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Setpoint Tracking vs. Disturbance Rejection for Stable and Unstable Processes

by

E. Zafiriou and M. Morari

SETPOINT TRACKING vs. DISTURBANCE REJECTION FOR STABLE AND UNSTABLE PROCESSES

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ABSTRACT

The issue of controller tuning for setpoint tracking versus disturbance rejection has been discussed in the process control literature for many years. The purpose of this paper is to review and explain the problem and to point out solution procedures. We do not make any claims about developing anything novel. We merely summarize what is available at different places in the literature and put it in perspective in a tutorial manner.

1. INTRODUCTION

In most papers on the subject of setpoint tracking vs. disturbance rejection, it is assumed implicitly that the setpoints have the form of steps and that the disturbances are steps entering at the process input. For example, in his book Smith (1972) provides two lists of tuning rules for PID controllers for a first order process with deadtime; one for step setpoint changes, the other one for disturbances entering as steps at the process input.

More recently, a number of authors (Hwang & Stephanopoulos, 1985; Wellons & Edgar, 1985; Svoronos, 1986; Yuan & Seborg, 1986) have criticized controller design techniques like IMC for what they perceived to be an inability to deal with disturbances. It is interesting to note that this topic of setpoints versus disturbances is almost absent from the rest of the control literature. This might seem surprising because the combined ability of command following and disturbance rejection is probably even more important in aerospace type applications than in process control where setpoint changes are generally quite rare. We hope that the presentation in this paper of a summary of results available in the literature will put the subject to rest once and for all.

2. ONE-DEGREE-OF-FREEDOM STRUCTURE

Consider the feedback structure shown in Fig. 1a with the disturbance entering at the process output. The relationship between the disturbance and setpoint inputs and the error is described by:

$$e = y - r = (I + PC)^{-1}(d - r) \triangleq E(d - r) \qquad (2.1)$$

Where E is referred to as the sensitivity function of the closed loop system. It is clear from this relationship that the response to disturbances and setpoint changes is identical apart from the sign. Thus, if disturbances and setpoint changes have the same form (e.g., both steps or both ramps), then the controller C can be designed "optimally" for *both* disturbances and setpoints. If, on the other hand, the disturbances have a very different form from the setpoints (e.g., disturbances are ramp-like, setpoints are steps) then it is generally not possible to design a controller C which works well for the disturbances *and* the setpoints. In those cases, with the structure shown in Fig. 1a, a compromise must be made and either disturbance or setpoint response must be sacrificed.

Let us study first how to design the controller for a particular form of input. If we use the Integral Square Error (ISE) as an objective function to optimize and restrict our attention to single-input single-output systems then the optimal controller can be expressed analytically the formulation of the problem is simpler in the Internal Model Control framework (Garcia & Morari, 1982). The IMC structure is shown in Fig. 1b. The two structures in Fig. 1 are mathematically equivalent through

$$Q = C(I + \tilde{P}C)^{-1} \qquad (2.2)$$

$$C = Q(1 - \tilde{P}Q)^{-1} \qquad (2.3)$$

where $\tilde{P}$ is the process model. Details on the Internal Stability requirements can be found in the literature (Vidyasagar, 1985; Zafiriou, 1987).

Factor $\tilde{P}$ as:

$$\tilde{P} = P_A P_M \qquad (2.4)$$

where P_A is an allpass containing all the nonminimum phase elements of $\tilde{P}$ and P_M is minimum phase. Let b_p be an allpass with the poles of $\tilde{P}$ in the open right half plane (open RHP) as zero. Let v represent the external input, i.e., $v = d$ or $v = r$, and define V_A, v_M, b_v in a similar way.

For a physical system, the set of poles of v in the open RHP will generally be of a subset of the set of open RHP poles of $\tilde{P}$. For the poles at $s = 0$, the assumption is made that v has at least as many poles at $s = 0$ as $\tilde{P}$. This assumption is not restrictive in the case of disturbances but it may be for setpoints, in which case the Two-degree-of- freedom structure has to be used (Section 3).

The ISE- optimal controller is given by

$$Q = b_r b_v^{-1} P_M^{-1} v_M^{-1} \{b_p^{-1} b_v P_A^{-1} v_M\}_* \tag{2.5}$$

where the operator $\{\cdot\}_*$ denotes that after a partial fraction expansion of the operand, all terms involving the poles of P_A^{-1} are omitted.

Note that in the case where P is open-loop unstable, the control system has to be implemented in the feedback structure of Fig. 1a. Equation (2.5) can still be used for the design of Q. C can then be obtained from (2.3) and the structure in Fig. 1a implemented. However special care has to be taken in the construction of C so that all the common RHP zeros of Q and $(1 - \tilde{P}Q)$ are cancelled in (2.3).

Note the following facts.

1. For minimum phase systems, the optimal controller Q is the inverse of the process transfer function. This controller is independent of the particular type of input to be controlled.
2. For nonminimum phase systems and step inputs, the optimal controller Q is the inverse of the minimum phase portion of the process defined in (2.4).
3. For nonminimum phase systems, the optimal controller depends, in general, on the type of process input. The optimal controller for step setpoint changes and for step disturbances entering at the plant input will generally have a different structure and different parameters.

For multivariable systems the disturbance model, i.e., the differential equations describing the form of the disturbance, can be augmented to the state space description of the plant. The resulting optimal control problem can be solved in a standard manner by solving two Riccati equations, one for the Kalman filter parameters and one for the optimal regulator parameters (Kwakernaak & Sivan, 1972). In the multivariable case the same conclusions about minimum and nonminimum phase systems made for SISO systems hold (Zafiriou & Morari, 1987). In addition, the direction (in a geometric sense) of the input vector determines the controller structure. For a certain type of input a decoupler might be the optimal solution, for other types of inputs decoupling might be detrimental to the overall control performance. The construction of regulators with structural constraints is discussed by Zafiriou and Morari (1986). Marino-Gallaraga, et al., 1987 also observed that the desirability of decoupling depends very much on the type of inputs and the input direction.

In summary, the design of ISE optimal controllers for specific inputs is straightforward using techniques which have been in the literature for more than two decades. Next the tradeoff between setpoint tracking and disturbance rejection will be addressed.

3. TWO-DEGREE-OF-FREEDOM STRUCTURE

Consider the general feedback structure of Fig. 2a with three controller blocks C_1, C_2 and C_3. C_2 is in the feedback path, C_3 is referred to as a prefilter in the aerospace literature or as a setpoint compensator (Ray, 1981). The closed-loop transfer functions relating the disturbance and setpoint input to the error are described by:

$$e = y - r = (1 + PC_1C_2)^{-1} d - (1 + PC_1C_2)^{-1} (I + PC_1(C_2 - C_3))r \tag{3.1}$$

Hence when C_2 and C_3 are not equal to each other, the response to disturbances is different from the response to setpoint changes.

We shall proceed to show that the degrees of freedom available in the structure of Fig. 2a are sufficient for designing a control system that produces independent compensation for setpoints and disturbances. Let us start from the point where a C has been found for the structure in Fig. 1a which produces satisfactory disturbance response (e.g., through (2.5), (2.3) for $v = d$). We can then split C into two blocks C_1 and C_2, such that C_1 is minimum phase and C_2 is stable (Vidyasagar, 1985). For the disturbance behaviour it is irrelevant if the controller is implemented as one block C as in Fig. 1a, or as two blocks C_1, C_2 as in Fig. 2a. Hence the achievable disturbance rejection is restricted both by the RHP zeros and poles of P as the quantitative results of Section 2 indicate, even when the two-degree-of-freedom structure is used.

Let us now consider the design of C_3. Define

$$u' \triangleq C_3 r \tag{3.2}$$

then

$$y = PC_1(1 + PC_1C_2)^{-1} u' \tag{3.3}$$

Now consider the stabilized system as a new plant P':

$$P' \triangleq PC_1(1 + PC_1C_2)^{-1} \tag{3.4}$$

Since C_1 is minimum phase and C_2 stable, P' is stable and its only RHP zeros are the RHP zeros of P. Thus C_3 can be designed without regard for the RHP poles of P and the achievable setpoint tracking is restricted by the RHP zeros of P only. C_3 can actually be designed as the IMC controller Q' for the plant P' and for $v = r$. Indeed, if we factor P' as in (2.4) we have $P_A' = P_A$ and since $b_p' = b_r = 1$ we get from (2.5):

$$C_3 = Q' = (P_M')^{-1} r_M^{-1} \{P_A^{-1} r_M\}_* \tag{3.5}$$

then (3.2), (3.3), (3.5) yield

$$y = P_A r_M^{-1} \{P_A^{-1} r_M\}_* \tag{3.6}$$

which explicitly shows that the setpoint response is limited only by the RHP zeros of P.

In the case of open-loop stable plants, the two-degree-of-freedom controller can be implemented in the IMC structure of Fig. 2b, where

$$Q_1 = (I + C_1C_2\tilde{P})^{-1} C_1 \tag{3.7}$$

4. CONCLUSIONS

When a one-degree-of-freedom control configuration (Fig. 1) is used, the disturbance and setpoint responses cannot be designed independently and they are both restricted by the plant RHP zeros and poles.

When a two-degree-of-freedom control configuration (Fig. 2) is used, the same restrictions apply to the disturbance rejection. However the setpoint response can be designed independently and it is limited only by the RHP zeros of the plant. In this case, the RHP poles of the plant impose no restrictions on the achievable setpoint tracking.

All these arguments hold in the absence of model error. If the process model is inaccurate then the controllers cannot be designed independently but have to be designed simultaneously to achieve the proper tradeoff between performance and robustness.

ACKNOWLEDGEMENT

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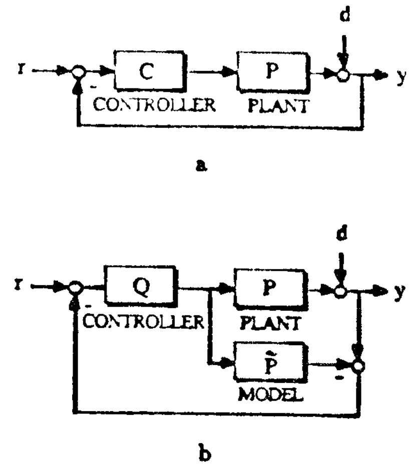

Figure 1. One-degree-of-freedom structure.

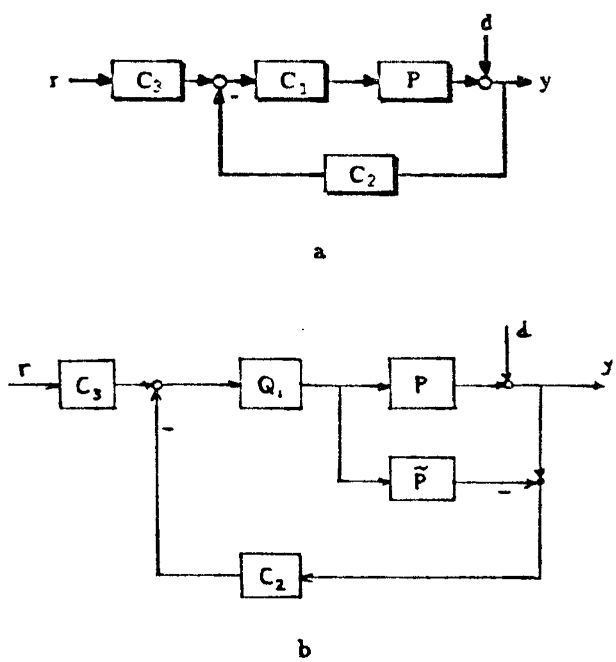

Figure 2. Two-degree-of-freedom structure.

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Design of the IMC Filter by Using the Structured Singular Value Approach

by

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DESIGN OF THE IMC FILTER BY USING THE STRUCTURED SINGULAR VALUE APPROACH

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Abstract

The Internal Model Control (IMC) structure has been widely recognized as very useful in clarifying the issues related to the mismatch between the model used for controller design and the actual process. The structure also gives rise to a two step controller synthesis procedure, of which the second step deals with the design of a low pass filter such that robustness with respect to model-plant mismatch is guaranteed. The Structured Singular Value (SSV) was introduced recently and it allowed the non-conservative quantification of the concept of robust performance. This paper deals with the design of the IMC filter by using the SSV and it demonstrates how this approach can be used with either an H_2- or an $H_\infty-$ optimal controller.

1. Internal Model Control

The IMC structure (Fig.1a), introduced by Garcia and Morari (1982) is mathematically equivalent to the classical feedback structure (Fig.1b). The IMC controller Q and the feedback C are related through

$$Q = C(I + \tilde{P}C)^{-1} \qquad (1.1)$$

$$C = Q(I - \tilde{P}Q)^{-1} \qquad (1.2)$$

where $\tilde{P}$ is the process model.

The advantage of using IMC can be seen by examining the structure for $P = \tilde{P}$ and for $P \neq \tilde{P}$.

$P = \tilde{P}$.

In this case the overall transfer function connecting the set-points r and disturbances d to the errors $e = y - r$, where y are the process outputs, is

$$e = y - r = (I - PQ)(d - r) \stackrel{\text{def}}{=} \tilde{E}(d - r) \qquad (1.3)$$

Hence the IMC stucture becomes effectively open-loop (Fig.2a) and the design of Q is simplified. Note that the IMC controller is identical to the parameter of the Q-parametrization (Zames, 1981). Also the addition of a diagonal filter F by writing

$$Q = \tilde{Q}F \qquad (1.4)$$

introduces parameters (the filter time constants) which can be used for adjusting on-line the speed of response for each process output.

$P \neq \tilde{P}$.

The model-plant mismatch generates a feedback signal in the IMC stucture which can cause performance deterioration or even instability. Since the relative modeling error is larger at higher frequencies, intuitively the addition of the low-pass filter F (Fig.2b) also adds robustness characteristics into the control system. In this case the closed-loop transfer function is

$$e = y - r = (I - P\tilde{Q}F)(I - (P - \tilde{P})\tilde{Q}F)^{-1}(d - r) \stackrel{\text{def}}{=} E(d - r) \qquad (1.5)$$

Hence the IMC structure gives rise rather naturally to a two step design procedure:

Step 1: Design $\tilde{Q}$, assuming $P = \tilde{P}$.

Step 2: Design F so that the closed-loop characteris-

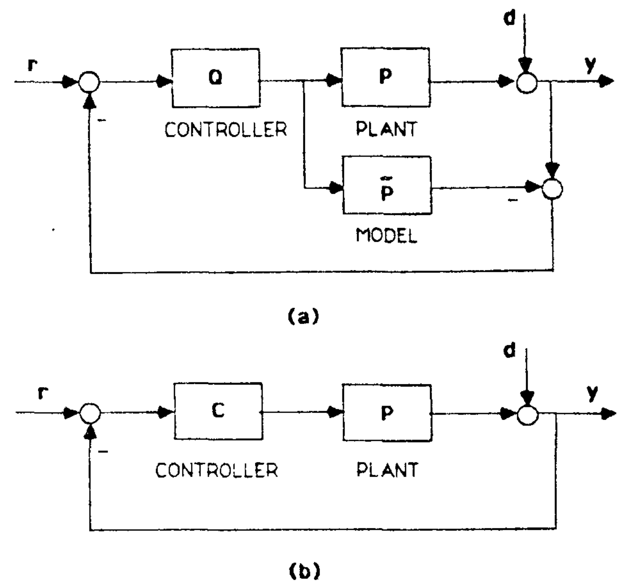

Figure 1.

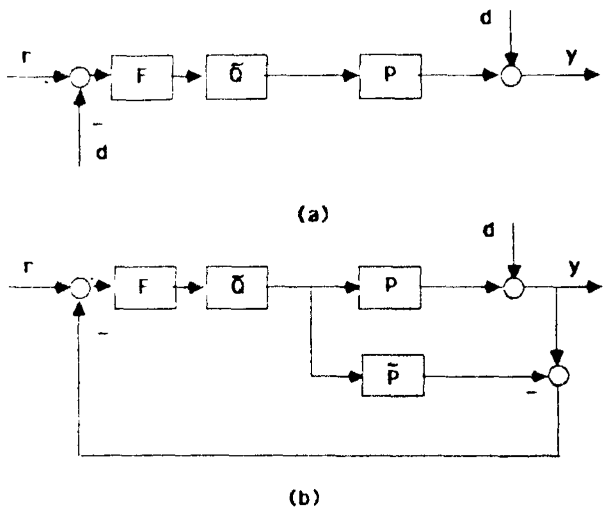

Figure 2.

tics that $\tilde{Q}$ produces in Step 1, are preserved in the presence of model-plant mismatch ($P \neq \tilde{P}$).

Finally note that the feedback controller C, given from (1.2), includes integral action if and only if Q inverts at steady-state the model $\tilde{P}$, i.e. *iff*

$$\tilde{Q}(\omega = 0) = \tilde{P}(\omega = 0)^{-1} \tag{1.6}$$

$$F(\omega = 0) = I \tag{1.7}$$

2. Structured Singular Value.

The SSV was introduced by Doyle (1982) and it allows the derivation of conditions for robust performance and stability for general structures of model uncertainty. For a constant complex matrix M the definition of the SSV $\mu_{\mathbf{\Delta}}(M)$ depends also on a certain set $\mathbf{\Delta}$. Each element Δ of $\mathbf{\Delta}$ is a block diagonal complex matrix with a specified dimension for each block, i.e.

$$\mathbf{\Delta} = \left\{diag(\Delta_1, \Delta_2, ..., \Delta_n) | \Delta_j \in \mathbf{C}^{m_j \times m_j}\right\} \tag{2.1}$$

Then

$$\frac{1}{\mu_{\mathbf{\Delta}}(M)} = \min_{\Delta \in \mathbf{\Delta}} \{\bar{\sigma}(\Delta) | det(I - M\Delta) = 0\} \tag{2.2}$$

and $\mu_{\mathbf{\Delta}}(M) = 0$ if $det(I - M\Delta) \neq 0 \quad \forall \Delta \in \mathbf{\Delta}$. Note that $\bar{\sigma}$ is the maximum singular value of the corresponding matrix.

Details on how the SSV can be used for studying the robustness of a control system can be found in Doyle (1985), where a discussion of the computational problems is also given. For three or fewer blocks in each element of $\mathbf{\Delta}$, the SSV can be computed from

$$\mu_{\mathbf{\Delta}}(M) = \inf_{D \in \mathbf{D}} \bar{\sigma}(DMD^{-1}) \tag{2.3}$$

where

$$\mathbf{D} = \{diag(d_1 I_{m_1}, d_2 I_{m_2}, ..., d_n I_{m_n}) | d_j \in \mathbf{R}_+\} \tag{2.4}$$

and I_{m_j} is the identity matrix of dimension $m_j \times m_j$. For more than three blocks, (2.3) still gives an upper bound for the SSV.

3. Filter Design

3.1. Block Structure

In order to effectively use the SSV for designing F, some rearrangement of the block structure is necessary. The IMC structure of Fig.1a can be written as that of Fig.3a, where $v = d - r$, $e = y - r$ and

$$G = \begin{pmatrix} 0 & 0 & \tilde{Q} \\ I & I & \tilde{P}\tilde{Q} \\ -I & -I & 0 \end{pmatrix} \tag{3.1.1}$$

where the blocks 0 and I have appropriate dimensions.

The structure in Fig.3a can always be transformed into that in Fig.3b, where Δ is a block diagonal matrix with the additional property that

$$\bar{\sigma}(\Delta) \leq 1 \qquad \forall \omega \tag{3.1.2}$$

The superscript u in G^u denotes the dependance of G^u not only on G but also on the specific uncertainty description available for the model $\tilde{P}$. Only some of the more common types will be covered here to demonstrate how this is done, but it is straightforward to apply the same concepts to other types of uncertainty descriptions, like parametric uncertainty.

i) Multivariable Additive Uncertainty.

The information on the model uncertainty is of the form

$$\bar{\sigma}(P - \tilde{P}) \leq l_a(\omega) \tag{3.1.3}$$

where l_a is a known function of frequency. In this case we can easily write $P - \tilde{P} = l_a \Delta$ where $\bar{\sigma}(\Delta) \leq 1$ and so obtain

$$G^u = G^a = \begin{pmatrix} l_a I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \tag{3.1.4}$$

ii) Multivariable Input Multiplicative Uncertainty.

$$\bar{\sigma}(\tilde{P}^{-1}(P - \tilde{P})) \leq l_i(\omega) \tag{3.1.5}$$

where l_i is known. Then

$$G^u = G^i = G \begin{pmatrix} l_i \tilde{P} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{3.1.6}$$

iii) Multivariable Output Multiplicative Uncertainty.

$$\bar{\sigma}((P - \tilde{P})\tilde{P}^{-1}) \leq l_o(\omega) \tag{3.1.7}$$

$$G^u = G^o = \begin{pmatrix} l_o \tilde{P} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \tag{3.1.8}$$

iv) Element by Element Additive Uncertainty.

For each element p_{ij} of P we have

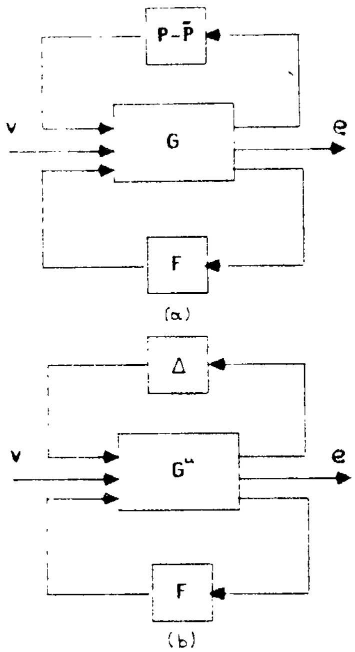

Figure 3.

$$|p_{ij} - \tilde{p}_{ij}| \le l_{ij}(\omega), \qquad i = 1,...,n; \quad j = 1,...,n \quad (3.1.9)$$

Then

$$P - \tilde{P} = J_1 \Delta L J_2 \qquad (3.1.10)$$

where

$$L = diag(l_{11}, l_{12}, ..., l_{1n}, l_{21}, ..., l_{nn}) \qquad (3.1.11)$$

$$J_1 = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 & \dots & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 & \dots & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & \dots & 1 & \dots & 1 \end{pmatrix} \qquad (3.1.12)$$

$$J_2 = \begin{pmatrix} I_n \\ I_n \\ \vdots \\ I_n \end{pmatrix} \qquad (3.1.13)$$

From (3.1.10) it follows that

$$G^u = G^{ebe} = \begin{pmatrix} LJ_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \begin{pmatrix} J_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \qquad (3.1.14)$$

Note that all the above relations yield a G^u already partitioned as

$$G^u = \begin{pmatrix} G^u_{11} & G^u_{12} & G^u_{13} \\ G^u_{21} & G^u_{22} & G^u_{23} \\ G^u_{31} & G^u_{32} & G^u_{33} \end{pmatrix} \qquad (3.1.15)$$

Then Fig.3b can be written as Fig.4 with

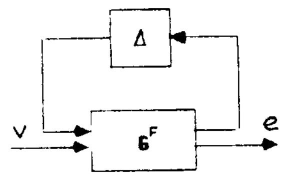

Figure 4.

$$G^F = \begin{pmatrix} G^u_{11} & G^u_{12} \\ G^u_{21} & G^u_{22} \end{pmatrix} + \begin{pmatrix} G^u_{13} \\ G^u_{23} \end{pmatrix} (I - FG^u_{33})^{-1} F \begin{pmatrix} G^u_{31} & G^u_{32} \end{pmatrix}$$
$$\stackrel{def}{=} \begin{pmatrix} G^F_{11} & G^F_{12} \\ G^F_{21} & G^F_{22} \end{pmatrix} \qquad (3.1.16)$$

3.2. Robustness Conditions

3.2.1. Robust Stability

The system is stable for any of the plants in the set defined from the bounds on the model uncertainty, if and only if (Doyle,1985)

$$\mu_{\mathbf{\Delta}}(G^F_{11}) < 1 \qquad \forall \omega \qquad (3.2.1)$$

3.2.2. Robust Performance

For performance, two cases will be examined; the $H_\infty-$ and the H_2-optimal. First the definitions of the L_2-norm of a vector and of the $L_\infty-$norm of a matrix will be given:

$$||v||_2 = \left(\int_{-\infty}^{+\infty} v^*(i\omega) v(i\omega) \quad d\omega \right)^{1/2} \qquad (3.2.2)$$

$$||G||_\infty = \sup_{||v||_2 = 1} ||Gv||_2 = \sup_\omega \bar{\sigma}(G(i\omega)) \qquad (3.2.3)$$

where the superscript $*$ indicates complex conjugate transpose.

i) $H_\infty-$optimal.

In this case, the IMC controller $\tilde{Q}$ designed in the first step, can be obtained by solving (Zames and Francis,1983,1984; Chang and Pearson, 1984; Doyle *et al*,1984):

$$\min_{\tilde{Q}} ||w\tilde{E}||_\infty \qquad (3.2.4)$$

where $\tilde{E}$ was defined in (1.3) and w is a weight reflecting the frequency range of interest for the external system input v ($v = d$ for $r = 0$; $v = -r$ for $d = 0$).

In the second step of the IMC design we wish to keep $\bar{\sigma}(wE)$ bounded by a known boung $b(\omega)$ in spite of modeling error, i.e.

$$||b^{-1}wE||_\infty < 1 \qquad \forall \Delta \in \mathbf{\Delta} \qquad (3.2.5)$$

Note that $E = \tilde{E}$ when $P = \tilde{P}$. The value of $\bar{\sigma}(w\tilde{E})$ for the optimal $\tilde{Q}$ obtained from (3.2.4) can serve as a guideline for the selection of the shape of $b(\omega)$. Then (Doyle,1985)

$$||b^{-1}wE||_\infty < 1 \qquad \forall \Delta \in \mathbf{\Delta} \iff \sup_\omega \mu_{\mathbf{\Delta}^0}(G^b) < 1 \qquad (3.2.6)$$

where

$$G^b = \begin{pmatrix} I & 0 \\ 0 & b^{-1}wI \end{pmatrix} G^F \qquad (3.2.7)$$

$$\mathbf{\Delta}^0 = \left\{ diag(\Delta, \Delta^0) | \Delta \in \mathbf{\Delta}, \Delta^0 \in \mathbf{C}^{n \times n} \right\} \qquad (3.2.8)$$

ii) H_2-optimal.

In the first step of the IMC design procedure, $\tilde{Q}$ is obtained by solving (Youla *et al*, 1976; Frank,1974; also see Morari *et al*,1986)

$$\min_{\tilde{Q}} ||W\tilde{E}v||_2 \qquad (3.2.9)$$

for a specified input v, which can be either a set-point r or a disturbance d. Note that $||W\tilde{E}v||_2$ is actually the Integral Squared Error (ISE), $||We||_2$ for this particular input v, where W is a diagonal matrix weighting each element of the error vector e differently. Also note that if one wishes the control system to behave well with both set-points and disturbances of different frequency content, then one has to implement a two-degree of freedom controller (see e.g. Morari *et al*,1986), each part of which is designed as presented here and in the corresponding references.

In the second step, the IMC filter F is designed so that the ISE ($||WEv||_2$) remains small even in the presence of model-plant mismatch. The following Theorem quantifies this objective.

Theorem 1:

For a specified v define

$$G^x \stackrel{\text{def}}{=} \begin{pmatrix} I & 0 \\ 0 & xW \end{pmatrix} G^F \begin{pmatrix} I & 0 \\ 0 & v \end{pmatrix} \qquad (3.2.10)$$

where x is a scalar function of ω and the blocks 0 have the appropriate dimensions (in general non-square). Augment G^x, which is in general a "tall" matrix, to obtain a square matrix:

$$G^x_{full} = \begin{pmatrix} G^x & 0 \end{pmatrix} \qquad (3.2.11)$$

Then

$$\mu_{\Delta^\circ}(G^x_{full}(i\omega)) = 1 \iff x(\omega) = x_0(\omega) \qquad \forall\omega \quad (3.2.12)$$

defines a function x_0 of frequency and

$$\sup_{\Delta\in\boldsymbol{\Delta}} ||WEv||_2 = ||x_0^{-1}||_2 \qquad (3.2.13)$$

Proof: For a matrix K partitioned as

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \qquad (3.2.14)$$

define

$$R(K,\Delta) \stackrel{\text{def}}{=} K_{22} + K_{21}\Delta(I - K_{11}\Delta)^{-1}K_{12} \qquad (3.2.15)$$

Then the transfer function relating v to e in Fig.4 is $R(G^F,\Delta)$ and since Fig.1a and Fig.4 are equivalent, we get by using (1.5)

$$E = R(G^F,\Delta) \qquad (3.2.16)$$

The properties of the SSV and (3.2.12) imply (Doyle,1985) that

$$\sup_{\Delta\in\boldsymbol{\Delta}} \bar{\sigma}(R(G^{x_0}_{full},\Delta)) = 1 \qquad (3.2.17)$$

From (3.2.10), (3.2.11), (3.2.15), (3.2.16), it follows after some algebra that

$$R(G^{x_0}_{full},\Delta) = \begin{pmatrix} x_0WEv & 0 \end{pmatrix} \qquad (3.2.18)$$

Then from (3.2.17),(3.2.18) and the definition of the singular values, it follows, since x_0WEv is a vector:

$$\sup_{\Delta\in\boldsymbol{\Delta}} (x_0^2 v^* E^* W^* W E v) = 1 \qquad \forall\omega$$

$$\Longrightarrow \sup_{\Delta\in\boldsymbol{\Delta}} \int_{-\infty}^{+\infty} v^*E^*W^*WEv \quad d\omega \quad = \int_{-\infty}^{+\infty} x_0^{-2} \quad d\omega$$

$\iff$ (3.2.13) $\qquad$ *QED*

Note that as it turned out $x_0^{-1} = \sup_{\Delta\in\boldsymbol{\Delta}} \bar{\sigma}(WEv)$, but the only way to compute it is through (3.2.12). Also without loss of generality x can be assumed to be positive since the value of $\mu_{\Delta^\circ}(G^x_{full})$ depends only on $|x|$.

An alternative to the above objective for designing F would be to design F with an H_∞ type of objective, even though $\tilde{Q}$ was obtained as an H_2−optimal controller in the first step of the IMC procedure. This is an interesting possibility that became available because of the two step IMC procedure and which experience showed to be of practical value. The idea behind it is that although one may expect inputs v of a particular type for which $\tilde{Q}$ is designed, one may still want to add some robustness characteristics not only with respect to model-plant mismatch but also with respect to different external inputs v entering the system. In this case one can select in (3.2.5) $w = 1$ and use as a guideline for selecting $b(\omega)$ the value of $\bar{\sigma}(\tilde{E})$ for the optimal $\tilde{Q}$ obtained from (3.2.9). From that point on, the procedure for designing F is the same as that described in the rest of this paper for the H_∞ type design.

3.3. Filter Parameter Optimization

The filter parameters can now be computed so that the robustness conditions that were discussed in §3.2 are satisfied. To do so, some structure will have to be assumed for F, which can be of any general type that the designer wishes. However in order to keep the number of variables in the optimization problem small, a rather simple structure like a diagonal F with first or second order terms would be recommended. In most cases this is not restrictive because the potentially higher orders of the model $\tilde{P}$ have been included in the controller $\tilde{Q}$ that was designed in the first step of the IMC procedure and which is in general a full matrix. The use of a full matrix F may be necessary in cases of extremely ill-conditioned systems ($\bar{\sigma}(\tilde{P})/\underline{\sigma}(\tilde{P})$ very large), but as mentioned the designer can specify such a structure for F if he so wishes. Note that F should still satisfy (1.7) for integral action. Also some additional restrictions on the filter exist in the case of an open-loop unstable plant (see Morari *et al*, 1986). Hence

$$F \stackrel{\text{def}}{=} F(s;\Lambda) \qquad (3.3.1)$$

where Λ is an array with the filter parameters. For example if an F of the form

$$F = \begin{pmatrix} 1/(\lambda_2 s^2 + \lambda_1 s + 1) & s/(\lambda_3 s + 1)^2 \\ s/(\lambda_4 s + 1)^2 & 1/(\lambda_5 s + 1) \end{pmatrix}$$

were selected, then $\Lambda = (\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5)^*$

3.3.1. Problem Formulation

The problems can now be formulated as minimization problems over the elements of the array Λ. A constraint is that Λ should be such that F is a stable transfer function. However the problem can be turned into an unconstrained one by writing the denominator of each element of F as a product of polynomials of degree 2 and one of degree 1 if the order is odd, with the constant terms of the polynomials equal to 1. Then the stability requirement translates into the requirement that the coefficients (elements of Λ) are positive, which is a constraint that can be eliminated by writing λ_k^2 instead of λ_k for the corresponding filter parameters.

i) H_∞.

In this case the goal if to satisfy (3.2.6). The filter parameters can be obtained by solving

$$\min_{\Lambda} \sup_{\omega} \mu_{\Delta^0}(G^b) \qquad (3.3.2)$$

It may be however that the optimum values for (3.3.2), still do not manage to satisfy (3.2.6). The reason may be that an F with more parameters is required, but more often that the performance requirements set by the selection of $b(\omega)$ in (3.2.5) are too tight to satisfy in the presence of model-plant mismatch. In this case one should choose a less tight bound b and resolve (3.3.2). Note that satisfaction of the Robust Performance condition (3.2.6) implies satisfaction of the Robust Stability condition (3.2.1) as well.

ii) H_2.

The objective is to minimize (3.2.13) for a specified v (set-point or disturbance). Hence the filter parameters are obtained by solving

$$\min_{\Lambda} ||x_0^{-1}||_2 \qquad (3.3.3)$$

An additional problem here is the computation of $x_0(\omega)$ for a given Λ. This computation will be made through (3.2.12) and (2.3) will be used for computing μ. The following Theorem simplifies the problem.

Theorem 2:

Let

$$M^x = \begin{pmatrix} M_{11} & M_{12} \\ xM_{21} & xM_{22} \end{pmatrix} \qquad (3.3.4)$$

where x a positive scalar.

Then $\inf_{D \in \mathbf{D}} \bar{\sigma}(DM^x D^{-1})$ is a non-decreasing function of x, where $\mathbf{D} = \{diag(D_1, D_2)\}$.

Proof: Let $0 < x_2 \le x_1$. Then we can write $x_2 = x_1\beta$, where $0 < \beta \le 1$. From (3.3.4) we have

$$DM^{x_2}D^{-1} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \beta I \end{pmatrix} M^{x_1} D^{-1}$$
$$= \begin{pmatrix} I & 0 \\ 0 & \beta I \end{pmatrix} DM^{x_1}D^{-1} \qquad (3.3.5)$$

Then the properties of the singular values yield

$$(3.3.5) \Longrightarrow \bar{\sigma}(DM^{x_2}D^{-1}) \le \bar{\sigma}\begin{pmatrix} I & 0 \\ 0 & \beta I \end{pmatrix} \quad \bar{\sigma}(DM^{x_1}D^{-1})$$
$$\Longrightarrow \bar{\sigma}(DM^{x_2}D^{-1}) \le \bar{\sigma}(DM^{x_1}D^{-1}) \qquad \forall D \in \mathbf{D}$$
$$\Longrightarrow \inf_{D \in \mathbf{D}} \bar{\sigma}(DM^{x_2}D^{-1}) \le \inf_{D \in \mathbf{D}} \bar{\sigma}(DM^{x_1}D^{-1}) QED$$

Note that G^x_{full} is a special case of M in the Theorem and so Theorem 2 applies to (3.2.12).

3.3.2. Computational Issues

i) H_∞.

The computation of μ in (3.3.2) is made through (2.3); details can be found in Doyle (1982). As it was pointed out in Doyle (1985), the minimization of the Frobenious norm instead of the maximum singular value yields D's which are very close to the optimal ones for (2.3). Note that the minimization of the Frobenious norm is a very simple task. In the computation of the supremum in (3.3.2) only a finite number of frequencies is considered. Hence (3.3.2) is transformed into

$$\min_{\Lambda} \max_{\omega \in \Omega} \inf_{D \in \mathbf{D}^0} \bar{\sigma}(DG^b D^{-1}) \qquad (3.3.6)$$

where Ω is a set containing a finite number of frequencies and $\mathbf{D}^0$ is the set corresponding to Δ^0 according to (2.1) and (2.4). Define

$$\Phi_\infty(\Lambda) \stackrel{\text{def}}{=} \max_{\omega \in \Omega} \inf_{D \in \mathbf{D}^0} \bar{\sigma}(DG^b D^{-1}) \qquad (3.3.7)$$

The analytic computation of the gradient of Φ_∞ with respect to Λ is in general possible. This is not the case when the two or more largest singular values of DG^bD^{-1} are equal. However this is quite uncommon and although the computation of a generalized gradient is possible, experience has shown the use of a mean direction to be satisfactory. A similar problem appears when the $\max_{\omega \in \Omega}$ is attained at more than one frequencies, but again the use of a mean direction seems to be sufficient. We shall now proceed to obtain the expression for the gradient of $\Phi_\infty(\Lambda)$ in the general case.

Assume that for the value of Λ where the gradient of $\Phi_\infty(\Lambda)$ is computed, the $\max_{\omega \in \Omega}$ is attained at $\omega = \omega_0$ and that the $\inf_{D \in \mathbf{D}^0} \bar{\sigma}(DG^b(i\omega_0)D^{-1})$ is obtained at $D = D_0$, where only one singular value σ_1 is equal to $\bar{\sigma}$. Let the singular value decomposition (SVD) be

$$D_0 G^b(i\omega_0) D_0^{-1} = (u_1 \quad U) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \Sigma \end{pmatrix} \begin{pmatrix} v_1^* \\ V^* \end{pmatrix} \qquad (3.3.8)$$

Then for the element of the gradient vector corresponding to the filter parameter λ_k we have under the above assumptions:

$$\frac{\partial}{\partial \lambda_k} \Phi_\infty = \frac{\partial}{\partial \lambda_k} \sigma_1(D_0 G^b(i\omega_0) D_0^{-1}) \qquad (3.3.9)$$

because $\nabla_{D_0}(\sigma_1) = 0$ since we are at an optimum with respect to the D's. To simplify the notation use

$$A = D_0 G^b(i\omega_0) D_0^{-1} = U_A \Sigma_A V_A^* \quad (3.3.10)$$

By using the properties of the SVD we obtain from (3.3.8)

$$AA^* = U_A \Sigma_A^2 U_A^* \Rightarrow u_1^* \frac{\partial}{\partial \lambda_k}(AA^*)u_1 = u_1^* U_A \frac{\partial}{\partial \lambda_k}(\Sigma_A^2) U_A^* u_1$$

$$\Rightarrow u_1^*(\frac{\partial}{\partial \lambda_k}(A)A^* + A\frac{\partial}{\partial \lambda_k}(A^*))u_1 = u_1^* U_A(2\Sigma_A \frac{\partial}{\partial \lambda_k}(\Sigma_A))U_A^* u_1$$

$$\Rightarrow u_1^* \frac{\partial}{\partial \lambda_k}(A)v_1\sigma_1 + \sigma_1 v_1^* \frac{\partial}{\partial \lambda_k}(A^*)u_1 = 2\sigma_1 \frac{\partial}{\partial \lambda_k}(\sigma_1)$$

$$\Rightarrow \frac{\partial}{\partial \lambda_k}(\sigma_1) = \mathrm{Re}\left[u_1^* \frac{\partial}{\partial \lambda_k}(D_0 G^b(i\omega_0) D_0^{-1})v_1\right] \quad (3.3.11)$$

Use of (3.3.9), (3.1.16), (3.2.7), (3.3.11), and of the property

$$\frac{d}{dz}(M(z)^{-1}) = -M(z)^{-1}\frac{d}{dz}(M(z))M(z)^{-1} \quad (3.3.12)$$

where M(z) is a matrix, yields after some algebra

$$\frac{\partial}{\partial \lambda_k}\Phi_\infty = \mathrm{Re}\left[u_1^* D_0 \begin{pmatrix} G_{13}^u \\ b^{-1} w G_{23}^u \end{pmatrix} (I - FG_{33}^u)^{-1} \frac{\partial}{\partial \lambda_k}(F(i\omega_0)) (I - FG_{33}^u)^{-1} \begin{pmatrix} G_{31}^u & G_{32}^u \end{pmatrix} D_0^{-1} v_1\right] \quad (3.3.13)$$

where F, G_{ij}^u, b, w are computed at $\omega = \omega_0$. The derivatives of F with respect to its parameters (elements of Λ) depend on the particular form that the designer selected and they can be easily computed.

ii) H_2.

• The first issue in this case is the computation of x_0. Note that this computation has to be made at every frequency ω. In practice only a set Ω with a finite number of frequencies is used, from which $||x_0^{-1}||_2$ can be computed approximately. Theorem 2 indicates that any basic descent method should be sufficient. The fact that it is possible to obtain an analytic expression for the gradient of $\mu_{\Delta^o}(G_{full}^z(i\omega))$ with respect to x, simplifies the problem even further. This is possible when (2.3) is used for the computation of μ and the two largest singular values of $DG_{full}^z D^{-1}$ for the optimal D's at the value of x where the gradient is computed, are not equal to each other. If this not the case a mean direction can be used as mentioned in the H_∞ case above.

Let the $\inf_{D\in\mathbf{D}^o} \bar{\sigma}(DG_{full}^z(i\omega)D^{-1})$ be attained for $D_0 = D_0(\omega; x)$ and let σ_1 be the maximum singular value and u_1, v_1 the corresponding singular vectors. Then the same steps for obtaining (3.3.11) are valid. Hence by using (3.2.10) and (3.2.11) we get after some algebra

$$\frac{\partial}{\partial x}\left(\mu_{\Delta^o}(G_{full}^z(i\omega))\right) = \mathrm{Re}\left[u_1^* D_0 \begin{pmatrix} 0 & 0 & 0 \\ WG_{21}^F & WG_{22}^F v & 0 \end{pmatrix} D_0^{-1} v_1\right] \quad (3.3.14)$$

• The second computational issue is the solution of (3.3.3). To obtain the gradient of $||x_0^{-1}||_2$ with respect to the filter parameters, we need to compute the gradient of $x_0(\omega)$ with respect to these parameters for every frequency $\omega \in \Omega$. From the definition of x_0 in (3.2.12) we see that as some filter parameter λ_k changes, $x_0(\omega)$ will also change so that $\mu_{\Delta^o}(G_{full}^z(i\omega))$ remains constantly equal to 1. Hence we can write

$$\frac{\partial \mu}{\partial x_0}\frac{\partial x_0}{\partial \lambda_k} + \frac{\partial \mu}{\partial \lambda_k} = 0 \Longrightarrow \frac{\partial x_0}{\partial \lambda_k} = -\frac{\partial \mu}{\partial \lambda_k} / \frac{\partial \mu}{x_0} \quad (3.3.15)$$

where μ is computed through (2.3). The denominator in the right hand side of (3.3.15) is given from (3.3.14). As for the numerator, it can be computed in the same way as (3.3.11) and (3.3.13) but with G_{full}^z instead of G^b:

$$\frac{\partial}{\partial \lambda_k}\left(\mu_{\Delta^o}(G_{full}^z(i\omega))\right) = \mathrm{Re}\left[u_1^* D_0 \begin{pmatrix} G_{13}^u \\ xWG_{23}^u \end{pmatrix} (I - FG_{33}^u)^{-1} \frac{\partial}{\partial \lambda_k}(F(i\omega))(I - FG_{33}^u)^{-1} \begin{pmatrix} G_{31}^u & G_{32}^u v & 0 \end{pmatrix} D_0^{-1} v_1\right] \quad (3.3.16)$$

Hence $\partial x_0 / \partial \lambda_k$ can be computed from (3.3.14), (3.3.15), (3.3.16).

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