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On Block Limiting Norm and Structured Singular Value

by

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Abstract

The notion of limiting norm, introduced by Pokrovskii (Soviet Math. Dokl., vol. 20, pp. 1314-1317, 1979), is generalized to that of block limiting norm. A resemblance of inequalities shared by both the block limiting norm and the structured singular value, introduced by Doyle (Proc. IEE, vol. 129, pp. 245-250, 1982), motivates further investigation of their relationships. To that effect, the concept of generalized spectral radius of a set of linear operators is introduced. It is then shown that, for block-structure of size less than 4, the block limiting norm is equal to the structured singular value and that, in the general case, the block limiting norm is always no less than the structured singular value. Finally, better bounds are obtained for both the block limiting norm and the structured singular value.

1 Introduction and Preliminaries

The notion of the *limiting norm* of a linear operator was introduced in 1979 by Pokrovskii [7]. It arises naturally in the study of operators acting in function spaces. The concept of *structured singular value* of a matrix in $\mathbb{C}^{n\times n}$ was introduced by Doyle in 1982 [1] as a tool for the analysis and synthesis of feedback systems with structured uncertainties. A resemblance of inequalities

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shared by both the limiting norm and the structured singular value motivates further investigation of the relationships between the two quantities. In the case of finite dimension linear operators, the definition of the limiting norm is first generalized for various structures and we call it the *block limiting norm.* Consequently, the usual notion of the limiting norm is the block limiting norm with respect to a particular structure. By introducing the concept of generalized spectral radius of a set of linear operators, it is shown that, for block-structures of size less than 4, the block limiting norm is equal to the structured singular value. For larger size, the block limiting norm is always no less than the structured singular value. Finally, better bounds are obtained for both the block limiting norm and the structured singular value.

Let B be a Banach space of functions u with values in $\mathbb{R}^n$ and defined on some set Ω. It is assumed that for any $u, v \in B$, the condition $\|u(t)\| \leq \|v(t)\|$ for all $t \in \Omega$ implies $\|u\|_B \leq \|v\|_B$, where $\|\cdot\|$ and $\|\cdot\|_B$ denote the norms in $\mathbb{R}^n$ and B respectively. For any $u \in B$, define

$$Q(u) = \{v \in B \ : \ \|v(t)\| \leq \|u(t)\| \ \forall t \in \Omega\} \ . \tag{1}$$

Analogously, for any set $\mathcal{S} \subset B$, define

$$Q(\mathcal{S}) = \bigcup_{u \in \mathcal{S}} Q(u) \ .$$

Let M be a linear operator mapping B into itself. Denote by $\mathcal{T}_0$ the unit ball in B. For $k = 1, 2, \ldots,$ consider the sequence of sets $\mathcal{T}_k = Q(M\mathcal{T}_{k-1})$ and define

$$d_k = \sup_{u \in \mathcal{T}_k} \|u\|_B \ .$$

Definition 1 [7]. The *limiting norm* of linear operator M is the nonnegative scalar

$$\pi(M) = \lim_{k \to \infty} (d_k)^{\frac{1}{k}} \ . \tag{2}$$

□

From this definition, it is easily checked that the limiting norm satisfies the inequalities

$$\rho(M) \leq \pi(M) \leq \|M\|_B \tag{3}$$

where $\rho(M)$ denotes the spectral radius of M [7]. An immediate application of the limiting norm is in relation with the fixed points of the compound operator Mf where M is linear and f is nonlinear and both act in B. The following two facts give sufficient conditions under which the equation $u = Mfu$ has either no nonzero solution or a unique solution.

Fact 1 [7]. Suppose that, for all $t \in \Omega$ and all $u \in B$, $\|fu(t)\| \leq a\|u(t)\|$ for some $a > 0$ and $a\pi(M) < 1$. Then the equation $u = Mfu$ has no nonzero solution. □

Fact 2 [7]. Suppose that, for all $t \in \Omega$ and all $u, v \in B$, $\|fu(t) - fv(t)\| \leq a\|u(t) - v(t)\|$ for some $a > 0$ and $a\pi(M) < 1$. Then the equation $u = Mfu$ has a unique solution u^*. Furthermore, for any $u_0 \in B$, the sequence $u_k = Mfu_{k-1}$, $k = 1, 2, \ldots,$ converges to u^*. □

In the sequel, given any square complex matrix M, we denote by $\rho(M)$ its spectral radius and by $\overline{\sigma}(M)$ its largest singular value. Given any complex vector x, $\|x\|$ indicates its Euclidean norm. A *block-structure* of size m is any m-tuple $\mathcal{K} = (k_1, \ldots, k_m)$ of positive integers.[1] Given a block-structure $\mathcal{K}$ of size m, we make use of the family of diagonal matrices

$$\mathcal{D} = \{ \text{ block diag } (d_1 I_{k_1}, \ldots, d_m I_{k_m}) \ : \ d_i \in (0, \infty)\} \ ;$$

of the family of block unitary matrices

$$\mathcal{U} = \{ \text{ block diag } (U_1, \ldots, U_m) \ : \ U_i \text{ is a } k_i \times k_i \text{ unitary matrix}\} \ ;$$

and of the projection matrices

$$P_i = \text{block diag } (O_{k_1}, \ldots, O_{k_{i-1}}, I_{k_i}, O_{k_{i+1}}, \ldots, O_{k_m}) \ ,$$

where, for any positive integer k, I_k is the $k \times k$ identity matrix and O_k the $k \times k$ zero matrix.

Definition 2 [4].[2] The *structured singular value* of a complex $n \times n$ matrix M *with respect to the block-structure* $\mathcal{K} = (k_1, \ldots, k_m)$ of size m, where $n = \sum_{i=1}^m k_i$, is the nonnegative scalar

$$\mu(M) \ = \ \max_{x \in \mathbb{C}^n} \{\|Mx\| \ : \ \|P_i x\| \ \|Mx\| = \|P_i Mx\|, \ i = 1, \ldots, m\}.$$

□

Notice in particular that, if $\mathcal{K} = (n)$, the structured singular value is equal to the largest singular value $\overline{\sigma}(M)$. It should be emphasized that $\mathcal{D}$, $\mathcal{U}$, P_i and $\mu(M)$ all depend on the underlying block-structure. For simplicity of notation however, we will not explicitly indicate this dependence.

The following two important properties of the structured singular value will be used below. The reader is referred to [1,2,4–6,9] for a complete exposition of this topic.

Fact 3 [1]. The structured singular value satisfies the relations

$$\rho(M) \leq \max_{U \in \mathcal{U}} \ \rho(UM) = \mu(M) \leq \inf_{D \in \mathcal{D}} \ \bar{\sigma}(DMD^{-1}) \leq \bar{\sigma}(M) \ . \tag{4}$$

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[1] This corresponds, in the terminology of [1], to structures with no repeated blocks.

[2] This definition of the structured singular value, while more simply expressed, is equivalent to that originally proposed by Doyle [1].

□

Fact 4 [1]. For block-structure of size less than 4, i.e., $m < 4$,

$$\mu(M) = \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}) \ . \tag{5}$$

□

2 Main Results

In this section, we consider the limiting norm of a linear operator in Euclidean space $\mathbb{C}^n$. To avoid any loss of continuity, all proofs are given in Appendix A.

Given $x \in \mathbb{C}^n$, let the definition of set $Q(x)$ in (1) be replaced by

$$Q(x) = \{y \in \mathbb{C}^n \ : \ \|P_i y\| \leq \|P_i x\| \ i = 1, \ldots, m\}$$

and define $Q(\mathcal{S})$, $\mathcal{T}_k$ and d_k accordingly, where $\mathcal{S} \subset \mathbb{C}^n$ and $\mathcal{T}_0$ is the unit ball of $\mathbb{C}^n$.

Definition 3. The *block limiting norm* of a complex $n \times n$ matrix M *with respect to the block-structure* $\mathcal{K} = (k_1, \ldots, k_m)$ of size m, where $n = \sum_{i=1}^{m} k_i$, is the nonnegative scalar

$$\nu(M) = \limsup_{k \to \infty} \ (d_k)^{\frac{1}{k}} \ .$$

□

Thus, the limiting norm is the block limiting norm with respect to block-structure $\mathcal{K} = (1, \ldots, 1)$. Note that $\nu(M)$ also depends on the underlying block-structure $\mathcal{K}$. Furthermore, similar to the limiting norm, the block limiting norm satisfies the inequalities

$$\rho(M) \leq \nu(M) \leq \bar{\sigma}(M). \tag{6}$$

for any block-structure $\mathcal{K}$.

A closer look at (3), (4) and (6) shows that similar properties are possessed by both $\nu(M)$ and $\mu(M)$ and this motivates further investigation on their relationships. A bold conjecture that $\nu(M)$ is another appearance of $\mu(M)$ is false, however.

In order to explore more properties of the block limiting norm, we now introduce a new measure for sets in $\mathbb{C}^{n \times n}$. It is well known that for any matrix $M \in \mathbb{C}^{n \times n}$,

$$\rho(M) = \lim_{k \to \infty} \bar{\sigma}(M^k)^{\frac{1}{k}} \ .$$

This property motivates a generalization of the spectral radius for sets in $\mathbb{C}^{n\times n}$. For $\mathcal{S} \subset \mathbb{C}^{n\times n}$ such that $\sup_{A\in\mathcal{S}} \bar{\sigma}(A) < \infty$, define

$$\gamma_k(\mathcal{S}) = \sup\{\bar{\sigma}(A_1 \cdots A_k)^{\frac{1}{k}} \ : \ A_1, \ldots, A_k \in \mathcal{S}\} \ .$$

It is easily checked that $\gamma_k(\mathcal{S})$ is finite and satisfies, for any k,

$$\sup_{A\in\mathcal{S}} \rho(A) \le \gamma_k(\mathcal{S}) \le \sup_{A\in\mathcal{S}} \bar{\sigma}(A) \ . \tag{7}$$

Definition 4. Let $\mathcal{S} \subset \mathbb{C}^{n\times n}$ and suppose $\sup_{A\in\mathcal{S}} \bar{\sigma}(A) < \infty$. The *generalized spectral radius* of $\mathcal{S}$ is the nonnegative scalar

$$\gamma(\mathcal{S}) = \limsup_{k\to\infty} \ \gamma_k(\mathcal{S}) \ .$$

□

The following theorem illustrates some properties of the generalized spectral radius.

Theorem 1. Let $M \in \mathbb{C}^{n\times n}$ and $\mathcal{S}, \mathcal{W} \subset \mathbb{C}^{n\times n}$. Also let $M\mathcal{S} = \{MA \ : \ A \in \mathcal{S}\}$ and $\mathcal{S}\mathcal{W} = \{A_1A_2 \ : \ A_1 \in \mathcal{S}, \ A_2 \in \mathcal{W}\}$. Suppose that $\sup_{A\in\mathcal{S}} \bar{\sigma}(A) < \infty$ and $\sup_{A\in\mathcal{W}} \bar{\sigma}(A) < \infty$. Then the following properties hold

1. For any k, $\gamma_k(\mathcal{S}) \ge \gamma(\mathcal{S})$. Therefore, $\lim_{k\to\infty} \gamma_k(\mathcal{S})$ exists and $\gamma(\mathcal{S}) = \lim_{k\to\infty} \ \gamma_k(\mathcal{S})$.
2. $\sup_{A\in\mathcal{S}} \rho(A) \le \gamma(\mathcal{S}) \le \sup_{A\in\mathcal{S}} \bar{\sigma}(A)$.
3. $\gamma(\mathcal{S}\mathcal{W}) = \gamma(\mathcal{W}\mathcal{S})$.
4. $\gamma(M\mathcal{S}) = \gamma(\mathcal{S}M)$.
5. For M nonsingular, $\gamma(M\mathcal{S}M^{-1}) = \gamma(\mathcal{S})$.

□

The following theorem shows that there is a very close relationship between the block limiting norm and the generalized spectral radius of a certain family of sets.

Theorem 2. Let $M \in \mathbb{C}^{n\times n}$ and let d_k be defined as before. Then for any k,

$$(d_k)^{\frac{1}{k}} = \gamma_k(\mathcal{U}M) \ , \tag{8}$$

so that

$$\nu(M) = \gamma(\mathcal{U}M)$$

and

$$\nu(M) = \lim_{k\to\infty} (d_k)^{\frac{1}{k}}.$$

□

Corollary 1. For any positive integer q,

$$\mu(M^q)^{\frac{1}{q}} \leq \nu(M) \ .$$

□

From properties 4, 5 in Theorem 1, Theorem 2 and the fact that for all $U \in \mathcal{U}$ and $D \in \mathcal{D}$, U and D commute, the proposition below follows immediately.

Proposition 1. For all $U \in \mathcal{U}$

$$\nu(M) = \nu(UM) = \nu(MU)$$

and, for all $D \in \mathcal{D}$,

$$\nu(M) = \nu(DMD^{-1}) \ . \tag{9}$$

□

Using (6) and (9), we can obtain a more easily checked but more conservative sufficient condition such that the claims in Facts 1 and 2 hold. Proposition 1 gives two classes of transformations under which the block limiting norm is preserved. Recall that the block limiting norm of matrix M is bounded above and below by its largest singular value and spectral radius, respectively. As a consequence of Fact 3, (6), Theorem 2 and Proposition 1, tighter bounds for $\nu(M)$ and $\mu(M)$ are obtained.

Theorem 3.

$$\rho(M) \leq \max_{U\in\mathcal{U}} \rho(UM) = \mu(M) \leq \nu(M) \leq \inf_{D\in\mathcal{D}} \bar{\sigma}(DMD^{-1}) \leq \bar{\sigma}(M).$$

□

By Fact 4 and Theorem 3, the following result is obvious.

Theorem 4. For block-structure of size less than 4,

$$\nu(M) = \mu(M) \ . \tag{10}$$

□

An example is exhibited in Appendix B with block-structure of size 4, for which (10) does not hold.

Corollary 2. For block-structure of size less than 4,

$$\gamma(\mathcal{U}M) = \max_{U \in \mathcal{U}} \rho(UM) .$$

□

Corollary 3. For block-structure of size less than 4 and for any positive integer q,

$$\mu(M^q)^{\frac{1}{q}} \leq \mu(M).$$

□

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3 Appendix A

We make use of the following three lemmas in proving Theorem 1.

Lemma 1. Suppose that $\{a_k\}$ is a bounded nonnegative sequence and $b > 0$. Then the sequence $\{c_k\}$ defined by

$$c_k = a_k^{\frac{k}{k+1}} b^{\frac{1}{k+1}} - a_k$$

converges to zero. □

Lemma 2. For any $\epsilon > 0$, there exists $K > 0$ such that, for all $k \geq K$,

$$\gamma_{k+1}(\mathcal{S}) \leq \gamma_k(\mathcal{S}) + \epsilon .$$

Proof. From the definition of $\gamma_{k+1}(\mathcal{S})$, it follows that

$$\begin{aligned} \gamma_{k+1}(\mathcal{S}) &= \sup \left\{ \bar{\sigma}(A_1 \cdots A_{k+1})^{\frac{1}{k+1}} \ : \ A_1, \ldots, A_{k+1} \in \mathcal{S} \right\} \\ &\leq \sup \left\{ \bar{\sigma}(A_1 \cdots A_k)^{\frac{1}{k+1}} \bar{\sigma}(A_{k+1})^{\frac{1}{k+1}} \ : \ A_1, \ldots, A_{k+1} \in \mathcal{S} \right\} \\ &= \sup \left\{ \bar{\sigma}(A_1 \cdots A_k)^{\frac{1}{k+1}} \ : \ A_1, \ldots, A_k \in \mathcal{S} \right\} \sup \left\{ \bar{\sigma}(A)^{\frac{1}{k+1}} \ : \ A \in \mathcal{S} \right\} \\ &= \gamma_k(\mathcal{S})^{\frac{k}{k+1}} \gamma_1(\mathcal{S})^{\frac{1}{k+1}} \\ &= \gamma_k(\mathcal{S}) + (\gamma_k(\mathcal{S})^{\frac{k}{k+1}} \gamma_1(\mathcal{S})^{\frac{1}{k+1}} - \gamma_k(\mathcal{S})) . \end{aligned}$$

Since $\gamma_k(\mathcal{S})$ is bounded, the result then follows from Lemma 1. □

Lemma 3. For any positive integers q and p,

$$\gamma_{qp}(\mathcal{S}) \leq \gamma_q(\mathcal{S}) .$$

Proof. From the definition of $\gamma_{qp}(\mathcal{S})$, it follows that

$$\begin{aligned}\gamma_{qp}(\mathcal{S}) &= \sup\left\{\bar{\sigma}(A_1\cdots A_{qp})^{\frac{1}{qp}} \;:\; A_1,\ldots,A_{qp}\in\mathcal{S}\right\}\\ &\le \left(\sup\left\{\bar{\sigma}(A_1\cdots A_q)^{\frac{1}{qp}} \;:\; A_1,\ldots,A_q\in\mathcal{S}\right\}\right)^p\\ &= \gamma_q(\mathcal{S})\ .\end{aligned}$$

□

Proof of Theorem 1.

1. By contradiction. Suppose for some integer q, $\gamma_q(\mathcal{S}) < \gamma(\mathcal{S})$. By Lemma 2, there exists $K > 0$ such that, for all $k > K$,

$$\gamma_{k+1}(\mathcal{S}) \le \gamma_k(\mathcal{S}) + \frac{\gamma(\mathcal{S}) - \gamma_q(\mathcal{S})}{2q}\ .$$

Without loss of generality, we assume that K is a multiple of q, i.e., $K = qp$ for some $p > 0$. Then for any $k \ge K$, express k as $k = qp_1 + l$ for some $p_1 \ge p$ and $l < q$, we have

$$\begin{aligned}\gamma_k(\mathcal{S}) &= \gamma_{qp_1+l}(\mathcal{S}) &&\le \gamma_{qp_1+l-1}(\mathcal{S}) + \tfrac{\gamma(\mathcal{S})-\gamma_q(\mathcal{S})}{2q}\\ &\le \gamma_{qp_1+l-2} + \tfrac{2(\gamma(\mathcal{S})-\gamma_q(\mathcal{S}))}{2q} &&\le \cdots\\ &\le \gamma_{qp_1} + \tfrac{l(\gamma(\mathcal{S})-\gamma_q(\mathcal{S}))}{2q} &&\le \gamma_{qp_1}(\mathcal{S}) + \tfrac{\gamma(\mathcal{S})-\gamma_q(\mathcal{S})}{2}\ .\end{aligned}$$

Thus, using Lemma 3 we have, for $k > K$,

$$\gamma_k(\mathcal{S}) \le \gamma_q(\mathcal{S}) + \frac{\gamma(\mathcal{S}) - \gamma_q(\mathcal{S})}{2} = \frac{\gamma(\mathcal{S}) + \gamma_q(\mathcal{S})}{2}$$

which implies

$$\gamma(\mathcal{S}) = \limsup_{k\to\infty} \gamma_k(\mathcal{S}) \le \frac{\gamma_q(\mathcal{S}) + \gamma(\mathcal{S})}{2} < \gamma(\mathcal{S})\ .$$

2. A direct consequence of (7).

3. The claim trivially holds if either $\mathcal{S}$ or $\mathcal{W}$ contains only the zero matrix. Thus assume $\gamma_1(\mathcal{S}) > 0$ and $\gamma_1(\mathcal{W}) > 0$. From the definition of $\gamma_k(\mathcal{S}\mathcal{W})$, it follows that

$$\begin{aligned}\gamma_k(\mathcal{S}\mathcal{W}) &= \sup\left\{\bar{\sigma}(A_1B_1\cdots A_kB_k)^{\frac{1}{k}} \;:\; A_1,\ldots,A_k\in\mathcal{S},\ B_1,\ldots,B_k\in\mathcal{W}\right\}\\ &\le \gamma_1(\mathcal{S})^{\frac{1}{k}}\sup\{\bar{\sigma}(B_1A_2B_2\cdots A_{k-1}B_{k-1}A_k)^{\frac{1}{k}} \;:\; A_1,\ldots,A_k\in\mathcal{S},\ B_1,\ldots,B_k\in\mathcal{W}\}\gamma_1(\mathcal{W})^{\frac{1}{k}}\\ &= \gamma_1(\mathcal{S})^{\frac{1}{k}}\gamma_{k-1}(\mathcal{W}\mathcal{S})^{\frac{k-1}{k}}\gamma_1(\mathcal{W})^{\frac{1}{k}}\ .\end{aligned}$$

Hence,

$$\gamma(\mathcal{S}\mathcal{W}) = \lim_{k\to\infty}\gamma_k(\mathcal{S}\mathcal{W}) \le \lim_{k\to\infty}\gamma_1(\mathcal{S})^{\frac{1}{k}}\gamma_{k-1}(\mathcal{W}\mathcal{S})^{\frac{k-1}{k}}\gamma_1(\mathcal{W})^{\frac{1}{k}} = \gamma(\mathcal{W}\mathcal{S})\ .$$

Similarly, $\gamma(\mathcal{W}\mathcal{S}) \leq \gamma(\mathcal{S}\mathcal{W})$.

4. A direct consequence of 3.

5. By using 4, $\gamma(M\mathcal{S}M^{-1}) = \gamma(\mathcal{S}M^{-1}M) = \gamma(\mathcal{S})$. □

In order to prove Theorem 2, we employ the following lemmas.

Lemma 4. For any bounded set $\mathcal{S} \subset \mathbb{C}^n$, $\sup_{x\in\mathcal{S}} \|x\| = \sup_{x\in\text{co}\mathcal{S}} \|x\|$. □

Lemma 5. Let $M \in \mathbb{C}^{n\times n}$ and $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{C}^n$. Then $\text{co}(\mathcal{S}_1 \cup \mathcal{S}_2) = \text{co}(\text{co}\mathcal{S}_1 \cup \text{co}\mathcal{S}_2)$ and $M\text{co}\mathcal{S}_1 = \text{co}M\mathcal{S}_1$. □

Lemma 6. Let $\mathcal{Z} = \{z \in \mathbb{R}^m : |z^i| = 1,\ i = 1,\ldots,m\}$ where z^i denotes the ith component of z. Let $w \in \mathbb{R}^m$ and suppose that, for $i = 1,\ldots,m$, $|w^i| \leq 1$. Then $w \in \text{co}\mathcal{Z}$. □

Lemma 7. Let $\mathcal{S} \subset \mathbb{R}^m$ and $w \in \text{co}\mathcal{S}$. Then there exist $s_1,\ldots,s_{m+1} \in \mathcal{S}$, $\lambda^1,\ldots,\lambda^{m+1} \in \mathbb{R}$ such that for $i = 1,\ldots,m+1$, $\lambda^i \geq 0$, $\sum_{i=1}^{m+1} \lambda^i = 1$ and $w = \sum_{i=1}^{m+1} \lambda^i s_i$. □

Lemma 8.

$$Q(\mathcal{S}) \subset \text{co}(\bigcup_{U\in\mathcal{U}} U\mathcal{S}) \ .$$

Proof. Let $\mathcal{W} = \bigcup_{U\in\mathcal{U}} U\mathcal{S}$. It suffices to prove that $x \in Q(\mathcal{S})$ implies $x \in \text{co}\mathcal{W}$. Suppose $x \in Q(\mathcal{S})$. Then there exists $y \in \mathcal{S}$ such that, for $i = 1,\ldots,m$, $\|P_i x\| \leq \|P_i y\|$. Since $y \in \mathcal{S} \subset \mathcal{W}$, there exists $U_1 \in \mathcal{U}$ such that $y_1 = U_1 y \in \mathcal{W}$ and, for $i = 1,\ldots,m$,

$$\alpha^i P_i y_1 = P_i x \tag{11}$$

for some $0 \leq \alpha^i \leq 1$. Let $\alpha = (\alpha^1 \cdots \alpha^m)^T$. By Lemmas 6 and 7, there exist $z_1,\ldots,z_{m+1} \in \mathcal{Z}$, $\lambda^1,\ldots,\lambda^{m+1} \in \mathbb{R}$ such that, for $j = 1,\ldots,m$, $\lambda_j \geq 0$, $\sum_{j=1}^{m+1} \lambda^j = 1$ and $\alpha = \sum_{j=1}^{m+1} \lambda^j z_j$ where $\mathcal{Z}$ is defined in Lemma 6. Therefore, for $i = 1,\ldots,m$,

$$\alpha^i P_i y_1 = \sum_{j=1}^{m+1} \lambda^j z_j^i P_i y_1 \ . \tag{12}$$

By using (11), we have

$$P_i x = \sum_{j=1}^{m+1} \lambda^j z_j^i P_i y_1 \ . \tag{13}$$

Summing (13) for $i = 1,\ldots,m$, we obtain

$$x = \sum_{j=1}^{m+1} \lambda^j (\sum_{j=1}^{m} z_j^i P_i y_1) \ .$$

Since for any i and j, z_j^i is either 1 or -1, it is clear that, for all j, $\sum_{j=1}^{m} z_j^i P_i y_1 \in \mathcal{W}$. Therefore, x could be expressed as a convex combination of points in $\mathcal{W}$, This implies that $x \in \text{co}\mathcal{W}$. □

Proof of Theorem 2. From the definition of $\gamma_k(\mathcal{U}M)$, it follows that

$$\begin{aligned}\gamma_k(\mathcal{U}M) &= \sup\{\bar\sigma(U_1M\cdots U_kM)^{\frac{1}{k}} \;:\; U_1,\ldots,U_k\in\mathcal{U}\}\\ &= \sup\{\|U_1M\cdots U_kMx\|^{\frac{1}{k}} \;:\; U_1,\ldots,U_k\in\mathcal{U},\; x\in\mathbb{C}^n,\; \|x\|=1\}\\ &= \sup\{\|y\|^{\frac{1}{k}} \;:\; y\in U_1M\cdots U_kM\tilde{\mathcal{T}}_0,\; U_1,\ldots,U_k\in\mathcal{U}\}\end{aligned}$$

where $\tilde{\mathcal{T}}_0$ denotes the unit sphere of $\mathbb{C}^n$. Define

$$\tilde{\mathcal{T}}_k = \{y : y\in U_1M\cdots U_kM\tilde{\mathcal{T}}_0,\; U_1,\ldots,U_k\in\mathcal{U}\}\ .$$

Therefore

$$\tilde{\mathcal{T}}_k = \bigcup_{U\in\mathcal{U}} UM\tilde{\mathcal{T}}_{k-1}$$

and

$$\gamma_k(\mathcal{U}M) = \sup_{x\in\tilde{\mathcal{T}}_k}\|x\|^{\frac{1}{k}}\ .$$

Now we want to show that, for all k,

$$\tilde{\mathcal{T}}_k \subset \mathcal{T}_k \subset \mathrm{co}\,\tilde{\mathcal{T}}_k \tag{14}$$

and therefore, by Lemma 4, the claim in (8) holds. We prove both inclusions in (14) by induction. It is clear that $\tilde{\mathcal{T}}_0\subset\mathcal{T}_0$. Suppose that for some k, $\tilde{\mathcal{T}}_k\subset\mathcal{T}_k$. Since, for all $U\in\mathcal{U}$, $UM\tilde{\mathcal{T}}_k\subset Q(M\mathcal{T}_k)$. Hence,

$$\tilde{\mathcal{T}}_{k+1} = \bigcup_{U\in\mathcal{U}} UM\tilde{\mathcal{T}}_k \subset Q(M\mathcal{T}_k) = \mathcal{T}_{k+1}\ .$$

For the second inclusion in (14), it is also clear that $\mathcal{T}_0\subset\mathrm{co}\,\tilde{\mathcal{T}}_0$. Suppose that for some k, $\mathcal{T}_k\subset\mathrm{co}\,\tilde{\mathcal{T}}_k$. Then by Lemmas 5 and 8, we have

$$\mathcal{T}_{k+1} = Q(M\mathcal{T}_k) \subset \mathrm{co}(\bigcup_{u\in\mathcal{U}} UM\mathcal{T}_k) = \mathrm{co}(\bigcup_{u\in\mathcal{U}} UM\mathrm{co}\,\tilde{\mathcal{T}}_k) = \mathrm{co}(\bigcup_{u\in\mathcal{U}} UM\tilde{\mathcal{T}}_k) = \mathrm{co}\,\tilde{\mathcal{T}}_{k+1}$$

□

Proof of Corollary 1. Let k be a positive integer. Then

$$\begin{aligned}\mu(M^q)^{\frac{1}{q}} &= \max_{U\in\mathcal{U}}\rho(UM^q)^{\frac{1}{q}}\\ &= \max_{U\in\mathcal{U}}\rho((UM^q)^k)^{\frac{1}{qk}}\\ &\le \max\left\{\rho(U_1M^q\cdots U_kM^q)^{\frac{1}{qk}} \;:\; U_1,\ldots,U_k\in\mathcal{U}\right\}\\ &\le \max\left\{\bar\sigma(U_1M^q\cdots U_kM^q)^{\frac{1}{qk}} \;:\; U_1,\ldots,U_k\in\mathcal{U}\right\}\\ &\le \max\left\{\bar\sigma(U_1M^q\cdots U_{qk}M^q)^{\frac{1}{qk}} \;:\; U_1,\ldots,U_{qk}\in\mathcal{U}\right\}\\ &= \gamma_{qk}(\mathcal{U}M)\ .\end{aligned}$$

Since k is arbitrary, it follows that

$$\mu(M^q)^{\frac{1}{q}} \leq \lim_{k\to\infty} \gamma_{qk}(\mathcal{U}M) = \nu(M) \ .$$

□

4 Appendix B

We give an example for which (10) does not hold. This example was constructed by Doyle to show that (5) may not hold for block-structure of size greater than 3 (see, e.g., [3] [8] for details).

Let $\mathcal{K} = (1,1,1,1)$ and $M = [u_1\ u_2]\,[v_1\ v_2]^H$ where the superscript H denotes the Hermitain operator,

$$u_1 = \begin{bmatrix} a \\ ab \\ ab \\ \sqrt{1-2a^2} \end{bmatrix},\ u_2 = \begin{bmatrix} 0 \\ ab \\ abj \\ \frac{-a^2(1+j)}{2\sqrt{1-2a^2}} \end{bmatrix},\ v_1 = \begin{bmatrix} 0 \\ ab \\ ab \\ \frac{a^2(1-j)}{2\sqrt{1-2a^2}} \end{bmatrix},\ v_2 = \begin{bmatrix} a \\ -ab \\ -abj \\ \sqrt{1-2a^2} \end{bmatrix},$$

$a = \sqrt{1-\frac{\sqrt{3}}{3}}$, $b = \frac{\sqrt{2}}{2}$ and $j = \sqrt{-1}$. It has been shown in [3] [8] that $\inf_{D\in\mathcal{D}} \bar{\sigma}(DMD^{-1}) = \bar{\sigma}(M) = 1$ and $\mu(M) < 1$. Here we show that $\nu(M) = 1$.

It is obvious that $\nu(M) \leq 1$. To show $\nu(M) \geq 1$, let $U_1, U_2 \in \mathcal{U}$ be defined as

$$U_1 = \text{diag}\ (1,-1,-j,1) \qquad U_2 = \text{diag}\ (1,1,-j,-\frac{1-j}{1+j}) \ .$$

Then, it is straightforward to check that

$$U_2 M U_1 M v_1 = v_1$$

which implies that $\nu(M) \geq 1$.

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