SRC TR 87-138

Singular Perturbation Method for the Solution of Kushner's Equation

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July 1987

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This research was sponsored by National Science Foundation under the grant CDR-85-00108.

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Abstract

In this paper we solve asymptotically Kushner's equation for the conditional probability density function of an one dimensional diffusion process measured in a low noise channel. We obtain the Stratonovich version and solve asymptotically this equation. The asymptotic solution agrees with the asymptotic solution of Zakai's equation.

In the second part of this paper we solve asymptotically Kushner's equation for a model of feedback channel and construct a sub-optimal filter for this model.

Key Words:

Kushner's equation, Stratonovich form, sub-optimal filter, feedback channel, diffusion process.

Singular Perturbations Methods for the Solution of Kushner's Equation

1. Introduction

In this paper we apply singular perturbations methods to solve Kushner's equation for the following model – a one dimensional diffusion process $x(t)$ transmitted through a low noise channel, our model is described by the Itô equations

$$dx(t) = m(x(t))dt + \sigma dw(t) \tag{1.1}$$

$$dy(t) = h(x)dt + \rho d\vartheta(t) \tag{1.2}$$

where $w(t)$ and $\vartheta(t)$ are independent standard Wiener processes, $m(x)$ is assumed to be an analytic function of x with polynomial growth at infinity and $h(x)$ is analytic monotone function of x, $h(x) > 0.$ for all $-\infty < x < \infty$.

The parameter ρ, the channel noise intensity is assume to be small. In terms of communications theory this asymption corresponds to high signal to noise ratio (SNR).

The filtering problem is to find the probability density function of $x(t)$ conditioned on the measurements $(y(s),\ 0 \le s \le t)$.

The conditional probability density of $x(t)$

$$P(x, t.\varrho) = P(x(t) = x/(y_0^t)$$

satisfies Kushner's equation ([KU1], [LS1]).

$$dP = LPdt + \frac{(h - \hat{h})P(dy - \hat{h}dt)}{\rho^2} \tag{1.3}$$

where $\hat{h}$ is the conditional expectation

$$\hat{h} = \int_{-\infty}^{\infty} h(x) P(x, t/y_0^t) dx. \tag{1.4}$$

So equation (1.3) is nonlinear integro differential equation. (1.3) is written in the Itô sense. An unnormalyzed version $\phi(x,t,\rho)$ of $P(x,t,\rho)$ is known to satisfy Zakai's equation [Z1]

$$d\phi(x,t,\rho) = L\phi dt + \frac{\phi h}{\rho^2} dy(t). \tag{1.5}$$

The model (1.1) – (1.2) had been studied recently, in [KBS1] the case where $h(x) = x$, that is linear channel, had been discussed. Using the Itô version of Kushner equation, a suboptimal filters were constructed. The suboptimal filters converge to the optimal one as $\rho \to 0$. But the method described in [KBS1] suffers from two problems – first it gives only finite number of approximations and not a consistant way of getting a sequence of approximated filters, and the method is valid only for linear measurements, and can't be expanded to the non-linear case.

The nonlinear case had been studied in [KBS2], [YBS1], [SBS1] where the Stratonovich form of Zakai's equation had been solved asymptotically as $\rho \longrightarrow 0$.

In this paper we apply similar methods for the solution of Kushner's equation for the case of nonlinear measurements. We solve Kushner's equation asymptotically using the Stratonovich version. The Stratonovich form of Kushner's equation appeared in [ST1], we derive this equation using the Wang Zakai correction [WZ1] for the Zakai's equation. Note that applying the Wang Zakai correction to Kushner's equation itself involves differentiation of nonlinear terms like $\hat{h}p$ with respect to p.

In the 2§ we derive the Stratonovich version of Kushner equation using the Zakai's equation. We believe that this derivation had not been published before.

In 3§ we solve Kushner's equation for the model (1.1) – (1.2) asymptotically as $\rho \to 0$. We use similar technique as in [KBS2] for Zakai's equation for the same model. The two asymptotic expansions agree up to a fartor that is a function of t and ρ - the normalization factor. Thus the same sequence of suboptimal filters that had been constructed using Zakai's equation, can be constructed using Kushner's equation.

Finally in 4§ we give example for which we use Kushner's equation in the Itô sense. In some engineering models the transmitted signal is function of $x - \hat{x}(t)$, in communication theory, we often use the following model to decrease bits per seconds transmition needed (see for example [A])

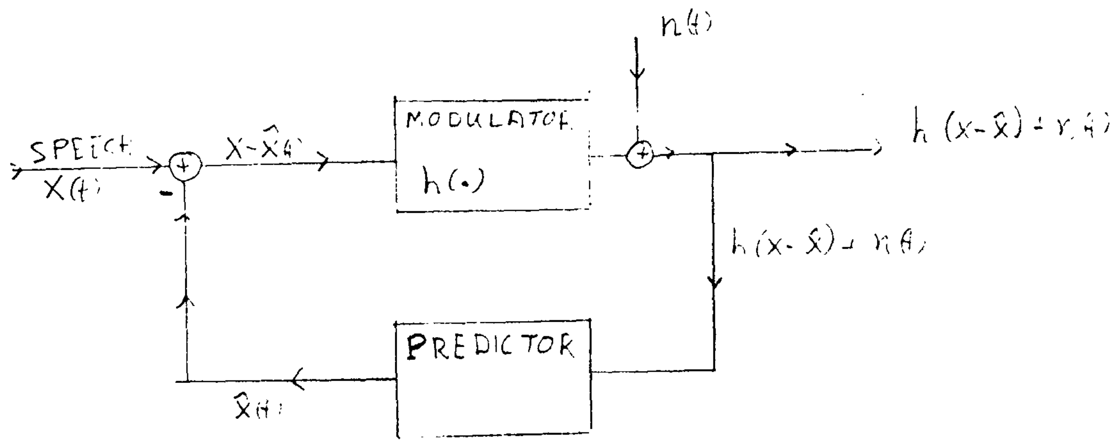

Figure 1. Block diagram of transmitter

The input to the filter is $h(x - \hat{x})$ and an adaptive noise, we describe the model by the following

$$dx = m(x)dt + \sigma dw(t) \tag{1.6}$$

$$dy = h(x - \hat{x})dt + \rho dv(t) \tag{1.7}$$

Where $h(\cdot)$ is an analytic function of it's argument. Since h is a function of $\hat{x}$ as well as x, Zakai's equation turns to be nonlinear equation as well as Kuhsner's equation, we solve Kushner's equation asymptotically for this case, and construct the first order sub-optimal filter for this model.

2. The Stratonovich form of Kushner's Equation

Consider again the model (1.1) – (1.2)

$$dx = m(x)dt + \sigma dw(t)$$

$$dy = h(x)dt + \rho d\vartheta(t),$$

denote by $p(x,t) = p(x(t) = x/y_o^t)$ the conditional probability density function and $\phi(x,t)$ the unnormalized version of p that obeys Zakai's equation, we have

$$p(x,t) = \phi(x,t) / \int_{-\infty}^{\infty} \phi(x,t)dx. \tag{2.1}$$

denote by

$$\eta(t) = \int_{-\infty}^{\infty} \phi(x,t)dx. \tag{2.2}$$

we have

$$p(x,t) = \phi(x,t)/\eta(t). \tag{2.3}$$

The Zakai's equation in the Stratonovich sense is given by the following (see [KBS2])

$$d\rho(x,t) = \left(-\frac{\partial}{\partial x}m(x)\phi + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\phi(x,t) - \frac{h^2\phi}{2\rho^2}\right)dt + \frac{\phi h dy}{\rho^2}. \tag{2.4}$$

First we write down the differential equation for $\eta(t)$, for that goal we assume that

$$\lim_{x\to\pm\infty}\frac{\partial\phi}{\partial x}(x,t) = 0 \qquad \forall t. \tag{2.5}$$

and that

$$\lim_{x\to\pm\infty} m(x)\phi(x,t) = 0. \tag{2.6}$$

(otherwise $\hat{m}(x)$ does not exist).

$$d\eta(t) = d\int_{-\infty}^{\infty}\rho(x,t)dx$$

changing the order of integration and differentiation, using integration by parts and (2.5), (2.6) we obtain

$$d\eta(t) = -\eta\frac{\hat{h}^2}{2\rho^2}dt + \eta\frac{\hat{h}}{\rho^2}dy(t) \tag{2.7}$$

(note that $\hat{F} = \int_{-\infty}^{\infty} F(x,t)\, p(x,t)dx/\eta(t)$)

so

$$\begin{aligned} dp &= d(\phi(x,t)/\eta(t)) = \\ \frac{d\phi}{\eta} - \frac{\phi d\eta}{\eta^2} &= \left(-\frac{\partial}{\partial x}mp + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}p - \frac{h^2 p}{2\rho^2}\right)dt \\ &\quad + \frac{ph}{\rho^2}dy + p\frac{\hat{h}^2}{2\rho^2}dt - \frac{p\hat{h}}{\rho^2}dy. \end{aligned} \tag{2.8}$$

or

$$dp = Lpdt - \frac{p}{2\rho^2}(h^2dt - \hat{h}^2dt) + \frac{p}{\rho^2}(h-\hat{h})dy(t). \tag{2.9}$$

or

$$(2.10) \qquad dp = Lpdt - \frac{p}{2\rho^2}\left((h-\hat{h})^2 - \widehat{(h-\hat{h})^2}\right)dt + \frac{(h-\hat{h})p(dy-\hat{h}dt)}{\rho^2}$$

equation 2.9 (or 2.10) is the Stratonovich version of Kushner's equation.

3. Asymptotic Expansion for Kushner's equation

In this section we construct an asymptotic expansion for the solution of Kushner's equation (2.9) for the model (1.1) – (1.2). We assume for simplicity that t is sufficiently large so that initial layers have decayed. Following the procedure of [KBS2] we adopt the followign ansatz

$$(3.1) \qquad p(x,t,\rho) = K(t,\rho)\exp(-\psi(x,t,\rho)/\rho)$$

as $p(x,t,\rho)$ is a density function, we have

$$(3.2) \qquad K(t,\rho)^{-1} = \int_{-\infty}^{\infty} \exp(-\psi(x,t,\rho/\rho)dx. \qquad \forall\rho > 0, \ \forall t.$$

Assume that for every t and ρ, ψ has a unique $\tilde{x}(t,\rho)$ such that

$$\psi(\tilde{x}(t,\rho),t,\rho) < \psi(x,t,\rho) \qquad \forall x \neq \tilde{x}.$$

Without losing generality we may assume that

$$\psi(\tilde{x},t,\rho) = 0$$

(Otherwise we take this term, as a function of t and ρ, into the K), and because of (3.2) it is clear that

$$(3.3) \qquad \psi(x,t,\rho) = \sum_{k=2}^{\infty} \frac{\psi^k(\tilde{x},t,\rho)}{k!}(x-\tilde{x})^k$$

Denote by

(3.4) $$q_k(t,\rho) = \psi^{(k)}(\tilde{x}(t,\rho),t,\rho) \qquad k = 2,3,\ldots$$

and

(3.5) $$\sqcap = 1/q_2(t,\rho)$$

We have the following

(3.6) $$p(x,t,\rho) = k(t,\rho)e^{-\left(\frac{(x-\tilde{x})^2}{2\sqcap} - q_3\frac{(x-\tilde{x})^3}{6} + q_4\frac{(x-\tilde{x})^4}{n!} + \ldots\right)/\rho}$$

Using the Laplace expansion for the integral we have the following approximations.

(3.7) $$\hat{x} = \int_{-\infty}^{\infty} xp(x,t,\rho)dx = \tilde{x} + \rho k\frac{q_3\sqcap^2}{2}\sqrt{\sqcap\rho}\sqrt{\frac{\pi}{2}} + \ldots$$

Note that

(3.8) $$k(t,\rho) = \frac{\sqrt{2}}{\sqrt{\pi\sqcap q}} + \ldots.$$

and we can express $\hat{x}$ and $k(t,\rho)$ up to any order in ρ as a function of $\sqcap$ and $\{q_i\}_{i=3}^{\infty}$. It is easy to verify that

(3.9) $$\begin{aligned}\hat{h} &= \int_{-\infty}^{\infty} h(x)p(x,t,\rho)dx = \\ &= h(\tilde{x}) + \rho k\frac{\sqrt{\pi\rho\sqcap}}{\sqrt{2}}\left(\frac{\sqcap h"(\tilde{x})}{2} - \frac{h'(\tilde{x})q_3\sqcap^2}{2}\right) + \ldots\end{aligned}$$

(3.10) $$\hat{h^2} = h^2(\tilde{x}) + \rho\left[-2h(\tilde{x})h'(\tilde{x})\frac{q_3}{2}\sqcap^2 + \frac{\sqcap}{2}(h'(\tilde{x})^2 + h'h"(\tilde{x})_2)\right]k\frac{\sqrt{\pi\rho\sqcap}}{\sqrt{2}} + \ldots$$

Substitute (3.1) in (2.9) we have

(3.11) $$\begin{aligned}K_t - \frac{K\psi_t}{\rho} &= K\left(-m + \frac{m\psi_x}{\rho} + \frac{1}{2}\sigma^2\left(-\frac{\psi_{xx}}{\rho} + \frac{\psi_x^2}{\rho^2}\right)\right) - \frac{K}{2\rho^2}(h^2 - \hat{h^2}) \\ &\quad + \frac{K}{\rho^2}(h - \hat{h})\frac{dy}{dt}(t)\end{aligned}$$

On the minimal path, $x = \tilde{x}$ we have

$$\begin{aligned}
\frac{K_t}{K} - \frac{\psi_t(\tilde{x})}{\rho} &= -m'(\tilde{x}) - \frac{1}{2}\sigma^2 \frac{\psi_{xx}(\tilde{x})}{\rho} \\
&\quad -\frac{1}{2\rho^2}\left[(\frac{\sqcap}{2}h_1^2 + h_1 h_2) - 2h\frac{h_1' q_3 \sqcap^2}{2}\right]\rho + \dots \\
&\quad +\frac{1}{\rho^2}\left[\rho(\frac{\sqcap h_2}{2} - \frac{h_1 q_3}{2}) + \dots\right]\frac{dy}{dt}.
\end{aligned} \tag{3.12}$$

where $h_i = \frac{\partial^i h}{\partial x^i}(\tilde{x})$, differentiate (3.11) with respect ot x we obtain

$$\begin{aligned}
-\frac{\psi_{x,t}}{\rho} &= -m"(x) + \frac{m'\psi_x}{\rho} + m\frac{\psi_{xx}}{\rho} + \frac{1}{2}\sigma^2\left(\frac{\psi_{xxx}}{\rho} + 2\frac{\psi_x\psi_{xx}}{\rho^2}\right) \\
&\quad -\frac{1}{2\rho^2}2hh' + \frac{1}{\rho^2}h'\frac{dy}{dt}.
\end{aligned} \tag{3.13}$$

Substitute the minimal path, using the identity

$$\begin{aligned}
0 &= \dot{\psi}_x = \psi_{xx} + \psi_{xx}\dot{\tilde{x}} \\
\frac{\dot{\tilde{x}}}{\rho\sqcap} &= -m'(\tilde{x}) + \frac{m}{\sqcap\rho} + \frac{1}{2}\sigma^2(\frac{q_3}{\rho}) - 2\frac{hh'}{\rho^2} + \frac{h'}{\rho^2}\frac{dy}{dt}
\end{aligned}$$

or, multiply by $\sqcap\rho$ we obtain

$$\dot{\tilde{x}} = -\rho\sqcap m"(\tilde{x}) + m + \frac{1}{2}\sigma^2 \sqcap q_3 - \frac{\sqcap h'}{\rho}\left(h - \frac{dy}{dt}\right) \tag{3.14}$$

equation (3.14) is the equation for the minimum path. The first order approximation is

$$\dot{\tilde{x}} = \frac{\sqcap h'(\tilde{x})}{\rho}\left(\frac{dy}{dt} - h(\tilde{x})\right). \tag{3.15}$$

In order to get approximated expansion for $\sqcap$ we differentiate (2.15) one more time

$$\begin{aligned}
-\psi_{xxt} &= -\rho m''' + \frac{m''\psi_x}{\rho} + 2\frac{m'\psi_{xx}}{\rho} + m\psi_{xxx} \\
&\quad +\frac{1}{2}\sigma^2\left(\psi_{xxxx} + 2\frac{\psi_{xx}^2}{\rho} + 2\frac{\psi_x\psi_{xxx}}{\rho}\right) - \frac{h'^2 + hh''}{\rho} + \frac{h''\dot{y}}{\rho}
\end{aligned} \tag{3.16}$$

using

$$\dot{\psi}_{xx} = \psi_{xxt} + \psi_{xxx}\dot{\tilde{x}} \tag{3.17}$$

we have

$$\begin{aligned} - \quad & \dot{\psi}_{xx} + q_3\dot{\tilde{x}} = -\rho m''' + \frac{2m'}{\sqcap} + mq_3 + \frac{1}{2}\sigma^2\left(q_4 + \frac{2}{\rho\sqcap^2}\right) \\ - \quad & \frac{h'^2 + hh''}{\rho} + \frac{h''\dot{y}}{\rho} \end{aligned}$$

but

$$\dot{q}_2 = (\dot{\frac{1}{\sqcap}}) = -\frac{\dot{\sqcap}}{\sqcap^2} \tag{3.18}$$

so

$$\begin{aligned} \frac{\dot{\sqcap}}{\sqcap^2} &= -\frac{q_3}{\rho}\sqcap h'\left(\frac{dy}{dt} - h\right) - \rho m''' + \frac{2m'}{\sqcap} + mq_3 \\ &\quad + \frac{1}{2}\sigma^2\left(q_4 + \frac{2}{\rho\sqcap^2}\right) - \frac{h'^2 + hh''}{\rho} + \frac{h''\dot{y}}{\rho}. \end{aligned} \tag{3.19}$$

Equation (3.19) is a stochastic equation for $\sqcap$ that involves q_3 and p_4, by proceeding as above, that is, differentiate (3.16) with respect to x and substitute the minimal path $x = \tilde{x}$, we obtain an infinite system of differential equation for $q_k(t,\rho)$, $(k \geq 3)$. For $k = 3$ we have

$$\begin{aligned} \dot{q}_3 &= -\frac{3}{\rho}\left(\frac{\sigma^2 q_3}{\sqcap} - h'h''\right) + \frac{1}{2}\sigma^2 q_5 - 3m'q_3 - 3\frac{m''}{\sqcap} + \frac{1}{2}\sigma^2 \sqcap q_3 q_4 \\ &\quad + \rho\left[m^{(h)} - q_4 \sqcap m''\right] + \frac{1}{\rho}\left[q_4 \sqcap h' - h'''\right](\dot{y} - h(x)) \end{aligned} \tag{3.20}$$

Following [KBS2] we expend the $\tilde{x}(t), \sqcap(t)$ and $q_k(t)$ $(k \geq 3)$ in the form

$$\begin{cases} \tilde{x}(t) \sim x_o(t) + \sum_{i=1}^{\infty} \rho^{(i+1)/2} x_i(t) \\ \sqcap(t) \sim r_o(t) + \sum_{i=1}^{\infty} \rho^{i/2} \sqcap_i(t) \\ q_k(t) \sim q_{k,o}(t) + \sum_{i=1}^{\infty} \rho^{i/2} q_{k,i}(t). \end{cases} \tag{3.21}$$

where the leading terms are given by

$$(3.22)\qquad \begin{cases} \sqcap_o(t) = \sigma/h'(x_o(t) \\ q_{k,o}(t) = h^{(k)}(x_o(t))/\sigma \end{cases}$$

Since $\hat{x}(t) = \tilde{x}(t) + 0(\rho)$, the first order filter is given by

$$(3.23)\qquad dx^*(t) = \sigma(dy - h(x^*(t)dt)/\rho.$$

The filter (3.24) is written in the Itô sense, this is the "constant gain" filter, this is a very easy to implement filter, however, for the performances of this filter see [BG1].

Using (3.7), (3.8) and the second order solution of $\tilde{x}$, and $\sqcap$ and the first order term in q_3 we obtain the second order filter as follows

$$(3.24)\qquad \begin{cases} dx^* = \left[-\frac{\sigma^2 h''(x^*)}{2h'(x^*)} + m(x^*)\right] dt + \sqcap_1^*(t)h'(x^*)\frac{dy - h(x^*)dt}{\rho} \\ d\sqcap_1^* = \frac{\sigma^2 - \sqcap_1^{*2} h'^2(x^*)}{\rho} dt. \end{cases}$$

Proceed as above, we can write infinite sequence of sub-optimal filters whose mean square estimation error (MSEE) are going closer to the error of the optimal filter (see [KBS2]).

4. Asymptotic analysis of the feedback channel

In this section we apply singular perturbations method for the solution of Kushner's equation for the case of feedback channel. The observation process here is a function of the optimal estimation too, in the following way – denote by $\hat{x}(t)$ the maximum square error estimator (MSEE), then the observation process is a function of $x - \hat{x}(t)$.

The signal that we analyze is the first order Bathoworth signal that is govern by the following Itô equation

$$dx(t) \quad = \quad -\beta x(t)dt + \sqrt{2\beta} dW_1(t)$$

$$x(0) = x_0 \tag{4.1}$$

where β is a constant (the band "width") and $W_1(t)$ is a standard Wiener process, x_o is a random variable with a given distribution, the observation process is given by

$$\begin{aligned} dy(t) &= h(x - \hat{x}(t))dt + \rho d\vartheta(t) \\ y(0) &= 0 \end{aligned} \tag{4.2}$$

where $h(\cdot)$ is an analytic function of $x - \hat{x}$, and $\vartheta(t)$ is another standard Wiener process independent of $w(t)$, ρ is a small paramter $0 < \rho < 1$. Kushner's equation for the conditional probability density function is given by

$$dp = (\beta x p_x + \beta p + \beta p_{xx})dt + \frac{(h - \hat{h})}{\rho^2} p(dy - \hat{h}dt) \tag{4.3}$$

in the Itô sense, or

$$\dot{p} = \beta x p_x + \beta p + \beta p_{xx} - \frac{p}{2\rho^2}\left((h - \hat{h})^2 - (\widehat{h - \hat{h})^2}\right) + \frac{p}{\rho^2}(h - \hat{h})(\dot{y} - \hat{h}) \tag{4.4}$$

in the Stratonovich sense. (where we use the notation of the ordinary calculus for the Stratonovich sense). We solve (4.3) asymptotically as $\rho \to 0$, we take for simplicity t large enough so that initial layers have decayed, $\hat{h}$ is the conditional expectation of h, and in general, for a function $F(x,t)$ we denote

$$\hat{F}(t, \rho) = \int_{-\infty}^{\infty} F(x,t)p(x,t,\rho)dx. \tag{4.5}$$

If the integral exists, using Kushner's equation, it is easy to obtain stochastic differential equation for $\hat{F}$, (see [J1]), for $F(x,t) = x$ we have the following

$$d\hat{x}(t) = -\beta\hat{x}(t)dt + \frac{\widehat{xh} - \hat{x}\hat{h}}{\rho^2}(dy - \hat{h}dt). \tag{4.6}$$

Denote by dJ the innovation process

$$dJ = \frac{dy - \hat{h}dt}{\rho}$$

it is well known that $J(t)$ is a standard Wiener process with respect to the same sigma algebra as $\vartheta(t)$.

Next we assume that $p(x,t,\rho)$ has the following form

$$p(x,t,\rho) = p(x - \hat{x}(t), t, \rho). \tag{4.7}$$

using Itô formula we differentiate p to obtain

$$dp(x - \hat{x}, t, \rho) = \frac{\partial p}{\partial \hat{x}} d\hat{x} + \frac{\partial p}{\partial t} dt + \frac{1}{2}\frac{\partial^2 p}{\partial \hat{x}^2} \cdot \left(\frac{\widehat{xh} - \hat{x}\hat{h}}{\rho}\right)^2 dt. \tag{4.8}$$

where $d\hat{x}$ is given by (4.6). Comparing (4.8) and (4.3), and using the following two relations

$$\frac{\partial p}{\partial \hat{x}} = -\frac{\partial p}{\partial x} \tag{4.9}$$

$$\frac{\partial^2 p}{\partial \hat{x}^2} = \frac{\partial^2 p}{\partial x^2} \tag{4.10}$$

We end up with the following equation for $p(x - \hat{x}, t, \rho)$

$$\begin{aligned} &\left(-p + (x - \hat{x})p_x + \left[\frac{\sigma^2}{2} - \frac{(\widehat{xh} - (\hat{x}\hat{h}))^2}{2\rho^2}\right] p_{xx} - p_t\right) dt \\ &+ \left(\frac{\widehat{xh} - \hat{x}\hat{h}}{\rho} p_x + p\frac{h - \hat{h}}{\rho}\right) dJ = 0 \end{aligned} \tag{4.11}$$

We solve (4.11) asymptotically as $\rho \to 0$, following [KBS1] we adapt the following ansatz.

$$p(x - \hat{x}, t, \rho) = \rho^{\frac{\alpha}{2}} K(x - \hat{x}, t, \rho) \exp(-\psi(x - \hat{x}, t, \rho)/\rho^{\alpha}) \qquad \alpha > 0 \tag{4.12}$$

where K and ψ are regular functions in $x - \hat{x}$, that is

$$K(x - \hat{x}, t, \rho) = \sum_{i=0}^{\infty} k_i(t,\rho)(x - \hat{x})^i \tag{4.13}$$

and

$$\psi(x-\hat{x},t,\rho)=\sum_{i=Z}^{\infty}A_i(t,\rho)(x-\hat{x})^i \tag{4.14}$$

such that

$$\lim_{\rho\to 0}k_i(t,\rho)<\infty$$

and

$$\lim_{\rho\to 0}A_i(t,\rho)<\infty.$$

Note that the sum in (4.14) starts at $i=2$ because $p(x-\hat{x},t,\rho)$ should satisfy

$$\int_{-\infty}^{\infty}p(x-\hat{x},t,\rho)dx=1 \qquad \text{for all } t \text{ and } \rho. \tag{4.15}$$

Substitute (4.12), (4.13) and (4.14) in (4.15) we obtain

$$\int_{-\infty}^{\infty}\rho^{\frac{\alpha}{2}}\left(\sum_{i=0}^{\infty}k_i(x-\hat{x})^i\right)\exp\left(-\sum_{i=Z}A_i(x-\hat{x})^i/\rho^{\alpha}\right)dx=1 \tag{4.16}$$

Using the Laplace expansion of the integral, expand the left hand side of (4.15) as a series in $\sqrt{\rho}$, and writing the right hand side as

$$1+\sum_{i=1}^{\infty}\rho^{\frac{i\alpha}{2}}\cdot 0=1 \tag{4.17}$$

we obtain a sequence of relations between A_i and k_i, more specific, the zero order term (in ρ) gives

$$k_0(t,\rho)=\sqrt{A_2(t,\rho)/\pi}, \tag{4.18}$$

the first order term is identically zero, but the second one yields

$$k_2=\frac{3a_3^2k_0}{8a_2^2}+\frac{3k_0a_4}{2a_2} \tag{4.19}$$

and so on. On the other hand, by definition

$$\widehat{(x-\hat{x})} = 0 \tag{4.20}$$

so

$$\int_{-\infty}^{\infty} (x-\hat{x}) P(x-\hat{x}, t, \rho) dx = 0 \tag{4.21}$$

expand (4.21) as we did for (4.16), using the Laplace expansion of the integral, we obtain another sequence of relations between a_i and the k_i, the first is

$$\frac{3a_3 k_0}{2a_2} = k_1 \tag{4.22}$$

and so on.

Next we choose α. We apply the technique used by [BZ1] to obtain lower and upper bounds for the error $\widehat{(x-\hat{x})^2}$ and conclude that

$$\widehat{(x-\hat{x})^2} = 0(\rho) \tag{4.23}$$

Expand

$$\widehat{(x-\hat{x})^2} = \int_{-\infty}^{\infty} (x-\hat{x})^2 p(x-\hat{x}, t, \rho) dx$$

as Laplace expansion of the integral we choose $\alpha = 1$, and in general we have:

for n even $n \geq 2$

$$\begin{aligned} \widehat{(x-\hat{x})^n} &= \int_{-\infty}^{\infty} (x-\hat{x})^n p(x-\hat{x}, t, \rho) dx = \rho^{\frac{n}{2}} \frac{(n-1)!!}{(2a_2)^{n/2}} + \\ &+\rho^{\frac{n}{2}+1} \left(-a_4 \frac{(n+3)!!}{(2a_2)^{2+\frac{n}{2}}} + \frac{a_3^2 (n+5)!!}{2(2a_2)^{3+n/2}} - \frac{k_1 a_3 (n+3)!!}{(2a_2)^{2+n/2}} \sqrt{\frac{\pi}{a_2}} \right. \\ &\left. + k_2 \frac{(n+1)!!}{(2a_2)^{1+n/2}} \sqrt{\frac{\pi}{a_2}} \right) + 0\left(\rho^{\frac{n}{2}+2}\right). \end{aligned} \tag{4.24}$$

and for n odd

$$(4.25)\quad \widehat{(x-\hat{x})^n} = \rho^{\frac{n+1}{2}}\left(k_1\frac{n!!}{(2a_2)^{(n+1)/2}}\sqrt{\frac{\pi}{a_2}}-\frac{a_3}{(2a_2)^{(3+n)/2}}\right)+ +\rho^{\frac{n+3}{2}}\left(k_3\frac{(n+2)!!}{(2a_2)^{(n+3)/2}}\sqrt{\frac{\pi}{a_2}}+\frac{a_5(n+4)!!}{(2a_2)^{(n+5)/2}}+ +\frac{2a_3a_4(n+6)!!}{(2a_2)^{(n+7)/2}}-\frac{a_3^3(n+8)!!}{(2a_2)^{(n+9)/2}}\right)+0\left(\rho^{\frac{n+5}{2}}\right).$$

We are interesting in asymptotic solution of (4.11) as $\rho \to 0$, for that aim, because of (4.23) we define a new set of variables

$$(4.26)\qquad x,t \longrightarrow u,t$$

where

$$(4.27)\qquad u=(x-\hat{x}(t))/\sqrt{\rho}.$$

Using (4.24), (4.25) and (4.27) it is easy to verify the following asymptotic equalities

$$(4.28)\qquad h-\hat{h}=L_{21}+u\sqrt{\rho}h_1+h_2\rho u^2+h_3\rho\sqrt{\rho}u^3+u^4(\quad)+\ldots$$

where $L_{21} \triangleq h(0)-\hat{h}$

$$(4.29)\quad \widehat{xh}-\hat{x}\hat{h} = \rho\frac{h_1}{2a_2}+\rho^2\left(h_1\left(\frac{105a_3^2}{32a_2^4}-\frac{15a_4}{8a_2^3}-\frac{15a_3k_1}{8a_3^3}\sqrt{\frac{\pi}{a_2}}-\right.\right. \left.\left.-\frac{3k_2}{4a_2^2}\sqrt{\pi a_2}\right)+h_2\left(\frac{3k_1}{4a_2^2}\sqrt{\frac{\pi}{a_2}}-\frac{15a_3}{8a_2^3}\right)+\frac{3h_3}{4a_2^2}\right)+0(\rho^3)\triangleq L_{22}$$

Substitute (4.24) – (4.28) in (4.11) and expand everything as a power series in u, we obtain the following equation:

$$(4.30)\quad \left\{\frac{\dot{a}_2}{2\sqrt{a_2\pi}}+u\sqrt{\rho}\left[\frac{3\dot{a}_3}{2\sqrt{\pi a_2}}-\frac{3a_3\dot{a}_2}{4a_2\sqrt{\pi a_2}}\right]+u^2\left[\rho k_{2,t}-k_o\dot{a}_2\right]\right.$$

$$+u^3[\qquad]+\ldots+(k_o+\sqrt{\rho}uk_1+\rho u^2k_2+\ldots)+$$

$$\sqrt{\rho}u\left(k_1+u\left(\frac{-2a_2k_o}{\sqrt{\rho}}+2k_2\sqrt{\rho}\right)+u^2[\qquad]+\ldots\right]+$$

$$+\frac{1}{2}\left(\sigma^2-\frac{L_{22}^2}{\rho^2}\right)\left[2k_2-\frac{2a_2k_o}{\rho}+u\left(6k_3\sqrt{\rho}-\frac{15a_3k_o}{\sqrt{\rho}}\right)+\right.$$
$$\left.+u^2\left(-27k_oa_4-\frac{87}{4}\frac{a_3^2k_o}{a_2}+\frac{4k_oa_2^2}{\rho}\right)+u^3[\qquad]+\ldots\right\}dt+$$
$$+\left\{(k_o+\sqrt{\rho}k_1u+\rho k_2u^2+\ldots)\left(\frac{L_{21}}{\rho}+\frac{uh_1}{\sqrt{\rho}}+h_2u^2+u^3h_3\sqrt{\rho}+u^4[\qquad]+\ldots\right)\right.$$
$$+\left[k_1+u\left(\frac{-2a_2k_o}{\sqrt{\rho}}+2k_2\sqrt{\rho}\right)+u^2(3k_3\rho-(3k_oa_3+2a_2k_1))\right.$$
$$\left.+u^3[\ldots]+\ldots\right]\left[\frac{L_{22}}{\rho}\right]dJ(t)=0$$

In order to solve asymptotically (4.30) we separate it into a sequence of stochastic differential equations for the coefficients of each power of u, that is from coefficients of u^o we get the following equation:

(4.31)
$$\left\{\frac{\dot{a}_2}{2\sqrt{\pi a_2}}+k_o+\frac{1}{2}\left[\sigma^2-\frac{L_{22}^2}{\rho^2}\right)\left[2k_2-\frac{2a_2k_o}{\rho}\right]\right\}dt+$$
$$\left\{\frac{k_oL_{21}}{\rho}+k_1\frac{L_{22}}{\rho}\right\}dJ(t)=0$$

similar equations can be obtained from any power of u, obviously the system is infinite coupled sequence of S.D.E. for the a_i and the k_i, indeed, using relations like (4.13), (4.19) and (4.22) we can get rid of the k_i and have a sequence of equations for a_i. The first equation (4.31) is for a_2, indeed all the others a_i appears in this equation, but the lugger i is, the coefficient of a_i is of higher power of ρ. This form let us to truncate the expansion and to construct sub-optimal finite dimensions filters.

Note that direct calculations yield that the terms in (4.28), (4.29) satisfy

(4.32)
$$L_{21}=-\left\{\rho\frac{h_2}{2a_2}+\rho^2\left[h_2\left(\frac{3a_3^2}{4a_2^4}-\frac{3a_4}{4a_2^3}\right)-\frac{3a_3h_3}{4a_2^3}\right.\right.$$
$$\left.+\frac{3h_4}{4a_2^2}\right]+\rho^3\left[F_{213}(a_2,a_3,a_4,a_5,a_6)\right]+\rho^4\left(F_{214}(a_2,\ldots,a_7,a_8)\right)$$
$$+0(\rho^5)$$

$$(4.33)\qquad L_{22} = \rho\frac{h_1}{2a_2} + \rho^2\left(h_1\left(\frac{2a_3^2}{4a_2^4} - \frac{3a_4}{4a_2^3}\right) - h_2\frac{3a_3}{4a_2^3} + \frac{3h_3}{4a_2^3}\right) + \rho^3 F_{223}(a_2,\ldots,a_5,a_6) + \rho^4 F_{224}(a_2,\ldots,a_7,a_8) + 0(\rho^5).$$

so

$$(4.34)\qquad L_{22}^2 = \rho^2\frac{h_1^2}{4a_2^2} + \rho^3\frac{h_1}{a_2}\left(h_1\left(\frac{2a_3^2}{4a_2^4} - \frac{3a_4}{4a_2^3}\right) - \frac{3a_3h_2}{4a_2^3} + \frac{3h_3}{4a_2^2}\right) + +\rho^4 F_{233}(a_2,\ldots,a_5,a_6) + \rho^5 F_{234}(a_2,\ldots,a_7,a_8) < 0(\rho^6).$$

where the F are rational functions.

Using (4.6) we construct a sub-optimal filter. Taking the zero order approximation only we obtain

$$(4.35)\qquad L_{21} \simeq -\rho\frac{h_2}{2a_2}$$

$$(4.36)\qquad L_{22} \approx \rho\frac{h_1}{2a_2}$$

$$(4.37)\qquad L_{22}^2 \approx \frac{\rho^2 h_1^2}{4a_2^2}$$

substitute (4.35) – (4.37) in (4.31) we obtain

$$(4.38)\qquad \left\{\frac{\dot{a}_2}{2\sqrt{\pi a_2}} + k_o + \frac{1}{2}\left(\sigma^2 - \frac{h_1^2}{4a_2^2}\right)\left(2k_2 - \frac{2a_2k_o}{\rho}\right)\right\}dt + + \left\{\frac{-k_oh_2}{2a_2} + \frac{k_1h_1}{2a_2}\right\}dJ = 0.$$

Looking at the high order terms in (4.38)

$$(4.39)\qquad \dot{a}_2 = \frac{2a_2^2}{\rho}\left(\sigma^2 - \frac{h_1^2}{4a_2^2}\right)$$

we see that the first order approximation of (4.38) is

$$(4.40)\qquad a_2 = \left|\frac{h_1}{2\sigma}\right|$$

so using (4.6) we obtain

$$
\begin{aligned}
(4.41) \qquad dx^* &= -\beta x^*(t)dt + \frac{h_1}{2a_2\rho}dy = \\
&= -\beta x^*(t)dt + \frac{\sigma}{\rho}dy.
\end{aligned}
$$

equation (4.41) is the zero order filter – the constant gain filter, similar constant gain filter for the non-feedback model is given in [KBS1], [KBS2] the two filters have the same gain but the drive terms are different.

Discussion and conclusions.

In this paper we derived the Stratonovich version for Kushner's equation. The Stratonovich version is easy to treat since it obeys the deterministic calculus rules. We use this version of Kuhsner's equation to solve asymptoticallly the filtering problem of an one dimensional diffusion process measured in a low noise channel. We solve the same model that was studied in [KBS2] using the Zakai's equation for the unnormalized conditional probability density function. The result agree with these in [KBS2].

In the second part of the work we solve Kushner's equation asymptotically for the feedback model, that is where the observation process is a function of $x - \hat{x}$ where $\hat{x}(t)$ is the optimal estimator at time t (with respect to minimum square error creterion) and we construct a simple constant gain filter to obtain $\hat{x}(t)$.

Acknowledgement

We thank Dr. B.Z. Bobrovsky for introducing some of the questions studied here.

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