Feedback Control for an Abstract Parabolic Equation

by

R. Rostamian, T.I. Seidman, and T. Nambu

FEEDBACK CONTROL FOR AN ABSTRACT PARABOLIC EQUATION

Rouben Rostamian[1]   and   Thomas I. Seidman[2]

Department of Mathematics

University of Maryland Baltimore County

Catonsville, Maryland 21228, U.S.A.

Takao Nambu[3]

Department of Mathematics

Kumamoto University

Kumamoto, Japan

1. Introduction

In our previous paper [6] we considered the problem of the exponential stabilization of the heat equation with Neumann boundary conditions

$$\frac{\partial u}{\partial t} = \Delta u \qquad \text{in } \Omega \times (0, \infty)$$

(1.1)
$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0, \infty)$$

$$u(x, 0) = u_0(x) \qquad x \text{ in } \Omega$$

in the smooth and bounded domain Ω in $\mathbf{R}^n$, where $\partial/\partial\nu$ denotes the differentiation in the direction normal to the boundary $\partial\Omega$. The solutions of this problem, although stable (in the sense of Liapunov, considered for any L^p space, say) are not asymptotically stable, because spatially homogeneous functions do not decay. A practical way to induce asymptotic stability is to introduce a feedback mechanism which observes the temperature at a patch of the boundary $\partial\Omega$ and causes an appropriate exchange of heat through the boundary or through an interior subset of Ω. A feedback mechanism of the latter type may be modeled by introducing a term of the form

(1.2)
$$\Phi(x, t) := -\epsilon \left(\int_{\partial\Omega} u(y, t)\varphi(y)\, dy \right) \sigma(x)$$

on the right hand side of the equation in (1.1). The functions φ and σ, prescribed on $\partial\Omega$ and Ω, respectively, are indicators of the sites where the observation and the feedback take place. The coefficient ϵ is the 'gain factor' of the feedback. Feedback mechanisms of the former type may also be treated by combining the method of section 6 of [6] and the estimates developed in the present paper.

More generally, we may consider a more complex diffusion processs governed by a general (autonomous) linear elliptic operator A, and several feedback mechanisms of the type above that act simultaneously and independently, so that the governing equation becomes

(1.3)
$$\frac{\partial u}{\partial t} = Au - \epsilon \sum_{k=1}^{p} \sum_{j=1}^{q} \gamma_{jk} \left(\int_{\partial\Omega} u(y, t)\varphi_k(y) \right) \sigma_j(x).$$

For instance,
$$Au = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}$$

may correspomd to a heat conduction problem in an anisotropic and inhomogeneous medium.

The method introduced in [6] for the stability analysis extends to this and more general situations. In this paper we study an abstract evolution equation which includes the problem in [6] and the problem corresponding to equation (1.3) as special cases. Our main result, in the context of the latter problem is, roughly, that if certain hypotheses are satisfied and if ϵ is sufficiently small then the system is exponentially asymptotically stable. Studies of similar kind, albeit with quite more complicated sufficient stability conditions, appear in various places in the literature; we refer the reader to the articles in [1], [5], [7], and [8].

The smallness of the feedback gain factor ϵ is essential for stability. In [6] we give an example where for a large ϵ the solution grows exponentially as $t \to \infty$. For a comprehensive study of the spectral properties and the spectral resolution of elliptic operators of the type occuring in (1.3) see the recent article of van Harten[3].

2. The Abstract Problem

Let A be a sectorial operator (cf. [4]) on a Hilbert space H, and denote by e^{tA} the analytic semigroup generated by A. Suppose that the null-space H_0 of A is finite-dimensional and non-trivial, and that the following hypotheses hold:

Hypotheses on A:

(i) There exists a continuous projection operator Q, not necessarily orthogonal, of H onto H_0, which commutes with A; thus $AQ = QA = 0$.

(ii) If $P = I - Q$, where I is the identity mapping in H, and $H_1 = \text{range}(P) = \text{null space}(Q)$, then there exist positive constants a and α such that

$$(2.1) \qquad \left| e^{tA} v \right| \le a e^{-\alpha t} |v|, \qquad v \in H_1.$$

Thus, in the dynamical system corresponding to e^{tA}, all orbits in H_1 are asymptotically exponentially stable. The points in H_0, on the other hand, are equilibria and remain stationary. We now introduce a feedback process, motivated by the concrete application in [6], which makes *all* orbits in H tend exponentially to zero as $t \to \infty$.

We suppose we are given q elements $\{\sigma_1, \cdots, \sigma_q\}$ of H and p linear (possibly unbounded) functionals $\{l_1, \cdots, l_p\}$ on H, for which there exists a $p \times q$ real matrix γ_{jk} such that the feedback operator B, formally defined by

$$(2.2) \qquad B(u) = -\sum_{k=1}^{p} \sum_{j=1}^{q} \gamma_{jk} l_k(u) \sigma_j,$$

satisfies the following hypotheses:

Hypotheses on B:

(i) For each $t > 0$ and $k = 1, 2, \cdots, p$, the expression $l_k\left(e^{tA}\right)$ defines a *bounded* linear functional on H.

(ii) There exists a $\delta \in (0, 1)$ such that

$$(2.3) \qquad \left| l_k\left(e^{tA}u\right) \right| \le a t^{-\delta} e^{-\alpha t} |u|, \qquad u \in H_1,$$

and

$$(2.4) \qquad |l_k(u)| \le a|u|, \qquad u \in H_0.$$

(iii) If we define the operator $B_0 : H_0 \to H_0$ by

$$(2.5) \qquad B_0 w = -\sum_{k=1}^{p} \sum_{j=1}^{q} \gamma_{jk} l_k(w) Q \sigma_j, \qquad w \in H_0,$$

then there exist positive constants b and β such that

$$(2.6) \qquad \left| e^{tB_0} w \right| \le b e^{-\beta t} |w|, \qquad w \in H_0.$$

Remark: The inequalities (2.4) and (2.6) imply the estimate

$$(2.7) \qquad \left| l_k\left(e^{tB_0} w\right) \right| \le a b e^{-\beta t} |w|, \qquad w \in H_0.$$

Note that, formally, $B_0 = QBQ$, and that the operator Be^{tA} is bounded on H for each $t > 0$. In view of this we are able to show that the equation

$$(2.8) \qquad \begin{aligned} \dot{u} &= Au + \epsilon Bu \\ u(0) &= u_0 \end{aligned}$$

makes sense in H. We prove the

Stabilization Theorem: *There exists a number $\epsilon_0 > 0$, depending on the constants $a, \alpha, b, \beta, \delta$ entering in estimates in* (2.1), (2.3), *and* (2.6) *as well as on p, q, γ_{jk}, and the norms of the σ_j's, such that, for any $\epsilon \in (0, \epsilon_0)$, the solution of* (2.8) *decays exponentially as $t \to \infty$.*

We defer the proof to the next section but first make some remarks:

(i) It is possible to obtain an *explicit* upper bound ϵ_0 in terms of the quantities mentioned above. The computation proceeds along the lines described in section 4 of [6].

(ii) It is worthwhile to observe that in our computation we use the estimates in (2.1), (2.3), (2.4), (2.6) very sparingly. In fact, the bulk of the computation in the following section, up to the derivation of the system of integral equations (3.7), is in terms of equalities rather than estimations. This has the advantage of keeping the results as sharp as possible; moreover, the general theorem on integral equations developed in [6] applies here directly.

(iii) The inequality in (2.1) applies in situations where the initial condition is in the space H. This assumption is of course unnecessarily strong; in concrete cases the initial conditions are often allowed to be outside H. For instance, H may be $L^2(\Omega)$ whereas u_0 is a distribution. The regularizing effect of the analytic semigroup puts the orbit in H in positive times. The estimate (2.1) can be modified to apply to such irregular data; see [6] for details.

(iv) The estimate in (2.6) is an assumption of the positivity of the operator $-B_0$ on the space H_0. One might also consider a possible order-preserving property of $-B_0$, in the form

$$w \geq 0 \;\; \Rightarrow \;\; -B_0 w \geq 0, \quad \forall w \in H_0$$

where the inequalities are understood in the sense of a partial order on H for which one has a maximum principle for A. It seems likely, but we have not verified this, that under this stronger condition the equation (2.8) is asymptotically stable with no restrictions on the magnitude of the coefficient ϵ. See [6] for further discussion and practical impications.

(v) It is possible to show, although we do not do it here, that the operator $A + \epsilon B$ entering in (2.8) is sectorial for arbitrary ϵ. Hence it generates an analytic semigroup on H.

3. The Proof of the Stabilization Result

We apply the projections P and Q to the evolution equation (2.8) and use the notation $v = Pu$, $w = Qu$, to arrive at the following coupled system of differential equations:

$$(3.1) \qquad \dot{v} = Av - \epsilon \sum_{k=1}^{p} \sum_{j=1}^{q} \gamma_{jk} l_k(u) P\sigma_j$$

$$(3.2) \qquad \dot{w} = -\epsilon \sum_{k=1}^{p} \sum_{j=1}^{q} \gamma_{jk} l_k(u) Q\sigma_j$$

We also set $v_0 = v(0) = Pu(0)$ and $w_0 = w(0) = Qu(0)$. Note that the equation (3.2) may be re-written, in view of the definition of the operator B_0 in (2.5) and the identity $u = v + w$, as:

$$\dot{w} + \epsilon B_0 w = -\epsilon \sum_{k=1}^{p} \sum_{j=1}^{q} \gamma_{jk} l_k(v) Q\sigma_j.$$

Applying the variation of constants formula, we get

$$(3.3) \qquad v(t) = e^{tA} v_0 - \epsilon \sum_{k=1}^{p} \sum_{j=1}^{q} \int_0^t \gamma_{jk} \left[e^{(t-s)A} P\sigma_j \right] l_k(v(s) + w(s))\, ds,$$

$$(3.4) \qquad w(t) = e^{\epsilon t B_0} w_0 - \epsilon \sum_{k=1}^{p} \sum_{j=1}^{q} \int_0^t \gamma_{jk} \left[e^{\epsilon(t-s)B_0} Q\sigma_j \right] l_k(v(s))\, ds.$$

Now define

$$\rho_i(t) = l_i(v(t)),$$

$$\lambda_{ik}(t) = \sum_{j=1}^{q} \gamma_{jk} l_i \left(e^{tA} P \sigma_j \right),$$

$$\eta_{ik}(t) = \sum_{j=1}^{q} \gamma_{jk} l_i \left(e^{\epsilon t B_0} Q \sigma_j \right),$$

and apply l_i to both of equations (3.3) and (3.4) to obtain

$$\rho_i(t) = l_i \left(e^{tA} v_0 \right) - \epsilon \sum_{k=1}^{p} \int_0^t \lambda_{ik}(t-s) \Big(\rho_k(s) + l_k(w(s)) \Big) \, ds,$$

$$l_i(w(t)) = l_i \left(e^{\epsilon t B_0} w_0 \right) - \epsilon \sum_{k=1}^{p} \int_0^t \eta_{ik}(t-s) \rho_k(s) \, ds.$$

Substitute the second of these equations in the first, interchange the order of integration in the resulting double-integral (justified by the estimates in (2.1) and (2.3),) and simplify, to arrive at

$$\rho_i(t) = l_i \left(e^{tA} v_0 \right) - \epsilon \sum_{k=1}^{p} \int_0^t \lambda_{ik}(t-s) l_k \left(e^{\epsilon s B} w_0 \right) ds$$

$$- \epsilon \sum_{k=1}^{p} \int_0^t \lambda_{ik}(t-s) \rho_k(s) \, ds$$

(3.5) $$+ \epsilon^2 \sum_{k,k'=1}^{p} \int_0^t \int_0^s \lambda_{ik}(t-s) \eta_{kk'}(s-\tau) \rho_{k'}(\tau) \, d\tau \, ds.$$

Further simplification is achieved by letting $\rho_i^*(t)$ denote the sum of the first two terms on the right hand side of (3.5) (i.e., the terms which contain the initial data) and setting

(3.6) $$\mu_{ij}(t) = \lambda_{ij}(t) - \epsilon \sum_{k=1}^{p} \int_0^t \lambda_{ik}(r) \eta_{kj}(t-r) \, dr.$$

After a bit of manipulation, (3.5) then yields the following system of integral equations:

(3.7) $$\rho_i(t) = \rho_i^*(t) - \epsilon \sum_{j=1}^{p} \int_0^t \mu_{ij}(t-\tau) \rho_j(\tau) \, d\tau$$

for $i = 1, 2, \cdots, p$.

It is easy to see that by the assumption $0 < \delta < 1$ in (2.3), the functions ρ_i^* and μ_{ik} have *integrable* singularities, hence the system of Volterra equations (3.7) has a unique solution. By a computation similar to the one in [6], it is possible to show that this solution decays exponentially as $t \to \infty$, if $\epsilon > 0$ is sufficiently small. Going

back to (3.4), recalling that $l_k(v(t)) = \rho_k(t)$, and using the estimations in (2.1) and (2.3), we conclude that the norm of $w(t)$ decays. Now since

$$l_k(u(t)) = l_k(v(t)) + l_k(w(t)) = \rho_k(t) + l_k(w(t))$$

we get exponential decay for $l_k(u(t))$. Finally, using this in (3.3) we obtain exponential decay for $v(t)$ and, consequently, for $u(t) = v(t) + w(t)$. This completes the proof of the stabilization result.

Remark: A variant of the control problem treated in this paper is that of the boundary observation and *boundary control*. In this situation the feedback mechanism involves interaction with Ω entirely through the boundary of Ω. The corresponding mathematical model is:

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) \qquad \text{in } \Omega \times (0, \infty)$$

$$\frac{\partial u}{\partial \nu} = -\epsilon \sum_{j,k}^{p} \gamma_{jk} \left(\int_{\partial\Omega} u(y,t)\varphi_k(y)\, dy \right) \sigma_j \qquad \text{on } \partial\Omega \times (0, \infty)$$

$$u(x,0) = u_0(x) \qquad x \in \Omega.$$

Abstractly, this corresponds to having $e^{tA}\sigma_j \in H$ for $t > 0$ rather than $\sigma_j \in H$; we must then assume an estimate like (2.3) for $\left| e^{tA}\sigma_j \right|$, corresponding to having $\sigma_j \in \mathcal{D}(A^\delta)$. It is possible to show that this system is also asymptotically exponentially stable provided that the coefficient ϵ is sufficiently small. The proof is along the lines of the argument in section 6 of [6], combined with the generalized frmework that we have developed in the resent paper.

References:

1. R. F. Curtain, Finite dimensional compensators for parabolic distributed systems with unbounded control and observations, SIAM J. Control Optim. **22** (1984), pp. 255–276.

2. D. Fujiwara, Concrete characterization of the domains of fractional powers of some elliptic differential operators of second order, Proc. Japan Acad. **43**(1967), pp. 82–86.

3. A. van Harten, On the spectral properties of a class of elliptic functional differential operators arising in feedback control theory for diffusion processes, SIAM J. Math. Anal. **17**(1986), pp. 352–383.

4. D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, no. 840, Springer-Verlag, 1981.

5. T. Nambu, On the stabilization of diffusion equations: Boundary observation and feedback, J. Diff. Euations **52**(1984), pp. 204–233.

6. T. Nambu, R. Rostamian and T. I. Seidman, Feedback stabilization of a marginally stable heat conduction problem, to appear in Applicable Analysis.

7. Y. Sakawa and T. Matsushita, Feedback stabilization of a class of distributed system and construction of a state estimator, IEEE Trans. Automatic Control AC-20(1975), pp. 748–753.

8. Y. Sakawa, Feedback stabilization of linear differential systems, SIAM J. Control Optim. **21**(1983), pp. 667–676.