SRC TR 87-118

An Impulse Control Problem for a Stochastic PDE Arising in Non Linear Filtering

by

J. S. Baras and A. Bensoussan

Proceedings of 24th Conference
on Decision and Control
Ft. Lauderdale, FL · December 1985

AN IMPULSE CONTROL PROBLEM FOR A STOCHASTIC PDE ARISING IN NON LINEAR FILTERING.

J.S. BARAS*, A. BENSOUSSAN**

(*) Electrical Engineering Dept., Univ. of Maryland, College Park, MD 20742

(**) Univ. Paris-Dauphine and INRIA, B.P 105, F78153 Le Chesnay

INTRODUCTION

We consider the nonlinear filtering problem of a vector diffusion process, when several noisy vector observations with possibly different dimension of their range space are available. At each time any number of these observations (or sensors) can be utililized in the signal processing performed by the nonlinear filter. The problem considered is the optimal selection of a schedule of these sensors from the available set, so as to optimally estimate a function of the state at the final time. Optimality is measured by a combined performance measure that allocates penalties for errors in estimation, switching between sensor schedules and for running a sensor. The solution is obtained in the form of a system of quasi-variational inequalities in the space of solutions of certain Zakai equations.

1 - PRELIMINARY DESCRIPTION OF THE PROBLEM

The problem considered is as follows. A signal (or state) process $x(\cdot)$ is given, modelled by the diffusion

$$
\begin{aligned}
&dx = f(x(t))dt + g(x(t))dw \\
&x(0) = \xi
\end{aligned}
\tag{1.1}
$$

in R^n. We further consider M noisy observations of $x(\cdot)$, described by

$$
\begin{aligned}
&dy^i = h^i(x(t))dt + R_i^{1/2}\, dv^i(t) \\
&y^i(0) = 0
\end{aligned}
\tag{1.2}
$$

with values in R^{d_i}. Here $w(\cdot)$, $v^i(\cdot)$ are independent, standard, Wiener processes in R^n, R^{d_i} respectively and $R_i = R_i^*$ are $d_i \times d_i$ symmetric, positive definite matrices.

The control concerns all possible sensor activation configurations. There are $N = 2^M$ possibilities (each sensor can be activated or not).

A schedule of sensors is a piecewise constant function $u(\cdot) : [0,T] \to [1,\ldots N]$. Let τ_j be the increasing sequence of switching times, and

$$\nu_j = u(\tau_j) \quad [1\ldots N]$$

the corresponding sequence of sensor configurations, hence

$$u(t) = \nu_j,\ t \in [\tau_j,\tau_{j+1})\ ,\ j=1,2\ldots$$

One can then make precise the observation process corresponding to a sensor schedule $u(\cdot)$.

Define indeed for $\nu = [1,\ldots N]$

$$
h(x;\nu) = \begin{bmatrix} h^1(x)\chi_\nu(1) \\ \vdots \\ h^M(x)\chi_\nu(M) \end{bmatrix}
\tag{1.3}
$$

where $\chi_\nu(i) = 1$ if i is activated under the configuration ν. Hence h is a R^D valued vector, where

$$D = d_1 + \ldots + d_M.$$

Define next

$$
v(t) = \begin{bmatrix} v^1(t) \\ \vdots \\ v^M(t) \end{bmatrix}
$$

which is a standard Wiener process in R^D, and $r(\nu) \in L(R^D;R^D)$ defined by

$$r(\nu) = \text{Block diagonal } \{R_i^{1/2}\chi_\nu(i)\}.$$

With this notation, the observation in the interval $[\tau_j,\tau_{j+1})$ is given by

$$h(x(t),\nu_j)dt + r(\nu_j)dv(t),\ t \in [\tau_j,\tau_{j+1}).$$

Therefore the observation corresponding to the schedule $u(\cdot)$ is described by

$$dy(t;u(\cdot)) = h(x(t),u(t))dt + r(u(t))dv(t). \tag{1.4}$$

In order to define the cost function, corresponding to a sensor schedule, one considers functions $k(x;\nu,\nu')$ and $c(x;\nu)$ representing the switching cost from the configuration ν to the configuration ν', and the running cost of the configuration ν. Typically they are of the form

$$c(x;\nu) = \sum_{j=1}^{M} c_j(x)\chi_\nu(j)$$

$$k(x;\nu,\nu') = \sum_{j=1}^{M} (k_j^o\, \chi_\nu(j) + k_j^1 \chi_{\nu'}(j))$$

where k_j^o represents the cost of switching off the sensor j, and k_j^1 the cost of switching on the sensor j.

The cost function corresponding to a schedule $u(\cdot)$, is written as

$$J(u(\cdot)) = E\{|x(T)-\hat{x}(T)|^2 + \int_0^T c(x(t),u(t))dt + \sum_j k(x(t),u(\tau_{j-1}),u(\tau_j))\chi_{\tau_j < T}\} \tag{1.5}$$

where $\hat{x}(T)$ is the best estimate of $x(T)$, corresponding to the observation process $y(t;u(\cdot))$.

2 - THE STOCHASTIC CONTROL FORMULATION

It remains to make precise the probabilistic set up, in particular the family of σ-algebras to which the sensor schedule should be adapted.

2.1. Setting of the model

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space, on which a filtration F_t is given, $\mathcal{A} = F_\infty$. Let $w(\cdot)$, $z(\cdot)$ be two independant, standard F_t-Wiener processes with values in R^n and R^D respectively, and ξ be a R^n-valued random variable, independant of $w(\cdot)$, $z(\cdot)$, with probability distribution π_o.

Let f,g such that

$$f : R^n \to R^n, \text{ bounded and Lipschitz}$$

$$g : R^n \to L(R^n;R^n), \text{ bounded and Lipschitz ;} \quad a = \frac{1}{2} gg^* \geq \alpha I. \tag{2.1}$$

The Lipschitz assumption simplifies some technicalities but are not essential

$$h^i : R^n \to R^{d_i}, \text{ bounded and Hölder continuous.} \tag{2.2}$$

Let A be the 2nd order differential operator

$$A = -\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} - \sum_i f_i(x) \frac{\partial}{\partial x_i} = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial}{\partial x_j}) + \sum_i a_i(x) \frac{\partial}{\partial x_i} \tag{2.3}$$

where

$$a_i(x) = -f_i(x) + \sum_{j=1}^n \frac{\partial a_{ij}}{\partial x_j}(x).$$

Consider an increasing sequence $\tau_1 < \tau_2 \ldots < \tau_k < \ldots$ of F_t stopping times. To each stopping time τ_i is associated a random variable ν_i with values in the set $\{1,2\ldots N\}$, and ν_i is F_{ν_i} measurable.

Moreover

$$\tau_i \uparrow T, \text{ as } i \uparrow \infty$$

and $\tau_o = 0$. Note that $\tau_k = T$ is possible. Define

$$u(t) = \nu_i \text{ for } t \in [\tau_i, \tau_{i+1}).$$

This process is a random schedule of sensors. Define $r(u(t))$ and $h(x(t),u(t))$ as in section 1, and the process $y(t;u(\cdot))$ by

$$y(t;u(\cdot)) = \int_0^t r(u(s))dz(s). \tag{2.4}$$

In order to derive (1.4) we proceed with a Girsanov transformation. First notice that although $r(\nu)$ is not invertible, one can write

$$h(x,\nu) = r(\nu)\tilde{h}(x;\nu) \tag{2.5}$$

where

$$\tilde{h}(x;\nu) = \begin{bmatrix} R_1^{-1/2} h^1(x)\chi_\nu(1) \\ \vdots \\ R_M^{-1/2} h^M(x)\chi_\nu(M) \end{bmatrix}$$

Consider then the process

$$\xi(t) = \exp\{\int_0^t \tilde{h}(x(s),u(s))\cdot dz(s) - \frac{1}{2}\int_0^t |\tilde{h}(x(s),u(s))|^2 ds\} \tag{2.6}$$

which is a F_t martingale.

Let us define a change of probability measure

$$\left.\frac{dP^{u(\cdot)}}{dP}\right|_{F_t} = \xi(t) \tag{2.7}$$

and consider also the process

$$v(t) = z(t) - \int_0^t \tilde{h}(x(s),u(s))ds. \tag{2.8}$$

By Girsanov's theorem, under the probability measure $P^{u(\cdot)}$ on $(\Omega, \mathcal{A})$, $v(\cdot)$ is a standard F_t-Wiener process with values in R^D. Note that $x(\cdot)$ retains its probability low under $P^{u(\cdot)}$. From (2.4) and (2.8) we see at once that under $P^{u(\cdot)}$, the process $y(t;u(\cdot))$ behaves according to the relation (1.4).

Let us now define what is the class of admissible controls. For any $u(\cdot)$, given the construction of $y(\cdot,u(\cdot))$ above we can consider $F^{y(\cdot,u(\cdot))}$ defined by

$$F_t^{y(\cdot,u(\cdot))} = \sigma(y(s,u(\cdot)),\ s \leq t).$$

We shall say that $u(\cdot)$ is <u>admissible</u> if $u(\cdot)$ is F_t^z and $F_t^{y(\cdot,u(\cdot))}$ measurable. Note that for an admissible control, $F_t^{y(\cdot,u(\cdot))} \subset F_t^z$.

Defining

$$\hat{x}(T) = E^{u(\cdot)}[x(T)\ F_T^{y(\cdot,u(\cdot))}]$$

we can write the cost function (1.5) more precisely as

$$J(u) = E^{u(\cdot)}\{|x(T)-\hat{x}(T)|^2 + \int_0^T c(x(t),u(t))dt + \sum_j k(x(\tau_j),u(\tau_{j-1}),u(\tau_j))\chi_{\tau_j < T}\} \tag{2.9}$$

The problem consists in minimizing $J(u)$ among the set of admissible controls.

2.2. The equivalent fully observed problem.

Consider as customary in the theory of non linear filtering, the operator

$$p(u(\cdot),t)(\psi) = E\{\xi(t)\psi(x(t))|F_t^{y(\cdot,u(\cdot))}\} \tag{2.10}$$

for each impulsive control $u(\cdot)$. One can view $p(u(\cdot),t)$ as a positive finite measure on R^n.

To obtain a simple form for the evolution equation of p, assume that

π_o has a density with respect to Lebesgue's measure $p_o \in L^2(R^n)$. (2.11)

Consider the Zakaï equation (controlled by $u(\cdot)$),

$$dp + A^* p\, dt = p\tilde{h}(\cdot,u(t))\cdot dz \qquad (2.12)$$
$$p(o) = p_o$$

whose solution is sought in the functional space

$$L^2(\Omega,\mathcal{A},P;C(0,T;R^n)) \cap L^2_{F^{y(\cdot,u(\cdot))}}(0,T;H^1(R^n)) \quad (2.13)$$

where the 2nd space means that the process p is adapted to the filtration $y(\cdot,u(\cdot))$. From PARDOUX [3] it follows that the solution of (2.12), (2.13) is unique, and moreover the correspondance between (2.10) and ((2.12) is given by

$$p(u(\cdot),t)(\psi) = \int_{R^n} \psi(x)p(u(\cdot),x,t)dx = (\psi,p(u(\cdot),t)) \qquad (2.14)$$

(scalar product in $L^2(R^n)$).

We can then rewrite the cost (2.9) in terms of the process $p(u(\cdot),t)$ (with values in $L^2(R^n)$)(*). However since we shall deal with unbounded functions, it is useful to consider, instead of $L^2(R^n)$, $H^1(R^n)$, Sobolev spaces with weights.

Let

$$\mu(x) = (1+|x|^2)^s \quad , \quad s > \frac{n+3}{4}$$

and $L^2(R^n;\mu)$ denotes the space of functions ϕ such that $\phi\mu \in L^2(R^n)$. Define in a similar way $L^1(R^n;\mu)$, $H^1(R^n;\mu)$. Then assume that

$$p_o \in L^2(R^n;\mu) \cap L^1(R^n;\mu) \qquad (2.15)$$

which is more stringent than (2.11). It follows that besides (2.13) the solution $p(u(\cdot),t)$ satisfies

$$p(u(\cdot),t) \in L^2(\Omega,\mathcal{A},P;C(0,T;L^2(R^n;\mu) \cap L^1(R^n;\mu)) \text{ and } L^2(0,T;H^1(R^n;\mu)). \qquad (2.16)$$

Consider the functional on $L^2(R^n;\mu)$

$$\psi(\theta) = \int \theta(x)x^2dx - \frac{[\int \theta(x)x\, dx]^2}{\int \theta(x)\, dx} \qquad (2.17)$$

which is well defined with the choice of the weight μ. Define also

$$c(\nu)(x) = c(x;\nu)$$
$$k(\nu,\nu')(x) = k(x,\nu;\nu')$$

then it is not difficult to convince oneself that the cost function J(u) can be written as

$$J(u) = E\{\psi(p(u(\cdot),T)) + \int_o^T (p(u(\cdot),t),C(u(t)))dt + \sum_{i=1}^{\infty} \chi_{\tau_i < T}\, (p(u(\cdot),\tau_i),K(u(\tau_{i-1}),u(\tau_i)))\}. \qquad (2.18)$$

(*) there is a slight abuse of notation here, since we denote in the same way the measure on R^n, $p(u(\cdot),t)$ and its density which belongs to $L^2(R^n)$.

Note that one can write (2.12) more directly in terms of dy (instead of dz), by noticing that

$$\tilde{h}(\cdot,u(t))\cdot dz = \delta(\cdot,u(t))\cdot dy(t;u(\cdot))$$

where

$$\delta(x,\nu) = \begin{bmatrix} R_1^{-1}\; h^1(x)\chi_\nu(1) \\ \vdots \\ R_M^{-1}\; h^M(x)\chi_\nu(M) \end{bmatrix}$$

3 - THE SOLUTION OF THE OPTIMIZATION PROBLEM

3.1. Setting up a system of quasi variational inequalities.

Let us consider the Banach space $H = L^2(R^n;\mu) \cap L^1(R^n;\mu)$ and the metric space H^+ of positive elements of H. Let

$\mathcal{B}$ = space of Borel, measurable, bounded functions on H^+

$\mathcal{C}$ = space of uniformly continuous, bounde functions on H^+.

Introduce also the subspaces $\mathcal{B}_1$ and $\mathcal{C}_1$ of functionals F such that

$$\|F\|_1 = \sup_{\pi \in H^+} \frac{|F(\pi)|}{1+|\pi|_\mu}$$

where $|\pi|_\mu = |\mu|_{L^1(R^n;\mu)}$.

The spaces $\mathcal{B}_1$ and $\mathcal{C}_1$ are also Banach spaces. Consider semi-groups $\phi_j(t)$ on $\mathcal{B}$ or $\mathcal{C}$, defined as follows. Freeze in (2.12) $u(t)$ as j and denote by p_j the corresponding density

$$p_j(t) = p(j,t)$$

then $p_j = p_{j,\pi}$ is the solution of

$$dp_j + A^*p_j\, dt = p_j\, \tilde{h}^j\cdot dz \qquad (3.1)$$
$$p_j(0) = \pi$$

where

$$\tilde{h}^j = \tilde{h}(\cdot,j).$$

We set

$$\phi_j(t)(F)(\pi) = E\{F(p_{j,\pi}(t))\}.$$

Then ϕ_j is a semi group on $\mathcal{B}$ or $\mathcal{C}$. It is not a semi group on $\mathcal{B}_1$, $\mathcal{C}_1$ but it has an important property. If one sets $\|\tilde{F}\|_1 = \sup_{\pi\geq 0} \frac{|F(\pi)|}{1+(\pi,1)}$ then

$$\|\tilde{\phi_h(t)(F)}\|_1 \leq \|\tilde{F}\|_1$$

which of course makes sense only for F such that $\|\tilde{F}\|_1 < \infty$. To simplify the writing we restrict ourselves to the case N=2, from now on, and we use the notation

$$c_i = c(i) \quad , \; i=1,2$$
$$k_1 = k(1,2)$$
$$k_2 = k(2,1)$$

c_1, c_2, k_1, k_2 which are bounded functions of x, correspond functionals on $\mathcal{C}_1$ via (for example)

$$c_1(\pi) = (c_1,\pi).$$

The functional $\psi(\pi)$ defined by (2.17), considered on ... belongs also to $\mathcal{C}_1$.

Consider now the set of functionals $U_1(\pi,t)$, $U_2(\pi,t)$ satisfying

$$
\begin{aligned}
&U_1, U_2 \quad C(0,T;\mathcal{C}_1)\\
&U_1(\cdot,t),\ U_2(\cdot,t) \geq 0\\
&U_1(\pi,T) = U_2(\pi,T) = \psi(\pi)\\
&U_1(\pi,t) \leq \Phi_1(s-t)U_1(s)(\pi) + \int_t^s \Phi_1(\lambda-t)C_1(\pi)d\lambda\\
&U_2(\pi,t) \leq \Phi_2(s-t)U_2(s)(\pi) + \int_t^s \Phi_2(\lambda-t)C_2(\pi)d\lambda \qquad (3.2)\\
&\forall s \geq t\\
&U_1(\pi,t) \leq k_1(\pi) + U_2(\pi,t)\\
&U_2(\pi,t) \leq k_2(\pi) + U_1(\pi,t)
\end{aligned}
$$

where we use the notation $U_i(s)(\pi) = U_i(\pi,s)$, $i=1,2$.

Then one can prove the following

Theorem 3.1 : We assume (2.1), (2.2), (2.15). Then the set of functionals U_1, U_2 satisfying (3.2) is not empty and has a maximum element, in the sense that if $\tilde{U}_1$, $\tilde{U}_2$ denotes this maximum element and U_1, U_2 satisfies (3.2) then

$$\tilde{U}_1 \geq U_1 \ , \ \tilde{U}_2 \geq U_2.$$

3.2. Interpretation of the maximum element.

Note now U_1, U_2 the maximum element, to save notation. Consider to fix the ideas $U_1(\pi,t)$ with $(\pi,1) = 1$ (π is a probability density).

One constructs a schedule as follows. Define

$$\tau_1^* = \inf_{t \leq T} \{U_1(p_1(t),t) = k_1(p_1(t)) + U_2(p_1(t),t)\}$$

and write

$$p^*(t) = p_1(t) \ , \ t \in [0,\tau_1^*].$$

Next define

$$\tau_2^* = \inf_{\tau_1^* \leq t \leq T} \{U_2(p_2(t),t) = k_2(p_2(t)) + U_1(p_1(t),t)\}$$

where $p_2(t)$ represents the solution of (3.1) with j=2 and initial condition given at τ_1^* with value $p_1/\tau_1^*)$. We then define

$$p^*(t) = p_2(t) \ , \ t \in [\tau_1^*,\tau_2^*].$$

Note that unless $\tau_1^* = T$, one has $\tau_2^* > \tau_1^*$.

One then proceeds in constructing a sequence of stopping times $\tau_1^* < \tau_2^* < \tau_3^* < \ldots$ and the process $p^*(\cdot)$. One then can prove the following

Theorem 3.2 : Under the assumptions of Theorem 3.1, one has

$$U_1(\pi,0) = \inf_{\substack{u(0)=1\\ p(0)=\pi}} J(u(\cdot))$$

and the sequence of stopping times $\tau_1^*,\tau_2^*,\ldots$ defines an optimal admissible sensor schedule.

Note that the functional $\psi(\pi)$ creates technical problems. The proof is carried over first for functionals satisfying

$$0 \leq \psi(\pi) \leq \bar{\psi}(\pi,1) \ , \text{ where } \bar{\psi} \text{ is a}$$

constant. Details can be found in BARAS-BENSOUSSAN [2].

REFERENCES

[1] A. Bensoussan, J.L. Lions, Contrôle impulsionnel et inéquations quasi-variationnelles, Dunod, Paris 1982.

[2] J.S. Baras - A. Bensoussan, Optimal sensor scheduling in non linear filtering of diffusion processes.

[3] E. Pardoux, Stochastic partial differential equations and filtering of diffusion processes, stochastics, vol. 9, N° 3 (1983).