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Well Posedness and Convergence of Some Regularization Methods for Nonlinear Ill-Posed Problems

by

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WELL-POSEDNESS AND CONVERGENCE OF SOME REGULARIZATION METHODS FOR NONLINEAR ILL-POSED PROBLEMS[1]

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ABSTRACT: In this paper we analyze two regularization methods for nonlinear ill-posed problems. The first is a penalty method called *Tikhonov regularization*, in which one solves an unconstrained optimization problem while the second is based on a constrained optimization problem. For each method we examine the well-posedness of the respective optimization problem. We then show strong convergence of the regularized 'solutions' to the true solution. (Note that this is well known for the application of these methods to linear problems.) In this analysis we consider such factors as the convergence of perturbed data to the true data, inexact solution of the respective optimization problems, and the choice of the regularization parameters.

Key words: *ill-posed, regularization, convergence, approximation*

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1. Introduction

Consider a nonlinear operator equation

$$\bar{F}(x) = \bar{y} \tag{1.1}$$

where $\bar{F} : \mathcal{X} \to \mathcal{Y}$ for Banach spaces $\mathcal{X}, \mathcal{Y}$. We assume that (1.1) has a unique solution $\bar{y}$ but is *ill-posed* in the sense that this solution does not depend continuously on the data, i.e., perturbed problems

$$F(x) = y \tag{1.2}$$

may have no solution or, if (1.2) has a (not necessarily unique) solution x, one may have a large change $(x - \bar{x})$ for the solution corresponding to very small variations in the data: $(y - \bar{y})$ and $[F(\cdot) - \bar{F}(\cdot)]$. Note that the precise specification of the nonlinear operator/function $F(\cdot)$ is considered part of the data. A 'typical' example might be a nonlinear Fredholm integral equation of first kind with $F(\cdot)$ of the form[4]:

$$[F(x)](t) := \int_0^1 ln \left[\frac{(t-\tau)^2 + H^2}{(t-\tau)^2 + [H - x(\tau)]^2} \right] d\tau \qquad (0 \leq t \leq 1) \tag{1.3}$$

which arises in inverse gravimetry [8], p.15. For other examples, see, e.g., [2], [4],[5].

The pejorative term *"ill-posed"* arose from Hadamard's attitude that such problems could not be treated usefully and would occur only by erroneously considering a problem in an unreasonable context. However, a variety of genuine applications (which, *a priori*, are reasonable contexts!) do force on us the consideration of such problems and, fortunately, techniques have been found for useful treatment.

In practical applications, the data $[F, y]$ are known only approximately, with perturbations/uncertainties arising from several sources. First, measurements are inherently inexact — although one usually can assume that this inexactitude can be arbitrarily reduced albeit with increasing cost for improved precision. Second, obtaining a tractable mathematical description of the problem introduces so-called 'modelling error'. Third, inaccuracies arise from computational limitations (e.g., approximation from a finite basis, finite-precision computer arithmetic, ...); again, these can usually be arbitrarily improved with increased cost. In effect, one must always deal with perturbed problems (1.2).

In order to obtain reasonable approximations to the 'true' solution $\bar{x}$ of (1.1), one must solve a 'regularized' problem constructed from the perturbed data:

$$[F, y, \mu] \longmapsto x \tag{1.4}$$

where μ parametrizes the approximation scheme in a way which reflects *a priori* information, estimates of the perturbation magnitude, etc. Such a regularized problem

[4] The value of the parameter $H > 0$ is experimentally determined. The 'true' function has a similar form with the 'true' value $\bar{H}$

should have the properties:

(1.5) (*i*) *well-posedness*: given the triple $[F, y, \mu]$, one can, indeed, obtain x in (1.4); further with μ fixed the x obtained is unique and depends stably (continuously) on $[F, y]$, at least for $[F, y]$ 'close to' $[\bar{F}, \bar{y}]$;

(*ii*) *convergence*: given a sequence $[F, y]_k \to [\bar{F}, \bar{y}]$, appropriate choice of $\mu = \mu_k$ gives $x_k \to \bar{x}$ via (1.4).

The condition (i) is of great practical importance. Since the regularized problem defined by $[F, y, \mu]$ is to be solved computationally, this problem should be robust. The condition (ii) is of obvious theoretical importance. It ensures that the true solution $\bar{x}$ can be obtained with arbitrary precision if one will 'pay the price' — this is the same result as for so-called 'well-posed problems' although here one expects a much more rapid increase of cost with demands for increased precision to the extent that requests for more than moderate accuracy become practically infeasible.

Perhaps the best known such regularization technique is the method of *Tikhonov regularization* [7] in which (taking the abstract parameter μ of (1.4) to be a number $\alpha > 0$) one obtains solutions x_α by solving the unconstrained minimization problem:

$$\|F(x) - y\|_{\mathcal{Y}}^2 + \alpha J(x) = \min \qquad (x \in \mathcal{X}) \tag{1.6}$$

where $J(\cdot)$ is a non-negative penalty function suitably chosen to incorporate *a priori* information about the true solution. (Often one takes $J(x) := \|\mathbf{L}x - z\|_{\mathcal{Z}}^2$ where the choice of $\mathbf{L} : \mathcal{X} \to \mathcal{Z}$ indicates assumptions as to the regularity of $\bar{x}$; see (2.2).)

Alternatively, one may choose an explicit bound β on the penalty term $J(x)$ and solve the constrained minimization problem

$$\|F(x) - y\|_{\mathcal{Y}}^2 = \min \qquad (x \in \mathcal{X},\ J(x) \leq \beta) \tag{1.7}$$

to obtain an approximate solution x_β (using (1.7) to define (1.4) with μ replaced by $\beta > 0$). See [8] where this regularization technique is used to solve the nonlinear integral equation given by using (1.3) in (1.2).

The object of this report is to examine the abstract techniques (1.6), (1.7) under reasonable hypotheses for which we can demonstrate the properties (1.5). A similar analysis was carried out in [3] for the *method of generalized interpolation* and variants.

2. Well-Posedness of Regularized Problems

We will be considering, abstractly, schemes of the forms (1.6) or (1.7) for which the basic ingredients are:

- the spaces $\mathcal{X}, \mathcal{Y}$;
- the penalty function $J(\cdot) : \mathcal{X} \to [0, \infty]$ (but $J \not\equiv \infty$);
- the form of $F(\cdot)$ with a topology for perturbations.

We begin with assumptions on $\mathcal{X}$, $J(\cdot)$:

(2.1) (*i*) $\mathcal{X}$ is a Banach space and there is given a continuous map $\mathbf{P}_0 : \mathcal{X} \to \mathcal{X}_0 =$ [another Banach space];

(*ii*) for each (finite) $\gamma \geq 0$ and any sequence $\{x_k\}$ in $\mathcal{X}$ such that $J(x_k) \leq \gamma$ and $\{\mathbf{P}_0 x_k\}$ is bounded in $\mathcal{X}_0$, there is a subsequence $\{x_{k(j)}\}$ converging weakly in $\mathcal{X}$ to some $\hat{x}$ for which $J(\hat{x}) \leq \gamma$.

The condition (*ii*) is essentially lower semicontinuity of $J(\cdot)$ together with a coercivity condition on $[J(x) + \|\mathbf{P}_0 x\|]$.

As a typical setting for applications, consider a Hilbert space $\mathcal{X}$, a closed (densely defined) linear operator $\mathbf{L} : \mathcal{X} \to \mathcal{Z} =$ [another Hilbert space], $\mathbf{P}_0$ the orthogonal projection: $\mathcal{X} \to \mathcal{X}_0 := \mathcal{N}(\mathbf{L})$ and set

$$J(x) := \{\|\mathbf{L}x - \bar{z}\|_{\mathcal{Z}}^2 \text{ for } x \in \mathcal{D}(\mathbf{L}); \ \infty \text{ else}\}. \tag{2.2}$$

For example, it might be convenient to take $\mathcal{X} = \mathcal{Y} = L^2(0,1)$ and let $\mathbf{L} = d/dt$ so J effectively penalizes the H^1-norm; here one would have $\mathcal{X}_0 = \{constants\}$.

Suppose $\mathbf{L}$ were to have a continuous right inverse $\mathbf{R} : \mathcal{Z} \to \mathcal{N}(\mathbf{P}_0)$ so $\mathbf{LR}z = z$ for $z \in \mathcal{Z}$. Then, given any sequence $\{x_k\} \in \mathcal{X}$, we may write $x_k = u_k + v_k$ with $u_k := \mathbf{P}_0 x_k \in \mathcal{N}(\mathbf{L})$ so $J(x_k) = J(v_k)$. If $J(x_k) \leq \gamma < \infty$ then $x_k, v_k \in \mathcal{D}(\mathbf{L})$ and we set $z_k := \mathbf{L}x_k = \mathbf{L}v_k \in \mathcal{Z}$. Note that $\mathbf{R}z_k = v_k$ since $\mathbf{LR}z_k = z_k = \mathbf{L}v_k$ gives $(\mathbf{R}z_k - v_k) \in \mathcal{N}(\mathbf{L})$ whereas v_k, $\mathbf{R}z_k \in \mathcal{N}(\mathbf{P}_0)$. We have $\|z_k\|_{\mathcal{Z}} \leq \gamma^{1/2} + \|\bar{z}\|_{\mathcal{Z}}$ so there is a subsequence $\{k(j)\}$ for which $z_{k(j)} \rightharpoonup \hat{z}$ (weak convergence in $\mathcal{Z}$) whence $v_{k(j)} = \mathbf{R}z_{k(j)} \rightharpoonup \mathbf{R}\hat{z} =: \hat{v}$ (weak convergence in $\mathcal{N}(\mathbf{P}_0) \subset \mathcal{X}$; with that $\hat{v} \in \mathcal{D}(\mathbf{L})$ since J is lower semicontinuous for this weak topology). Since we assume $\{u_k\}$ is bounded, we may extract further a subsubsequence $\{k'(j)\}$ for which also $u_{k'(j)} \rightharpoonup \hat{u}$ (weak convergence in $\mathcal{N}(\mathbf{L}) \subset \mathcal{X}$) so $x_{k'(j)} \rightharpoonup (\hat{u} + \hat{v}) =: \hat{x}$. Clearly $\mathbf{L}\hat{x} = \mathbf{L}\hat{v} = \hat{z} = w - \lim z_{k(j)}$ so $J(x_n) = \|z_k - \bar{z}\|_{\mathcal{Z}}^2 \leq \gamma$ gives $J(\hat{x}) = \|\hat{z} - \bar{z}\|_{\mathcal{Z}}^2 \leq \gamma$ and we have demonstrated (2.1)(ii). Note, also, that if we can only verify the surjectivity of $\mathbf{L}$, hence the surjectivity of the restriction $\mathbf{L}_0$ of $\mathbf{L}$ to $\mathcal{N}(\mathbf{P}_0)$, then it follows from the Closed Graph Theorem that a continuous $\mathbf{R}$ exists as above.

We have proved the following:

Lemma 1: *Let $\mathcal{X}, \mathcal{Z}$ be Hilbert spaces and $\mathbf{L} : \mathcal{X} \supset \mathcal{D}(\mathbf{L}) \to \mathcal{Z}$ a closed linear operator such that $\mathbf{L}x = z$ is solvable for each $z \in \mathcal{Z}$. Then, defining $J(\cdot)$ by (2.2) and taking $\mathbf{P}_0$ to be the orthogonal projection on $\mathcal{X}_0 := \mathcal{N}(\mathbf{L})$, we have (2.1).*

□

Clearly, using $J(\cdot)$ as in Lemma 1 for (1.6) or (1.7) gives x_α (resp., x_β) in $\mathcal{D}(\mathbf{L})$. Similar arguments permit the construction of such $J(\cdot)$ for more general $\mathcal{X}$ than Hilbert spaces — essentially, one needs reflexivity of $\mathcal{N}(\mathbf{L})$ and of $\mathcal{Z}$ and the existence of a closed complement to $\mathcal{N}(\mathbf{L})$ in $\mathcal{X}$.

Next, we consider assumptions on $\mathcal{Y}$ and on the admissible functions $F(\cdot)$:

(2.3) (i) $\mathcal{Y}$ is a Banach space; $F(\cdot) : \mathcal{X} \supset \mathcal{D}(F) \longrightarrow \mathcal{Y}$ with $\{x \in \mathcal{D}(F) \subset \mathcal{X} : J(x) < \infty\}$ nonempty;

(ii) for any sequence $\{x_k\}$ in $\mathcal{D}(F)$ such that $x_k \rightharpoonup \hat{x}$ weakly in $\mathcal{X}$ with $\{F(x_k)\}$ bounded in $\mathcal{Y}$, we have $\hat{x} \in \mathcal{D}(F)$ and $F(x_k) \rightharpoonup F(\hat{x})$ weakly in $\mathcal{Y}$;

(iii) $F(\cdot)$ is '$\mathbf{P}_0$ – coercive': i.e., if $\{F(x_k)\}$ is defined and bounded in $\mathcal{Y}$ then $\{\mathbf{P}_0 x_k\}$ is bounded in $\mathcal{X}_0$.

For (i), (iii) we are, of course, considering the same $J(\cdot)$, $\mathbf{P}_0$ as in (2.1). We remark that (ii) and (iii) will only be applied to sequences for which $\{J(x_k)\}$ is also bounded and so need be verified only in that context. Note that, for reflexive $\mathcal{Y}$, the condition (ii) is equivalent to assuming that the graph of $F(\cdot)$ is closed with respect to sequential weak convergence in $\mathcal{X} \times \mathcal{Y}$.

As a typical setting for applications, consider $\mathcal{X}, \mathcal{Y}$ reflexive and $F(\cdot)$ of the form:

$$F(x) := \mathbf{A}(z)x + G(z) \qquad \text{with } z := \mathbf{B}x \tag{2.4}$$

where $\mathbf{B}$ is a compact linear map: $\mathcal{X} \to \mathcal{Z} =$ [another Banach space] and
$\mathbf{A}(\cdot) : \mathcal{Z} \to \mathcal{B}(\mathcal{X}, \mathcal{Y}) :=$ [continuous linear maps: $\mathcal{X} \to \mathcal{Y}$],
$G : \mathcal{Z} \to \mathcal{Y}_w :=$ [$\mathcal{Y}$ topologized by sequential weak convergence]
are continuous nonlinear maps. If $x_k \rightharpoonup \hat{x}$ weakly in $\mathcal{X}$, then one easily sees that $z_k := \mathbf{B}x_k \to \hat{z} := \mathbf{B}\hat{x}$ strongly in $\mathcal{Z}$ whence $\mathbf{A}_k := \mathbf{A}(z_k) \to \mathbf{A} := \mathbf{A}(\hat{z})$ in $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $g_k := G(Z_k) \rightharpoonup \hat{g} := G(\hat{z})$ weakly in $\mathcal{Y}$. Now

$$\mathbf{A}_k x_k - \mathbf{A}\hat{x} = (\mathbf{A}_k - \mathbf{A})x_k + \mathbf{A}(x_k - \hat{x})$$

and the first term on the right goes strongly to 0 (as $\|\mathbf{A}_k - \mathbf{A}\| \to 0$ and $\{x_k\}$ is bounded) while the second term goes weakly to 0 in $\mathcal{Y}$ as $x_k \rightharpoonup \hat{x}$ in $\mathcal{X}$. Thus one has (2.3)(i), (ii); verification of (2.3)(iii) is likely to be more application-specific.

Theorem 1: *Suppose (2.1), (2.3) hold. Then, for arbitrary $y \in \mathcal{Y}$, each of the problems (1.6) (for arbitrary $\alpha > 0$) and (1.7) (for large enough $\beta > 0$) has a solution, i.e., the minimum is attained in each case.*

PROOF : We first consider the constrained problem (1.7). Set

$$S_\beta := \{x \in \mathcal{D}(F) \subset \mathcal{X} : J(x) \leq \beta\}$$

and note that $\mathcal{S}_\beta$ must be nonempty for large enough (finite) β by (2.1)(i). Suppose $\{x_k\}$ is any minimizing sequence for $\|F(x)-y\|_\mathcal{Y}$ on $\mathcal{S}_\beta$. Certainly $\{F(x_k)\}$ is bounded in $\mathcal{Y}$ so, by (2.3)(iii), $\{\mathbf{P}_0 x_k\}$ is also bounded in $\mathcal{X}_0$. By (2.1)(ii), it then follows that there is a subsequence (again denoted by $\{x_k\}$ for simplicity) which converges weakly in $\mathcal{X}$, i.e., $x_k \rightharpoonup \hat{x}$ with $\mathcal{J}(\hat{x}) \leq \beta$. Now, by (2.3)($ii$) we have $\hat{x} \in \mathcal{D}(F)$ (so $\hat{x} \in \mathcal{S}_\beta$) and $F(x_k) \rightharpoonup F(\hat{x})$ weakly in $\mathcal{Y}$. By the lower semicontinuity of the $\mathcal{Y}$-norm, it then follows that

$$\|F(\hat{x}) - y\|_\mathcal{Y}^2 \leq \liminf \|F(x_k) - y\|_\mathcal{Y}^2 = \inf\{\|F(x) - y\|_\mathcal{Y}^2 : x \in \mathcal{S}_\beta\}.$$

Thus the minimum over $\mathcal{S}_\beta$ is attained at $\hat{x}$.

The proof for (1.6) is much the same. Given any minimizing sequence $\{x_k\}$ for the unconstrained problem (1.6), we necessarily have both $\{\|F(x_k) - y\|\}$ and $\{\mathcal{J}(x_k)\}$ bounded since each is non-negative and $\alpha > 0$. As above, there is a subsequence converging weakly in $\mathcal{X}$ to some $\hat{x}$ and we again obtain $\hat{x} \in \mathcal{D}(F)$ and the minimum is attained at $\hat{x}$. □

This shows the existence part of (1.5)(i) for these two regularization methods. Uniqueness is rather more difficult, independent of any difficulties due to ill-posedness of (1.2). One possible way to obtain uniqueness (and continuous dependence on y) for (1.6) would be to require that the functional $\|F(x) - y\|_\mathcal{Y}^2$ be convex in x, making its derivative (assuming sufficient regularity) a monotone operator so the regularizing term can provide strict monotonicity and so uniqueness of the minimizer. While this is immediate for linear $F(\cdot)$ in Hilbert space settings, it is likely to be somewhat restrictive for nonlinear functions, yet can, on occasion, prove useful; see [9]. A certain weaker continuity property with respect to perturbation of $y \in \mathcal{Y}$ can be obtained without imposing any new assumptions.

Theorem 2: *Assume (2.1), (2.3). Fixing $\alpha > 0$ in (1.6), let x_n be a minimizer for (1.6), i.e., for (1.6) with $y = y_n$ where $y_n \to y_*$ in $\mathcal{Y}$. Then there is some subsequence $\{x_{n(k)}\}$ converging weakly in $\mathcal{X}$ to some minimizer x_* for (1.6), i.e., (1.6) with $y = y_*$; if (1.6) has a unique minimizer, then $x_n \rightharpoonup x_*$. A similar result holds for (1.7).*

PROOF : For (1.6), set

$$Q(x, y) := \|F(x) - y\|_\mathcal{Y}^2 + \alpha \mathcal{J}(x)$$

for $x \in \mathcal{D}(F) \subset \mathcal{X}$ and $y \in \mathcal{Y}$. We have, for any $\varepsilon > 0$, existence of C_ε such that

$$(a + b)^2 \leq [1 + \varepsilon]a^2 + C_\varepsilon b^2 \qquad (a, b \in \mathbb{R})$$

whence, as $\alpha\mathcal{J} \geq 0$, one has

$$\text{(2.5)} \qquad Q(x, y) \leq (1 + \varepsilon)Q(x, y') + C_\varepsilon\|y - y'\|_\mathcal{Y}^2 \qquad (x \in \mathcal{D}(F);\ y, y' \in \mathcal{Y}).$$

From this we obtain, with x_0 some minimizer for (1.6),

$$\begin{aligned} Q(x_n, y_*) &\leq (1+\varepsilon)Q(x_n, y_n) + C_\varepsilon \|y_n - y_*\|_{\mathcal{Y}}^2 && \text{by (2.5)} \\ &\leq (1+\varepsilon)Q(x_0, y_n) + C_\varepsilon \|y_n - y_*\|_{\mathcal{Y}}^2 && \text{by minimality for } x_n \\ &\leq (1+\varepsilon)^2 Q(x_0, y_*) + (2+\varepsilon) C_\varepsilon \|y_n - y_*\|_{\mathcal{Y}}^2 && \text{by (2.5).} \end{aligned}$$

The right hand side can be made arbitrarily close to the minimum $Q(x_0, y_*)$ by first taking $\varepsilon > 0$ small and then $\|y_n - y_*\|$ small enough. Hence $\{x_n\}$ is a minimizing sequence for (1.6). The existence of the subsequence $x_{n(k)} \rightharpoonup x_*$ then follows as in the proof of Theorem 1 which also shows that x_* minimizes (1.6). Convergence of the full sequence when the limit of convergent subsequences is unique is a standard argument. The proof for (1.7) is essentially the same. □

3. Convergence of Regularized Solutions

For our convergence analysis we assume a sequence of perturbed problems

$$F_k(x) = y_k \tag{3.1}$$

with data $[F_k, y_k]$ converging, in a sense to be made precise below, to the 'true' data $[\bar{F}, \bar{y}]$ of (1.1). We do not, of course, consider (3.1) directly (i.e., as an equation determining possible solutions x_k) but wish to use the data to consider a corresponding sequence of regularized problems — either using Tikhonov regularization as in (1.6) or constrained minimization as in (1.7). From the regularized problems we obtain a sequence $\{x_k\}$ and, under reasonable assumptions on the problem and our approaches, we wish to demonstrate convergence to the 'true solution' $\bar{x}$:

$$x_k \to \bar{x} \qquad \text{strongly in } \mathcal{X}. \tag{3.2}$$

For the method of Tikhonov regularization, we modify[5] (1.6) slightly and consider the unconstrained *approximate* minimization problem:

$$\|F_k(x) - y_k\|_{\mathcal{Y}}^2 + \alpha_k \mathcal{J}(x) \leq \inf + \delta_k \qquad (x \in \mathcal{D}(F_k) \subset \mathcal{X}) \tag{3.3}$$

where $\delta_k > 0$ is a small parameter. (We will later take $\alpha_k \to 0$ and $\delta_k \to 0$ as $[F_k, y_k] \to [\bar{F}, \bar{y}]$). Similarly, we modify[6] the regularization given through (1.7) and consider the approximate problem:

$$\|F_k(x) - y_k\|_{\mathcal{Y}}^2 \leq \inf + \delta_k \qquad (x \in \mathcal{D}(F_k) \subset \mathcal{X};\ \ \mathcal{J}(x) \leq \beta_k). \tag{3.4}$$

[5]This modification reflects computational reality: one never actually expects to obtain the exact minimum (even when it is attained) but can find x giving values arbitrarily close to that. A side effect of this modification is that (3.3) makes sense, theoretically and computationally, even when the minimum may *not* be attained; this will permit us to relax somewhat the restrictions on $F(\cdot)$ which would be imposed by (2.3).

[6]The significance of the modification (replacing "$=$ min" by "$\leq \inf + \delta_k$") is the same here as indicated above for (1.6), (3.3).

We assume $\mathcal{D}(F_k)$ is nonempty and β_k is large enough that

(3.5) $$S_k := \{x \in \mathcal{D}(F_k) \subset \mathcal{X} : J(x) \leq \beta_k\} \neq \phi \qquad (k = 1, 2, \ldots)$$

so (approximate) minimization over S_k is meaningful in (3.4).

Most applications can be formulated so $\mathcal{D}_k := \mathcal{D}(F_k) = \mathcal{D}(\bar{F}) =: \mathcal{D}_*$, independent of k — indeed, usually with $\mathcal{D}_k = \mathcal{D}_* = \mathcal{X}$. Occasionally, however, it is convenient to incorporate partly in the specification of $\mathcal{D}_k$ a computational implementation of F_k defined, e.g., only for 'mesh functions'. For simplicity[7] we take $\mathcal{X}$ as fixed and embed $\mathcal{D}_k$ (e.g., as a subspace) in $\mathcal{X}$ with suitable approximation properties familiar from the numerical analysis literature; similarly, we assume the codomain of each $F_k(\cdot)$ is (embedded in) the fixed Banach space $\mathcal{Y}$. Note that in applications the approximating nature of $F_k(\cdot)$ as a perturbation of $\bar{F}(\cdot)$ includes the treatment of modelling errors, the specification of relevant parameter values (in some general form of $F(\cdot)$ — e.g., as the value of H in (1.3)), and also the nature of the computational implementation to be used.

The relevant notion of convergence, $F_k(\cdot) \to \bar{F}(\cdot)$, is most easily viewed as a geometric notion of convergence for the graphs, considered as subsets of $\mathcal{X} \times \mathcal{Y}$. Specializing the definition from [3] to the present case, we have:

DEFINITION : We say "$\{F_k(\cdot)\}$ is *graph-subconvergent* to $\bar{F}(\cdot)$" if:

(3.6) Given any subsequence $\{k(j)\}$ and a sequence $\{\hat{x}_j\}$ in $\mathcal{X}$ such that
$\hat{x}_j \in \mathcal{D}(F_{k(j)})$; $\hat{x}_j \rightharpoonup \hat{x}$ weakly in $\mathcal{X}$ and
$\hat{y}_j := F_{k(j)}(\hat{x}_j) \longrightarrow \hat{y}$ strongly in $\mathcal{Y}$,
we have $\hat{x} \in \mathcal{D}(\bar{F})$ and $\hat{y} = \bar{F}(\hat{x})$.

If, in addition, we have:

(3.7) For each $\tilde{x} \in \mathcal{D}(\bar{F})$ there is some sequence $\{\tilde{x}_k\}$ in $\mathcal{X}$
such that $\tilde{x}_k \in \mathcal{D}(F_k)$ with $\tilde{x}_k \to \tilde{x}$ strongly in $\mathcal{X}$
and $F_k(\tilde{x}_k) \to F(\tilde{x})$ strongly in $\mathcal{Y}$

then we say "$\{F_k(\cdot)\}$ is *graph-convergent* to $\bar{F}(\cdot)$".

See [3], [4] for further discussion and some examples. We remark here that (3.7) will be needed only for $\tilde{x} = \bar{x}$, the (unique) solution of (1.1), and need only be verified for $\bar{x}$.

We will continue to impose (2.1), as before, but now adjoin an additional assumption regarding the penalty function $J(\cdot)$:

(3.8) For any weakly convergent sequence $\{x_k\}$ in $\mathcal{X}$ (so $x_k \rightharpoonup x_*$),
if $J(x_k) \to J(x_*) < \infty$ then the sequence $\{x_k\}$ is strongly convergent in $\mathcal{X}$.

[7] Generalization of this framework is possible but, for our present purposes, seems an unwarranted complication.

It is not difficult to relate this to a geometric condition on the level surfaces of $\mathcal{J}(\cdot)$: (3.8) is implied by the condition:

(3.9) Given $\hat{x} \in \mathcal{X}$ with $\mathcal{J}(\hat{x}) < \infty$ and given $\varepsilon > 0$, there is a $\delta > 0$ and a 'cylinder set' $\mathcal{C} \subset \mathcal{X}$ of the form
$\mathcal{C} := \{x \in \mathcal{X} : |\langle \xi_k, x - \hat{x}\rangle| < \delta \quad \text{for } k = 1, \ldots, K\}$
(with each $\xi_k \in \mathcal{X}^*$) such that $\hat{x} \in \mathcal{C}$ and
$\mathcal{C} \cap \{x : |\mathcal{J}(x) - \mathcal{J}(\hat{x})| < \delta\} \subset \mathcal{B}_\varepsilon(\hat{x}) := \{x \in \mathcal{X} : \|x - \hat{x}\|_{\mathcal{X}} < \varepsilon\}.$

It is well known that, in any uniformly convex Banach space $\mathcal{X}$, the norm (or any strictly increasing function of it) has this property (3.9) — indeed, one can take $K = 1$ and ξ_1 to be the support functional at $\hat{x}$ to $\{x \in \mathcal{X} : \|x\| \le \|\hat{x}\|\}$ in constructing $\mathcal{C}$ — and so $\mathcal{J}(x) := \|x\|_{\mathcal{X}}^2$ satisfies (3.8).

Returning to (2.2) in the setting of Lemma 1, we note that $\mathcal{D}(\mathbf{L})$ becomes a Hilbert space (*a fortiori* uniformly convex) under the norm $[\|\mathbf{P}_0 x\|^2 + \mathcal{J}(x)]^{1/2}$; the topologies of weak convergence are compatible. If $\mathcal{X}_0 := \mathcal{N}(\mathbf{L})$ is finite dimensional then weak convergence in $\mathcal{X}$ already gives strong convergence in $\mathcal{X}_0$ and (3.8) is easily verified; similar considerations apply to more general convex penalty functions.

In view of the modifications of (1.6), (1.7) and our somewhat different present perspective, we discard the earlier assumptions (2.3) as possibly applying to each of the $F_k(\cdot)$. Instead, we introduce new assumptions on the limit problem (1.1) and on the sequence $\{F_k(\cdot)\}$:

(3.10)
(*i*) $\mathcal{Y}$ is a Banach space; $\bar{F}(\cdot) : \mathcal{X} \supset \mathcal{D}(\bar{F}) \longrightarrow \mathcal{Y}$; $F_k(\cdot) : \mathcal{X} \supset \mathcal{D}(F_k) \to \mathcal{Y}$ for $k = 1, 2, \ldots$; $\{F_k(\cdot)\}$ is graph-subconvergent to $\bar{F}(\cdot)$ in the sense of (3.6);
(*ii*) there is some sequence $\{\tilde{x}_k\}$ with $\tilde{x}_k \in \mathcal{D}(F_k)$ such that $\tilde{x}_k \to \bar{x}$ strongly in $\mathcal{X}$, $\mathcal{J}(\tilde{x}_k) \to \bar{\beta}$, and $F_k(\tilde{x}_k) \to \bar{y}$ strongly in $\mathcal{Y}$;
(*iii*) the sequence $\{F_k(\cdot)\}$ is '$\mathbf{P}_0$-coercive': if $\{F_k(x_k)\}$ is defined and bounded in $\mathcal{Y}$ then $\{\mathbf{P}_0 x_k\}$ is bounded in $\mathcal{X}_0$.

Typically, in applications one has $\mathcal{D}(F_k) \supset \mathcal{D}(\bar{F})$ (e.g., $\mathcal{D}(F_k) = \mathcal{X}$) and (3.10)(*iii*) is automatic: one simply takes $\tilde{x}_k = \bar{x}$ for $k = 1, 2, \ldots$. Note that the earlier set of assumptions (2.3) just corresponds to (3.10) with $F_k(\cdot) = \bar{F}(\cdot) = F(\cdot)$.

We are now ready to demonstrate, separately, the convergence (3.2) for each of the regularization techniques: (3.3) and (3.4). We consider first the approach by constrained minimization.

Theorem 3: *Assume (2.1), (3.8). Assume $[\bar{F}, \bar{y}]$ is such that (1.1) has a unique solution $\bar{x} \in \mathcal{X}$; assume (3.10) and that $y_k \to \bar{y}$ strongly in $\mathcal{Y}$. Let $\beta_k \to \bar{\beta}$ with each β_k large enough to give (3.5); this is always possible. Let $\{x_k\}$ be any sequence satisfying (3.4) for each k with $0 < \delta_k \to 0$; such sequences always exist. Then one has strong convergence: $x_k \to \bar{x}$ strongly in $\mathcal{X}$.*

PROOF : By (3.10)(*ii*) one need only take, e.g., $\beta_k \geq P(\tilde{x}_k) \to \bar{\beta}$ to have (3.5); the form of (3.4) with $0 < \delta$ then ensures existence[8] of x_k satisfying (3.4) for each k.

Clearly $\{\beta_k\}$ is bounded so $\{J(x_k)\}$ is bounded. Since (3.4) gives

$$\|F_k(x_k) - y_k\|_{\mathcal{Y}}^2 \leq \left(\|F_k(\tilde{x}_k) - \bar{y}\|_{\mathcal{Y}} + \|\bar{y} - y_k\|_{\mathcal{Y}}\right)^2 + \delta_k$$

and the properties of $\{\tilde{x}_k\}, \{y_k\}, \{\delta_k\}$ make the right hand side go to 0, we see that $\{F_k(x_k)\}$ is not only bounded but, also, $F_k(x_k) \to \bar{y}$ strongly in $\mathcal{Y}$. By (3.10)(*iii*), we then have $\{\mathbf{P}_0 x_k\}$ bounded in $\mathcal{X}_0$ as well. The assumption (2.1)(*ii*) then applies and we have existence of a subsequence $\{x_{k(j)}\}$ such that $\hat{x}_j := x_{k(j)} \rightharpoonup \hat{x}$ weakly in $\mathcal{X}$ for some $\hat{x}$. Applying (3.10)(*i*), the definition (3.6) of graph-subconvergence ensures that $\hat{x} \in \mathcal{D}(\bar{F})$ and $\bar{F}(\hat{x}) = \lim F_{k(j)}(\hat{x}_j) = \bar{y}$. The assumed uniqueness of the solution of (1.1) then implies that $\hat{x} := w - \lim x_{k(j)}$ must be $\bar{x}$ and so, by a standard argument, that $x_j \rightharpoonup \bar{x}$ weakly in $\mathcal{X}$.

Now suppose $\liminf J(x_k) < \bar{\beta}$. We could then find $\alpha < \bar{\beta}$ with $J(x_k) \leq \alpha$ for large k. Applying (2.1)(*ii*) to $\{x_k : k \geq K\}$ gives a subsequence $\{x_{k(j)}\}$ such that $x_{k(j)} \rightharpoonup \hat{x}$ weakly in $\mathcal{X}$ with $J(\hat{x}) \leq \alpha < \bar{\beta}$. Since we have already shown $x_k \rightharpoonup \bar{x}$, this is a contradiction. Thus, $\liminf J(x_k) \geq \bar{\beta}$. On the other hand, $J(x_k) \leq \beta_k$ by (3.4) and $\beta_k \to \bar{\beta}$ so $\limsup J(x_k) \leq \bar{\beta}$. This shows $J(x_k) \to \bar{\beta} := J(\bar{x})$ and (3.8) gives the desired strong convergence. □

The argument in the case of Tikhonov regularization is rather similar. In this case we will need a condition on the sequence $\{\alpha_k\}$ of regularizing parameters — it must go to 0 but must do so 'slowly enough'. We will be assuming $y_k \to \bar{y}$ in $\mathcal{Y}$ and existence of a sequence $\{\tilde{x}_k\}$ as in (3.10)(*ii*). Set

(3.11) $$\tilde{\nu}_k := \|F_k(\tilde{x}_k) - y_k\|_{\mathcal{Y}} \quad , \quad \tilde{\gamma}_k := J(\tilde{x}_k).$$

We will require

(3.12) $$(i) \quad 0 < \alpha_k \longrightarrow 0; \qquad (ii) \quad \tilde{\nu}_k^2/\alpha_k \longrightarrow 0$$

and remark here that such sequences $\{\alpha_k\}$ always exist since $\tilde{\nu}_k \to 0$, noting that $F_k(\tilde{x}_k) \to \bar{y}$ by (3.10)(*ii*) and $y_k \to \bar{y}$.

[8] At this point we remark that the mere existence of such x_k is not really at issue — after all, we were willing to assume (3.10)(*ii*). The point is that, given (3.5), we can expect a feasible implementation (for each k) enabling us actually to *compute* explicitly an x_k satisfying (3.4). It is the convergence to $\bar{x}$ of this computed sequence which is the real point of this theorem.

Theorem 4: *Assume (2.1), (3.8). Assume $[\bar{F},\bar{y}]$ is such that (1.1) has a unique solution $\bar{x} \in \mathcal{X}$; assume (3.10) and that $y_k \to \bar{y}$ strongly in $\mathcal{Y}$. Let α_k satisfy (3.12) and take $0 < \delta_k$ with $\delta_k/\alpha_k \to 0$. Then for any sequence $\{x_k\}$ satisfying (3.3) we have strong convergence: $x_k \to \bar{x}$ strongly in $\mathcal{X}$.*

PROOF : The form of (3.3) with $\alpha_k > 0$ ensures existence of a solution x_k for each (3.3). As above for (3.4), we remark that we are interested in the particular x_k obtained by some explicit computational procedure; for this $\{x_k\}$ set

$$\nu_k := \|F_k(x_k) - y_k\|_{\mathcal{Y}} \ ,\gamma_k := J\,(x_k).$$

The minimality property (3.3) gives

$$\nu_k^2 + \alpha_k\gamma_k \le \tilde{\nu}_k^2 + \alpha_k\tilde{\gamma}_k + \delta_k. \tag{3.13}$$

The right side goes to 0 as $\tilde{\nu}_k \to 0$, $\alpha_k \to 0$, $\delta_k \to 0$, $\tilde{\gamma}_k \to \bar{\beta}$; hence, as in the previous proof, we have $\{F_k(x_k)\}$ bounded with $F_k(x_k) \to \bar{y}$ strongly in y and, again by (3.10)(iii), $\{\mathbf{P}_0x_k\}$ is bounded in $\mathcal{X}_0$ as well.

To bound $\{\gamma_k\}$, we observe that the estimate (3.10) can be divided by α_k to give

$$\gamma_k \le (\tilde{\nu}_k^2/\alpha_k) + \tilde{\gamma}_k + (\delta_k/\alpha_k).$$

We have assumed $\delta_k/\alpha_k \longrightarrow 0$ and, by (3.12)(ii), that $\tilde{\nu}_k^2/\alpha_k \longrightarrow 0$; by (3.10)$(ii)$ we have $\tilde{\gamma}_k \to \bar{\beta}$. Thus, we have

$$\limsup J\,(x_k) \le \bar{\beta}. \tag{3.14}$$

In particular, $\{J\,(x_k)\}$ is bounded and (2.1)(ii) applies to give existence of a subsequence $\{x_{k(j)}\}$ with $x_{k(j)} \rightharpoonup \hat{x}$ weakly in $\mathcal{X}$. The graph-subconvergence (3.10)(i) ensures that $\hat{x} \in \mathcal{D}(\bar{F})$ with $\bar{F}(\hat{x}) = \lim F_{k(j)}(x_{k(j)}) = \bar{y}$ (since $\nu_{k(j)} \to 0$ by (3.12)).

The argument that $\liminf J\,(x_k) \ge \bar{\beta}$ is exactly as in the proof of Theorem 3 and, with (3.11), again gives $J\,(x_k) \to \bar{\beta} := J\,(\bar{x})$. Again, application of (3.8) gives the desired strong convergence (3.2). □

4. An Exemplary Application

In this section we will discuss an important ill-posed inverse problem arising in 'remote sensing' of the atmosphere. Here one wishes to estimate the atmospheric temperature profile from infrared radiation measurements taken by a satellite at the top of the atmosphere.

The relationship between radiative intensity I and temperature T is modelled by the nonlinear integral equation

$$I(\nu) = f_S(\nu, T(a)) + \int_a^b f(\nu, p, T(p))\,dp \tag{4.1}$$

Here ν is the wavenumber and p denotes atmospheric pressure, a monotonically decreasing function of height above the surface. The function f_S models the contribution from the surface and is continuous in both its arguments. The function f is nonlinear in T and depends smoothly on all three arguments[9]. Making the identifications

$$y = I(\cdot), \quad x = T(\cdot), \text{ and } F(x) = f_S(\cdot, x(a)) + \int_a^b f(\cdot, p, x(p))\, dp$$

gives a nonlinear operator equation: $F(x) = y$. In practice, measurements of y (i.e., of $I(\cdot)$) are subject to instrument error. Moreover, simplifying assumptions are used to derive the model equation (4.1). Thus we must deal with a perturbation of the underlying 'true' problem $\bar{F}(x) = \bar{y}$.

Since we expect atmospheric temperature to vary smoothly with height, we assume that x lies in $\mathcal{X} := H^1(a, b)$. We will make the (physically reasonable) assumption that the measurements y lie in $\mathcal{Y} := L^2(c, d)$. Under these assumptions, the problem: $F(x) = y$ is ill-posed. From the continuity of f_S, the smoothness of the kernel f, and the fact that weak convergence in $H^1[a, b]$ implies strong convergence in $C[a, b]$, one can easily show that the operator $F : \mathcal{X} \to \mathcal{Y}$ is weakly continuous. If we define the penalty functional $\mathcal{J}(x)$ as in (2.2) with $\mathcal{Z} := L^2(a, b)$ and $\mathbf{L}x := dx/dt$, then the results of the previous sections hold. Numerical results using the method of Tikhonov regularization (1.6) have been obtained for this problem by O'Sullivan and Wahba [2].

Now let us consider a modification of this. In practice, at a certain height the gradient of the temperature of the atmosphere may vary quite rapidly: the dependence of temperature on height may still be smooth when viewed on a microscopic scale, but on a macroscopic scale, it is convenient to assume a jump in the derivative of x. Both the location τ and the magnitude m of this jump are unknown, and both are to be estimated. In this case we may consider the 'parametrization':

$$x(t) = u(t) + m \cdot (t - \tau) H(t - \tau), \tag{4.2}$$

where $u \in H^2(a, b)$ and $H(\cdot)$ denotes the Heaviside function. The inverse problem may now be reformulated as solving $F(u, m, \tau) = y$ for the triple $[u, m, \tau]$ in $\mathcal{X}' := H^2(a, b) \times \mathbb{R} \times [a, b]$, where

$$F : \mathcal{X}' \to L^2(c, d) : [u, m, \tau] \mapsto y$$

is given by

$$F(u, m, \tau) = f_S(\cdot, u(a)) + \int_a^b f(\cdot, p, u(p) + m \cdot (t - \tau) H(p - \tau))\, dp. \tag{4.3}$$

[9]Details, including the exact form of f_S and of the kernel $f(\nu, T, p)$, appear in [1].

A penalty functional $J(x)$ such as (2.2) is no longer appropriate. Instead, we may consider[10]

$$J(u,m,\tau) = ||u||^2_{H^2(a,b)} + m^2 + (\tau - a)^2.$$

The results of the previous sections now apply to this example. To obtain (2.1), note that when $\{J(u_k, m_k, \tau_k)\}$ is bounded we can extract a subsequence $\{k(j)\}$ for which $\{u_{k(j)}\}$ converges weakly to some u in $H^2(a,b)$, $\{m_{k(j)}\}$ converges to some $m \in I\!R$, and $\{\tau_{k(j)}\}$ converges to some $\tau \in [a,b]$. To obtain (2.3)(*ii*), consider

$$\begin{aligned} x_k(t) &:= u_k(t) + m_k \cdot (t - \tau_k)H(t - \tau_k), \\ x(t) &:= u(t) + m \cdot (t - \tau)H(t - \tau). \end{aligned} \tag{4.4}$$

If $\{u_k\}$ converges weakly to u in $H^2[a,b]$, $m_k \to m$, and $\tau_k \to \tau$, we can easily show that $\{x_k\}$ converges to x in $C[a,b]$ (uniform convergence) and, from the continuity of f_S and the smoothness of f, show that $\{F(x_k\})$ converges to $F(x)$ in $\mathcal{Y} = L^2(c,d)$. Theorem 1 now applies and, subject to the existence/uniqueness assumptions for the solution[11], so does Theorem 2. Similarly, one may verify conditions (3.8) and (3.10) to apply Theorems 3 and 4.

It is instructive to consider the contingency that the true solution does *not*, in fact, involve a jump. The representation above will cover this case by taking $m = 0$ but we note that this leaves τ indeterminate and so introduces a spurious nonuniqueness[12] for the 'true' solution. Looking more carefully at, e.g., the proof of Theorem 3, we observe that the uniqueness was used only to ensure that the convergent subsequences extracted all converged to the same limit — *the* true solution. That would remain the case here even though different such subsequences might involve convergence to different *representations* of that solution.

Numerical implementation of these ideas is straightforward; see Shiau [5], [6].

Extension of this analysis to the possibility of several jumps (with a bound on the number) is immediate. The extension of these ideas for solutions in two or three variables appears likely.

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[1] Liou, K-N., *An Introduction to Atmospheric Radiation*, Academic Press, New York, 1980.

[10] Actually, to have $\mathcal{X}'$ a Banach space as in the earlier statements, we may take its last factor to be $I\!R$, rather than $[a,b]$, and then define J to be ∞ when $\tau \notin [a,b]$. This leaves J lower semicontinuous, etc.

[11] Verification of the uniqueness is 'application-specific' and is typically the technically most difficult part of justifying these approaches.

[12] Actually, it would be possible to deepen the earlier analysis to note that any computational implementation would here 'automatically' select $\tau = a$ for the limit (due to our particular choice of penalty function) so there would really be no problem.

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