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François Bacelli and
Armand Makowski

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François Baccelli[1] and Armand M. Makowski[2]

ABSTRACT

This paper presents several relationships between the notion of *associated* random variables and notions of *stochastic ordering* which have appeared in the literature over the years. More concretely, the discussion centers around the following question: Under which conditions does the association of the $I\!R$-valued RV's $\{X_1, ..., X_n\}$ imply a possible ordering in some stochastic sense between the $I\!R^n$-valued RV $X := (X_1, ..., X_n)$ and its independent version $\overline{X} := (\overline{X}_1, ..., \overline{X}_n)$? Some of the results in that direction are as follows: (i) These $I\!R^n$-valued RV's are comparable in either one of the orderings $\leq_{st}$, $\leq_{ci}$ and $\leq_{cv}$ iff they are identical in law, and (ii) If the RV's $\{X_1, ..., X_n\}$ are associated, certain comparison properties hold for the stochastic orderings $\leq_D$, $\leq_K$ and $\leq_L$ defined in Stoyan [8, p. 27]. Strengthening of result (i) leads to the following results on the stochastic ordering properties of $I\!R^n$-valued RV's X and Y with identical mean: (j) The RV's X and Y are comparable for $\leq_{st}$ iff they are identical in law, and (jj) If $X \leq_D Y$ (resp. $X \leq_K Y$) , then X and Y are comparable for $\leq_{ci}$ (resp. $\leq_{cv}$) iff they are identical in law. These and related results are given direct applications to queueing theory and to the assymptotics of associated random variables.

[1] INRIA, Domaine de Voluceau, Rocquencourt, B.P. 105, 78153 Le Chesnay, Cedex, France. The work of this author was supported partially through a grant AT&T Bell Laboratories.

[2] Electrical Engineering Department, University of Maryland, College Park, Maryland 20742, U.S.A. The work of this author was supported partially through ONR Grant N00014-84-K-0614, partially through NSF Grant ECS-83-51836 and partially through a grant from AT&T Bell Laboratories.

1. INTRODUCTION

As amply demonstrated in the literature on system reliability, the notion of *associated* random variables is a useful one for establishing bounds on the maximum and minimum of correlated random variables. This paper explores possible relationships of this concept to various notions of *stochastic ordering* which have been given in the literature over the years. The need for establishing such connections suggested itself naturally in the work on Fork-Join systems reported by the authors in [1], where several upper bounds on the system response time statistics were obtained through a variety of methodologies. More concretely, the discussion centers around the following question (Q), where

(Q) Under which conditions does the association of the $\mathbb{R}$-valued RV's $\{X_1, ..., X_n\}$ imply a possible ordering in some stochastic sense between the $\mathbb{R}^n$-valued RV $X := (X_1, ..., X_n)$ and its independent version $\overline{X} := (\overline{X}_1, ..., \overline{X}_n)$?

In the process of answering this question, several results were obtained that indicate how multi-dimensional probability distributions are determined by conditions on their one-dimensional marginal distributions in the event of stochastic comparisons.

A few words on the notation and terminology used throughout this paper: For any two vectors x and y in $\mathbb{R}^n$, the ordering $x \leq y$ is interpreted *componentwise* and is thus equivalent to $x_i \leq y_i$ for all $1 \leq i \leq n$. A mapping $f : \mathbb{R}^n \to \mathbb{R}$ is then said to be *monotone non-decreasing* (resp. *non-increasing*) if $x \leq y$ in $\mathbb{R}^n$ implies $f(x) \leq$ (resp. $\geq$)$f(y)$; this notion of monotonicity is equivalent to *componentwise* monotonicity. A mapping $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* whenever $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all λ in the interval $[0,1]$ and every pair of vectors x and y in $\mathbb{R}^n$. In contrast with the notion of monotonicity introduced earlier, this notion of convexity does *not* reduce to componentwise convexity.

All the random variables (RV) considered in this paper are defined on some fixed probability triple $(\Omega, \mathcal{F}, P)$. Equality in distribution (or in law) between two $\mathbb{R}^n$-valued RV's is denoted by $=_{st}$. The definitions of the various stochastic orderings used here are recalled when needed and the reader is invited to consult the monographs by Ross [6] and Stoyan [8] for additional information.

Following Barlow and Proschan [3], the $\mathbb{R}$-valued RV's $\{X_1, ..., X_n\}$ are said to be *asso-*

ciated if

$$E[f(X)g(X)] \geq E[f(X)]E[g(X)] \tag{1.1}$$

for all *monotone non-decreasing* mappings $f, g : I\!R^n \to I\!R$ for which the expectations $E[f(X)], E[g(X)]$ and $E[f(X)g(X)]$ exist. The following elementary property of associated RV's [4, Thm. 5.1, pp. 1472] provided the initial impetus for the work reported here.

Lemma 1. *If the RV's* $\{X_1, ..., X_n\}$ *are associated, then the inequalities*

$$P[\ X_i > a_i,\ 1 \leq i \leq n\ ] \geq \prod_{i=1}^{n} P[\ X_i > a_i\ ] \tag{1.2a}$$

and

$$P[\ X_i \leq b_i,\ 1 \leq i \leq n\ ] \geq \prod_{i=1}^{n} P[\ X_i \leq b_i\ ] \tag{1.2b}$$

hold true for all $a = (a_1, ..., a_n)$ *and* $b = (b_1, ..., b_n)$ *in* $I\!R^n$.

An alternative and useful way of expressing (1.2) is obtained as follows: The $I\!R$-valued RV's $\{\overline{X}_1, ..., \overline{X}_n\}$ are said to form *independent versions* of the RV's $\{X_1, ..., X_n\}$ if

(i) The RV's $\{\overline{X}_1, ..., \overline{X}_n\}$ are *mutually independent* , and

(ii) For every $1 \leq i \leq n$, the RV's X_i and $\overline{X}_i$ have the *same* probability distribution.

With this notion, Lemma 1 takes the following form

Lemma 1bis. *If the RV's* $\{X_1, ..., X_n\}$ *are associated, then the inequalities*

$$P[\ X_i > a_i,\ 1 \leq i \leq n\ ] \geq P[\ \overline{X}_i > a_i,\ 1 \leq i \leq n\ ] \tag{1.3a}$$

and

$$P[\ X_i \leq b_i,\ 1 \leq i \leq n\ ] \geq P[\ \overline{X}_i \leq b_i,\ 1 \leq i \leq n\ ] \tag{1.3b}$$

hold true for all a *and* b *in* $I\!R^n$.

Upon specializing (1.3) with $a_1 = \cdots = a_n = a$ and $b_1 = \cdots = b_n = b$, it is plain that when the RV's $\{X_1, ..., X_n\}$ are associated, the inequalities

$$P[\ \min_{1 \leq i \leq n} X_i > a\ ] \geq P[\ \min_{1 \leq i \leq n} \overline{X}_i > a\ ] \tag{1.4a}$$

and

$$P[\max_{1\leq i\leq n} X_i \leq b] \geq P[\max_{1\leq i\leq n} \overline{X}_i \leq b] \quad (1.4b)$$

hold true.

Following Stoyan [8], with minor variations in the notation and terminology, the $I\!R^n$-valued RV Y is said to be ***stochastically larger*** than the $I\!R^n$-valued RV X if and only if the inequality

$$E[f(X)] \leq E[f(Y)] \quad (1.5)$$

holds for all *monotone non-decreasing* mappings $f : I\!R^n \to I\!R$ for which the expectations exist; this is denoted in short by $X \leq_{st} Y$. For the special case $n = 1$, the inequality $X \leq_{st} Y$ holds if and only if

$$P[X > t] \leq P[Y > t] \quad (1.6)$$

for all t in $I\!R$.

With this notation, it is now easy to conclude that when the RV's $\{X_1, ..., X_n\}$ are associated, the inequalities (1.4) are equivalent to the order relations

$$\min_{1\leq i\leq n} \overline{X}_i \leq_{st} \min_{1\leq i\leq n} X_i \quad (1.7a)$$

and

$$\max_{1\leq i\leq n} X_i \leq_{st} \max_{1\leq i\leq n} \overline{X}_i. \quad (1.7b)$$

These properties (1.7) above already explain the usefulness of the notion of association of RV's in a wide variety of situations, for they suggest a natural way of generating ***computable bounds*** for the statistics on the maximum and minimum of the RV's $\{X_1, ..., X_n\}$. These statistics are typically very difficult to compute in the presence of inter-variable ***correlations*** , as is the case in many stochastic models of interest [1] .

2. THE RESULTS

At this stage, the reader might wonder in view of (1.7) whether the property that the RV's $\{X_1, ..., X_n\}$ are associated, implies a possible ordering in some stochastic sense for the $I\!R^n$-valued RV's $X := (X_1, ..., X_n)$ and $\overline{X} := (\overline{X}_1, ..., \overline{X}_n)$. For instance, is it possible under some conditions that $X \leq_{st} \overline{X}$, a fact which would be compatible with (1.7b)? The main results along these lines are *negative* for all three orderings $\leq_{st}$, $\leq_{ci}$ and $\leq_{cv}$. This is the basic content of Theorems 1 and 2 below.

Theorem 1. *Assume the $\mathbb{R}$-valued RV's $\{X_1, ..., X_n\}$ to be associated, with corresponding independent versions $\{\overline{X}_1, ..., \overline{X}_n\}$. Under the foregoing assumptions, the statements (i)-(iii) below are equivalent, where*

(i) *The RV's X and $\overline{X}$ are comparable for the stochastic ordering $\leq_{st}$,*

(ii) *The RV's $\{X_1, ..., X_n\}$ are mutually independent, and*

(iii) *The RV's X and $\overline{X}$ coincide in law.*

A proof for Theorem 1 is provided in Section 4. This result is not as surprising as it may initially appear. Indeed, if the RV's X and $\overline{X}$ were comparable, say $X \leq_{st} \overline{X}$, then elementary properties of the stochastic ordering $\leq_{st}$ readily imply the inequality $\min_{1\leq i\leq n} X_i \leq_{st} \min_{1\leq i\leq n} \overline{X}_i$, since the mapping $\mathbb{R}^n \to \mathbb{R} : x \to \min_{1\leq i\leq n} x_i$ is monotone non-decreasing. The fact that the RV's $\{X_1, ..., X_n\}$ are associated now implies via (1.7a) that the equality $\min_{1\leq i\leq n} X_i =_{st} \min_{1\leq i\leq n} \overline{X}_i$ must hold, thus adding plausibility to Theorem 1.

A first extension of Theorem 1 is given in Theorem 2 below where it is shown that the stochastic ordering $\leq_{st}$ in the statement (i) can be replaced by a weaker ordering condition.

To that end, recall that the $\mathbb{R}^n$-valued RV Y is said to be *larger* than the $\mathbb{R}^n$-valued X in the *stochastic convex* (resp. *concave*) *increasing* order if (1.5) holds for all *monotone non-decreasing convex* (resp. *concave*) mappings $f : \mathbb{R}^n \to \mathbb{R}$ for which the expectations exist; this is denoted in short by $X \leq_{ci} Y$ (resp. $X \leq_{cv} Y$).

Theorem 2. *Assume the $\mathbb{R}$-valued RV's $\{X_1, ..., X_n\}$ to be associated and to have finite mean. Under the foregoing assumptions, the statements (i)-(iii) below are equivalent, where*

(i) *The RV's X and $\overline{X}$ are comparable for the stochastic ordering $\leq_{ci}$ (resp. $\leq_{cv}$),*

(ii) *The RV's $\{X_1, ..., X_n\}$ are mutually independent, and*

(iii) *The RV's X and $\overline{X}$ coincide in law.*

Note that whenever the RV's $\{X_1, ..., X_n\}$ are associated, so are the RV's $\{X_{i_1}, ..., X_{i_k}\}$, with $1 \leq i_1 < i_2 < ... < i_k \leq n$ for every $2 \leq k \leq n$. Theorems 1 and 2, for the orderings $\leq_{st}$, $\leq_{ci}$ and $\leq_{cv}$, respectively, thus state that no subvector of X is comparable to the corresponding subvector of $\overline{X}$, unless the involved components are already mutually independent.

Despite these negative results, it is noteworthy that the association of the RV's $\{X_1, ..., X_n\}$ implies certain comparison properties for the stochastic orderings $\leq_D$ and $\leq_K$ for $I\!R^n$-valued RV's defined in Stoyan [8, p. 27]. More precisely, the $I\!R^n$-valued RV Y is said to be *greater* than the $I\!R^n$-valued RV X *in the orders* $\leq_D$ and $\leq_K$, respectively, if and only if the inequalities

$$P[\, X_i \leq a_i, \ 1 \leq i \leq n \,] \geq P[\, Y_i \leq a_i, \ 1 \leq i \leq n \,] \tag{2.1}$$

and

$$P[\, X_i > b_i, \ 1 \leq i \leq n \,] \leq P[\, Y_i > b_i, \ 1 \leq i \leq n \,] \tag{2.2}$$

hold for all a and b in $I\!R^n$, respectively. This is denoted in short by $X \leq_D Y$ and $X \leq_K Y$, respectively, and as pointed out in [8, Prop. 1.10.1, p.27] and in [5, Thms 2 and 3, pp. 1314-1315], the former is equivalent to

$$E\left[\prod_{i=1}^{n} g_i(X_i)\right] \geq E\left[\prod_{i=1}^{n} g_i(Y_i)\right] \tag{2.3}$$

for every collection $\{g_1, ..., g_n\}$ of *monotone non-increasing* mappings $I\!R \to I\!R_+$, whereas the latter is equivalent to

$$E\left[\prod_{i=1}^{n} f_i(X_i)\right] \geq E\left[\prod_{i=1}^{n} f_i(Y_i)\right] \tag{2.4}$$

for every collection $\{f_1, ..., f_n\}$ of *monotone non-decreasing* mappings $I\!R \to I\!R_+$. It is easy to see from these characterizations that $X \leq_{st} Y$ implies both $X \leq_D Y$ and $X \leq_K Y$.

A simple rephrasing of the inequalities (1.3) now leads to the following comparison results.

Theorem 3. *Whenever the* $I\!R$*-valued RV's* $\{X_1, ..., X_n\}$ *are associated, both inequalities*

$$X \leq_D \overline{X} \quad \text{and} \quad \overline{X} \leq_K X \tag{2.5}$$

hold true.

Owing to Theorem 3, if the associated RV's $\{X_1, ..., X_n\}$ are all *non-negative*, then specializing (2.3) to the negative exponentials $g_i : I\!R \to I\!R_+ : t \to e^{-s_i t}$, with $0 \leq s_i$ for all $1 \leq i \leq n$, readily gives

$$E[\exp(-\sum_{i+1}^{n} s_i X_i)] \geq E[\exp(-\sum_{i=1}^{n} s_i \overline{X}_i)] = \prod_{i=1}^{n} E[\exp(-s_i X_i)]. \tag{2.6}$$

This remark can be generalized as follows [8, Defn. 1.8.1, p. 22].

Corollary 3.1. *If the RV's* $\{X_1, ..., X_n\}$ and $\{Y_1, ..., Y_n\}$ *are all non-negative with* $X \leq_D Y$*, then for any* $a = (a_1, ..., a_n)$ *in* IR^n_+*, the inequality*

$$\sum_{i=1}^{n} a_i X_i \leq_L \sum_{i=1}^{n} a_i Y_i \tag{2.7}$$

holds with $\leq_L$ *denoting the stochastic order on one-dimensional distributions induced by the Laplace transform.*

In particular, if $a_1 = \cdots = a_n = 1$, the inequality

$$\sum_{i=1}^{n} X_i \leq_L \sum_{i=1}^{n} Y_i \tag{2.8}$$

holds. Therefore, with $Y = \overline{X}$, (2.8) asserts that the sum of associated non-negative RV's is smaller in the ordering $\leq_L$ than the corresponding quantity for their independent version, a fact very much in the spirit of (1.7).

The remaining of this section is devoted to the discussion of several extensions of Theorems 1 and 2. Theorem 2 admits a first strengthening in the form of Theorem 2bis whose proof is delayed to Section 4.

Theorem 2bis. *Assume the* IR^n*-valued RV's* X *and* Y *to have finite identical means, namely*

$$E[X_i] = E[Y_i], \quad 1 \leq i \leq n \tag{2.9}$$

and to satisfy the stochastic ordering relation

$$X \leq_D Y \quad (\text{resp.} \quad Y \leq_K X). \tag{2.10}$$

Under these conditions, the statements (i) and (ii) below are equivalent, where

(i) *The RV's* X *and* Y *are comparable for the stochastic ordering* $\leq_{ci}$ *(resp.* $\leq_{cv}$*); and*

(ii) *The RV's* X *and* Y *coincide in law.*

That Theorem 2bis contains Theorem 2 can be seen as follows: Assume the RV's $\{X_1, \cdots, X_n\}$ to be associated and to have finite mean. By Theorem 3, the pair of RV's

X and $Y = \overline{X}$ satisfy the assumptions of Theorem 2bis, and Theorem 2 is obtained from it as an immediate corollary. It should be emphasized that the situation covered by Theorem 2bis is more general in several ways. Indeed, comparison in the stochastic ordering $\leq_{ci}$ (resp. $\leq_{cv}$) is assumed between two $I\!R^n$-valued RV's X and Y with the RV Y not necessarily an independent version of the RV X; the RV's $\{X_1, \cdots, X_n\}$ may not be associated and the condition that X and $\overline{Y}$ have the same one-dimensional marginal distributions is replaced by the condition that X and Y have the same mean.

The last results of this section are given in Theorems 4 and 4bis below, and consitute a strengthening of Theorem 1.

Theorem 4. *Assume the $I\!R^n$-valued RV's X and Y to have finite identical means, namely*

$$E[X_i] = E[Y_i], \quad 1 \leq i \leq n \tag{2.11}$$

Under this condition, the equality in law

$$X =_{st} Y \tag{2.12}$$

holds if and only if the RV's X and Y are comparable for the stochastic ordering $\leq_{st}$,i. e.,

$$X \leq_{st} Y \quad \text{or} \quad Y \leq_{st} X \tag{2.13}$$

holds.

Proof. In order to prove the result, it is clearly sufficient to show that (2.13) implies (2.12) under the assumption (2.11). If $X \leq_{st} Y$, then necessarily $X \leq_D Y$ and $X \leq_{ci} Y$, and the conclusion (2.12) follows from Theorem 2bis. □

Theorem 4 is to Theorem 1 for the ordering $\leq_{st}$ what Theorem 2bis is to Theorem 2 for the orderings $\leq_{ci}$ and $\leq_{cv}$, under the *finite mean* condition. This last technical restriction is now removed.

Corollary 4.1. *Assume the $I\!R^n$-valued RV's X and Y to have the same one-dimensional marginal distributions, namely*

$$X_i =_{st} Y_i \quad 1 \leq i \leq n. \tag{2.14}$$

Under this condition, the equivalence in law (2.12) holds if and only if the RV's X and Y are comparable for the stochastic ordering $\leq_{st}$.

Proof. Again, it suffices to show that (2.13) implies (2.12) under the assumption (2.14). To that end, consider the mapping $\psi : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2}) : x \to \mathrm{Arctg}\, x$, and define the $\mathbb{R}^n$-valued RV's $\tilde{X}$ and $\tilde{Y}$ componentwise by

$$\tilde{X}_i = \psi(X_i) \quad \text{and} \quad \tilde{Y}_i = \psi(Y_i) \tag{2.15}$$

for all $1 \leq i \leq n$. It is plain that the *bounded* RV's $\{\tilde{X}_1, \cdots, \tilde{X}_n\}$ and $\{\tilde{Y}_1, \cdots, \tilde{Y}_n\}$ all have a *finite* mean, a property which the original RV's may not have possessed. Moreover, the condition (2.11) is automatically satisfied for the $\mathbb{R}^n$-valued RV's $\tilde{X}$ and $\tilde{Y}$ as a consequence of (2.14).

Since the mapping ψ is monotone increasing, comparability of the RV's X and Y under the ordering $\leq_{st}$ implies comparability of the RV's $\tilde{X}$ and $\tilde{Y}$ under the same ordering. A direct application of Theorem 4 yields the equality $\tilde{X} =_{st} \tilde{Y}$, a fact which is clearly equivalent to $X =_{st} Y$ due to the strict monotonicity of the mapping ψ. □

It is also possible to give a *direct* proof of Corollary 4.1 through an argument by induction on the dimension n. The case $n = 2$ follows readily from a remark by Rüschendorf in [7, Thm. 3 , p. 344] which contains a strengthened version of Corollary 4.1 when $n = 2$. Another version of this result was also given by Mosler [5, Thm. 6, p. 1316].

Theorem 4bis. *Assume the $\mathbb{R}^2$-valued RV's $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ to have the same one-dimensional marginal distributions, namely*

$$X_1 =_{st} Y_1 \quad \text{and} \quad X_2 =_{st} Y_2. \tag{2.16}$$

Under this condition, the equivalence in law

$$X =_{st} Y \tag{2.17}$$

holds if and only if both inequalities

$$X \leq_K Y \quad \text{and} \quad X \leq_D Y \tag{2.18}$$

hold.

A comparison of the assumptions of Theorems 4 and 4bis naturally suggests the following question: Assume the RV's X and Y to have the same one-dimensional marginal distributions as in Theorem 4. For $n = 2$, Theorem 4bis provides a strengthening to Theorem 4, since the condition (2.11) of Theorem 4, originally given in the ordering $\leq_{st}$, implies the weaker condition (2.18) in the orderings $\leq_D$ and $\leq_K$. It is then tempting to ask whether Theorem 4bis holds true for all $n \geq 2$, and not merely for $n = 2$ as shown here. As to the writing of this paper, this is still an open question.

3. APPLICATIONS

Several interesting consequences of Theorems 1-4 are now presented in this section.

Bounds for Fork-Join queues

The first application is given in the context of Fork-Join (FJ) queue models which arise in many application areas, including flexible manufacturing and parallel processing. As discussed in [1], a K-dimensional FJ queue is a queueing system operated by *parallel* K servers with *synchronized* arrival streams. Each server is attended by a buffer of infinite capacity and individually operates according to the FIFO discipline. Customers arrive into the system in bulks of size K and are processed according to the following discipline: Upon arrival, a bulk of size K, bringing customers to the K servers, is immediately split and each one of the customers composing it is allocated to exactly one server (the so-called Fork primitive). As soon as all the K customers constituting a bulk have been serviced, the bulk is immediately recomposed (the so-called Join primitive) and leaves the system at once.

In this section only, the k-th component RV of any $\mathbb{R}^K$-valued RV is denoted by the same symbol as this RV but superscripted by k. The probability triple $(\Omega, \mathcal{F}, P)$ is assumed to simultaneously carry the sequences $\{\tau_{n+1}\}_0^\infty$ and $\{\sigma_n\}_0^\infty$ of $\mathbb{R}_+$-valued and $\mathbb{R}_+^K$-valued RV's,respectively.

The FJ queue of interest is then generated by the constituting sequence

$$\langle \sigma_n, \tau_{n+1}, \ n = 0, 1, ... \rangle \tag{3.1}$$

in the following way: the RV's $\{\tau_n\}_1^\infty$ model the interarrival times of bulk customers, in that arrivals to the queues are taking place along the time sequence $\{A_n\}_0^\infty$ defined by

$$A_{n+1} = \sum_{m=0}^{n} \tau_{m+1} \qquad n = 0, 1... (3.2)$$

with $A_0 = 0$. The customers which arrive at the k-th queue at such times are called type k customers hereafter. The n-th customer of type k brings an amount of processing time σ_n^k to be executed by the k-th server.

Consider the sequence of $I\!R_+^K$-valued RV's $\{W_n\}_0^\infty$ generated componentwise by the recursions

$$W_{n+1}^k = [W_n^k + \sigma_n^k - \tau_{n+1}]^+, \quad 1 \le k \le K, \qquad n = 0, 1, ... \quad (3.3)$$

with $W_0 = 0$. The RV W_n^k represents the *waiting time* of the n-th customer of type k, whereas its *response time* is the RV R_n^k given by

$$R_n^k := W_n^k + \sigma_n^k, \quad 1 \le k \le K. \qquad n = 0, 1, ... \quad (3.4)$$

The *system response times* for the n-th bulk customer is denoted by T_n and is given by

$$T_n := \max_{1 \le k \le K} R_n^k. \qquad n = 0, 1, ... \quad (3.5)$$

The difficulty of analyzing these queueing systems with synchronization comes from the fact that these K parallel channels are not independent even under standard renewal assumptions because of their common arrival pattern. It is this very lack of independence that makes the computation of the joint statistics of the RV's $\{R_n^k\}_0^\infty, 1 \le k \le K$, and thus of $\{T_n\}_0^\infty$, extremely hard. In view of these difficulties, the authors in [1] have used stochastic ordering methods to derive several *computable bounds* on these statistics. The main results are summarized in the next three propositions, where for sake of simplicity, the constituting sequence (3.1) is assumed to satisfy the standard renewal assumptions, i. e., the sequences $\{\tau_n\}_1^\infty$, $\{\sigma_n^1\}_0^\infty$, $\cdots$ $\{\sigma_n^K\}_0^\infty$ form *mutually independent* renewal sequences.

Theorem 5. *Let the RV's* $\{\underline{R}_n^k\}_0^\infty$ and $\{\underline{T}_n\}_0^\infty$ *be defined through (3.3)-(3.5) for the constituting sequence*

$$\langle \sigma_n, E[\tau_{n+1}], \ n = 0, 1, ... \rangle \quad (3.6)$$

instead of (3.1) .Under the foregoing assumptions the RV's $\{\underline{R}_n^1, \cdots, \underline{R}_n^K\}$ *are mutually independent for all* $n = 0, 1...$ *and the inequalities*

$$\underline{R}_n \le_{ci} R_n \qquad n = 0, 1, ... \quad (3.7)$$

and

$$\underline{T}_n \leq_{ci} T_n \qquad n = 0, 1, ... (3.8)$$

hold.

Theorem 6 below provides upper bounds when the input process is *divisible* in the sense that the following conditions (D.1)-(D.2) are satisfied by the renewal sequence $\{\tau_{n+1}\}_0^\infty$, where

(D.1) *There exists a sequence $\{\tilde{\tau}_{n+1}\}_0^\infty$ of $\mathbb{R}_+^K$-valued RV's such that*

$$\tau_{n+1} = \frac{1}{K}\sum_{k=1}^{K} \tilde{\tau}_{n+1}^k \qquad n = 0, 1, ... (3.9)$$

where the K sequences $\{\tilde{\tau}_{n+1}^1\}_0^\infty, \cdots, \{\tilde{\tau}_{n+1}^K\}_0^\infty$ of i.i.d. $\mathbb{R}_+$-valued RV's are mutually independent and statistically indistinguishable; and

(D.2) *The families of RV's $\{\tilde{\tau}_{n+1}\}_0^\infty$ and $\{\sigma_n\}_0^\infty$ are mutually independent.*

For $1 \leq k \leq K$, let $\{\tilde{R}_n^k\}_0^\infty$ be the response times in a GI/GI/1 queue with interarrival sequence $\{\tilde{\tau}_{n+1}^k\}_0^\infty$ and service times $\{\sigma_n^k\}_0^\infty$, and define the RV $\tilde{T}_n$ by

$$\tilde{T}_n := \max_{1\leq k\leq K} \tilde{R}_n^k. \qquad n = 0, 1, ... (3.10)$$

Theorem 6. *Under the foregoing assumptions, the RV's $\{\tilde{R}_n^1, \cdots, \tilde{R}_n^K\}$ are mutually independent for all $n = 0, 1...$, and the inequalities*

$$R_n \leq_{ci} \tilde{R}_n \qquad n = 0, 1, ... (3.11)$$

and

$$T_n \leq_{ci} \tilde{T}_n \qquad n = 0, 1, ... (3.12)$$

hold.

The association of some key RV's is now used to provide improved upper bounds on the statistics of the system response times $\{T_n\}_0^\infty$. In agreement with the notation of Section 1, the RV's $\{\overline{R_n^1}, \cdots, \overline{R_n^K}\}$ denote independent versions of the RV's $\{R_n^1, \cdots, R_n^K\}$ for all $n = 0, 1, \cdots$

Theorem 7. *Under the foregoing assumptions, the RV's $\{R_n^1, \cdots, R_n^K\}$ are associated for all $n = 0, 1, ..,$ and consequently*

$$T_n \leq_{st} \overline{T}_n := \max_{1\leq k\leq K} \overline{R_n^k}. \qquad n = 0, 1, ... (3.13)$$

The proofs of Theorems 5-7 are available in [1], together with several other more general results. In view of the developments of Section 2, several remarks are in order on Theorems 5-7. It can be shown that $\overline{T}_n \leq_{ci} \tilde{T}_n$ for all $n = 0, 1, \cdots$, i. e., Theorem 7 is an improvement on the upper bound provided by Theorerm 6 for the *system response times* statistics. However, Theorems 1 and 2 show that the bounds obtained in Theorems 5 and 6 are the only ones that hold in the *vector* sense. More precisely, there is no possible vector ordering relation between R_n and $\overline{R}_n$ like (3.7) or (3.11), under either one of the orderings $\leq_{st}$ and $\leq_{ci}$.

Bounds on the tail behavior of the maximum of associated RV's

The tail behavior of an $\mathbb{R}$-valued RV X is said to be *exponentially bounded* of parameter $\alpha > 0$ if

$$P[X > t] = o(e^{-\alpha t}) \tag{3.14}$$

when t goes to $+\infty$. It is then natural to say that the RV X has an *exponentially bounded* tail behavior if its tail behavior is *exponentially bounded* for some parameter $\alpha > 0$.

Theorem 8. *Assume the $\mathbb{R}$-valued RV's $\{X_1, \cdots, X_n\}$ to be associated. The tail behavior of the RV $\max_{1 \leq i \leq n} X_i$ is exponentially bounded if and only if the RV's $\{X_1, \cdots, X_n\}$ all have an exponentially bounded tail behavior. If the RV X_i has an exponentially bounded tail of parameter α_i for all $1 \leq i \leq n$, then the RV $\max_{1 \leq i \leq n} X_i$ has an exponentially bounded tail of parameter $\alpha = \min_{1 \leq i \leq n} \alpha_i$.*

Proof. These results are immediate consequences of the inequalities

$$P[X_i > t] \leq P[\max_{1 \leq i \leq n} X_i > t] \leq 1 - \prod_{i=1}^{n} P[X_i \leq t] \tag{3.15}$$

for all t in $\mathbb{R}$ and all $1 \leq i \leq n$. The first inequality follows from the fact that $X_i \leq \max_{1 \leq i \leq n} X_i$ for all $1 \leq i \leq n$, whereas the second one is a mere rephrasing of (1.7b). □

A fact on monotone functions

An interesting and somewhat unexpected by-product to the comparison results obtained so far is now given. Denote by $\mathcal{M}_n$ (resp. $\mathcal{M}_n^+$) the collection of all *monotone non-decreasing* mappings $\mathbb{R}^n \to \mathbb{R}$ (resp. $\mathbb{R}^n \to \mathbb{R}_+$). Moreover, let $\mathcal{P}_n^+$ denote the collection of product-form elements in $\mathcal{M}_n^+$ with factors in $\mathcal{M}_1^+$, i. e., an element f of $\mathcal{M}_n^+$ belongs to $\mathcal{P}_n^+$ if and only f it has the form

$$f = \prod_{i=1}^{n} f_i \tag{3.16}$$

with f_i in $\mathcal{M}_1^+$ for all $1 \le i \le n$.

Theorem 9. *The collection of mappings $\mathcal{P}_n^+$ cannot generate the collection $\mathcal{M}_n^+$ through (positive) linear combinations and through limiting operations .*

To the best of the authors' knowledge, there does not seem to be any direct proof in the literature for establishing this result.

Proof. Let $\{X_1, \cdots, X_n\}$ be a set of associated RV's with the property that the RV X is not identical in law with the independent version $\overline{X}$. Note that Fork-Join queue systems provide a rich class of examples of such non-trivial collections of associated RV's.

Therefore, Theorem 3 asserts that $X \le_K \overline{X}$ or equivalently that

$$E[f(X)] \le E[f(\overline{X})] \tag{3.17}$$

for *every* mapping f in $\mathcal{P}_n^+$. On the other hand, it follows from Theorem 1 that the RV's X and $\overline{X}$ are not comparable under the ordering $\le_{st}$ since not identical in law, and a simple argument by contradiction readily shows that there must exists an element g in $\mathcal{M}_n^+$ (and not merely in $\mathcal{M}_n$) such that

$$E[g(X)] > E[g(\overline{X})]. \tag{3.18}$$

Details are left to the interested reader and the proof of Theorem 9 is now completed upon combining (3.17) and (3.18). □

4. PROOFS

Auxiliary results

The following technical facts will be used in the proofs of Theorems 1 and 2bis.

Proposition 1. *The inequalities (1.3) hold in equality form for all a (resp. b) in $\mathbb{R}^n$ if and only if the RV's $\{X_1, ..., X_n\}$ are mutually independent.*

Proposition 2. *If the non-negative RV's X and Y have finite identical means, then the ordering $X \le_{ci} Y$ holds if and only if the inequality*

$$E[h(X)] \le E[h(Y)] \tag{4.1}$$

holds for all convex mappings $h : I\!R^+ \to I\!R$.

Proposition 1 is a simple rephrasing of the notion of independence and is given for easy reference. The characterization of the ordering $\leq_{ci}$, given in Proposition 2, can be found in Corollary 8.5.2 of [6, p. 271].

Note that the ordering $\leq_{ci}$ has the ***anti-symmetry*** property on the collection of finite mean probability distributions with support in $I\!R^+$ [8, p. 8]. Therefore, *strict* inequality in (4.1) for *some convex* mapping $h : I\!R^+ \to I\!R$ leads to the conclusion that the RV's X and Y do *not* coincide in law, and consequently there must exist *at least one* convex non-decreasing mapping $h : I\!R^+ \to I\!R$ for which the expectations exist and which gives (1.4) in *strict* inequality form.

Proof of Theorem 1

Since the RV's $\{X_1, ..., X_n\}$ are associated, it follows readily from Lemma 1bis and Proposition 1 that the statements (ii) and (iii) are equivalent, and it thus suffices to show that (i) and (ii) are equivalent. Note that if (ii) holds, then by (iii), the RV's X and $\overline{X}$ are trivially comparable for the stochastic ordering $\leq_{st}$ owing to its antisymmetry property and (i) thus holds. The proof of the second equivalence will be completed if (i) is shown to imply (ii). Although this last implication is an immediate consequence of Corollary 4.1, a direct argument of independent interest is now provided.

Assume that the RV's X and $\overline{X}$ are comparable for the stochastic ordering $\leq_{st}$, say $X \leq_{st} \overline{X}$ for the sake of discussion. This implies that $X \leq_K \overline{X}$ whereas the assumed association of the RV's $\{X_1, ..., X_n\}$ implies $\overline{X} \leq_K X$ by Theorem 3. Therefore, $X =_K \overline{X}$ or equivalently (1.3b) holds with equality for all $b = (b_1, \cdots, b_n)$ in $I\!R^n$, and the conclusion $X =_{st} \overline{X}$ is immediate. In the event the RV's X and $\overline{X}$ were comparable for the stochastic ordering $\leq_{st}$ with $\overline{X} \leq_{st} X$, a similar argument using the first inequality in (2.5) would yield the result.

$\square$

Proof of Theorem 2bis.

It is plain that $X \leq_D Y$ if and only if $-Y \leq_K -X$, whereas the comparability of the RV's X and Y under the ordering $\leq_{ci}$ is equivalent to the comparability of the RV's $-X$ and $-Y$ under the ordering $\leq_{cv}$. Consequently, it is sufficient to prove the part of Theorem 2bis related to the ordering $\leq_{ci}$ for the part related to the ordering $\leq_{cv}$ will follow by changing X (resp. Y) in $-X$ (resp. $-Y$).

Condition (ii) readily implies (i) and the proof of Theorem 2bis will be completed if it can be shown that (i) implies (ii). This will be done in two steps.

Step 1: The desired implication is first established in the particular case where the RV's $\{X_1, ..., X_n\}$ and $\{Y_1, ..., Y_n\}$ are all non-negative.

In order to show that (i) implies (ii), the RV's X and Y are assumed to be comparable under the ordering $\leq_{ci}$ without being identical in law for otherwise, the result would be trivially true. This last assumption, the ordering $X \leq_D Y$ and Corollary 3.1 combine to imply the existence of $s \neq 0$ in $I\!R^n_+$ such that the *strict* inequality

$$E[\exp(-\sum_{i=1}^{n} s_i X_i)] > E[\exp(-\sum_{i=1}^{n} s_i Y_i)] \tag{4.2}$$

holds. With the RV's U and V defined by

$$U := \sum_{i=1}^{n} s_i X_i \quad \text{and} \quad V := \sum_{i=1}^{n} s_i Y_i, \tag{4.3}$$

the inequality (2.6) takes the form

$$E[h(U)] > E[h(V)] \tag{4.4}$$

where the mapping $h : I\!R^+ \to I\!R : t \to \exp(-t)$ is clearly *convex*.

Since the RV's X and Y are assumed comparable under $\leq_{ci}$, so are the RV's U and V and direction of the comparison must necessarily be the one between the RV's X and Y. To determine the direction of this comparison, note that the RV's U and V are *non-negative* , with *identical* and *finite* means by virtue of (2.9). Consequently, Proposition 2 asserts that $V \leq_{ci}$ (resp. $\geq_{ci}$) U if and only if the inequality

$$E[h(V)] \leq (\text{ resp. } \geq) E[h(U)] \tag{4.5}$$

holds for all *convex* mappings $h : I\!R^+ \to I\!R$ for which the expectations exist. The *strict* inequality (4.4) and the remarks following Proposition 2 now imply that necessarily

$$V \leq_{ci} U, \tag{4.6}$$

with the RV's U and V *not* coinciding in law.

The comparability of the RV's X and Y in the stochastic ordering $\leq_{ci}$, previous remarks and the inequality (4.6) lead to the conclusion that the inequality

$$Y \leq_{ci} X \tag{4.7}$$

must hold. In turn this implies that the relation

$$\max_{1 \leq i \leq n} (Y_i - b_i) \leq_{ci} \max_{1 \leq i \leq n} (X_i - b_i). \tag{4.8}$$

holds for all $b = (b_1, \cdots, b_n)$ in $I\!R^n$, since the mapping $I\!R^n \to I\!R : x \to \max_{1 \leq i \leq n}(x_i - b_i)$ is convex and monotone non-decreasing.

On the other hand, the relation $X \leq_D Y$ readily implies via (2.1) that

$$P[\max_{1 \leq i \leq n} (X_i - b_i) \leq t\ ] \geq P[\max_{1 \leq i \leq n} (Y_i - b_i) \leq t\ ] \tag{4.9}$$

for all t in $I\!R$ and all b in $I\!R^n$. This last family of inequalities is equivalent to

$$\max_{1 \leq i \leq n} (Y_i - b_i) \geq_{st} \max_{1 \leq i \leq n} (X_i - b_i) \tag{4.10}$$

and since the ordering $\leq_{st}$ is stronger than the ordering $\leq_{ci}$, the inequality

$$\max_{1 \leq i \leq n} (Y_i - b_i) \geq_{ci} \max_{1 \leq i \leq n} (X_i - b_i). \tag{4.11}$$

immediately follows for all b in $I\!R^n$. Upon combining (4.8) and (4.11), the equality

$$\max_{1 \leq i \leq n} (Y_i - b_i) =_{st} \max_{1 \leq i \leq n} (X_i - b_i). \tag{4.12}$$

is obtained for *all* b in $I\!R^n$, or equivalently $X =_{st} Y$, a conclusion which contradicts the assumption that (ii) does not hold. In short, (i) implies (ii) and the proof of Theorem 2bis is now complete in the case of non-negative RV's.

Step 2: To treat the case of general $I\!R^n$-valued RV's, consider the mapping $\phi : I\!R \to I\!R^+$ given by

$$\phi(x) = \begin{cases} \exp(x) & \text{if } x \leq 0; \\ x + 1 & \text{if } x \geq 0 \end{cases} \tag{4.13}$$

and define the $\mathbb{R}^n_+$-valued RV's $\tilde{X}$ and $\tilde{Y}$ componentwise by

$$\tilde{X}_i = \phi(X_i) \quad \text{and} \quad \tilde{Y}_i = \phi(Y_i) \tag{4.14}$$

for all $1 \leq i \leq n$. The finite mean assumption on the RV's $\{X_1, \cdots, X_n\}$ (resp. $\{Y_1, \cdots, Y_n\}$) implies that the RV's $\{\tilde{X}_1, \cdots, \tilde{X}_n\}$ (resp. $\{\tilde{Y}_1, \cdots, \tilde{Y}_n\}$) also have finite mean. The mapping ϕ being monotone increasing and convex, it is easy to check that the relation $X \leq_D Y$ implies the relation $\tilde{X} \leq_D \tilde{Y}$ and that the RV's X and Y are comparable in the stochastic ordering $\leq_{ci}$ whenever the RV's $\tilde{X}$ and $\tilde{Y}$ are assumed comparable in this ordering. Under the comparability condition (i), the first part of the proof immediately implies that the RV's $\tilde{X}$ and $\tilde{Y}$ coincide in law, but this is equivalent to the RV's X and Y coinciding in law since the mapping ϕ is strictly monotone, thus invertible. In other words, (i) implies (ii) and the proof of Theorem 2bis is completed. □

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