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# Crossing Minimization in Linear Embeddings of Graphs

by

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**Index Terms -**

Circuit layout

Computational complexity

Crossing minimization

Graph embedding

NP-completeness

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**Abstract**

The problem of embedding a graph in the plane with the minimum number of edge-crossings arises in some circuit layout problems. It has been known to be NP-hard in general. Recently, in the area of book embedding, this problem was shown to be NP-hard even when the vertices are placed on a straight line $l$ and the edges are drawn completely on either side of $l$. In this paper, we show that the problem remains NP-hard even if, in addition to these constraints, the positions of the vertices on $l$ are predetermined.

## 1. Introduction

The *crossing number problem* [4,5] is the problem of determining, for a given integer $K$, whether a graph $G$ can be embedded in the plane with $K$ or fewer pair-wise crossings of the edges (not including the intersections of the edges at their common endpoints). The *crossing minimization problem* is that of embedding a graph in the plane with the minimum number of edge-crossings. This problem arises in the layout of integrated circuits in certain technologies [13]. Furthermore, a solution to the problem may lead to a small area embedding of the graph in a VLSI circuit [10,14]. Unfortunately, the crossing number problem was recently proven to be NP-complete [5]. This implies that the crossing minimization problem is very likely to be intractable.

Nicholson [12] developed a heuristic algorithm for the crossing minimization problem. His algorithm finds a special type of embedding of a given graph, namely, (i) the vertices are placed on a horizontal line $l$, and (ii) the edges are drawn by semicircles (see Fig. 1). We call this type of embedding a *linear embedding.* Despite the restrictions, Nicholson's algorithm finds embeddings for small complete graphs and complete bipartite graphs that are nearly optimal for any style of embedding. However, it does not always produce good results for general graphs. One way to improve his solution for the general case is to reassign the semicircles to either side of $l$ after the positions of the vertices on $l$ are determined. This motivates us to consider the crossing minimization problem with an additional constraint, (iii) the positions of the vertices on $l$ are predetermined.

The *fixed linear crossing minimization problem* is such a problem of finding a linear embedding of a graph with the minimum number of edge-crossings under a specified vertex ordering. We call the problem of finding such an embedding with no vertex ordering specified the *free linear crossing minimization problem.* Furthermore, we define the *free linear crossing* *number problem* to be that of determining, for a given integer $K$, whether there is a linear embedding of a graph with $K$ or fewer edge-crossings. When a vertex ordering is specified, we call it the *fixed linear crossing number problem.*

The linear crossing number problems are related to the *book embedding problems* [1,3] which have recently attracted considerable attention. A *book embedding* of a graph is an embedding in a book with the vertices placed on the spine and the edges on the pages such that no two edges drawn on the same page cross each other. Similar to the linear crossing number problems, we call the book embedding problems *fixed* or *free* depending on whether the vertex ordering on the spine is specified or not. The *fixed book thickness* of a graph $G$ is the least integer $k$ such that $G$ can be embedded into $k$ pages under a specified vertex ordering. For example, the fixed book thickness of the graph of Fig. 1(a) is 4 if the vertices must be placed on the spine in ascending order of their subscripts (see Fig. 2).

It is clear that a graph $G$ is 2-page free (resp., fixed) embeddable if and only if there exists a free (resp., fixed) linear embedding of $G$ with no edge-crossings. Since testing the 2-page free embeddability of graphs is NP-complete [3], the free linear crossing number problem is NP-complete. Determining the fixed book thickness of graphs is also NP-hard, since it is equivalent to coloring circle graphs [3], which was proven to be NP-hard in [6]. On the other hand, it is easy to show that the 2-page fixed embeddability problem, which is equivalent to the fixed linear crossing number problem with $K=0$, is solvable in linear time. The situation is analogous to that for arbitrary embeddings, with *graph thickness* [4,9] corresponding to fixed book thickness and crossing number corresponding to fixed linear crossing number. Both of the graph thickness and crossing number problems are NP-complete for the general case [5,9], and they have the planarity testing problem, which is solvable in linear time [2,8], as their common subproblem. Therefore, it is interesting to investigate the computational complexity of the fixed linear crossing number problem.

Another problem which may be related to the fixed linear crossing number problem is the so-called topological via minimization problem [7,11]. Slightly modifying Marek-Sadowska's formulation [11], we can formulate the via minimization problem as that of finding a maximum subgraph having a fixed linear embedding with no edge-crossings under a specified vertex ordering.

In this paper, we show that the fixed linear crossing number problem is NP-complete. In fact, we prove that the problem is NP-complete even if each connected component of a given graph is a single edge. In the next section, we define some terms and give a formal description of our problem. In Section 3, we first show a polynomial transformation from the *set splitting problem* [4] to our problem when parallel edges are allowed in the graph. We then modify the graph so as to show the NP-completeness of the problem for the restricted case.

## 2. Definitions

Let $G=(V, E)$ be an undirected graph with no self-loops. If two edges $e$ and $e'$ connect vertices $u$ and $v$, we say that $e$ and $e'$ are *parallel edges* and call each of them a *copy* of edge $(u, v)$. If $G$ has no parallel edges, it is called a *simple graph*; otherwise called a *multi-graph*. In this paper, we deal with both types of graphs.

Let $f : V \rightarrow \{1, 2, ..., |V|\}$ be a one-to-one function. We call an embedding $\overline{G}$ of $G$ in the plane an *f-fixed linear embedding*, or simply an *f-linear embedding*, if

(a) Each vertex $v \in V$ is placed on the x-axis $l$ with x-coordinate $f(v)$,

(b) The edges in $E$ are drawn by semi-ellipses, with one edge joining any adjacent pair of vertices drawn as a semicircle, and

(c) The semi-ellipses for non-parallel edges intersect in at most one point.

For an $f$-linear embedding $\overline{G}$ of $G = (V, E)$ and two edges $e$ and $e'$ in $E$, we say that $e$ and $e'$ *cross* each other in $\overline{G}$ if they intersect in $\overline{G}$ but are not incident in $G$. Such an unordered pair of edges $e$ and $e'$ is called an ***edge-crossing***, or simply a ***crossing***, of $\overline{G}$. We denote by $\nu_f(G)$ the least total number of crossings among all $f$-linear embeddings of $G$. Our problem is now formally described as follows.

**FIXED LINEAR CROSSING NUMBER**

**Instance:** Graph $G = (V, E)$, integer $K \geq 0$ and one-to-one function $f : V \rightarrow \{1, 2, ..., |V|\}$.

**Question:** $\nu_f(G) \leq K$ ? □

The special case of this problem in which $K = 0$ can easily be solved. Suppose $V = \{v_1, v_2, ..., v_n\}$ and $f(v_i) = i$ for $i = 1, 2, ..., n$. It is obvious that $\nu_f(G) = 0$ if and only if graph $(V, E \cup \{(v_i, v_{i+1}) \mid i = 1, 2, ..., n-1\} \cup \{(v_n, v_1)\})$ is planar. Thus, one can solve the problem in linear time by using one of the existing graph planarity testing algorithms [2,8]. In the next section, we show that FIXED LINEAR CROSSING NUMBER is in general NP-complete.

## 3. NP-Completeness of FIXED LINEAR CROSSING NUMBER

It is clear that FIXED LINEAR CROSSING NUMBER belongs to the class NP. In order to prove its NP-hardness, we use the following problem, which is known to be NP-complete [4].

**SET SPLITTING**

**Instance:** Collection $S = \{S_1, S_2, ..., S_n\}$ of subsets of a finite set $X = \{x_1, x_2, ..., x_m\}$.

**Question:** Is there a *set splitting* of $X$ with respect to $S$, that is, a partition of $X$ into two subsets $X_1$ and $X_2$ such that no subset in $S$ is entirely contained in either $X_1$ or $X_2$ ?

□

It is known that this problem remains NP-complete even if each subset $S_j$ contains either two or three elements [4]. Suppose that $S = \{S_1, S_2, ..., S_n\}$ and $X = \{x_1, x_2, ..., x_m\}$ are given as an instance of this restricted SET SPLITTING, where $S$ has $n_2$ 2-element sets and $n_3$ 3-element sets, and $n_2+n_3 = n$. For $j = 1, 2, ..., n$ and $k = 1, 2, ..., |S_j|$, let $s_{j,k}$ denote the $k$-th element of $S_j$. Corresponding to $S$ and $X$, we construct a multi-graph $G = (V^*, E^*)$, an integer $K$, and a numbering $f$ of $V^*$ such that $\nu_f(G) \leq K$ if and only if $X$ has a set splitting with respect to $S$.

The graph $G$ has an induced subgraph $G_i = (V_i, E_i)$ for each $x_i \in X$, and an induced subgraph $H_j = (W_j, F_j)$ for each $S_j \in S$, and $V^* = \bigcup_{x_i \in X} V_i \cup \bigcup_{S_j \in S} W_j$. The numbering $f$ is determined to give contiguous assignments to the vertices of each of these subgraphs, in the order indicated by the vertex names defined below; beyond that the ordering of the subgraphs is immaterial. Let $E = \bigcup_{x_i \in X} E_i$ and $F = \bigcup_{S_j \in S} F_j$. The set of edges $E^*$ is $E \cup F \cup A$, where $A$ will be defined to have $2n_2+9n_3$ edges connecting the vertices in $V_i$'s and those in $W_j$'s. Let $M = \binom{n}{2} + \binom{n_2+8n_3}{2} + 1$ [1]; we will create the edges in $G_i$'s and $H_j$'s with multiplicity $M$. As will be shown later, this value of $M$ exceeds the number of crossings we may have from the edges in $A$ in a particular $f$-linear embedding of $G$.

For $i = 1, 2, ..., m$, the graph $G_i = (V_i, E_i)$ has three vertices $v_i^{(1)}$, $v_i^{(2)}$, $v_i^{(3)}$, and its edge set has $M$ copies of $(v_i^{(1)}, v_i^{(3)})$. See Fig. 3. For $j = 1, 2, ..., n$, $H_j = (W_j, F_j)$ is defined depending on whether $S_j$ contains two or three elements. If $|S_j| = 2$, $W_j$ has six vertices $w_j^{(1)}, w_j^{(2)}, ..., w_j^{(6)}$, and $F_j$ consists of $M$ copies each of $(w_j^{(1)}, w_j^{(4)})$ and $(w_j^{(3)}, w_j^{(6)})$. These two

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[1] $\binom{a}{b}$ denotes the number of combinations of $a$ objects taken $b$ at a time.

subsets of edges are called $F_j^{(1)}$ and $F_j^{(2)}$, respectively. See Fig. 4. If $|S_j| = 3$, $H_j$ has twenty-one vertices $w_j^{(1)}, w_j^{(2)}, \ldots, w_j^{(21)}$ and $M$ copies each of $(w_j^{(1)}, w_j^{(5)})$, $(w_j^{(4)}, w_j^{(7)})$, $(w_j^{(8)}, w_j^{(12)})$, $(w_j^{(11)}, w_j^{(14)})$, $(w_j^{(15)}, w_j^{(19)})$ and $(w_j^{(18)}, w_j^{(21)})$. We call these subsets of edges $F_j^{(1)}$, $F_j^{(2)}$, $F_j^{(3)}$, $F_j^{(4)}$, $F_j^{(5)}$, and $F_j^{(6)}$, respectively. See Fig. 5.

We now define the set of edges $A$ as $\bigcup_{S_j \in S} A_j$. $A_j$ contains two or nine edges depending on whether $S_j$ has two or three elements. Specifically, if $|S_j| = 2$, $A_j = \{(v_{i_1}^{(2)}, w_j^{(2)}), (v_{i_2}^{(2)}, w_j^{(5)})\}$, where $x_{i_1} = s_{j,1}$ and $x_{i_2} = s_{j,2}$, and if $|S_j| = 3$, $A_j = \{(v_{i_1}^{(2)}, w_j^{(2)}), (v_{i_1}^{(2)}, w_j^{(13)}), (v_{i_1}^{(2)}, w_j^{(17)}), (v_{i_2}^{(2)}, w_j^{(3)}), (v_{i_2}^{(2)}, w_j^{(9)}), (v_{i_2}^{(2)}, w_j^{(20)}), (v_{i_3}^{(2)}, w_j^{(6)}), (v_{i_3}^{(2)}, w_j^{(10)}), (v_{i_3}^{(2)}, w_j^{(16)})\}$, where $x_{i_1} = s_{j,1}$, $x_{i_2} = s_{j,2}$ and $x_{i_3} = s_{j,3}$. Thus, $|A| = 2n_2 + 9n_3$. For $i = 1, 2, \ldots, m$, the set of edges in $A$ incident upon $v_i^{(2)}$ is called $A_i'$.

Finally, we set $K = M \cdot (2n_3 + 1) - 1$. It is easy to see that $G$ has $3m + 6n_2 + 21n_3$ vertices and $(m + 2n_2 + 6n_3) \cdot M + 2n_2 + 9n_3$ edges. Therefore, we have the following lemma.

**Lemma 1.** $G$, $f$ and $K$ can be constructed from $S$ and $X$ in polynomial time with respect to $m$ and $n$. □

To complete the NP-hardness proof, we will show the equivalence of the following statements.

(I) $X$ has a set splitting with respect to $S$.

(II) $\nu_f(G) \leq K$.

**Theorem 1.** If Statement (I) holds for $S$ and $X$, then Statement (II) holds for $G$, $f$ and $K$.

*Proof.* Let $X_1$ and $X_2$ be a set splitting of $X$ with respect to $S$. We will construct a natural $f$-linear embedding of $G$ with at most $K$ crossings.

For $i = 1, 2, \ldots, m$, we draw all edges in $E_i$ below (resp., above) the x-axis $l$ and all edges in $A_i'$ above (resp., below) $l$ if and only if $x_i \in X_1$ (resp., $X_2$). For each $j$ such that $|S_j| = 2$, if $s_{j,1} \in X_1$ (resp., $X_2$), then we draw the edges in $F_j^{(1)}$ below (resp., above) $l$ and those in $F_j^{(2)}$ above (resp., below) $l$. In this way, we can draw the edges in $F_j$ in such a way that they do not cross the edges in $A$.

For each $j$ such that $|S_j| = 3$, we can draw the edges in $F_j$ in such a way that the number of crossings between the edges in $F_j$ and those in $A$ is equal to $2M$. For instance, if $s_{j,1}, s_{j,2} \in X_1$ and $s_{j,3} \in X_2$, by drawing all edges in $F_j^{(2)} \cup F_j^{(3)} \cup F_j^{(5)}$ above $l$ and all edges in $F_j^{(1)} \cup F_j^{(4)} \cup F_j^{(6)}$ below $l$, we can obtain an embedding with the desired number of crossings. The other cases can be treated in a similar way.

In the resultant $f$-linear embedding, there are no crossings involving any edges in $E$, or between any two edges in $F$. Furthermore, from the above argument, the number of crossings between the edges in $F$ and those in $A$ is $2M \cdot n_3$. Finally, consider the crossings among the edges in $A$. Recall that $|A| = 2n_2+9n_3$. Because the sets are split, the edges in $A$ are partitioned with at least $n$ and at most $n_2+8n_3$ edges on each side of $l$. Even if every pair of these edges on the same side form a crossing, we have at most $\binom{n}{2}+\binom{n_2+8n_3}{2} = M - 1$ crossings among them. Therefore, the total number of crossings in the embedding is less than or equal to $M \cdot (2n_3+1) - 1 = K$. □

**Theorem 2.** If Statement (II) holds for $G$, $f$ and $K$, then Statement (I) holds for $S$ and $X$.

*Proof.* Let $\overline{G}$ be an $f$-linear embedding of $G$ with $K$ or fewer crossings. Suppose that two copies $e$ and $e'$ of an edge are drawn on opposite sides of $l$ and $e$ crosses at least as many edges as $e'$. Since copies of any edge drawn on the same side cross the same number of edges, we can switch every such copy $e$ of the same edge to the other side of $l$ without increasing the number of crossings. Thus, we may assume that all copies of any edge are drawn on the same side of $l$ in $\overline{G}$. This enables us to partition $X$ into two disjoint subsets $Y_1$ and $Y_2$ such that $x_i \in Y_1$ (resp., $Y_2$) if and only if the edges in $E_i$ are drawn below (resp., above) $l$.

Let $j$ be an integer such that $|S_j| = 2$. Let $x_{i_1} = s_{j,1}$ and $x_{i_2} = s_{j,2}$. Assume that $S_j$ is entirely contained in either $Y_1$ or $Y_2$. If the edges in $F_j^{(1)}$ and $F_j^{(2)}$ lie on the same side of $l$, we immediately have $M^2$ crossings. Even if these sets are placed on opposite sides of $l$, there are $M$ or more crossings among the edges in $F_j \cup A_j \cup E_{i_1} \cup E_{i_2}$. For example, suppose that $S_j \subset Y_1$, and the edges in $F_j^{(1)}$ and $F_j^{(2)}$ are drawn above and below $l$, respectively. See Fig. 6. Since $V_{i_1}$, $V_{i_2}$ and $W_j$ are separated in the ordering $f$, the edge $(v_{i_1}^{(2)}, w_j^{(2)})$ crosses the edges in $F_j^{(1)}$ or those in $E_{i_1}$. The other cases are similarly treated.

Next, let $j$ be an integer such that $|S_j| = 3$. Since either $Y_1$ or $Y_2$ contains at least two elements of $S_j$, using the above argument twice proves that there are at least $2M$ crossings among the edges in $F_j \cup A_j \cup \bigcup_{x_i \in S_j} E_i$. Furthermore, if $S_j$ is entirely contained in either $Y_1$ or $Y_2$, by applying the same argument three times, we can show that $\overline{G}$ has $3M$ or more crossings among those edges.

Even if all $S_j$'s are split into $Y_1$ and $Y_2$, $\overline{G}$ has $2M \cdot n_3$ crossings. Moreover, if some $S_j$ is entirely contained in either $Y_1$ or $Y_2$, we have additional $M$ crossings. Since $K < M \cdot (2n_3+1)$, we know that $Y_1$ and $Y_2$ form a set splitting of $X$ with respect to $S$. □

As mentioned earlier, SET SPLITTING with sets of size at most three is NP-complete, and FIXED LINEAR CROSSING NUMBER belongs to the class NP. Therefore, from Lemma 1, Theorems 1 and 2, we have the following theorem.

**Theorem 3.** FIXED LINEAR CROSSING NUMBER is NP-complete. □

We now consider the FIXED LINEAR CROSSING NUMBER problem for the restricted case of simple graphs. The graph $G$ constructed above can easily be converted to a simple graph by splitting each vertex $v$ into $d(v)$ vertices corresponding to the incident edges, where $d(v)$ denotes the degree of $v$ in $G$. The newly created vertices each will be connected to exactly one vertex which was adjacent to $v$, and they will be placed on $l$ in such an order that their incident edges do not cross each other in any linear embedding. We can repeat this operation until the resultant graph is composed of isolated edges. It is easy to see that embeddings of $G$ and embeddings of the new graph are in an obvious one-to-one correspondence that preserves crossings. Thus, we have the following theorem.

**Theorem 4.** FIXED LINEAR CROSSING NUMBER remains NP-complete even if a given graph is simple and each of its connected components is a single edge. □

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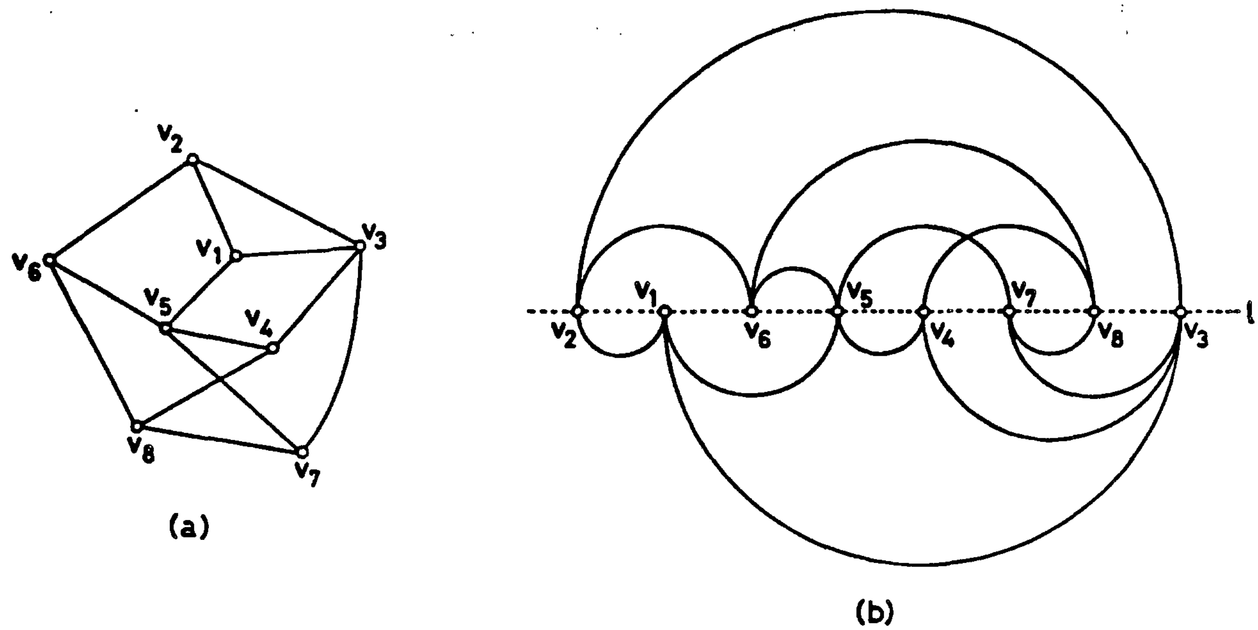


Fig. 1. (a) A simple graph $G$.
(b) A linear embedding of $G$.

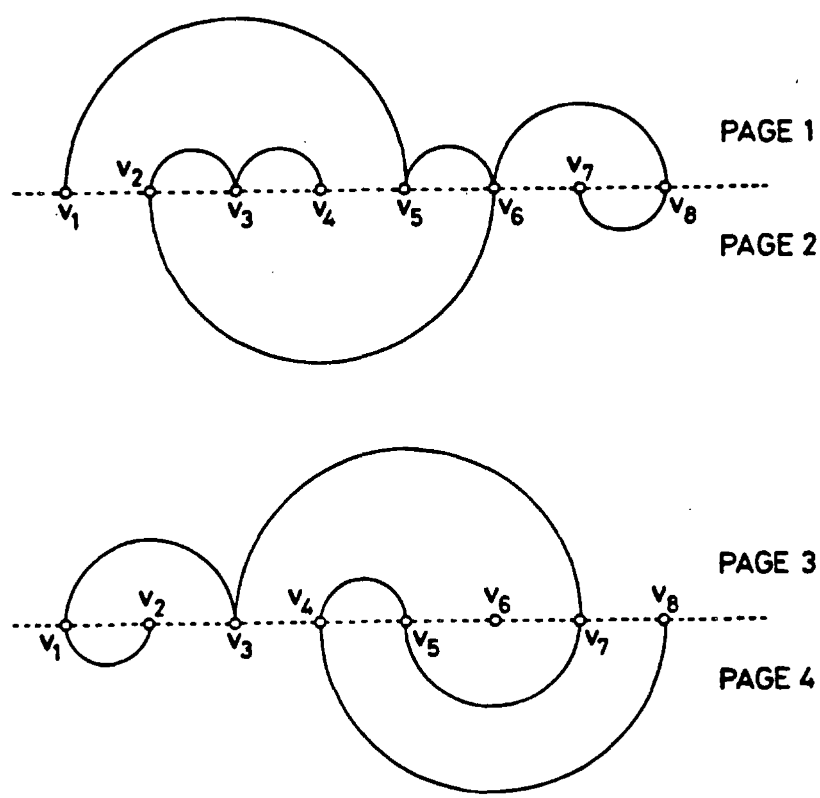


Fig. 2. A 4-page embedding of the graph shown in Fig. 1(a).

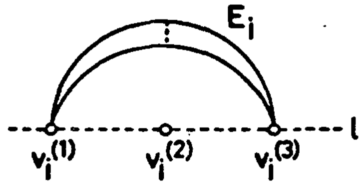


Fig. 3. A subgraph $G_i$.

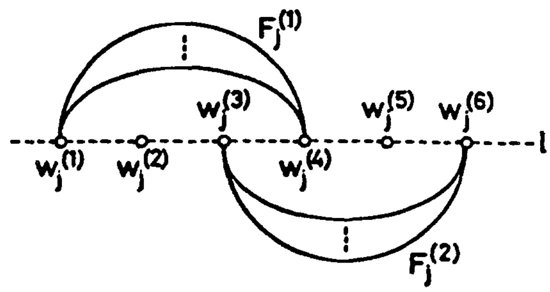


Fig. 4. A subgraph $H_j$ such that $|S_j| = 2$.

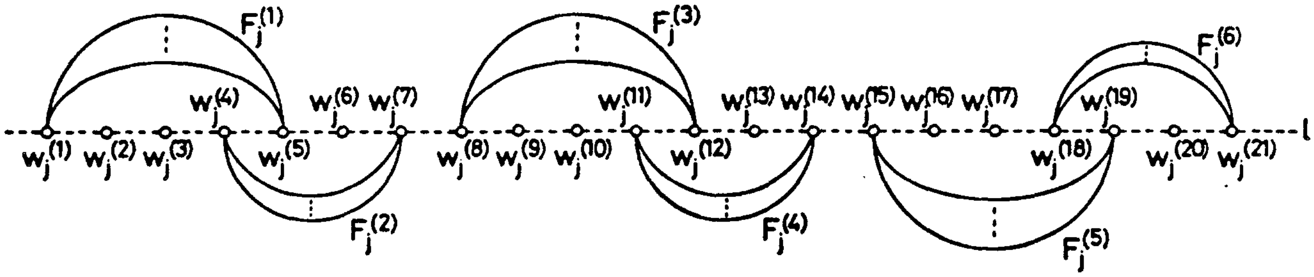


Fig. 5. A subgraph $H_j$ such that $|S_j| = 3$.

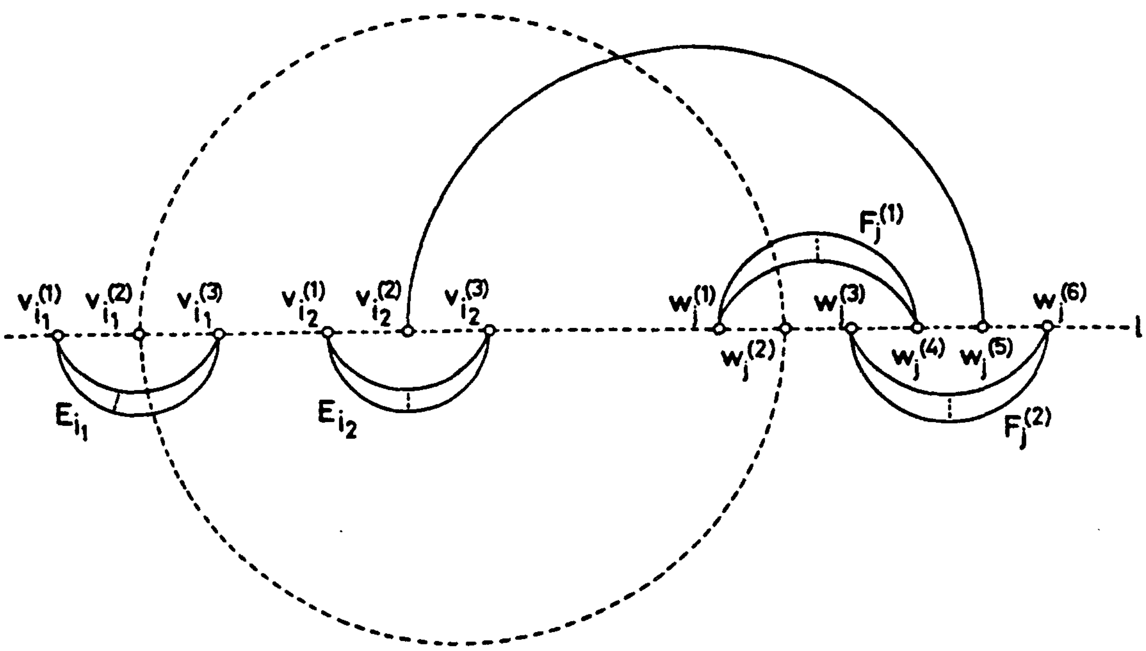


Fig. 6. An illustration for the proof of Theorem 2.