched filter is given by (8). Thus, we need to know the statistics of the incident light and noises exactly in order to design an appropriate filter. More realistically, we might assume that the quantities represented by λ_s and R_n are known only to lie within some classes S and N, respectively, of elements of H^+, but are otherwise unknown. This characterization allows for the modeling of channel and photodectector distortion and of possible uncertainties in the behavior of the thermal noise process. Given this type of uncertainty model, the design criterion $\max_h \rho(h;\lambda_s, R_n)$ is no longer useful since it is quite unlikely that a single filter will maximize $\rho(h;\lambda_s, R_n)$ for all elements of any realistic classes S and N. The alternative objective should be to find a filter that gives at least a guaranteed minimum level of output SNR regardless of which elements of S and N are actually present during $[0,T]$. Thus the performance measure $\rho(h;\lambda_s, R_n)$ is more appropriately replaced by its minimum value over S and N; i.e., a better performance measure is

$$\inf_{(\lambda_s, R_n)\in S\times N} \rho(h;\lambda_s, R_n), \tag{10}$$

which now gives a performance measure dependent only on the *classes* S and N and not on the specific characteristics λ_s and R_n. Ideally, we would like to find the filter for which the quantity of (10) achieves its maximum value. That is, we would like to solve the problem

$$\max_{h\in H} \inf_{(\lambda_s, R_n)\in S\times N} \rho(h;\lambda_s, R_n). \tag{11}$$

Such a filter has the best worst-case performance over the classes of allowable statistics, and thus its performance will suffer the least possible performance degradation. Also, in similar design problems (see [1]), it turns out that filters designed in this way preserve good (near-optimum) performance at nominal operating conditions when S and N represent neighborhoods of some nominal operating point. Thus, the solution to (11) can be termed a *robust optical matched filter*.

To seek solutions to the maximin robust design game of (11), we look for a *saddlepoint* of $\rho(h;\lambda_s, R_n)$ over $H\times S\times N$. Recall that $(h_R; \lambda_L, R_L) \in H\times S\times N$ is a saddlepoint solution to (11) if it satisfies

$$\min_{(\lambda_s, R_n) \in S \times N} \rho(h_R; \lambda_s, R_n) = \rho(h_R; \lambda_L, R_L) = \max_{h \in H} \rho(h; \lambda_L, R_L). \tag{12}$$

Note that the right-hand equality of (12) implies that h_R is the optical matched filter (8) for the specific model (λ_L, R_L). The left-hand equality implies that h_R achieves its worst-case SNR at the model (λ_L, R_L) for which it is optimum. The quantity $\rho(h_R; \lambda_L, R_L)$ is thus the *worst-case* SNR for the filter h_R and is called the *value* of the game (11). Concerning such solutions, we have the following result:

Lemma 1: Suppose S and N are convex, $(\lambda_L, R_L) \in S \times N$ and h_R is the optimum filter (8) for (λ_L, R_L). Then $(h_R; \lambda_L, R_L)$ is a saddlepoint for (11) if and only if the following inequality holds for all $(\lambda_s, R_n) \in S \times N$:

$$2 < \overline{h}_R, \lambda_s > - <\overline{h}_R, \lambda_L> - <\overline{h}_R, (\lambda_s + R_n)h_R > \geq 0 \quad . \tag{13}$$

Proof: The proof of this result parallels that of Lemma 1 of [3], and will be omitted.

Lemma 1 characterizes saddlepoints of (11). To seek such solutions we introduce the following notion: A pair (λ_L, R_L) is said to be *least favorable* for $S \times N$ if

$$<\lambda_L, (\lambda_L + R_L)^{-1}\lambda_L> = \min_{(\lambda_s, R_n) \in S \times N} <\lambda_s, (\lambda_s + R_n)^{-1}\lambda_s>. \tag{14}$$

It follows easily from (11), (12), and (14) that $(\lambda_L, R_L) \in S \times N$ is least favorable for $S \times N$ if $(h_R; \lambda_L, R_L)$ with h_R optimum for (λ_L, R_L) is a saddlepoint solution to (11) (see [9]). We also have the following result:

Lemma 2: Suppose S and N are convex and $(\lambda_L, R_L) \in S \times N$. Then (λ_L, R_L) is least favorable for $S \times N$ if and only if (13) holds for all $(\lambda_s, R_n) \in S \times N$.

Proof: The proof parallels those of Lemmas 1 and 2 of [3], and will be omitted.

Lemmas 1 and 2 imply that the triple $(h_R; \lambda_L, R_L)$ with $(\lambda_L, R_L) \in S \times N$ and $h_L = (\lambda_L + R_L)^{-1}\lambda_L$ is a saddlepoint solution to (11) *if and only if* (λ_L, R_L) is least favorable for S and N. The usefulness of this result lies in the fact that the search for a saddlepoint of (11) is reduced to a straightforward minimization problem of the functional $<\lambda_s, (\lambda_s + R_n)^{-1}\lambda_s> = \int_0^T \{\lambda_s^2(t)/[\lambda_s(t) + R_n(t)]\}dt$. On the other hand, Lemma 1 provides us with a direct condition (13) which if satisfied guarantees the existence of a saddlepoint for (11). In the following sections, we use these results to find saddlepoints $(h_R; \lambda_L, R_L)$ for specific uncertainty classes modeling uncertainty in the incident rate function and in the noise (dark current and thermal noise).

IV. DESIGN OF ROBUST OPTICAL MATCHED FILTERS - SOLUTION METHODS

In this section we apply the general results of Section III to derive solution methods for the robust optical matched filtering problem for two general types of uncertainty models. The models that we consider allow a fairly general range of uncertainty behavior, and so should cover most types of uncertainty encountered in applications. Some specific solutions are discussed in Section V.

A. *Uncertainty Based on Mean-Square Distortion.*

We consider first the case in which the noise (dark current and thermal noise) statistics as described by R_n are known to be given by, say, $R_o(t)$, and there is uncertainty only in the rate function of the incident light. Such a situation arises in practice when the dark current and the thermal noise are fairly well-modeled while the incident light, having passed through a distorting channel and been converted by the PD, is less well modeled. In this case condition (13) reduces to

$$<\bar{h}_R(2 - \bar{h}_R), \lambda_* - \lambda_L> \geq 0, \text{ for all } \lambda_* \in S. \tag{15}$$

A good model for the rate function of the incident light is that it satisfies the condition

$$[\int_0^T [\lambda_*(t)-\lambda_o(t)]^2 dt]^{1/2} \leq \Delta_*, \tag{16}$$

where λ_o is a nominal rate function (for example, it might be a transmitted modulation waveform) and where Δ_* is a degree of possible distortion caused by nonideal effects in the channel and photodetector. In the formulation introduced in the previous sections, we can write this uncertainty model as

$$S = \{\lambda_* \in H^+ \mid ||\lambda_*-\lambda_o|| \leq \Delta_*\} \tag{17}$$

where $|| \cdot ||$ stands for the $L_2[0,T]$ norm defined by $|| a || = <a,a>^{1/2}$ for $a \in H$.

To characterize the robust optical matched filter in this case, we define

$$\lambda_L = \lambda_o - \Delta_* \bar{h}_R(2-\bar{h}_R)/||h_R(2-h_R)|| \tag{18}$$

where $h_R \overset{\Delta}{=} (\bar{\lambda}_L+\bar{R}_o)^{-1}\lambda_L$ satisfies the equation

$$R_o\overline{h}_R = (1-\overline{h}_R)[\lambda_o - \Delta_*\overline{h}_R \cdot (2-\overline{h}_R)/||h_R(2-h_R)||].\tag{19}$$

Then, if a solution to (19) exists we have the following.

Proposition 1: The triple $(h_R;\lambda_L,R_o)$ is a saddlepoint solution to (11) for S of (17) and $N = \{R_o\}$.

Proof: From (18) and (19) we have

$$<\overline{h}_R(2-\overline{h}_R),\lambda_*-\lambda_L)> = <\overline{h}_R(2-\overline{h}_R),\lambda_*-\lambda_o> + \Delta_*||h_R(2-h_R)||.\tag{20}$$

From the Schwarz Inequality, (18) and (19), we have

$$|<\overline{h}_R(2-\overline{h}_R),\lambda_*-\lambda_o>| \leq ||h_R(2-h_R)|| \cdot ||\lambda_*-\lambda_o|| \leq ||h_R(2-h_R)||\Delta_*,\tag{21}$$

which together with (20), implies (15). Lemma 1 assures that (15) is sufficient for $(h_R;\lambda_L,R_o)$ to be a saddlepoint for (11).

Equation (19), which gives the robust matched filter for our problem, can be solved iteratively. Iterative solutions to related equations for the continuous-observations case have been treated in [10].

Next let us suppose alternatively that the incident rate function is completely known to be, say, λ_o and that there is uncertainty about the noise model $R_n(t)$. In this case $h_R = (\lambda_o + R_L)^{-1}\lambda_o$ and condition (13) reduces to

$$<\overline{h}_R,(R_L - R_n)h_R> \geq 0, \quad \text{for all } R_n \in N.\tag{22}$$

Suppose that the noise model is known only to satisfy a model analogous to (16); i.e.,

$$[\int_0^T [R_n(t)-R_o(t)]^2 dt]^{1/2} \leq \Delta_n,\tag{23}$$

where R_o is a nominal noise model and Δ_n is a degree of uncertainty placed on this nominal model. Then we have a noise uncertainty class

$$N = \{R_n \in H^+ | \, ||R_n-R_o|| \leq \Delta_n\}.\tag{24}$$

On defining

$$R_L = R_o + \Delta_n\overline{h}_R^2/||h_R^2||\tag{25}$$

where $h_R \overset{\Delta}{=} (\bar{\lambda}_o + \bar{R}_L)^{-1}\bar{\lambda}_0$ satisfies the equation

$$\bar{h}_R = (\lambda_o + R_o + \Delta_n \bar{h}_R^2 / ||h_R^2||)^{-1}\lambda_o \, , \tag{26}$$

we have the following when a solution to (26) exists.

Proposition 2: The triple $(h_R; \lambda_o, R_L)$ is a saddlepoint solution of (11) for the N of (24) and $S = \{\lambda_o\}$.

Proof: Through Lemma 1, (23) and (26) it is sufficient to show that

$$\langle \bar{h}_R, (R_L - R_n)h_R \rangle = \langle \bar{h}_R, (R_n - R_o)h_R \rangle + \Delta_n ||h_R^2|| \geq 0 \, , \tag{27}$$

which is satisfied since

$$|\langle \bar{h}_R, (R_o - R_n)h_R \rangle| \leq ||h_R^2|| \cdot ||R_o - R_n|| \leq ||h_R^2||\Delta_n. \tag{28}$$

Again solutions to (27) must be found iteratively.

Now we may combine the results of Propositions 1 and 2 to give a solution for the robust optical matched filter for the case in which both λ_s and R_n are allowed to vary through classes S and N of the form (17) and (24). In particular we note that it is straightforward to show that the conditions $\langle \bar{h}_R(2 - \bar{h}_R), \lambda_s - \lambda_L \rangle \geq 0$ (i.e., (15)) and $\langle \bar{h}_R, (R_L - R_n)h_R \rangle \geq 0$ (i.e., (23)) together imply (13). Thus, we may state the following result for simultaneous uncertainty of the L_2 type.

Proposition 3: If a solution h_R to the following equation exists

$$(1 - \bar{h}_R)[\lambda_o - \Delta_s \bar{h}_R(2 - h_R)/ ||h_R(2 - h_R)||] = \bar{h}_R(R_o + \Delta_n \bar{h}_R^2/ ||h_R^2||) \, , \tag{29}$$

then the triple $(h_R; \lambda_L, R_L)$ is a saddlepoint solution (13) for S and N of (17) and (24), where λ_L and R_L are given by (18) and (25), respectively.

Proof: Since λ_L and R_L are defined in (18) and (25), respectively, conditions (15) and (22) are satisfied and therefore so is (13). Then the optimality of (29) follows from the fact that the condition $\bar{h}_R = (\lambda_L + R_L)^{-1}\lambda_L$ is equivalent to the equation $(1 - \bar{h}_R)\lambda_L = \bar{h}_R R_L$.

B. *Solutions Based on Results from Robust Hypothesis Testing.*

In several other problems of robust signal processing it has been possible to find solutions by applying results developed for analogous problems in robust hypothesis testing. This approach has been particularly useful in the problems of filtering [11,12], prediction [12,13,14] and interpolation [15,16] of time series in which uncertainties are defined in terms of power spectra of relevant signals and noise. It turns out that because of the structure of the optical matched filtering problem this approach can also be applied here, even though its utility in the conventional robust matched filtering problem has been limited.

To formulate this problem, we restrict attention to uncertainty classes in which the total power of all members of the classes is fixed; i.e., we consider classes S and N for which the quantities

$$\int_0^T \lambda_s(t)dt \stackrel{\Delta}{=} P_s T \tag{30}$$

and

$$\int_0^T R_n(t)dt \stackrel{\Delta}{=} P_n T , \tag{31}$$

are constants. (This is not an unrealistic assumption since total energy is an easily estimated quantity.) Then, noting that λ_s and R_n are nonnegative functions, we can define two classes of *probability density functions* (pdf's) on $[0,T]$ via

$$P_S = \{p_s \mid p_s(t) = \lambda_s(t)/P_s T, \ 0 \le t \le T, \ \lambda_s \in S\} \tag{32}$$

$$P_N = \{p_n \mid p_n(t) = R_n(t)/P_n T, \ 0 \le t \le T, \ R_n \in N\} \tag{33}$$

Note that the elements of (32) and (33) are pdf's because of (30) and (31).

We now consider the following composite statistical hypothesis testing problem about a random variable X taking values in $[0,T]$

$$H_0 : X \text{ has pdf } p_n \in P_N$$

versus $\tag{34}$

$$H_1 : X \text{ has pdf } p_s \in P_S.$$

This problem is an example of a robust hypothesis testing problem as studied by Huber in [17]. A robust test

for (34) can, for many uncertainty classes of interest, be obtained by finding certain least-favorable pdf's $(q_S,q_N) \in P_S \times P_N$. These pdf's have the property that the ratio $q_S(x)/q_N(x)$ is stochastically largest over P_N at q_N and it is stochastically smallest over P_S at q_S. This implies, among other things, that the pair (q_S, q_N) minimizes over $P_S \times P_N$ all functionals of the form

$$\int_0^T C(p_s(t)/p_n(t))p_n(t)dt, \quad (p_s,p_n) \in P_S \times P_N \tag{35}$$

with C concave. (see, e.g., [12,18,19]).

Returning to the robust optical matched filtering problem for S and N satisfying (30) and (31), we know from Lemma 2 that a pair $(\lambda_L,R_L) \in S \times N$ together with its optimal filter gives a saddlepoint for (11) if and only if (λ_L,R_L) maximizes

$$\int_0^T \lambda_s^2(t)/[\lambda_s(t)+R_n(t)]dt \tag{36}$$

over the uncertainty classes S and N. Note that (36) can be rewritten in the form (35), where $p_s(t) = \lambda_s(t)/P_s T$, $p_n(t) = R_n(t)/P_n T$, and C is the concave function

$$C(x) = \frac{P_s^2 T^2 x^2}{P_s x/P_n + 1} , x \geq 0. \tag{37}$$

Thus we conclude that, if $(q_S,q_N) \in P_S \times P_N$ is least favorable for the hypothesis test of (34), then

$$\lambda_L(t) = P_s T q_S(t) , 0 \leq t \leq T \tag{38}$$
and

$$R_L(t) = P_n T q_N(t) , 0 \leq t \leq T \tag{39}$$

are the least favorable statistics for the robust optical matched filtering problem with the uncertainty models of S and N. We summarize this result in the following.

Proposition 4: Suppose that S and N satisfy the power constraints (30) and (31) and that there exists a pair of least-favorable pdf's $(q_S, q_N) \in P_S \times P_N$. Then the least-favorable statistics for optical matched filtering in $S \times N$ are given by (38) and (39).

The utility of the above result lies in the fact that least-favorable pairs of pdf's are known for many uncertainty models that can be adapted easily to the optical communications problem. A number of such solutions are found in [17,20,21,22], and one of the most useful ones is discussed in the following section. It should be noted that the reason this approach works here is because the statistical quantities λ_s and R_n are real *nonnegative* functions, and the constraints (30) and (31) are realistic physical constraints. These factors are not present in the conventional matched filtering problem since the signal quantity there is possibly complex and negative, and also since its energy is proportional to the integral of its *squared* value.

V. AN EXAMPLE OF ROBUST OPTICAL MATCHED FILTER DESIGN

In this section we illustrate the approach outlined in Section IV B with a specific example of robust optical matched filter design. We assume the normalization (30) and (31). Many uncertainly models other than the one given here can be treated in an analogous manner, and a variety of examples is found in the references given above. In Section A we give the general form of the robust matched filter for our example model, and in Section B we explore the performance in a specific case numerically.

A. *Robust Filters for ε-contamination Models*

Suppose the functions specifying the statistical model (i.e., λ_s and R_n), are known to lie in classes of the following form:

$$\lambda_s(t) = (1-\varepsilon_S) \, \lambda_o(t)+\varepsilon_S \lambda(t) \quad , \, 0 \leq t \leq T \, , \tag{40}$$

and

$$R_n(t) = (1-\varepsilon_N)R_o(t)+\varepsilon_N R(t) \quad ,0 \leq t \leq T \, , \tag{41}$$

where λ_o and R_o represent a known, nominal model, λ and R are arbitrary nonnegative functions "contaminating" the nominal model, and where $\varepsilon_S \in [0,1]$ and $\varepsilon_N \in [0,1]$ are constants that quantify the degree of uncertainty placed on the nominal model by the designer. This type of model is known as an ε-*contamination model or* ε-*mixture model*, and similar models for other statistical quantities have been used quite frequently throughout the study of robust statistics and signal processing (see the survey [1] for many examples). This is the most commonly used model in this context.

Using the approach of Section IV B, we can obtain the least-favorable model (λ_L, R_L) for this case from the least favorables for the analogous hypothesis testing problem given by Huber in [17]. In particular, it follows straightforwardly from [17] and Proposition 4 that the least-favorables are

$$\lambda_L(t) = \begin{cases} (1-\varepsilon_S)\lambda_o(t), & \text{if } \lambda_o(t) \leq a''R_o(t) \\ (1-\varepsilon_S)R_o(t)a'', & \text{if } \lambda_o(t) > a''R_o(t) \end{cases}, \tag{42}$$

and

$$R_L(t) = \begin{cases} (1-\varepsilon_N)R_o(t), & \text{if } \lambda_o(t) \geq a''R_o(t) \\ (1-\varepsilon_N)\lambda(t)/a', & \text{if } \lambda_o(t) < a''R_o(t) \end{cases} \tag{43}$$

where $a' < a''$ are constants chosen (uniquely) so that λ_L and R_L satisfy the energy constraints.

From Lemma 2, the robust filter for the signal and noise model of (40) and (41) is given by

$$h_R(t) = \lambda_L(T-t)/[\lambda_L(T-t) + R_n(T-t)], \ 0 \leq t \leq T \ .$$

The form of this filter is particularly interesting for the case of $\varepsilon_S = \varepsilon_N$, for which we have

$$h_R(t) = \begin{cases} b'', & \text{if } h_o(t) < b' \\ h_o(t), & \text{if } b' \leq h_o(t) \leq b'' \\ b'', & \text{if } h_o(t) > b'' \end{cases} \tag{44}$$

where $h_o(t)$ is the filter matched to the nominal model, i.e.,

$$h_o(t) = \lambda_o(T-t)/[\lambda_o(T-t) + R_o(T-t)], \ 0 \leq t \leq T \ , \tag{45}$$

and where $b' = a'/(a'+1)$ and $b'' = a''/(a''+1)$. Thus, we see from (44) that the robust filter h_R limits the response of the nominal filter h_o for those values of t for which h_o is very small or very large. The limiting from above desensitizes the filter to unexpectedly large noise contributions in regions where they would not be expected under the nominal model, but where they may very likely occur due to the contamination. The limiting of the impulse response from below prevents the filter from missing signal counts that might occur rarely in the nominal model but that are more likely under the uncertainty. (This might occur, for example, if the receiver timing were slightly off). A similar interpretation can be given to h_R for general ε_S and ε_N, in which the above two effects are more or less pronounced depending on the relative values of ε_S and ε_N.

Note also that for any value of ε_S and ε_N, if the original nominal filter was an integrate-and-dump (i.e., constant impulse response) type filter, then the robust filter is also of this type, but with possibly different gain. (This follows from the fact that λ_o is proportional to R_o in this case, which implies that λ_L will be proportional to R_L.) Since an overall gain factor is irrelevant in the detection problem (i.e., we need only adjust the decision

threshold if the gain is changed), we see that for this particular case the nominal filter is itself robust against ε-contamination. This is a useful result since many systems operate with a nominally rectangular pulse waveform and under the assumptions of constant dark current rate and thermal noise level. *Under these conditions the conventional integrate-and-dump receiver is robust against arbitrary ε-contamination.*

B. *Performance Analysis of a Robust Optical Matched Filter*

The above example illustrates the structure of robust optical matched filters for one useful model for statistical uncertainty. In this section we analyze the design and performance of the robust matched filter for a specific example of this model. Recall that the objective of our design is to provide near-optimal performance under nominal operating conditions and to guard against undesirable performance degradion at operating points away from nominal. The following numerical example illustrates the effectiveness of our design method in achieving these goals.

Consider the situation in which we have a known constant dark current/thermal noise level, $R_o(t) = P_n, 0 \leq t \leq T$, and in which, nominally, we have a raised-cosine amplitude modulation on the optical source so that the nominal signal rate is given by

$$\lambda_o(t) = P_s[1-\cos{(2\pi t/T)}], \; 0 \leq t \leq T. \tag{46}$$

The filter matched to the nominal model is thus (note that $\cos{(2\pi(T-t)/T)} = \cos{(2\pi t/T)}$)

$$h_o(t) = \frac{P_s[1-\cos{(2\pi t/T)}]}{P_s[1-\cos{(2\pi t/T)}]+P_n}, \; 0 \leq t \leq T. \tag{47}$$

The nominal signal rate (46) and matched filter (47) are depicted in Figs. 2 and 3, respectively.

We now assume that the actual signal rate is known only to lie in an ϵ-contamination neighborhood (40) of the nominal rate (46). From (42) we see that the least-favorable signal rate is given by

$$\lambda_L(t) = \max\{g', (1-\epsilon_S)\lambda_o(t)\}, \; 0 \leq t \leq T, \tag{48}$$

with $g' = (1-\epsilon_s)P_n/a''$, where a'' is from (42). This least-favorable signal rate is depicted in Fig. 4. Note that, as ϵ_S increases, the least-favorable signal rate becomes more like the background noise rate.

The corresponding robust filter impulse response, $h_R(t) = \lambda_L(T-t)/[\lambda_L(T-t)+P_n]$, is depicted in Fig. 5. The parameter g' in (48) can be determined from the requirement that $\int_0^T \lambda_L(t)dt = P_s T$. Referring to Fig. 4, we have

$$P_s T = \int_0^T \lambda_L(t)dt = 2g'\int_0^{t'} dt + 2P_s(1-\epsilon_S)\int_{t'}^{T/2} [1-\cos{(2\pi t/T)}]dt, \tag{49}$$

where t' is as depicted in Fig. 4. Equation (49) reduces straightforwardly to

$$(1-\varepsilon_S)^{-1} = 1+\frac{1}{\pi}\sin(\pi x')-x'\cos(\pi x') , \tag{50}$$

where $x'\overset{\Delta}{=}2t'/T$ is the fraction of the observation interval for which λ_L is constant. Equation (50) can be solved for x' when ε_S is specified. Note that a value of $x' = 1$ corresponds to a degree of contamination $\varepsilon_S=\frac{1}{2}$, and thus, the least favorable signal rate for $\varepsilon_S=\frac{1}{2}$ in this model is constant. Figure 6 shows x' as a function of ε_S.

To examine the performance of the robust filter h_R, we are interested in three quantities $\rho(h_R; \lambda_L, R_o)$, $\rho(h_R; \lambda_o, R_o)$, and $\rho(h_o; \lambda_o, R_o)$. The first of these quantities is the worst-case performance of the robust filter (i.e., the value of the maximin game), the second is the performance of the robust filter under nominal conditions, and the third is the performance of the nominal filter under nominal conditions. The first of these three quantities, for example, is given by

$$\rho(h_R; \lambda_L, R_o) \doteq \int_0^T \lambda_L^2(t)/[\lambda_L(t)+R_o(t)]dt$$

$$= P_s\text{Tr}[x'(h')^2/(rh'+1)+(1-\varepsilon_S)^2[\int_{x'}^1 \frac{[1-\cos(\pi x)]^2}{r[1-\cos(\pi x)+1]}dx], \tag{51}$$

where $h'=g'/P_s$ and $r = P_s/P_n$. Since h' and x' depend only on ε_S, we see from (51) that the saddle value of (11) for this case depends only on these quantities: P_sT, the average number of signal counts per observation interval; ε_S, the fraction of uncertainty in the model; and r, the ratio of the signal level to the noise level - a measure of input signal-to-noise ratio. Similarly, the other two performance measures of interest can be shown straightforwardly to depend only on these three quantities.

Figures 7 and 8 show the behavior of the robust filter as a function of input signal-to-noise ratio for values of ε_S of 0.1 and 0.3, respectively. The quantity P_sT is set equal to 100 in these figures. Note that the maximin filter does indeed achieve the desired performance goals in this case.
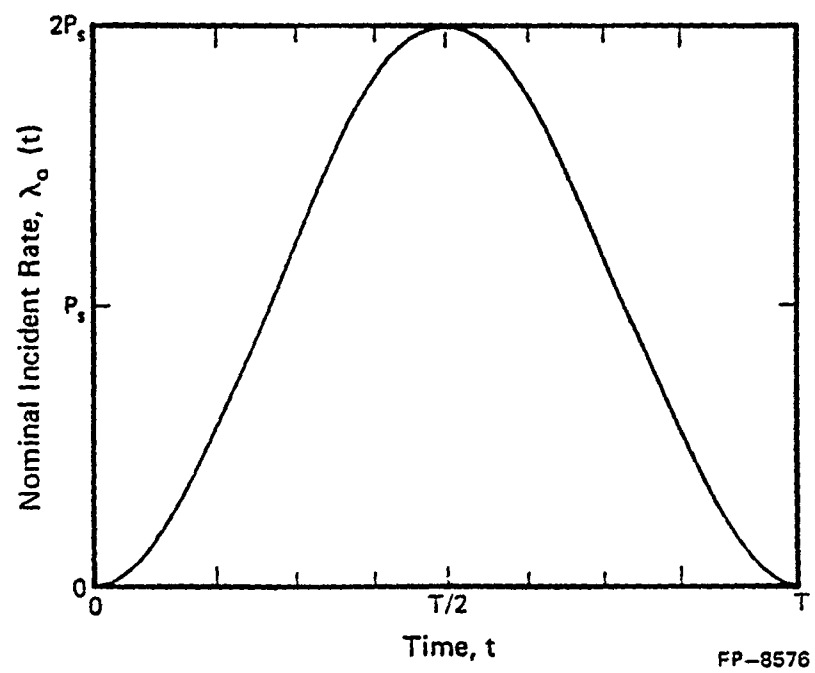
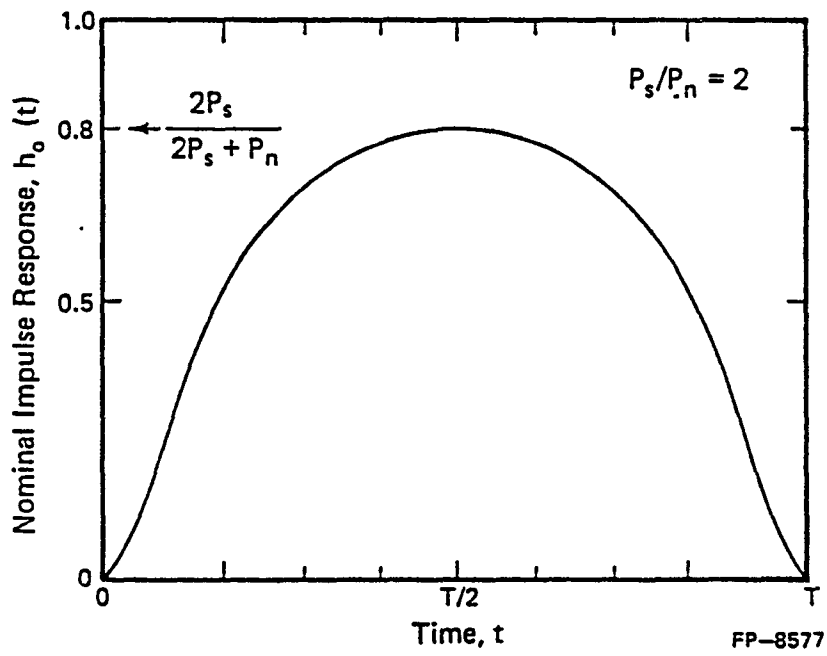*Figure 2: Nominal Raised-cosine Incidence Rate*

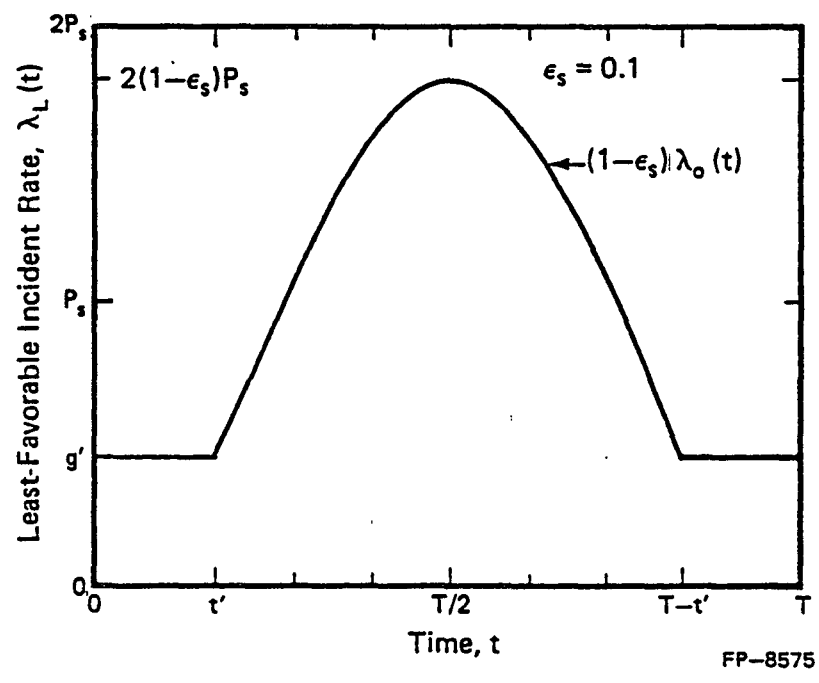*Figure 3: Optical Matched Filter for a Nominal Raised-cosine Incidence Rate*

$2P_s$

$2(1-\epsilon_s)P_s$

$\epsilon_s = 0.1$

Least-Favorable Incident Rate, $\lambda_L(t)$

$\leftarrow (1-\epsilon_s)\lambda_o(t)$

P_s

g'

0

0   t'   T/2   T-t'   T

Time, t

FP—8575

Figure 4: *Least-favorable Incidence Rate for an* ϵ-*contaminated Raised-cosine Pulse*
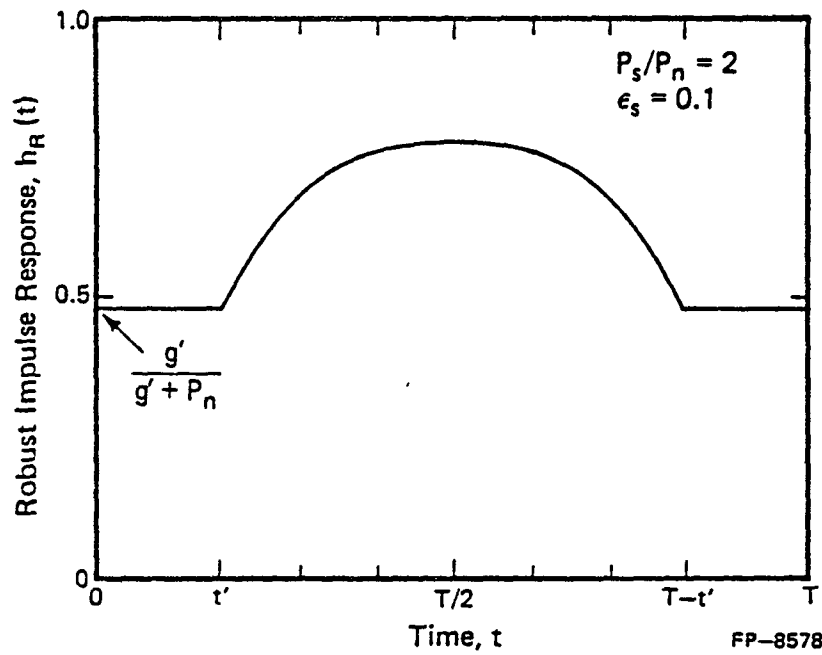
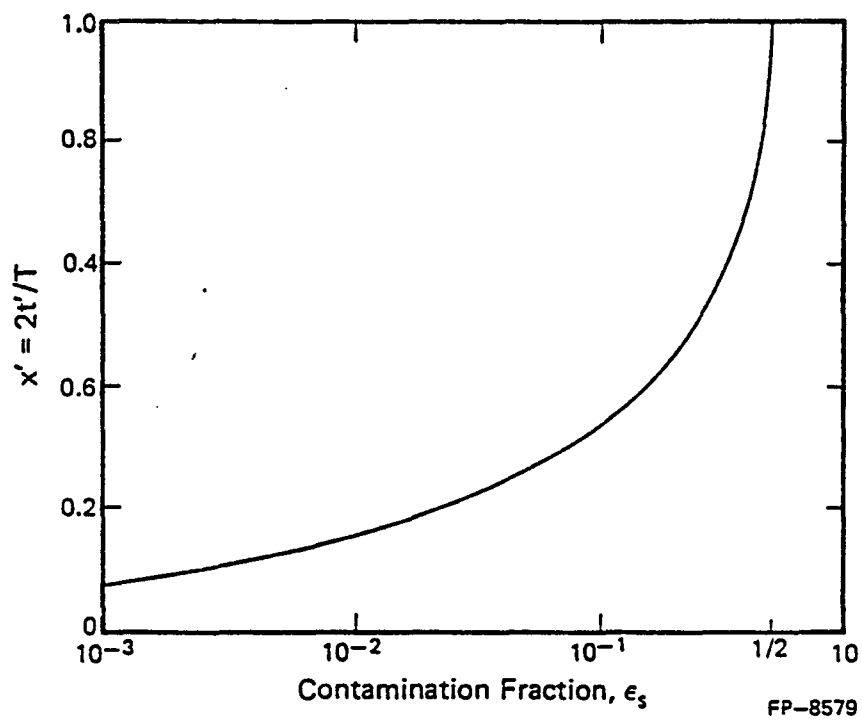Figure 5: *Impulse Response of Robust Matched Filter for an ε-contaminated Raised-cosine Incidence Rate*

Figure 6: The Fraction of the Observation Interval with Constant
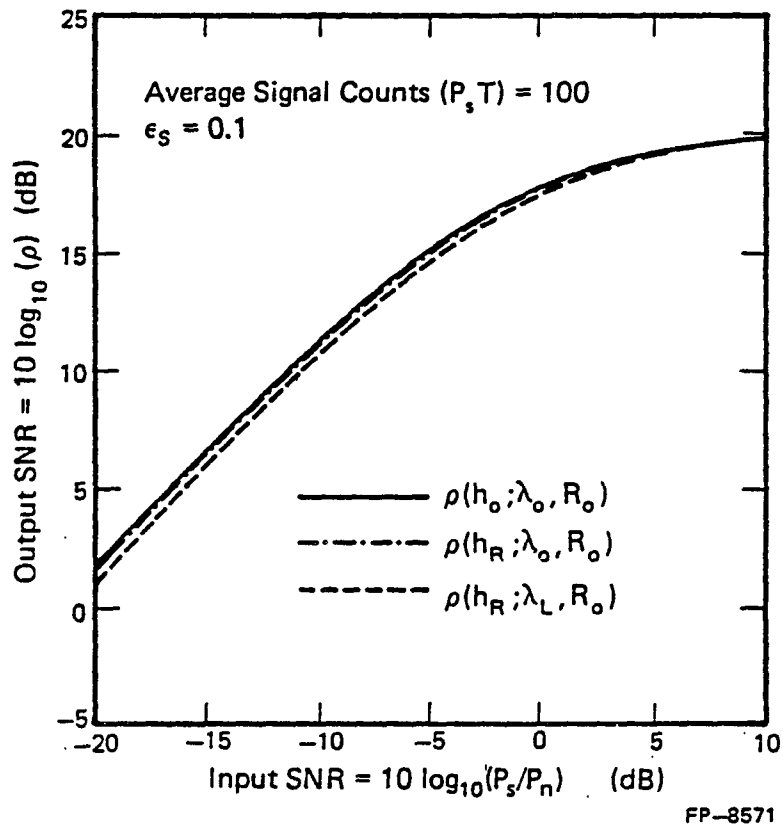
Least-favorable Rate versus the Degree of Uncertainty

Figure 7: *Nominal Performance of the Nominal and Robust Filters and*

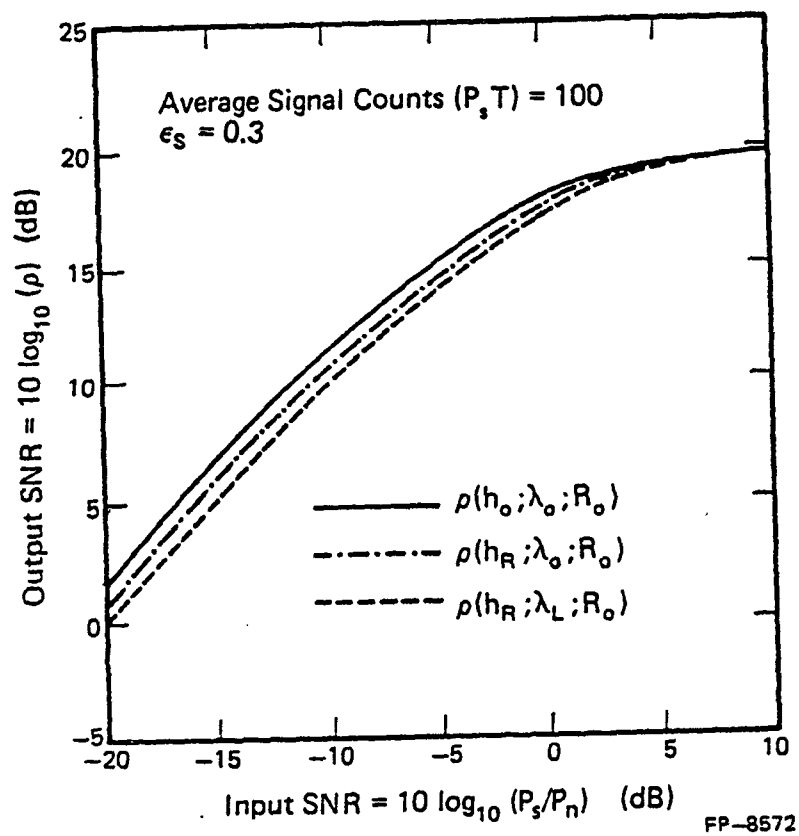*Worst-case Performance of the Robust Filters* (ϵ_s = 0.1)

Figure 8: Nominal Performance of the Nominal and Robust Filters and

Worst-case Performance of the Robust Filters ($\epsilon_s = 0.3$)

VI. SUMMARY

In this paper, we have considered the problem of designing optimum post-detection filters for optical receivers operating under uncertain statistical conditions. We have given a general formulation of this problem in terms of a maximin game on the output SNR of the receiving filter. In Section III we have shown that the corresponding robust filter is the optical matched filter for the least-favorable statistics, a result analogous to results for robust design in other statistical signal processing problems. Two general solution techniques have been discussed in Section IV, with the solution for a specific uncertainty model being given in Section V. The numerical example of Section V illustrates the effectiveness of the maximin filter in achieving the design goals of near-optimum performance under nominal conditions and acceptable performance under worst-case conditions.

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