SRC TR 87-125

Local Bifurcation Control

by

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To appear in "Dynamical Systems Approaches to
Nonlinear Problems in Systems and Circuits",
M. Levi and F.M.A. Salam, Eds., Siam Publ.,
Philadelphia, 1987

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Abstract. Local feedback stabilization of bifurcated solution branches is studied. Two cases are considered: that in which the nominal system undergoes a Hopf bifurcation as a parameter is varied, and the case of a stationary bifurcation from a simple zero eigenvalue. For each case, results on the existence of a stabilizing feedback are given. Moreover, simple synthesis techniques for the stabilizing controllers are discussed. A concept of "proximity stabilization" is introduced as an alternative to stabilization in the ordinary sense for systems that are not locally stabilizable. A result is stated on the genericity of proximity stabilizability. Motivation for further research in several areas is given.

1. Introduction

A standard preliminary step in the analysis and design of control systems is the linearization of the model dynamics about a nominal reference trajectory. The analysis and/or design are then performed for the resulting approximate linear model. The success of this technique in many applications can be attributed to the result of Liapunov that, if the linearized system is locally asymptotically stable, so too is the original nonlinear model. This is the celebrated "principle of linearized stability." This principle holds for finite systems of ordinary differential equations, as well as for some infinite dimensional problems. Situations in which the local stability properties of the nonlinear model cannot be inferred from its linearization are referred to as *critical cases* in stability theory [15] .

* Department of Electrical Engineering and Systems Research Center, University of Maryland, College Park, MD 20742 USA. Research supported in part by the National Science Foundation under Grants ECS-84-04275 and CDR-85-00108, and by AFOSR-URI Grant No. AFOSR-87-0073

Early results on stability in critical cases were obtained by Liapunov [25] , Malkin [26] , Pliss [33] and Krasovskii [23] . This topic has continued to be an active area of investigation. Recently, the related question of stabilization in critical cases has received significant attention. The papers [12, 8, 35, 4, 1, 2, 16, 7] represent a broad spectrum of approaches to this problem.

Our aim in this lecture is to summarize some recent results of the author and J.-H. Fu [1, 2] on the closely related problem of local *bifurcation control*, to present an extension to one case where the results of [1, 2] do not apply, and to indicate some directions for further work in this area. By "local bifurcation control" we mean the local stabilization of bifurcated solutions. Two types of bifurcation from an equilibrium point of a one-parameter family of ordinary differential equations are considered. First, the standard Hopf bifurcation of a periodic solution; and second, stationary (or static) bifurcation, involving only equilibrium points. The connection between local stabilization in critical cases and local bifurcation control becomes transparent given some basic facts about bifurcations of equilibria of differential equations. There are advantages to analyzing these two problems simultaneously. Often results on stabilization in critical cases can be directly applied also to problems in the control of bifurcations. Correspondingly, local bifurcation control problems provide added motivation for the study of stabilization in critical cases.

Our approach to the local feedback stabilization problem has the novel feature that it facilitates the derivation of generally valid analytical criteria for stabilizability, as well as specific stabilizing feedback controls. This is possible through use of bifurcation formulae which involve only Taylor series expansion of the vector field and eigenvector computations. The bifurcation formulae used in [1] for the study of control of Hopf bifurcations were obtained by Howard [19] by a harmonic balance approach using the Fredholm Alternative. These formulae significantly simplify similar formulae obtained by Hopf [18] . In our study of stationary bifurcations, we derive bifurcation formulae following the Projection Method, as outlined in Iooss and Joseph [21] . This is essentially the same technique used by Howard [19] in his study of Hopf bifurcations.

Previous work on the feedback control of bifurcations includes that of Mehra [28] and Mehra, Kessel and Carroll [29] . See also the account in Casti [10] . These studies tend to be concerned with the problem of globally *removing* bifurcations by state feedback. The general results apply only to stationary bifurcations, since they are obtained by appealing to a global implicit function theorem. This is in contrast to the *local* bifurcation control problems considered here, where one seeks only to modify the stability properties of the bifurcated solutions.

The development of the paper is as follows. The problems of local feedback stabilization and local bifurcation control are discussed in Section 2. In Section 3, the main results on control of Hopf bifurcations are recalled from [1] without proof. In Section 4 known results on stationary bifurcations are recalled, and bifurcation formulae are derived for this case. These results are applied to the stationary bifurcation control problem in Section 5, summarizing the main results of [2] . A notion of "proximity stabilization" is introduced in Section 6, and shown to be useful in problems for which no stabilizing control law exists in the usual sense. Section 7 contains two worked examples, one illustrating the results and the other indicating an area for further investigation related to the proximity stabilization issue of Section 6.

2. Problem Setting

Consider a one-parameter family of nonlinear control systems

$$\dot{x} = f_{\mu}(x, u) \tag{1}$$

where $x \in R^n$, u is a scalar control, μ is a real-valued parameter, and the vector field f_{μ} is sufficiently smooth. Suppose that for $u \equiv 0$ Eq. (1) has an equilibrium point $x_0(\mu)$ which depends smoothly on μ. In the sequel the system

$$\dot{x} = f_0(x, u), \tag{2}$$

which is simply (1) with $\mu = 0$, will also be of interest.

This paper is concerned with the synthesis of feedback controls $u = u(x)$ achieving certain stability properties for each of the descriptions (1) and (2).

The linearization of Eq. (2) at $x = 0$, $u = 0$ is given by

$$\dot{x} = Ax + bu \tag{3}$$

where $A := \frac{\partial f}{\partial x}(0,0)$ and $b := \frac{\partial f}{\partial u}(0,0)$. If the pair (A, b) is controllable,[1] then a standard linear systems result [22] asserts the existence of a linear feedback $u = -kx$ such that the resulting system $\dot{x} = (A - bk)x$ is asymptotically stable. Applying this feedback in the original nonlinear system (2) renders the origin locally asymptotically stable. Moreover, the same conclusion applies if the uncontrollable modes of (3) are asymptotically stable. In contrast, if (A, b) has an unstable uncontrollable mode, then the origin of (2) remains unstable regardless of the applied feedback. These considerations imply that the only essentially nonlinear situation encountered in local feedback stabilization occurs when some uncontrollable modes of (3) are pure imaginary, and any other uncontrollable modes are asymptotically stable. However, even if there are modes which are pure imaginary and controllable, it may be important to study the existence of *nonlinear* stabilizing feedback control laws [34, 31, 32] .

It is natural to question the utility of results giving nonlinear stabilizing feedback controls when a linear stabilizing control law exists. There are indeed several reasons for using nonlinear feedbacks, two of which are noted next. First, the effect of a linear feedback control designed to stabilize the linearized version of the critical system (2) on the one-parameter family of systems (1) may be difficult to determine. Indeed, at least for small feedback gains, one can expect that the bifurcation will reappear at a different value of the parameter μ. The stability of this new bifurcation is not easily determined. Hence, simply using a linear stabilizing feedback may be unacceptable if the goal is to stabilize a bifurcation and not merely to stabilize an equilibrium point for a fixed

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[1]This means that the state x of Eq. (3) can be "steered" from any initial condition x_0 to any prescribed terminal condition x_1 within a prespecified time, by appropriate choice of the control $u(t)$. Similarly, one speaks of controllability of modes of Eq. (3) (eigenvalues of A), depending on whether or not these modes are affected by state feedback.

parameter value. Second, it should not be surprising that in some situations a linear feedback which locally stabilizes an equilibrium may result in globally unbounded behavior, whereas nonlinear feedbacks exist which stabilize the equilibrium both locally and globally. For an example, see Moon and Rand [32] . Hence, even if stabilization, rather than bifurcation control, is the issue being studied, nonlinear feedback controls can be superior.

In the light of the foregoing discussion, it is appropriate to assume that the matrix A of Eq. (3) possesses at least one eigenvalue with zero real part. The type of results obtained will depend heavily on the number of eigenvalues of A which are assumed pure imaginary, their multiplicity, and whether they are zero or have nonzero imaginary parts. The results of this paper address situations in which either of the following two hypotheses is satisfied. The first implies that Eq. (1) undergoes a Hopf bifurcation to periodic solutions at $\mu = 0$ when $u \equiv 0$, while the second ensures that new stationary solutions of (1) bifurcate from $x_0(\mu)$ at $\mu = 0$ when $u \equiv 0$.

(H) Eq. (1) has an equilibrium $x_0(\mu)$ when $u = 0$. Furthermore, the linearization of (1) near x_0, $\mu = 0$ possesses a pair of simple, complex conjugate eigenvalues $\lambda_1(\mu)$, $\lambda_2(\mu) = \overline{\lambda_1(\mu)}$, with $\lambda_1(0) = i\omega_c$, $\omega_c > 0$, Re $\lambda_1'(0) \neq 0$, with the remaining eigenvalues $\lambda_3(0), \ldots, \lambda_n(0)$ in the open left half complex plane.

(S) Eq. (1) has an equilibrium $x_0(\mu)$ when $u = 0$. Furthermore, the linearization of (1) near x_0, $\mu = 0$ possesses a simple eigenvalue $\lambda_1(\mu)$ with $\lambda_1(0) = 0$, $\lambda_1'(0) \neq 0$, with the remaining eigenvalues $\lambda_2(0), \ldots, \lambda_n(0)$ in the open left half complex plane.

The assumption that $\lambda_1'(0) \neq 0$ is the familiar strict-crossing (transversality) condition introduced by Hopf [18] .

Two stabilization problems are considered in the sequel. Both are studied separately for hypotheses (H) and (S). In each case, one of these problems pertains to Eq. (1) and the other to Eq. (2). For Eq. (1), the goal is to ensure local asymptotic stability of the bifurcated solutions. This will be referred to as the *local Hopf bifurcation control problem* or the *local stationary bifurcation control problem*, depending on which hypothesis is in force. For the description (2), it is desired to solve the standard *local feedback stabilization problem* at the equilibrium point $x_0(0)$. Note that under either hypothesis (H) or (S), Eq. (2) with $u \equiv 0$ is an example of a critical nonlinear system since its linearization possesses an eigenvalue with zero real part.

3. Control of Hopf bifurcations

Under hypothesis (H), the Hopf Bifurcation Theorem asserts the existence of a one-parameter family $\{ p_\epsilon , 0 < \epsilon \leq \epsilon_0 \}$ of nonconstant periodic solutions of Eq. (1) emerging from $x = 0$ at $\mu = 0$. (This assumes $u \equiv 0$, of course.) Here ϵ is a measure of the amplitude of the periodic solutions and ϵ_0 is sufficiently small. The periodic solutions $p_\epsilon(t)$ have period near $2\pi\omega_c^{-1}$ and occur for parameter values μ given by a smooth function $\mu(\epsilon)$. Exactly one of the characteristic exponents of p_ϵ is near 0, and it is given by a real, smooth and even function

$$\beta(\epsilon) = \beta_2\epsilon^2 + \beta_4\epsilon^4 + \cdots \tag{4}$$

Moreover, $p_\epsilon(t)$ is orbitally asymptotically stable with asymptotic phase if $\beta(\epsilon) < 0$ but is unstable if $\beta(\epsilon) > 0$. Denote by β_{2K} the first nonvanishing coefficient in the expansion (4). Checking the sign of β_{2K} is sufficient for determining stability. Generically, $K = 1$ so that locally the stability of the bifurcated periodic solutions p_ϵ is typically decided by the sign of the coefficient β_2.

An algorithm for the computation of β_2 can be useful in the solution of local feedback stabilization problems under hypothesis (H). In [4] the evaluation of (a scaled version of) β_2 is performed using a formula which applies to two-dimensional systems. The original n-dimensional system is reduced to a two-dimensional system by appealing to the Center Manifold Theorem. Use is then made of the fact [9, 14, 27, 11] that the stability properties of an equilibrium on the center manifold coincide with its stability in R^n. In fact, the value of $\beta(\epsilon)$ is known [27, 17, 3] to be the same for the original and the reduced systems. The approach taken in this work (cf. Abed and Fu [1]) differs from that of [4] mainly in the choice of algorithm for computing β_2. The implications for the type of results one obtains are nontrivial.

Now suppose $\beta_2 \neq 0$. Besides locally determining the stability of the bifurcated periodic solutions $p_\epsilon(t)$, it is known that the sign of the coefficient β_2 also *determines the stability of the equilibrium $x_0(\mu)$ at criticality* (i.e. at $\mu = 0$). This fact implies that if a feedback control $u = u(x)$ can be found such that $\beta_2 < 0$ for the Hopf bifurcation occurring in the controlled system

$$\dot{x} = f_\mu(x, u(x)) \tag{5}$$

then the local feedback stabilization problem described in Section 2 is solved. Simply use the feedback $u = u(x)$ in Eq. (1). Indeed, such a feedback solves *both* the local smooth feedback stabilization problem for Eq. (1) *and* the local Hopf bifurcation stabilization problem for *any* parametrized version of (1) of the form (5). This establishes the connection between local feedback stabilization and Hopf bifurcation control.

Rewrite Eq. (1) in the series form

$$\dot{x} = L_0x + u\gamma + uL_1x + Q_0(x,x)$$
$$+ C_0(x,x,x) + \cdots \tag{6}$$

where the terms not written explicitly are of higher order in x, u and μ than those which are. Thus L_0 and L_1 are square matrices, γ is a constant vector, $Q_0(x,x)$ is a quadratic form generated by a symmetric bilinear form $Q_0(x,y)$ giving the second order (in x) terms at $u = 0$, $\mu = 0$, and $C_0(x,x,x)$ is a cubic form generated by a symmetric trilinear form $C_0(x,y,z)$ giving the third order (in x) terms at $u = 0$, $\mu = 0$. (Note that L_0 is simply A of Eq. (3), and γ corresponds to b.)

Denote by r the right (column) and by l the left (row) eigenvector of L_0 with eigenvalue $i\omega_c$. Normalize by setting the first component of r to 1 and then choose l so that $lr = 1$.

It is well known that only the quadratic and cubic terms occurring in a nonlinear system undergoing a Hopf bifurcation influence the value of β_2. Thus only the linear, quadratic and cubic terms in an applied feedback $u(x)$ have potential for influencing β_2. To simplify the analysis and to emphasize the influence of nonlinear terms in the feedback control, we require $u(x)$ to be of the form

$$u(x) = x^T Q_u x + C_u(x, x, x), \tag{7}$$

where Q_u is a real *symmetric* $n \times n$ matrix, and C_u is a cubic form generated by a scalar valued *symmetric* trilinear form.

Theorems 1 and 2 below give sufficient conditions for local stabilizability of a Hopf bifurcation by feedback of the form (7). Both are positive results. Theorem 1 applies under the hypothesis that $l\gamma \neq 0$, while in Theorem 2 one $l\gamma = 0$ is assumed. By the well known Popov-Belevitch-Hautus (PBH) eigenvector test for controllability [22] of modes of linear time-invariant systems, the former case corresponds to the critical modes being controllable for the linearized system (3), while in the latter case these modes are uncontrollable. The theorems were proved in [1] using the formula for β_2 derived by Howard [19] . Reference [1] also contains specific formulae for stabilizing feedback controls.

Theorem 1. *Let hypothesis* (H) *hold and assume that* $l\gamma \neq 0$. *That is, the critical eigenvalues are controllable for the linearized system. Then there is a smooth feedback* $u(x)$ *with* $u(0) = 0$ *which solves the local Hopf bifurcation control problem for Eq.* (1) *and the local feedback stabilization problem for Eq.* (2). *Moreover, this can be accomplished with only third order terms in* $u(x)$, *leaving the critical eigenvalues unaffected.*

Theorem 2. *Suppose that hypothesis* (H) *is satisfied and that* $l\gamma = 0$. *Then there is a smooth feedback* $u(x)$ *with* $u(0) = 0$ *which solves the local Hopf bifurcation control problem for Eq.* (1) *and the local feedback stabilization problem for Eq.* (2) *provided that*

$$\begin{aligned} 0 \neq \mathrm{Re}\, \{ & -2lQ_0(r, \frac{1}{2}L_0^{-1}\gamma) \\ & + lQ_0(r, \frac{1}{2}(2i\omega_c I - L_0)^{-1}\gamma) \\ & + \frac{1}{4}l[2L_1 r + L_1\bar{r}] \}. \end{aligned} \tag{8}$$

4. Stationary bifurcations: Analysis

Under the stationary bifurcation hypothesis (S), it is well known [19] that Eq. (1) exhibits a stationary (or static) bifurcation from x_0 at $\mu = 0$. That is, new stationary solutions (i.e. equilibrium points) bifurcate from x_0 at $\mu = 0$. The stability characteristics of the new solutions are intimately related to those of $x_0(\mu)$ at criticality, i.e. at $\mu = 0$. It is this intrinsic relationship that allows the joint consideration of local stabilization for Eq. (2) and bifurcation control for Eq. (1).

To establish this relationship and motivate the derivations to follow, consider a general one-parameter family of nonlinear ordinary differential equations

$$\dot{x} = F_\mu(x) \tag{9}$$

having an equilibrium point $x_0(\mu)$ at which hypothesis (S) holds. Then near $x_0(0)$, $\mu = 0$ in (x, μ) space there exists a locally unique curve of points $(x(\epsilon), \mu(\epsilon))$, distinct from the μ axis and passing through (0,0), such that for all sufficiently small $|\epsilon|$, $x(\epsilon)$ is an equilibrium point of (9) when $\mu = \mu(\epsilon)$. Moreover, the parameter ϵ may be chosen so that $x(\epsilon)$ and $\mu(\epsilon)$ are smooth.

Denote the series expansions of $\mu(\epsilon)$, $x(\epsilon)$ by

$$\mu(\epsilon) = \mu_1\epsilon + \mu_2\epsilon^2 + \cdots, \tag{10}$$

$$x(\epsilon) = x_1\epsilon + x_2\epsilon^2 + \cdots \tag{11}$$

respectively. Generically, $\mu_1 \neq 0$, and there is a second equilibrium point besides $x_0(\mu)$ for all small $|\mu|$. However, if $\mu_1 = 0$ and $\mu_2 > 0$ (resp. $\mu_2 < 0$), then there are two new equilibrium points, one for positive and negative values of ϵ. These occur only for sufficiently small positive (resp. negative) values of μ. The new equilibrium points also have an eigenvalue β which vanishes at $\mu = 0$, with a series expansion

$$\beta(\epsilon) = \beta_1\epsilon + \beta_2\epsilon^2 + \cdots \tag{12}$$

Moreover, the exchange of stability formula [19, 18]

$$\beta_1 = -\mu_1\lambda_I'(0) \tag{13}$$

holds. If $\mu_1 = 0$ and $\mu_2 \neq 0$, the appropriate exchange of stability formula is [18]

$$\beta_2 = -2\mu_2\lambda_I'(0). \tag{14}$$

(Note: Eqs. (13) and (14) may be derived using the Factorization Theorem in Iooss and Joseph [15, pp. 90-91].) Suppose $\lambda_I'(0) > 0$. Then these facts imply that supercritical solution branches are stable while subcritical branches are unstable.

The following result follows from an application of the Center Manifold Theorem to a suspended version of Eq. (9) at $x_0(0)$, $\mu = 0$.

Theorem 3. *Let hypothesis* (S) *hold. If* $\mu_1 \neq 0$, *then the equilibrium point* $x_0(0)$ *is unstable for Eq.* (9). *If* $\mu_1 = 0$ *and* $\mu_2 \neq 0$, *then* $x_0(0)$ *is asymptotically stable if* $\beta_2 < 0$ *but is unstable if* $\beta_2 > 0$.

Thus, the equilibrium point $x_0(0)$ will be assured asymptotically stable if one can arrange that $\beta_1 = 0$ and $\beta_2 < 0$. If explicit formulae can be derived for β_1 and β_2, this provides a starting point for the construction of locally stabilizing feedback controls for the critical system (2). In fact, by the exchange of stability formulae, it is clear that this also ensures the stability of the bifurcated stationary solution, by ensuring that the bifurcation is a supercritical *pitchfork bifurcation*. This is a desirable outcome, as compared to the *transcritical bifurcation* which would occur if $\beta_1 \neq 0$, in which the bifurcated equilibrium point is stable on one side of $\mu = 0$ and unstable on the other. Indeed, under hypothesis (S), a supercritical pitchfork bifurcation ensures that, even though the nominal equilibrium solution $x_0(\mu)$ loses stability as μ varies through 0, the new equilibrium solution attracts a neighborhood of initial conditions about $x_0(0)$. This can also be shown through an application of the Center Manifold Theorem and the theory of normal forms (see [14] for details).

Next, bifurcation formulae for Eq. (9) are derived. The results will be the main tool in the construction of stabilizing feedbacks for Eqs. (1) and (2) in Section 5. The Projection Method, as elaborated in [21] , will be employed in the derivation.

By assumption, the Jacobian matrix $D_x F_0(x_0(0))$ of (9) at criticality possesses a simple zero eigenvalue $\lambda_1(0)$. Denote by r (resp. l) the right column (resp. left row) eigenvector of the critical Jacobian matrix corresponding to this eigenvalue. Using the fact that 0 is a simple eigenvalue, it is not difficult to see that the vectors l and r may be chosen to have only real elements. To be more specific, set the first component of r to 1 and then choose l so that $lr = 1$.

Without loss of generality, assume that for small $|\mu|$ the known equilibrium point $x_0(\mu)$ of (9) is the origin, i.e. $x_0(\mu) \equiv 0$ for small $|\mu|$. This can always be achieved by a smooth change of variables $x \to x + x_0(\mu)$. Rewrite (9) in the series form

$$\begin{aligned} \dot{x} &= L(\mu)x + Q_\mu(x,x) + C_\mu(x,x,x) + \cdots \\ &= L_0x + \mu L_1x + \mu^2 L_2x + \cdots \\ &\quad + Q_0(x,x) + \mu Q_1(x,x) + \cdots \\ &\quad + C_0(x,x,x) + \cdots \end{aligned} \tag{15}$$

Here, $L(\mu)$, L_1, L_2 are $n \times n$ matrices, $Q_\mu(x,x)$, $Q_0(x,x)$, $Q_1(x,x)$ are vector valued quadratic forms generated by symmetric bilinear forms $Q_\mu(x,y)$, $Q_0(x,y)$, $Q_1(x,y)$, respectively, and $C_0(x,x,x)$ is a vector valued cubic form generated by a symmetric trilinear form $C(x,y,z)$. The terms not explicitly written in (15) are of higher order in x and μ than those which are.

A convenient outcome of this representation is the formula

$$\lambda_1'(0) = lL_1r. \tag{16}$$

See [19] or [21] for a proof.

If x is any real (unknown) solution of $F_\mu(x) = 0$, define the parameter ϵ by

$$\epsilon := lx, \tag{17}$$

and attempt a series expansion of the form

$$\begin{pmatrix} x(\epsilon) \\ \mu(\epsilon) \end{pmatrix} = \sum_{k=1}^{\infty} \epsilon^k \begin{pmatrix} x_k \\ \mu_k \end{pmatrix}. \tag{18}$$

Substituting the expansion (18) in the equation obtained by equating the right side of (15) to 0, and equating coefficients of like powers of ϵ yields the following relationships.

$$0 = L_0x_1, \tag{19}$$

$$0 = L_0x_2 + \mu_1L_1x_1 + Q_0(x_1, x_1), \tag{20}$$

$$0 = L_0 x_3 + \mu_1 L_1 x_2 + \mu_2 L_1 x_1 + \mu_1^2 L_2 x_1$$

$$+ 2Q_0(x_1, x_2) + \mu_1 Q_1(x_1, x_1) + C_0(x_1, x_1, x_1). \tag{21}$$

By Eqs. (17) and (18),

$$\epsilon = lx(\epsilon)$$

$$= \epsilon l x_1 + \epsilon^2 l x_2 + \epsilon^3 l x_3 + \cdots \tag{22}$$

Hence,

$$lx_1 = 1, \text{ and } lx_k = 0 \text{ for } k \geq 2. \tag{23}$$

Eqs. (19) and (23), and the assumption that 0 is a simple eigenvalue of L_0, now imply

$$x_1 = r. \tag{24}$$

Substituting this in Eq. (20) gives the following equation, which should evidently be solved for *both* x_2 and μ_1:

$$L_0 x_2 = -\mu_1 L_1 r - Q_0(r, r). \tag{25}$$

Recall that L_0 is singular. From elementary linear algebra (or the Fredholm Alternative), this equation has a solution x_2 if and only if the right side of (25) is orthogonal to all left eigenvectors of L_0 corresponding to the zero eigenvalue. Since zero is a simple eigenvalue of L_0, one need only require that

$$\mu_1 l L_1 r + l Q_0(r, r) = 0, \tag{26}$$

so that μ_1 is determined as

$$\mu_1 = -\frac{1}{\lambda_1'(0)} l Q_0(r, r) \tag{27}$$

where Eq. (16) has been employed.

Since the Fredholm Alternative conditions are now satisfied, Eqs. (20) and (23) for x_2 have a solution. This solution is easily verified to be unique. Equations (20), (23) are conveniently expressed as the single equation

$$\begin{pmatrix} L_0 \\ l \end{pmatrix} x_2 = \begin{pmatrix} -\mu_1 L_1 r - Q_0(r, r) \\ 0 \end{pmatrix}. \tag{28}$$

Since (28) has a unique solution, the coefficient matrix

$$R := \begin{pmatrix} L_0 \\ l \end{pmatrix} \tag{29}$$

is full rank. Hence, $R^T R$ is a nonsingular square matrix and x_2 is given by

$$x_2 = (R^T R)^{-1} R^T \begin{pmatrix} -\mu_1 L_1 r - Q_0(r, r) \\ 0 \end{pmatrix}. \tag{30}$$

With x_2 now available, one applies the Fredholm Alternative to Eq. (21) to solve for the coefficient μ_2. Multiplying both sides of (21) by l and solving for μ_2, one obtains

$$\mu_2 = -\frac{1}{\lambda_1'(0)} \{ \mu_1 l L_1 x_2 + \mu_1^2 l L_2 r + 2lQ_0(r, x_2)$$

$$+ \mu_1 l Q_1(r, r) + lC_0(r, r, r) \}. \tag{31}$$

Using the exchange of stability formula (13) and Eq. (27) for μ_1, the coefficient β_1 is found to be

$$\beta_1 = lQ_0(r, r). \tag{32}$$

If $\mu_1 = 0$ (implying also $\beta_1 = 0$), then the exchange of stability formula (14) is valid. In that case, one finds that β_2 is given by

$$\beta_2 = 2l \; (2Q_0(r, x_2) + C_0(r, r, r)), \quad (\text{if } \mu_1 = 0 \;\; \mathit{only}). \tag{33}$$

The formulae (27), (30)-(33) will be employed in the next section to obtain sufficient conditions for bifurcation controllability and local stabilizability for Eqs. (1) and (2), respectively.

5. Stationary bifurcations: Control

Motivated by the bifurcation formulae derived above, and by the results of Section 3, one expands the vector field of Eq. (1) as

$$\begin{aligned}\dot{x} &= f_\mu(x, u) \\ &= L_0 x + \mu L_1 x + u\tilde{L}_1 x + u\gamma + Q_0(x, x) \\ &\quad + \mu^2 L_2 x + \mu Q_1(x, x) + u\tilde{Q}_1(x, x) \\ &\quad + C_0(x, x, x) + \cdots \end{aligned} \tag{34}$$

The notation here is similar to that in Eq. (15). As in Section 3, a feedback control consisting of quadratic and cubic terms is assumed. That is, $u = u(x)$ is taken as

$$u(x) = x^T Q_u x + C_u(x, x, x), \tag{35}$$

where Q_u is a real symmetric $n \times n$ matrix and $C_u(x, x, x)$ is a cubic form generated by a scalar valued symmetric trilinear form. Note that, as in Section 3, $u(x)$ contains no terms linear in x. This ensures that the left and right eigenvectors corresponding to the zero eigenvalue, and the value of μ at criticality, will be unaffected by the feedback control. The closed loop dynamics with a feedback of the form (35) become (starred quantities below denote values after feedback)

$$\begin{aligned}\dot{x} &= L_0^* x + Q_0^*(x, x) + C_0^*(x, x, x) \\ &\quad + \mu L_1^* x + \mu^2 L_2^* x + \mu Q_1^*(x, x) + \cdots \end{aligned} \tag{36}$$

where the matrices L_i^*, $i = 0,1,2$, the quadratic forms $Q_0^*(x, x)$, $Q_1^*(x, x)$ and the cubic form $C_0^*(x, x, x)$ are

$$L_i^* = L_i,\ i = 0,1,2, \tag{37a}$$

$$Q_0^*(x,x) = (x^T Q_u x)\gamma + Q_0(x,x) \tag{37b}$$

$$Q_1^*(x,x) = Q_1(x,x) \tag{37c}$$

and

$$C_0^*(x,x,x) = C_u(x,x,x)\gamma + C_0(x,x,x) + (x^T Q_u x)\tilde{L}_1 x. \tag{37d}$$

Symmetric bilinear and trilinear forms $Q_0^*(x,y)$, $C_0^*(x,y,z)$ generating the quadratic and cubic forms $Q_0^*(x,x)$ and $C_0^*(x,x,x)$, respectively, are now chosen:

$$Q_0^*(x,y) = (x^T Q_u y)\gamma + Q_0(x,y). \tag{38}$$

$$C_0^*(x,y,z) = C_u(x,y,z)\gamma + C_0(x,y,z) + \frac{1}{3}\{ (y^T Q_u z)\tilde{L}_1 x + (x^T Q_u y)\tilde{L}_1 z + (z^T Q_u x)\tilde{L}_1 y \}. \tag{39}$$

After feedback, the coefficient β_1 becomes, using Eq. (32),

$$\begin{aligned} \beta_1^* &= l Q_0^*(r,r) \\ &= l\ \{ Q_0(r,r) + (r^T Q_u r)\gamma \} \\ &= \beta_1 + (r^T Q_u r) l \gamma, \end{aligned} \tag{40}$$

where β_1 denotes the value of β_1 with no feedback, i.e. with $u(x) \equiv 0$. From (40) it is clear that a sufficient condition for the existence of a feedback $u(x)$ driving β_1 to 0 is

$$l\gamma \neq 0. \tag{41}$$

Indeed, if (41) holds, then any feedback control of the form (35) with

$$r^T Q_u r = -\frac{\beta_1}{l\gamma} \tag{42}$$

results in $\beta_1^* = 0$.

Recall from the Popov-Belevitch-Hautus (PBH) eigenvector test for controllability of modes of linear time-invariant systems [22] that $l\gamma \neq 0$ is equivalent to controllability of the zero eigenvalue of the linearized system corresponding to Eq. (2) near the origin. Thus, controllability of the critical zero eigenvalue for the linearized system is sufficient for the existence of a feedback ensuring $\beta_1^* = 0$. Indeed, if $\beta_1 \neq 0$, this condition is also necessary, as can be seen from Eq. (40).

As outlined in Section 4, the next step after ensuring that $\beta_1^* = 0$ is to arrange, if possible, that $\beta_2^* < 0$. By Eq. (33), β_2^* is given by

$$\beta_2^* = 2l\ (2Q_0^*(r,x_2^*) + C_0^*(r,r,r)). \tag{43}$$

To proceed, it is necessary to evaluate x_2^*, according to the formula (30) derived in Section 4. Since $u(x)$ contains no linear terms, the matrix R occurring in Eq. (30) is the same before and after feedback, as is clear from Eq. (29). The fact that $\mu_1^* = 0$ also simplifies the expression for x_2^*. One has

$$
\begin{aligned}
x_2^* &= -(R^T R)^{-1} R^T \begin{pmatrix} Q_0^*(r,r) \\ 0 \end{pmatrix} \\
&= -(R^T R)^{-1} R^T \begin{pmatrix} Q_0(r,r) + (r^T Q_u r)\gamma \\ 0 \end{pmatrix} \\
&= x_2 - (r^T Q_u r)(R^T R)^{-1} R^T \begin{pmatrix} \gamma \\ 0 \end{pmatrix}. \qquad (44)
\end{aligned}
$$

From (44), it is clear that only the quadratic terms (i.e., Q_u) in the feedback control influence the value of x_2^*. Using (38) and (39), one evaluates $\tilde{\beta}_2^*$:

$$
\begin{aligned}
\beta_2^* &= 2l \ \{ 2Q_0(r, x_2^*) + 2(r^T Q_u x_2^*)\gamma + (r^T Q_u r) L_1 r \\
&\qquad + C_0(r,r,r) + C_u(r,r,r)\gamma \} \\
&= 2l\gamma \, C_u(r,r,r) + \phi, \qquad (45)
\end{aligned}
$$

where ϕ is a quantity which is fixed once Q_u is chosen, and, moreover, does not depend on the trilinear form $C_u(x,y,z)$.

From Eq. (45) it follows that if $l\gamma \neq 0$, the value of β_2^* may be assigned arbitrarily by appropriate choice of the scalar valued trilinear form $C_u(x,y,z)$. This holds regardless of the choice made for the symmetric matrix Q_u. The following theorem summarizes these results.

Theorem 4. *Let hypothesis* (S) *hold and assume $l\gamma \neq 0$, that is, the critical zero eigenvalue is controllable for the linearized version of Eq.* (2) *near the origin. Then there is a smooth feedback control $u = u(x)$ with $u(0) = 0$, containing only quadratic and cubic terms in x, which solves the local stationary bifurcation control problem for Eq.* (1) *and the local smooth feedback stabilization problem for Eq.* (2)*. Moreover, the quadratic terms in $u(x)$ can be used to ensure that $\beta_1 = 0$ for the controlled system, and the cubic terms can then be used to ensure that $\beta_2 < 0$.*

Theorem 4 should be compared with Theorem 1, which contains the analogous results for local Hopf bifurcation control. The assumption $l\gamma \neq 0$ was sufficient for stabilizability in that setting as well. However, a stabilizing feedback consisting of only cubic terms was needed, while both quadratic and cubic terms are required in Theorem 4. This is due to the need for a two-stage control design in the stationary bifurcation control case.

Now consider the case $l\gamma = 0$, i.e. let the critical (zero) eigenvalue be *uncontrollable* for the linearized system. In the setting of Section 3, under the analogous assumption it was found that *generically* local feedback stabilization of the nonlinear system is achievable. However, Eq. (40) reveals that in the present setting feedback has *no effect* on the value of β_1 in case $l\gamma = 0$. The discussion in Section 4 therefore implies that the local feedback stabilization problem for Eq. (2) will then be *unsolvable*, unless perhaps it happens that $\beta_1 = 0$ in the absence of a control effort (a nongeneric assumption). Similarly, the local stationary bifurcation control problem is also generically unsolvable in case $l\gamma = 0$.

Theorem 5. *Let hypothesis* (S) *hold and assume* $l\gamma = 0$, *that is, the critical zero eigenvalue is uncontrollable for the linearized version of* (2). *Then if* $\beta_1 \neq 0$ *for Eq.* (1) *with* $u(x) \equiv 0$, *both the local stationary bifurcation control problem for Eq.* (1) *and the local feedback stabilization problem for Eq.* (2) *are not solvable by a smooth feedback control with vanishing linear part.*

Note that the negative conclusion of this theorem does not exclude the possibility that "nearly stabilizing" feedback controls might be constructed for the case $l\gamma = 0$. In the next section we consider this point, and introduce precise notions of "proximity stabilization" and "proximity stabilizability" as an alternative to local stabilization for this case.

6. Proximity stabilization

Under the assumptions that a simple zero eigenvalue is linearly uncontrollable and $\beta_1 \neq 0$, Theorem 5 above asserts that smooth local feedback stabilization is unattainable within the class of purely nonlinear feedback controls. This fact may lead one to search for other acceptable forms of local stability besides the standard notion of asymptotic stability in the sense of Liapunov. In this context, recall that Brockett [8] has obtained easily verifiable necessary conditions for stabilizability by any smooth feedback control. Consider the following definition.

Definition 1. The origin is said to be *proximity stabilizable* for Eq. (2) if for any $\epsilon > 0$ there is a smooth feedback $u = u(x)$ rendering the ball centered at the origin of radius ϵ in R^n a locally attracting set.

To illustrate the nature of proximity stabilization, we consider a simple scalar example. Suppose x satisfies

$$\dot{x} = x^2 - ux. \tag{46}$$

Then $\beta_1 \neq 0$ and $l\gamma = 0$, so there does not exist a smooth feedback $u(x)$ containing no linear terms stabilizing the origin. Indeed, it is easily checked that no smooth feedback can render the origin asymptotically stable, even with linear terms in x. However, this system is proximity stabilizable in the sense of Definition 1. This is easily seen by noting that $\{\, |x| < \epsilon \,\}$ is attracting for (46) if $u(x) = kx^2$ with $k > \epsilon^{-1}$.

The following result states that under the assumptions of Theorem 5, generically there is a smooth feedback control containing only cubic terms in the state x solving the proximity stabilization problem. Our proof is not difficult but in the interest of brevity it will be presented elsewhere.

Theorem 6. *Under the hypotheses of Theorem* 5, *the proximity stabilization problem for Eq.* (2) *is solvable with a cubic feedback control under generic assumptions on the function* f.

7. Examples and directions for further research

The first example considered below was studied previously by Su, Meyer and Hunt [35] . They derived a locally stabilizing feedback control law using feedback linearization as an intermediate step. In principle, this technique allows one to construct an infinite family of stabilizing controllers for the example. This example is one of a nonlinear critical system, and does not involve a bifurcation parameter. The second example is that of a parametrized scalar equation, and is studied mainly to motivate consideration of an issue of proximity stabilization (in a sense to be discussed below) of bifurcations.

Example 1. Following Su, Meyer and Hunt [35] , consider the control system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \sin x_2 \\ 0 \end{pmatrix} + v \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{47}$$

where v is the control. Note that the linearization of (47) possesses a double zero eigenvalue. Thus the theory of Section 5 cannot be applied directly. To circumvent this problem, we define a new control u by

$$u = v + \alpha x_2$$

where α is an arbitrary positive number. Then (47) becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \sin x_2 \\ -\alpha x_2 \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{48}$$

A suitably parametrized version of (48) would, if $u = 0$, undergo a stationary bifurcation from a simple zero eigenvalue. Thus the results of Section 5 are applicable to Eq. (48). Note that no quadratic terms appear in the series expansion of the right side of (48) for $u = 0$. Hence $Q_0(x,y) \equiv 0$. By (27), this implies $\mu_1 = 0$, and hence also $\beta_1 = 0$ for the uncontrolled system. Theorem 4 then implies that, if $l\gamma \neq 0$, only cubic terms would be needed to stabilize the origin of Eq. (48). Indeed, it is easily seen that $r = (1\ 0)^T$, $l = (1\ \ \alpha^{-1})$, and since $\gamma = (0\ 1)^T$,

$$l\gamma = \alpha^{-1} \neq 0. \tag{49}$$

The coefficient β_2 for the uncontrolled system can be found using Eq. (33). We have

$$\begin{aligned} \beta_2 &= 2l\ C_0(r,r,r) \\ &= 2\,(1\ \ \alpha^{-1})\,(0) \\ &= 0. \end{aligned} \tag{50}$$

Hence stability cannot be determined based only on knowledge of β_1 and β_2. However, using Eq. (45) and the preceding comments, we find that the value of β_2 after control is applied is

$$\beta_2^* = 2\,\alpha^{-1}\, C_u(r,r,r) \tag{51}$$

and we are free to choose the cubic terms C_u in the feedback control. (Recall that the quadratic terms Q_u have been set to 0.) It is easy to check that the choice

$$u = C_u(x,x,x) = -\ 0.5\,\alpha\,\delta\, x_1^3, \tag{52}$$

$\delta > 0$, renders $\beta_2^* < 0$. Hence, for the original system (47), we have the stabilizing control laws

$$v = -\alpha x_2 - \delta x_1^3 \tag{53}$$

for any α, $\delta > 0$. The essential distinction between this control law and that obtained in [35] is that linear feedback in only one variable appears in our control law, while linear feedback in both variables forms part of the control obtained in [35] . Computations involving the nonlinear terms are essential in deriving the control law (53) above.

Example 2 (Motivational). Consider the parametrized scalar equation

$$\dot{x} = \mu x - \alpha x^3 + 2\alpha^2 x^9 \tag{54}$$

where μ is the bifurcation parameter and α is a design parameter. We wish to view (54) as arising from a specific choice of the form of a feedback controller in a parametrized scalar control system. The term $2\alpha^2 x^9$ has been retained since we are now mainly interested in more global considerations. It is easy to check that any choice of $\alpha > 0$ will stabilize the (stationary) bifurcation occurring at $\mu = 0$, $x = 0$.

Suppose, however, that we are also interested in ensuring the persistence of some sort of local asymptotic stability for a wide range of values of the parameter $\mu > 0$. Since the origin is unstable for $\mu > 0$, we will be satisfied with rendering a neighborhood of the origin attracting. If no further restrictions are placed on the problem, the solution is quite simple: given an interval $0 \leq \mu \leq \mu_0$ of parameter values for which attractivity of a neighborhood of the origin is desired, there is an α sufficiently large achieving this goal. This follows by a simple Liapunov function argument, viewing (54) as a perturbed version of the stable equation

$$\dot{z} = -\alpha z^3. \tag{55}$$

The difficulty, of course, is that the *size* of the neighborhood which is attracting may shrink drastically for large μ. In a practical sense, therefore, this approach may prove unacceptable.

Consider the following alternative formulation: Given a "stability region tolerance" $\epsilon_0 > 0$, find $\alpha > 0$ to maximize the value μ_1 such that the set $|x| < \epsilon_0$ is locally attracting for $0 \leq \mu \leq \mu_1$. The following analysis addresses this issue. Before proceeding, it should be noted that the important goal of maximizing the size of the region of attraction of the ϵ_0 neighborhood is also an important goal, but is not incorporated into this problem formulation.

Define the Liapunov function candidate

$$V(x) = x^2/2. \tag{56}$$

The derivative of V along trajectories of (54) is

$$\dot{V}(x) = x^2(\mu - \alpha x^2 + 2\alpha^2 x^8). \tag{57}$$

Now $\dot{V} < 0$ at $x = \pm\epsilon_0$ if and only if

$$\mu - \alpha\epsilon_0^2 + 2\,\alpha^2\,\epsilon_0^8 < 0, \qquad (58)$$

i.e., if and only if

$$\mu < \mu_{\max}(\alpha) := \alpha\epsilon_0^2\,(\,1 - 2\,\alpha\,\epsilon_0^6). \qquad (59)$$

Now

$$\max_{\alpha > 0} \mu_{\max}(\alpha) = \mu_{\max}(\alpha^*), \qquad (60)$$

where

$$\alpha^* = 0.25\,\epsilon_0^{-6} \qquad (61)$$

is the best value of α that could be obtained with the Liapunov function V. Incidentally, μ_1 will then be given by

$$\mu_1 = \epsilon_0^{-4}/8. \qquad (62)$$

This simple example is given as motivation for studying more general versions of this proximity stabilization issue for bifurcations. The basic issues noted for Example 2 recur in the analysis of the general case. The stability of critical systems depending on parameters has received much attention in the past. Parameter values for which the origin is stable for the critical system are termed *safe* while those for which the origin is unstable are known as *dangerous* [15, 5] . This terminology arose from the consideration of the stability properties of maneuvering military aircraft. This application is still an active area of investigation [29, 30, 20] . The author is continuing these efforts by investigating application of the results discussed here to the aircraft high angle of attack flight control problem.

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