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Singular Perturbation and Order Reduction for Filtering Problem

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Abstract

We consider the problem of optimal filtering of two dimensional diffusion process measured in a noisy channel. We approximate the solution of Zakai equation for the two dimensional process by a solution of Zakai equation for one dimensional process for two models. The first one is fast and slow variables, that is where one element of the process changes much more rapidly than the second one. The second model is the quasi-deterministic case for which the fast element has a small diffusion term. In both cases a simple approximated equations for the filtering problem are given that make numerical solution simpler.

1. Introduction

We study filtering of nonlinear diffusion process given by the Itô equation

$$dX(t) = F(X(t), t)dt + \sigma(X(t))dw(t) \tag{1.1}$$

from the observation process

$$dZ(t) = h(X(t))dt + dv \tag{1.2}$$

lead to the solution of the Zakai equation for an unnormalized conditional probability density of $X(t)$ given $\{y(s),\ 0 \le s \le t\}$. Here $F(X, t) : I\!R^n \times [0\,\infty) \longrightarrow I\!R^n$, $X(t = 0) = X_0$, $Z(t = 0) = Z_0$, $\sigma(x, t)$ is a $n \times m$ matrix, $w(t)$ is a m-dimensional standard Brownian motion independent of the initial value $x_0, v(t)$ is an one dimensional standard Brownian motion independent of Z_0 and $w(t)$. The Zakai equation is a linear stochastic partial differential equation given by

$$du(x, t) = Lu(x, t)dt + h(x, t)\, u(x, t)\, dZ(t) \tag{1.3}$$

where L is the Fokker-Planck operator

$$L(\cdot) = \frac{1}{2}\Sigma_{i,j} \frac{\partial^2}{\partial x_j \partial x_i} \left[(\sigma\sigma^T)_{ij} \cdot\right] - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[F_i(x\, t)\cdot\right]. \tag{1.4}$$

Numerical methods to solve (1.3) on line for a given realization $Z(t)$ could not be implemented until recently. Indeed, such numerical methods involve the real time solution of parabolic partial differential equations whose right hand side is a function of t, and hence a calculation for an elliptic difference scheme during the time between two successive measurements of the function $Z(t)$. However, the recent progress in VLSI special purpose array processors and especially the advent of systolic VLSI array made it possible to implement fast algorithms for the solution of the discretized Zakai equation [6].

In [7] La Vigna showed that sequential detection of diffusion signals for some cost function is solved by a real time solution of Zakai equation, and presented a discretized scheme that converges to the exact solution of Zakai equation as well as a fast algorithm to solve this scheme and a VLSI architecture for implementation of the algorithm. However this was done only for the one dimension case. Generalization of the result to, say, two dimensions requires calculation of a two dimensional elliptic partial differential equation scheme between two successive measurements which is too slow unless one increases the mesh size and hence the error due to the discritization scheme. In this paper we will present an approximation solution for the Zakai equation (1.3) for a two dimensional process (1.1) and a one dimensional observation process (1.2) by the following procedure:

Let $u(\underline{x}, t)$ be the exact solution for Zakai equation (1.3). Let $\tilde{u}(\underline{x}, t)$ be an approximation, in some sense, to $u(\underline{x}, t)$ of the form

$$\tilde{u}(\underline{x}, t) = A(x, t) \cdot B(x, t) \tag{1.5}$$

such that A as well as B is either a solution to the Zakai equation corresponding to a one dimensional model or is a solution for a Fokker-Planck equation that can be pre-calculated before the real-time procedure starts. Although there are two sources of error. (One due to the approximation of $u(x, t)$ by $\tilde{u}(x, t)$ and the second due to the discretization scheme), there are models for which $\tilde{u}(x, t)$ is close to $u(x, t)$, and the total error is small.

Note that in discretized schemes the process $Z(t)$ is measured only at mesh points so one can assume that $Z(t)$ is a Hölder continuous function of t and solves the Zakai equation as a deterministic parabolic partial differential equation.

In this work we consider two dimensional models that have separable approximations, that is, approximations that can be written as (1.5).

The first model is of fast and slow variables, that is the two dimensional process is governed by the equations

$$dx = \frac{1}{\epsilon} F_1(x, y) dt + \frac{\sigma_1}{\sqrt{\epsilon}} dW_1(t) \tag{1.6}$$

$$dy = G(x, y) dt + \sigma_2 \, dW_2(t) \tag{1.7}$$

The observation process is given by (1.2) where $\underline{X} = (x, y)$. The small parameter ϵ indicates that x is a fast variable related to y, that is, the changes in y are $0(1)$ when y has a value $y(t)$, while after time $0(\epsilon)$ x becomes a stationary - like random process $x(t) = x(y(t))$.

The Zakai equation for this case is given by

$$du(x, y, t, \epsilon) = L \cdot u(x, y, t, \epsilon) dt + h(x, y) \, u(x, y, t, \epsilon) dZ(t) \tag{1.8}$$

where

$$Lu \equiv \left\{ \frac{1}{\epsilon}(\frac{\sigma_1^2}{2}U_{xx}(x,y,t,\epsilon) - \frac{\partial}{\partial x}(F_1(x,y)U(x,y,t,\epsilon))+ \right.$$

$$\left. \frac{\sigma_2^2}{2}U_{yy}(x,y,t,\epsilon) - \frac{\partial}{\partial y}(G(x,y)u(x,y,t,\epsilon)) \right\} . \tag{1.9}$$

In this case we write an asymptotic approximation for (8) in the form

$$\tilde{u}(x,y,t) = A_0(y,t)\ exp\left(\int_0^x \frac{2F(s,y)}{\sigma_1^2}ds\right) + \sum_{n=1}^{\infty} A_n(y,1)\rho_n(x,y)e^{-\lambda_n t/\epsilon} \tag{1.10}$$

where $A_i(y,t)$ are the solutions of the corresponding one-dimensional Zakai equations, and $\rho_n(x,y)$ can be precalculated.

In the second model a large parameter multiplies the drift coefficient of x. The process x is now given by

$$dx = \frac{1}{\epsilon}F_1(x,y)dt + \sigma_1 dW_1(t) \tag{1.11}$$

and the process y is governed by (1.7). In this case we assume that:

$$\frac{\partial F_1(x,y,\epsilon)}{\partial y} = 0(\epsilon) \qquad \frac{\partial F_1(x,y,\epsilon)}{\partial x} = 0(1) \qquad \text{as } \epsilon \longrightarrow 0.$$

Although we must have an uniform expansion in t, (thus, as for the previous model, defining the two scale time variables (t,τ) where $\tau = t/\varepsilon$), we only calculate the zero order term, that is, $t = 0(1)$. We show that

$$\tilde{u}(x,y,\epsilon,t) \approx A(y,t,\epsilon)\ exp\left[\frac{1}{\epsilon}T(x,y)\right] \tag{1.12}$$

where

$$T(x,y) = \frac{2}{\sigma_1^2}\int_0^x F_1(s,y)ds \tag{1.13}$$

and $A(y,t,\epsilon)$ is a solution of a Zakai equation.

After this work was complete it came to our attention that the first models were treated by Marchetti in his Ph.D. Thesis [8].

2. Fast and Slow Variables

Consider the model (1.6), (1.7), (1.2), assume that $F_1(x,y), G(x,y), h(x,y)$ are such that there exists unique solution to Zakai's equation, (see for example Pardoux, [1], Baras, Blankership and Mitter [2], Baras, Blankenship and Hopkins [3]). Zakai's equation is given by

$$
\begin{aligned}
du(x,y,t) &= \left\{\frac{1}{\epsilon}(\frac{\sigma_1^2}{2}U_{xx} - \frac{\partial}{\partial x}F_1(x,y)u) + \frac{1}{2}\sigma_2^2 U_{yy} - \frac{\partial}{\partial y}G(x,y)u\right\} dt \\
&\quad + h(x,y)u\, dZ(t) \qquad (2.1)
\end{aligned}
$$

in the Itô sense and by

$$
\begin{aligned}
du(x,y,t) &= \left\{\frac{1}{\epsilon}(\frac{\sigma_1^2}{2}u_{xx} - \frac{\partial}{\partial x}F_1(x,y)u) + \frac{1}{2}\sigma_2^2 u_{yy} - \frac{\partial}{\partial y}G(x,y)u - \right. \\
&\quad \left. -\frac{1}{2}h^2(x,y)u\right\} dt + h(x,y)u dZ(t) \qquad (2.2)
\end{aligned}
$$

in the Stratonovich sense. Using the Stratonovich version it is easy to construct a robust version of the problems which to overcome the problems arise because of the $dZ(t)$ term.

Formally, equation (2.1) can be written as

$$\frac{\partial u}{\partial t} = \frac{1}{\epsilon}L_1 u + L_2 u + L_3 u \qquad (2.3)$$

where

$$L_1 = \frac{\sigma_1^2}{2}\frac{\partial^2 \cdot}{\partial x^2} - \frac{\partial}{\partial x}F_1(x,y)\,\cdot \qquad (2.4)$$

$$L_2 = \frac{\sigma_1^2}{2}\frac{\partial^2 \cdot}{\partial y^2} - \frac{\partial}{\partial y}G(x,y)\,\cdot \qquad (2.5)$$

$$L_3 = h(x,y)\frac{dZ(t)}{dt} \qquad (2.6)$$

with the following condition: for $0 \le t \le T$

$$\int_x\int_y u(x,y,t)dx\,dy = \gamma(t) \qquad 0 < \gamma(t) < \infty \qquad (2.7)$$

and

$$u(x,y,0) = \sqcap(x,y) \qquad (2.8)$$

where $u_0(x,y)$ is given non negative function and $u(x,y,t) \ge 0$ for all $(x,y) \in I\!R^2$ and $t \in [0,T]$.

(2.3) is a singular perturbation problem. In order to have a uniform expansion in time for the solution of (2.3) we represent a multi-timed version of (2.3) by defining the following two scaled time variables:

$$t \longrightarrow (t,\tau) \qquad \text{where } \tau = \frac{t}{\epsilon}. \tag{2.9}$$

so

$$\frac{\partial u}{\partial t} + \frac{1}{\epsilon}\frac{\partial u}{\partial \tau} = \frac{1}{\epsilon}L_1 u + L_2 u + L_3 u. \tag{2.10}$$

Now we expand $u = u(t,\tau,x,y,\epsilon)$ as a series in ϵ, that is, we formally assume that

$$u(t,\tau,x,y,\epsilon) = u_0(t,\tau,x,y) + \epsilon u_1(t,\tau,x,y) + \dots \tag{2.11}$$

If we substitute (2.11) in (2.10) and collect terms of order $\frac{1}{\epsilon}$, we obtain

$$\frac{\partial u_0}{\partial \tau} = L_1 u_0 \tag{2.12}$$

where the variables t and y appear as parameters. Following (Risken [4]) we have the following ansatz for u_0

$$u_0 = A(y,t)\rho(x,y,t)e^{-\lambda\tau}. \tag{2.13}$$

This ansatz leads to the following eigenvalue problem

$$L_1\rho + \lambda\rho = 0 \tag{2.14}$$

and

$$\lim_{x \longrightarrow \pm\infty} \rho(x,y,t) = 0 \tag{2.15}$$

where L_1 is the Fokker-Planck operator (2.4) with respect to x. We have the following results ([4]):

i) All eigenvalues of (2.14) are real and nonnegative.

ii) The first eigenvalue $\lambda = 0$ corresponding to the stationary solution

$$\rho_0 = exp(-\phi_0(x)) \tag{2.16}$$

where

$$\phi_0(x) = -\int_0^x \frac{2F_1(s,y)}{\sigma_1^2}ds. \tag{2.17}$$

iii) Defining the Hermitian operator $\tilde{L} = e^{\phi_0/2} L_1 e^{-\phi_0/2}$, we see that both $\tilde{L}$ and L_1 have the same eigenvalues and that the corresponding eigenfunctions satisfy $\phi_n = e^{\phi_0/2} \cdot \rho_n$ where ϕ_n are the eigenfunctions of $\tilde{L}$ and ρ_n are of L_1.

iv) The eigenvalues form either a non-negative discrete sequence or a non-negative continuous set; in either case the eigenfunctions that corresponding to different eigenvalues are orthogonal.

v) Assuming that λ is a discrete sequence (otherwise the sum sign is replaced by the integral sign) and that $\sqcap(x,y)$ is smooth, we have

$$e^{\frac{\phi(x)}{2}} \sqcap (x,y) = \Sigma_n a_n(y) \psi_n(xy) \tag{2.18}$$

where

$$a_n(y) = \int_{-\infty}^{\infty} e^{\frac{\psi(x)}{2}} \sqcap (x,y)\, \psi_n(x,y) dx, \quad \text{and}$$

$$\sqcap(x,y) = \Sigma_n a_n(y) \rho_n(x,y), \qquad \text{so that}$$

$$u_0(t,\tau,x,y) = \sum_{n=0}^{\infty} A_n(y,t) \rho_n(x,y) e^{-\lambda_n \tau} \tag{2.19}$$

where λ_n are the eigenvalues, ρ_n are the eigenfunctions of $L_1, A_n(y,0) = a_n(y)$ as defined in (2.18), ρ_0 is the stationary solution. $F_1(x,y)$ must satisfy the appropriate conditions to ensure that

$$\int_{-\infty}^{\infty} \rho_i(x,y) dx < \infty \qquad \forall\, i. \tag{2.20}$$

$A_n(y,t)$ will be determined by the Fredholm Alternative when we look at the next order terms, that is the ϵ^0 order:

$$-\frac{\partial u_1}{\partial \tau} + L_1 u_1 = L^* u_1 = \frac{\partial u_0}{\partial t} - L_2\, u_0 - L_3\, u_0 \stackrel{\text{def}}{\equiv} M u_0 \tag{2.21}$$

Let $\mathcal{L}$ be the Banach space of all bounded function of x with the inner product

$$\langle u, v \rangle = \int_{-\infty}^{\infty} u(x)\, v(x)\, dx. \tag{2.22}$$

Assuming the same ansatz as (2.13) for $u_1(t,\tau,x,y)$, the solvability conditions for the existence of solutions for (2.21) are

$$\int_{-\infty}^{\infty} M\left[A_k(y,t)\rho_k(x,t)\right]\rho_k^*(x,t)\,dx = 0 \tag{2.23}$$

Note that for $n \neq k$ one has $\lambda_n \neq \lambda_k$; hence $e^{\lambda_k\tau}$ is an independent sequence of functions.

We have

$$\int_{-\infty}^{\infty} M\,(A_x(yt)\rho_k(x,y)\;\rho_k^*(x,y)\,dx = 0 \qquad\qquad \forall\, k$$

but direct calculation yields

$$\rho_n^* = e^{\phi(x)}\cdot\rho_n(x,y) \tag{2.24}$$

so

$$\int_{-\infty}^{\infty} M(A_k(y,t)\;\rho_k(x,y))e^{\phi(x)}\rho_k(x,y)\,dx = 0. \tag{2.25}$$

Recall that

$$M[A_k(y,t)\rho_k(x,y)] = \frac{\partial A_k}{\partial t}\rho_k(x,y) - \frac{\sigma_2^2}{2}\frac{\partial^2}{\partial y^2}A_k\rho_k - \frac{\partial}{\partial y}[GA_k\rho_k] - h(x,y)A_k\rho_k(x,y)\frac{dZ(t)}{dt} \tag{2.26}$$

and substitute (2.26) in (2.25). Then for each k we have a stochastic differential equation for $A_k(y,t)$ with the initial conditions

$$A_k(Y,0) = a_k(Y). \tag{2.27}$$

For $k=0$, one has a stationary solution, $\rho_0(x,y) = e^{-\phi}$; hence we have

$$\int_{-\infty}^{\infty} M[A_0(y,t)e^{-\phi(x,y)}]dx = 0. \tag{2.28}$$

Denoting by $\hat{E}$ the integral

$$\hat{E}(y) = \int_{-\infty}^{\infty} E(x,y)\,\exp(-\phi(x,y))dx, \tag{2.29}$$

we have the following equation for $A_0(y,t)$:

$$\frac{\partial A_0}{\partial t} = \frac{\sigma_2^2}{2}A_{0yy} + D_1A_{0y} + D_2A_0 + D_3A_0 \tag{2.30}$$

where

$$D_1 = -\left(\frac{\partial \widehat{\phi(xy)}}{\partial y} + G\widehat{(xy)}\right) \cdot (\hat{1})^{-1}, \tag{2.31}$$

$$D_2 = \left((\frac{\partial \widehat{\phi(xy)}}{\partial y})^2 - \frac{\widehat{\partial^2 \phi}}{\partial y^2} - \frac{\widehat{\partial G}}{\partial y} + \frac{\widehat{\partial \phi}}{\partial y} G\right)(\hat{1})^{-1}, \tag{2.32}$$

$$D_3 = h\widehat{(xy)}(\hat{1})^{-1}\frac{dZ(t)}{dt}, \tag{2.33}$$

and

$$A_0(Y,0) = a_0(y). \tag{2.34}$$

Similar equations can be obtain for each k. The first approximation for $u(x,y,t)$ is, then,

$$u(x,y,t) = \sum_{k=0}^{\infty} e^{-\lambda_k t/\epsilon} A_k(y,t)\rho_k(x,y) + 0(\epsilon) \tag{2.35}$$

uniformly in time.

That is, in order to find an approximate solution for 2.3 one have to solve for each ρ_i a deterministic differential equation (this can be done off line) and then calculating (2.31) – (2.33) and solving (2.30) in a real time procedure.

3. The quasi deterministic case.

Consider the model:

$$dx = \frac{1}{\epsilon}F_1(x,y,\epsilon)dt + \sigma_1 dW_1, \tag{3.1}$$

$$dy = G_1(x,y)dt + \sigma_2 dW_2, \tag{3.2}$$

$$dZ = h(x,y)dt + dv(t). \tag{3.3}$$

Assume that $F(x,y), G(x,y), h(x,y)$ are such that there exists an unique solution to Zakai's equation and assume that

$$\frac{\partial F_1(x,y,\epsilon)}{\partial y} = 0(\epsilon), \qquad \frac{\partial F(x,y,\epsilon)}{\partial x} = 0(1),$$
$$\frac{\partial G(x,y)}{\partial x} = 0(1), \qquad \frac{\partial G(x,y)}{\partial y} = 0(1). \tag{3.4}$$

Zakai's equation for the model (3.1) – (3.3) is then given by

$$du = (\frac{1}{\epsilon}L_1 u + L_2 u + L_3 u)dt \tag{3.5}$$

where

$$L_1 = L_1(\epsilon) = \epsilon\frac{\sigma_1^2}{2}\frac{\partial^2 \cdot}{\partial x^2} - \frac{\partial}{\partial x}F(x,y,\epsilon), \tag{3.6}$$

$$L_2 = \left[\frac{\sigma_2^2}{2}\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial y}\,G(x,y)\right], \tag{3.7}$$

$$L_3 = h(x,y)\,\frac{dZ(t)}{dt}. \tag{3.8}$$

We write an asymptotic approximation for the solution of (3.5) but, at least formally, our approximation is not valid in $I\!R^2$ but only in $\{I\!R^2 \setminus A\}$ where the Borel measure of A is zero. To illustrate the situation we present the following linear case:

$$dx = (-\frac{1}{\epsilon}x + y)dt + \sqrt{2}dW_1, \tag{3.9}$$

$$dy = (-2y + x)dt + \sqrt{2}dW_2 \tag{3.10}$$

The Fokker Planck equation for the probability density function for the model (4.9) – (4.10) is given by

$$\frac{\partial P}{\partial t} = P_{xx} + \frac{\partial}{\partial x}(\frac{x}{\epsilon} - y)P + P_{yy} + \frac{\partial}{\partial y}(2y - x)P. \tag{3.11}$$

The stationary solution is given by

$$P(x,y,\epsilon) = \frac{\sqrt{2-\epsilon}}{2\pi\sqrt{\epsilon}}\exp(-(\frac{x^2}{2\epsilon} - xy + y^2)). \tag{3.12}$$

On the other hand we want to expand P as

$$P(x,y,\epsilon) = P_0(x,y,\epsilon) + \epsilon\,P_1(x,y,\epsilon) + \ldots \tag{3.13}$$

with

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_i(x,y,\epsilon)dx\,dy = 1-\epsilon \qquad i = 0,1,2,\ldots \tag{3.14}$$

(so that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{i=0}^{\infty}\epsilon^i P_i(x,y,\epsilon)dx\,dy = 1$).

First, we write (3.11) in a different form:

$$\frac{\partial p}{\partial t} = \frac{1}{\epsilon}L_1 P + L_2 P \tag{3.15}$$

where

$$L_1 P = \epsilon P_{xx} + \frac{\partial}{\partial x}(x - \epsilon y)P, \tag{3.16}$$

$$L_2 P = P_{yy} + \frac{\partial}{\partial y}(2y - x)P. \tag{3.17}$$

We substitute (3.13) in (3.15), assume that $P_i(x, y, \epsilon) = 0(1)$ and collect terms of order $\frac{1}{\epsilon}$ to obtain.

$$L_1 P_0 = 0 \tag{3.18}$$

so that

$$P_0(x, y, \epsilon) = A(y) \cdot \exp(-\frac{x^2}{2\epsilon} + xy). \tag{3.19}$$

Note that we are looking for stationary solutions since the stationary conditions on (3.9) – (3.10) are obviously satisfied.

At the next order we see that:

$$L_1 P_1 = -L_2 P_0. \tag{3.20}$$

Using the same argument as above, the solvability condition for (3.20) is

$$\int_{-\infty}^{\infty} L_2 P_0 \, dx = 0 \tag{3.21}$$

which gives

$$A_{yy} + \epsilon y A_y + 2yA_y + 2A + 2\epsilon y^2 A = 0 \tag{3.22}$$

The bounded solution is then given by

$$A(y) = A_0 \exp(-y^2) \tag{3.23}$$

Thus, using (3.14), we obtain

$$P_0(x, y, \epsilon) = \frac{(1-\epsilon)\sqrt{2-\epsilon}}{2\pi \cdot \sqrt{\epsilon}} \exp(-\frac{x^2}{2\epsilon} + xy - y^2). \tag{3.24}$$

Substituting (3.24) in (3.20) we obtain

$$L_1 P_1 = -L_2 P_0 = 0 \tag{3.25}$$

and, because of the solvability condition for P_2 and the normalization condition (3.14), we see that

$$P_1 = P_0. \tag{3.26}$$

It follows that

$$P_i = P_0 \qquad i = 1, 2, \dots \ . \tag{3.27}$$

Returning to the assumption that

$$P_i(x, y, \epsilon) = 0(1) \qquad i = 0, 1, 2, \dots \ ,$$

we note that this assumption is not satisfied for the set $A = \{(x, y) : \ x = 0\}$. Thus the approximation is only valid on $\{I\!R^2 \setminus A\}$. Note that for every fixed $0 < \epsilon < 1$ and every $(x, y) \in I\!R^2$ (including $x = 0$) the power series (3.13) converges to the stationary solution (3.12), (which is unbounded on A as $\epsilon \longrightarrow 0$).

Coming back to the more general filtering problem, equations (3.1) – (3.3) and the conditions (3.4), Zakai's equation is given by (3.5). Again in order to have a uniform expansion in time we must apply two different time scales, for simplicity we present only the "zeroth" order term in τ, that is, $t > 0$.

In that case we assume that

$$u = u_0(x, y, \epsilon, t) + \epsilon u_1(x, y, \epsilon, t) + \dots \tag{3.28}$$

where

$$u_i(x, y, \epsilon, t) = 0(1) \qquad \text{as } \epsilon \to 0 \text{ in a subset } D \text{ of } I\!R^2 \times [0, T].$$

Proceeding as above we obtain

$$u_0(x, y, \epsilon, t) = \ A(y, t, \epsilon) \ \exp[\frac{1}{\epsilon} T(x, y)] \tag{3.29}$$

where

$$T(x, y) = \frac{2}{\sigma_1^2} \int_0^x F_1(s, y) ds. \tag{3.30}$$

We assume that for every $(x, y) \in R^2$, $\epsilon > 0$ one has $T(x, y, \epsilon) \leq 0$ so that

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \exp[\frac{1}{\epsilon} T(x, y, \epsilon)] dx < \infty. \tag{3.31}$$

In general, $A(y,t,\epsilon)$ will be determined by the solvability condition

$$L_1 u_1 = \frac{\partial u_v}{\partial t} - L_2 u_0 - L_3 u_0 = M u_0 \tag{3.32}$$

but unnder the assumption (3.31), this solvability condition reduces to

$$\int_{-\infty}^{\infty} M u_0 dx = 0. \tag{3.33}$$

Assume that for each y and ϵ there is a unique $x = \tilde{x}(y,\epsilon)$ for which $T(x,y,\epsilon)$ achieves its maximum as a function of x. Thus, for every y and ϵ,

$$T(x,y,\epsilon) < T(\tilde{x}(y,\epsilon),y,\epsilon) \leq 0 \qquad x \neq \tilde{x}(y,\epsilon). \tag{3.34}$$

We also assume that

$$\frac{\partial^2 T}{\partial x^2}(\tilde{x}(y,\epsilon),\, y,\epsilon) < 0.$$

Next we expand $T(x,y,\epsilon)$ in Taylor's series:

$$T(x,y,\epsilon) = T(\tilde{x}(y,\epsilon),y,\epsilon) + \sum_{k=2}^{\infty} \frac{\partial^k}{\partial x^k} T(\tilde{x}(y,\epsilon),y,\epsilon) \frac{[x - \tilde{x}(y,\epsilon)]^k}{k!} \tag{3.35}$$

and $T(\tilde{x}(y,\epsilon),y,\epsilon) \leq 0$. Then

(3.36) $$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} A(y,\epsilon,t) \ \exp(\frac{1}{\epsilon} T(x,y))dx = A_t(y,\epsilon,t) \cdot \sqrt{\epsilon} \cdot e^{T(\tilde{x}(y,\epsilon),y,\epsilon)/\epsilon} \cdot$$

$$\cdot \left(\sqrt{2\pi / \frac{\partial^2}{\partial x^2} T(\tilde{x},y,\epsilon)} + 0(\epsilon) \right) = A_t(y,\epsilon,t) \cdot \ \ E(\epsilon,y)$$

where

(3.37) $$E(\epsilon,y) = \int_{-\infty}^{\infty} e^{\frac{1}{\epsilon}T(xy)} dx$$

and

(3.38) $$E(\epsilon,y) = 0(1) \qquad \forall \, y \in I\!R.$$

Proceeding as above, we obtain

(3.39) $$\frac{\partial u_0}{\partial y} = \left[Ay + \frac{1}{\epsilon} A \frac{\partial T}{\partial y} \right] \exp \left[\frac{1}{\epsilon} T(x,y,\epsilon) \right]$$

and

(3.40) $$\frac{\partial^2 u_0}{\partial y^2} = \left[A_{yy} + \frac{1}{\epsilon} \frac{\partial^2 T}{\partial y^2} A + \frac{2}{\epsilon} A_y \frac{\partial T}{\partial y} + \frac{1}{\epsilon^2} (\frac{\partial T}{\partial y})^2 A \right].$$

Thus,

(3.41) $$\int_{-\infty}^{\infty} A_{yy} \exp \left[\frac{1}{\epsilon} T(x,y,\epsilon) \right] dx = A_{yy}(y,t,\epsilon) \cdot E(\epsilon,y)$$

and

(3.42) $$\int_{-\infty}^{\infty} \frac{1}{\epsilon} A_y T_y \ \ \exp \left[\frac{1}{\epsilon} T(x,y,\epsilon) \right] ds$$

$$= Ay \cdot \int_{-\infty}^{\infty} \frac{1}{\epsilon} \frac{\partial T}{\partial y}(x,y,\epsilon) \cdot \exp[\frac{1}{\epsilon} T(x,y,\epsilon)]dx$$

$$= \sqrt{\epsilon} \cdot A_y(t,y,\epsilon) \cdot \left(\frac{1}{\epsilon} \frac{\partial T}{\partial y} \right) (\tilde{X}(y,\epsilon)y,\epsilon) e^{-T(\tilde{x}(y,\epsilon),y,\varsigma)} / \epsilon.$$

$$\cdot \left(\sqrt{2\pi / T_{xx}(\tilde{x}(y,\epsilon),y,\epsilon)} + 0(\epsilon) \right).$$

Since $\frac{1}{\epsilon}\frac{\partial T}{\partial y} = 0(1)$, we denote it by $\frac{\partial T}{\partial y^*}$.

Similar calculations yield that:

(3.43) $$\int_{-\infty}^{\infty} A \cdot \frac{1}{\epsilon^2} (\frac{\partial T}{\partial y})^2 \ \ \exp[\frac{1}{\epsilon} T(x,y,\epsilon)]dx =$$

$$A\sqrt{\epsilon} \cdot (\frac{\partial T}{\partial y^*})^2 (\tilde{x},y,\epsilon) e^{-T(\tilde{x},y,\epsilon)} (\sqrt{2\pi / T''} + 0(\epsilon)),$$

$$(3.44)\qquad \int_{-\infty}^{\infty} \frac{1}{\epsilon} A \frac{\partial^2 T}{\partial y^2} \ \exp\left[\frac{1}{\epsilon} T(x,y,\epsilon)\right] dx =$$

$$= \sqrt{\epsilon} \frac{\partial}{\partial y} \frac{\partial T}{\partial y^*}(\tilde{x}, y, \epsilon) \cdot A \cdot e^{\frac{1}{\epsilon}T(\tilde{x},y,\epsilon)} (\sqrt{2\pi/T''} + 0(\epsilon),)$$

$$(3.45)\qquad \int_{-\infty}^{\infty} \frac{\partial}{\partial y} G(x,y) u_0 dx =$$

$$\int_{-\infty}^{\infty} \left[G_y(x,y) A + G(x,y)(A_y + \frac{1}{\epsilon} A \frac{\partial T}{\partial y}) \right] \exp\left[\frac{1}{\epsilon} T(x,y,\epsilon)\right] dx =$$

$$= \sqrt{\epsilon} \cdot \left(\sqrt{2\pi/T''} + 0(\epsilon)\right) \cdot e^{T(x,y,\epsilon)/\epsilon} \cdot [A(y,t,\epsilon) G_y(\tilde{x}, y, \epsilon) +$$

$$+ A_y(x,t,\epsilon) G(\tilde{x}, y) + A(y,t,\epsilon) \cdot \frac{\partial T}{\partial y^*}(\tilde{x}, y, \epsilon) \cdot G(\tilde{x}, y, \epsilon)]$$

Let us denote

$$(3.46)\qquad \hat{R}(y,t,\epsilon) = E(y,\epsilon)^{-1} \cdot \int_{-\infty}^{\infty} R(x,y,\epsilon,t) \ \exp\left[\frac{1}{\epsilon} T(x,y,\epsilon)\right] dx$$

and note that if $\partial R / \partial X = 0$ for all x then

$$(3.47)\qquad \hat{R}(y,t,\epsilon) = R(y,t,\epsilon).$$

In that case condition (3.33) has the form:

$$(3.48)\qquad A_t(y,\epsilon,t) = A_{yy} + A \frac{1}{\epsilon} \widehat{\frac{\partial^2 T}{\partial y^2}} + \frac{2}{\epsilon} A_y \widehat{\frac{\partial T}{\partial y}} + \frac{1}{\epsilon^2} A (\widehat{\frac{\partial T}{\partial y}})^2$$

$$- A \hat{G}_y - A_y \hat{G} - A \widehat{\frac{\partial T}{\partial y} G} + \hat{h}(xy) A \frac{dZ(t)}{dt}$$

or

$$(3.49)\qquad \frac{\partial A}{\partial t} = A_{yy} + B(y,\epsilon,t) A_y + C(y,\epsilon,t) A$$

where B and C are $0(1)$ in ϵ. Assume that $dZ(t)/dt$ in (3.8) is a Hölder continuous function of t. If we assume proper smoothness conditions on F, G and h, it is then clear that $B(y,t,\epsilon)$ and $C(y,t,\epsilon)$ are Lipschitz continuous with respect to y and t for any fixed $\epsilon > 0$ so the solution A of (3.49) exists and is continuous, as are $\frac{\partial A}{\partial t}, \frac{\partial A}{\partial y}, \frac{\partial^2 A}{\partial y^2}$.

The function u_0 is an asymptotic approximation to u if $u_1(x,y,\epsilon,t)$ is bounded as $\epsilon \to 0$ where u_1, is the solution of

$$(3.50)\qquad L_1 u_1 = \frac{\partial u_0}{\partial t} - L_2 u_0 - L_3 u_0.$$

Let D be any compact subset of $\mathbb{R}^2$. Then the right hand side of (3.50) is bounded and locally Lipschitz in D. Thus u_1 exists, is bounded in D, and u_0 is an asymptotic approximation for u.

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